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Mapping hierarchy for dendrites

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Abstract

Let a family \mathcal{S} of spaces and a class \mathbb{F} of mappings between members of \mathcal{S} be given. For two spaces X and Y in \mathcal{S} we define $Y \leq_{\mathbb{F}} X$ if there exists a surjection $f \in \mathbb{F}$ of X onto Y . We investigate the quasi-order $\leq_{\mathbb{F}}$ in the family of dendrites, where \mathbb{F} is one of the following classes of mappings: retractions, monotone, open, confluent or weakly confluent mappings. In particular, we investigate minimal and maximal elements, chains and antichains in the quasi-order $\leq_{\mathbb{F}}$, and characterize spaces which can be mapped onto some universal dendrites under mappings belonging to the considered classes.

1. Introduction

Two spaces are topologically different if they are not homeomorphic, i.e., all homeomorphic spaces are identified from the topological point of view. However, the difference between two nonhomeomorphic spaces can be measured in many various ways. One of the possible methods is to consider the behaviour of the spaces with respect to a given class of mappings.

The idea of classification of topological spaces from the point of view of mapping theory is certainly not new. It can be considered as a continuation of the concept of Felix Klein presented in 1872 and known as the Erlangen Program. The reader can find various examples of such approach in the literature. In particular, K. Borsuk in [5] (and later in several other papers, in particular in [6]) developed this idea, applying it to classify spaces with respect to r -mappings. We use the same method, but consider other classes of mappings.

We restrict our attention to a rather narrow family of curves, namely to dendrites. Their structural as well as mapping properties were extensively studied in the thirties, and numerous important results were obtained then. However, many interesting and important problems remain open.

After some preliminaries, a hierarchy of spaces from the standpoint of theory of mappings is presented in the third chapter. Its contents can be considered as a research program, and can be applied not only to dendrites, as in the present paper, but also to various families of topological spaces as well as to various classes of mappings between them. In particular, mapping hierarchy of locally connected metric continua seems to be a nice area for further study, and other classes of mappings, larger than those discussed in the present paper, should be taken into consideration.

In the fourth chapter, several theorems concerning the structure of dendrites and their behaviour under some special mappings are collected. In particular, basic properties of universal dendrites are either recalled or proved.

Chapters 5 and 6 contain the main results of the paper. The study of monotone mappings between dendrites is the main subject of the fifth chapter. Furthermore, in that chapter we also discuss problems regarding confluent mappings and r -mappings. Some results concern other classes of mappings, e.g. weakly confluent ones. The sixth chapter is devoted to open mappings.

Unsolved problems are recalled at the end of the paper.

2. Preliminaries

All *spaces* considered in this paper are assumed to be metrizable and separable. Since each such space is embeddable in the Hilbert cube, one can assume that all spaces under consideration are subsets of this cube. Given a subset A of a space X , we denote by $\text{cl } A$ the closure, by $\text{bd } A$ the boundary, and by $\text{int } A$ the interior of A in X . A *compactum* means a compact metric space, and a *continuum* means a connected compactum. A property of a continuum is said to be *hereditary* if every subcontinuum of the continuum has the property. In particular, a continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected.

A family of subsets of a metric space X is said to be a *null-family* if for any $\varepsilon > 0$ at most a finite number of members of the family have diameter greater than ε . In particular, a sequence of subsets of X is a *null-sequence* if the diameters of its members tend to zero.

A *mapping* means a continuous function. In this paper we do not consider constant mappings: if a mapping $f : X \rightarrow Y$ is surjective, then Y is nondegenerate. A surjective mapping $f : X \rightarrow Y$ is said to be:

- *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$;
- *open* if the images of open sets under f are open;
- *confluent* if for each subcontinuum Q in Y each component of $f^{-1}(Q)$ maps onto Q under f ;
- *weakly confluent* if for each subcontinuum Q in Y some component of $f^{-1}(Q)$ maps onto Q under f ;
- *light* if $f^{-1}(y)$ has one-point components for each $y \in Y$ (note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional).

Obviously, each monotone mapping is confluent, each confluent mapping is weakly confluent, and (see [37], Theorem 7.5, p. 148) open mappings of compact spaces are confluent.

A mapping $f : X \rightarrow Y$ is said to be *interior at* $x \in X$ if for every open set U in X containing x , the point $f(x)$ is in the interior of $f(U)$. The following fact is immediate.

2.1. FACT. *A mapping is open if and only if it is interior at each point of its domain.*

A mapping $f : X \rightarrow Y \subset X$ is a *retraction*, and Y is a *retract* of X , if $f|_Y : Y \rightarrow Y$ is the identity (equivalently, if $f(f(x)) = f(x)$ for each $x \in X$). A surjective mapping $f : X \rightarrow Y$ is an *r-mapping* if there exists a mapping $g : Y \rightarrow X$ which is a right inverse of f , that is, $f(g(y)) = y$ for each $y \in Y$. The following is shown in [5], Section 11, Theorem, p. 1085:

2.2. THEOREM. *r-mappings coincide with compositions of the form $h \circ r$, where r is a retraction and h is a homeomorphism.*

Let \mathbb{F} be a class of mappings between compacta X and Y . We say that a mapping $f : X \rightarrow Y$ is *hereditarily* \mathbb{F} provided that $f|_K : K \rightarrow f(K) \subset Y$ is in \mathbb{F} for every continuum $K \subset X$.

We denote by \mathbb{M} , \mathbb{O} , \mathbb{C} , \mathbb{W} , and \mathbb{R} the classes of monotone, open, confluent, weakly confluent and *r-mappings*, respectively.

3. Hierarchy of spaces

A class \mathbb{F} of mappings between topological spaces is said to be *neat* if it contains all homeomorphisms and it is transitive, i.e. for any two mappings $f_1, f_2 \in \mathbb{F}$ such that the range of f_1 is the domain of f_2 , the composition $f_2 \circ f_1$ belongs to \mathbb{F} . Let a neat class \mathbb{F} of mappings be given. Then we write $Y \leq_{\mathbb{F}} X$ if there exists a surjection $f \in \mathbb{F}$ of X onto Y , and we put $X =_{\mathbb{F}} Y$ if and only if $Y \leq_{\mathbb{F}} X$ and $X \leq_{\mathbb{F}} Y$.

Let \mathcal{S} be a family of spaces. The relation $\leq_{\mathbb{F}}$ is a *quasi-ordering* on \mathcal{S} , which means that it is reflexive and transitive. It follows that $=_{\mathbb{F}}$ is an equivalence relation on \mathcal{S} . In other words, two spaces X and Y are said to be *equivalent with respect to* \mathbb{F} if there are mappings in \mathbb{F} from X onto Y and from Y onto X . Note that if X and Y are homeomorphic, then $X =_{\mathbb{F}} Y$ for each neat class \mathbb{F} but not conversely (in general), so the quasi-ordering $\leq_{\mathbb{F}}$ need not be an *ordering* (i.e. a quasi-ordering for which $Y \leq_{\mathbb{F}} X$ and $X \leq_{\mathbb{F}} Y$ implies $X = Y$ up to homeomorphism). The equivalence class of X with respect to \mathbb{F} will be denoted by $[X]_{\mathbb{F}}$.

Consider the quotient family $\mathcal{S}^* = \mathcal{S}/=_{\mathbb{F}}$, and observe that if $X_1, X_2 \in [X]_{\mathbb{F}}$ and $Y_1, Y_2 \in [Y]_{\mathbb{F}}$, then $X_1 \leq_{\mathbb{F}} Y_1$ if and only if $X_2 \leq_{\mathbb{F}} Y_2$. Therefore the relation $\leq_{\mathbb{F}}^*$ on \mathcal{S}^* given by

$$[X]_{\mathbb{F}} \leq_{\mathbb{F}}^* [Y]_{\mathbb{F}} \quad \text{if and only if} \quad X \leq_{\mathbb{F}} Y$$

is well defined, and moreover, it is an ordering of \mathcal{S}^* . The reader is referred to [5], Sections 1 through 8, pp. 1082–1084 for more information on this subject.

To simplify terminology and notation, we omit stars in notation; we also omit the phrase “on \mathcal{S} ” for \mathcal{S} fixed.

If $Y \leq_{\mathbb{F}} X$ and if $X \leq_{\mathbb{F}} Y$ does not hold, then we write $Y <_{\mathbb{F}} X$ and we call Y *\mathbb{F} -smaller* than X , and X *\mathbb{F} -greater* than Y .

A subfamily of \mathcal{S} is called a *chain* (with respect to $\leq_{\mathbb{F}}$) if for any two elements X and Y of the subfamily we have either $Y \leq_{\mathbb{F}} X$ or $X \leq_{\mathbb{F}} Y$. If neither $Y \leq_{\mathbb{F}} X$ nor $X \leq_{\mathbb{F}} Y$, then X and Y are *\mathbb{F} -incomparable*. An *antichain* (with respect to $\leq_{\mathbb{F}}$) is a subfamily of \mathcal{S} with any two members \mathbb{F} -incomparable.

According to the usual terminology, we say that a member X_0 of \mathcal{S} is *minimal* (resp. *maximal*) in \mathcal{S} with respect to \mathbb{F} if for each Y in \mathcal{S} with $Y \leq_{\mathbb{F}} X_0$ (resp.

with $X_0 \leq_{\mathbb{F}} Y$) we have $Y =_{\mathbb{F}} X_0$. Further, we say that $X_0 \in \mathcal{S}$ is the *least* (resp. *greatest*) element in \mathcal{S} with respect to \mathbb{F} if $X_0 \leq_{\mathbb{F}} Y$ (resp. $Y \leq_{\mathbb{F}} X_0$) for each $Y \in \mathcal{S}$.

We shall also consider a stronger version of these concepts. A space X is said to be *unique with respect to* \mathbb{F} provided the class of X consists of X only up to homeomorphism. In other words, X is unique with respect to \mathbb{F} if and only if for each space Y the existence of two mappings in \mathbb{F} , one from X onto Y and the other from Y onto X , implies that X and Y are homeomorphic.

Therefore we say that an element X_0 of \mathcal{S} is the *unique minimal element* (resp. *unique maximal element*) in \mathcal{S} with respect to \mathbb{F} if each $Y \in \mathcal{S}$ with $Y \leq_{\mathbb{F}} X_0$ (resp. with $X_0 \leq_{\mathbb{F}} Y$) is homeomorphic to X_0 . Similarly, X_0 is the *unique least element* (resp. *unique greatest element*) in \mathcal{S} with respect to \mathbb{F} if it is the least (the greatest) element in \mathcal{S} with respect to \mathbb{F} and if its equivalence class consists of one element only, i.e., $X_1 =_{\mathbb{F}} X_0$ implies that X_1 is homeomorphic to X_0 .

For X and Y in \mathcal{S} we write $X \simeq_{\mathbb{F}} Y$ if there exist finite sequences of spaces P_1, \dots, P_n, P_{n+1} and Q_1, \dots, Q_n in \mathcal{S} such that $P_1 = X$ and $P_{n+1} = Y$ and finite sequences of surjective mappings $f_i : P_i \rightarrow Q_i$ and $g_i : P_{i+1} \rightarrow Q_i$ in \mathbb{F} for each $i \in \{1, \dots, n\}$:

$$X = P_1 \xrightarrow{f_1} Q_1 \xleftarrow{g_1} P_2 \xrightarrow{f_2} Q_2 \dots Q_n \xleftarrow{g_n} P_{n+1} = Y.$$

One can verify that $\simeq_{\mathbb{F}}$ is an equivalence relation. The equivalence class of $X \in \mathcal{S}$ with respect to $\simeq_{\mathbb{F}}$ will be denoted by $\{X\}_{\mathbb{F}}$. Obviously, $X =_{\mathbb{F}} Y$ implies $X \simeq_{\mathbb{F}} Y$, thus $[X]_{\mathbb{F}} \subset \{X\}_{\mathbb{F}}$.

Fix a family \mathcal{S} of spaces and a neat class \mathbb{F} of mappings between elements of \mathcal{S} . Then, to describe the order structure of $(\mathcal{S}, \leq_{\mathbb{F}})$, i.e., the hierarchy of spaces in \mathcal{S} with respect to \mathbb{F} , one can try to answer a number of questions:

Q1. Describe (if there exist) the greatest, least, maximal, minimal elements in $(\mathcal{S}, \leq_{\mathbb{F}})$.

Q2. If a space is the greatest (least, maximal, minimal) element in $(\mathcal{S}, \leq_{\mathbb{F}})$, verify if it is unique (up to homeomorphism).

Q3. What is the maximal cardinality of (a) antichains, (b) chains? Do there exist uncountable chains?

Q4. (a) Does every chain have a lower (upper) bound? (b) Does every bounded chain have an infimum (a supremum)?

Q5. Does there exist a chain whose order structure is (a) dense, (b) similar to a segment?

Q6. Does there exist, for any two distinct elements X and Y , an element Z which is their common (a) lower bound (i.e. $Z \leq_{\mathbb{F}} X$ and $Z \leq_{\mathbb{F}} Y$), (b) upper bound (i.e. $X \leq_{\mathbb{F}} Z$ and $Y \leq_{\mathbb{F}} Z$)?

Q7. Does the infimum (supremum) exist for any two distinct elements?

Q8. Is $(\mathcal{S}, \leq_{\mathbb{F}})$ a lattice?

4. Dendrites

We shall use the notion of *order of a point* in the sense of Menger–Urysohn (see e.g. [20], §51, I, p. 274), and we denote by $\text{ord}(p, X)$ the order of the space X at a point $p \in X$. A *dendrite* is a locally connected continuum containing no simple closed curve. Given two points p and q of a dendrite X , we denote by pq the unique arc from p to q in X .

The following property of dendrites is well known ([37], Chapter 5, (1.3), (i), p. 89).

(4.1) *Each subcontinuum of a dendrite is again a dendrite.*

Since each dendrite is a hereditarily locally connected continuum ([20], §51, VI, Theorem 4, p. 301 and IV, Theorem 2, p. 283) and since each such continuum contains no nondegenerate continuum of convergence ([20], §50, IV, Theorem 2, p. 269), we obtain the next known result.

(4.2) *No dendrite contains a nondegenerate continuum of convergence.*

A metric space X equipped with a metric d is said to be *convex* (and then d is called a *convex metric* on X) if for any two distinct points x and y of X there exists a point $z \in X$ different from x and y and such that $d(x, y) = d(x, z) + d(z, y)$. It is well known that each locally connected continuum admits a convex metric (see Bing [2], Theorem 8, p. 1109; [4], Theorem 6, p. 546; and Moise [28], Theorem 4, p. 1119; see also [29] and [3]). Thus, in particular, we have the following fact, which can also be deduced from an earlier result in [22], p. 324.

4.3. FACT. *Each dendrite admits a convex metric.*

The following property characterizes dendrites (see [37], (1.1), (iv), p. 88; cf. [20], §51, VI, Theorem 6, p. 302).

4.4. THEOREM. *A continuum X is a dendrite if and only if the order of X at $p \in X$ and the number of components of $X \setminus \{p\}$ are equal for every $p \in X$ for which either of these is finite.*

Points of order 1 in X are called *end points* of X ; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points* of X . It is known that the set of all ordinary points is a dense subset of a dendrite. For $m \in \{3, 4, \dots, \omega\}$, points of order m are called *ramification points* of X ; the set of all ramification points is denoted by $R(X)$. It is known that $R(X)$ is at most countable for each dendrite X .

Given a dendrite X we decompose $R(X)$ into the subsets of points of finite and of infinite orders:

$$R_{\mathbb{N}}(X) = \{p \in R(X) : \text{ord}(p, X) \text{ is finite}\} = \{p \in R(X) : \text{ord}(p, X) \in \mathbb{N}\},$$

$$R_{\omega}(X) = \{p \in R(X) : \text{ord}(p, X) \text{ is infinite}\} = \{p \in R(X) : \text{ord}(p, X) = \omega\}.$$

In a dendrite X each point of order ω is an accumulation point of $E(X)$ (cf. [11], Lemma 2.1, p. 166). Thus we have the following fact.

4.5. FACT. *If, for a dendrite X , the set $E(X)$ is closed, then each point of X is of finite order.*

The following result will be useful in our further study of mapping and structural properties of dendrites (see Theorem 2.4 of [11], p. 167, where the result is proved under the weaker assumption that the considered continuum is a local dendrite).

4.6. THEOREM. *For each dendrite X the following conditions are equivalent:*

(4.7) $E(X)$ is dense in X ;

(4.8) $R(X)$ is dense in X ;

(4.9) for each arc $A \subset X$ the set $A \cap R(X)$ is dense in A .

Given $A \subset X$, the symbol A^d stands for the derived set of A , i.e. the set of all accumulation points of A in X (see e.g. [19], §9, pp. 75–80). Further, for each ordinal α we define (by transfinite induction) the α th derived set $A^{(\alpha)}$ as follows:

(4.10) $A^{(0)} = \text{cl } A$; $A^{(\alpha+1)} = (A^{(\alpha)})^d$; and, for a limit ordinal β , we put $A^{(\beta)} = \bigcap \{A^{(\alpha)} : \alpha < \beta\}$.

Then the following consequence of (4.10) is well known.

(4.11) $A \subset B$ implies $A^{(\alpha)} \subset B^{(\alpha)}$ for each ordinal α .

(4.12) If $f : X \rightarrow Y$ is a mapping of a compactum X , then for each $A \subset X$ and for each ordinal α we have

$$(f(A))^{(\alpha)} \subset f(A^{(\alpha)}).$$

The following proposition can easily be shown using (4.2).

4.13. PROPOSITION. *For every dendrite X we have*

$$[E(X)]^d = [R(X)]^d \cup R_\omega(X).$$

4.14. PROPOSITION. *If A and B are dendrites, then*

$$A \subset B \text{ implies } [E(A)]^d \subset [E(B)]^d.$$

PROOF. Indeed, by Proposition 4.13 and (4.11) we have

$$[E(A)]^d = [R(A)]^d \cup R_\omega(A) \subset [R(B)]^d \cup R_\omega(B) = [E(B)]^d.$$

One can use monotone mappings to characterize dendrites. Recall that a mapping $f : X \rightarrow Y$ is said to be *hereditarily monotone* if $f|K$ is monotone for each subcontinuum $K \subset X$. Since a locally connected continuum is a dendrite if and only if it is hereditarily unicoherent (compare [37], Chapter 5, Theorem 1.1, (v), p. 88) and since a continuum X is hereditarily unicoherent if and only if any monotone mapping of X is hereditarily monotone ([24], Corollary 3.2, p. 126), we have the next result.

4.15. THEOREM. *A locally connected continuum X is a dendrite if and only if every monotone mapping defined on X is hereditarily monotone.*

If a retraction $f : X \rightarrow Y \subset X$ is monotone, we say that Y is a *monotone retract* of X . It is known ([23], Theorem 2.1, p. 332) that each subcontinuum of a dendrite X is a monotone retract of X , and moreover, this property characterizes dendrites among arbitrary metric continua ([18], Theorem, p. 157):

4.16. THEOREM. *A continuum X is a dendrite if and only if each subcontinuum of X is a monotone retract of X .*

Theorems 4.16 and 2.2 imply the following.

4.17. COROLLARY. *For any two dendrites X and Y the relation $Y \leq_{\mathbb{R}} X$ holds if and only if X contains a homeomorphic copy of Y .*

4.18. COROLLARY. *Let \mathbb{F} be any of the following classes of mappings between dendrites: r -mappings, monotone, confluent, weakly confluent. Then for any two dendrites X and Y we have $X \not\approx_{\mathbb{F}} Y$.*

Proof. Indeed, by Theorem 4.16 one can find two monotone r -mappings $f : X \rightarrow A$ and $g : Y \rightarrow A$, where A is an arc. So $X \simeq_{\mathbb{R}} Y$ and $X \simeq_{\mathbb{M}} Y$. Since any monotone mapping is confluent, thus weakly confluent, we have $X \simeq_{\mathbb{C}} Y$ and $X \simeq_{\mathbb{W}} Y$.

It is known (compare [8], Corollary 1, p. 219) that

4.19. PROPOSITION. *The image of a dendrite under a confluent (thus under a monotone) mapping is again a dendrite.*

The same holds for arcs ([9], Corollary 20, p. 32). Furthermore, end points of an arc are mapped to end points of the range under a monotone mapping of the arc (see e.g. [37], Chapter 9, Theorem 1.1, p. 165). This is no longer true if the domain space is a dendrite. However, any end point of the range is the image of an end point of the domain. Namely, Theorem 4.15 implies the following (easy, but important) result.

4.20. PROPOSITION. *If a mapping $f : X \rightarrow Y$ between dendrites X and Y is a monotone surjection, then $E(Y) \subset f(E(X))$.*

Given a family \mathcal{S} of spaces, a member X of \mathcal{S} is said to be *universal in \mathcal{S}* if for each $Y \in \mathcal{S}$ there exists a homeomorphism h such that $h : Y \rightarrow h(Y) \subset X$. In particular, a dendrite is said to be *universal* if it contains a homeomorphic image of any other dendrite. Similarly, if the order of each point of a dendrite X is bounded by a number $m \in \{3, 4, \dots, \omega\}$, and X contains homeomorphic copies of all dendrites whose points have orders not greater than m , then X is called a *universal dendrite of order m* . Thus, since no dendrite contains points of order exceeding ω ([20], §51, VI, Theorem 4, p. 301), a universal dendrite of order ω is universal according to the former definition.

Observe that if a dendrite X contains a universal dendrite Y , then X is universal itself. The same holds for universal dendrites of order m . Hence, to avoid confusion, we shall consider some special universal dendrites whose definition is taken from Section 6 of [14].

For a given set $S \subset \{3, 4, \dots, \omega\}$ we denote by D_S any dendrite X satisfying the following two conditions:

$$(4.21) \quad \text{if } p \in R(X), \text{ then } \text{ord}(p, X) \in S;$$

$$(4.22) \quad \text{for each arc } A \subset X \text{ and for every } m \in S \text{ there is a point } p \in A \text{ with } \text{ord}(p, X) = m.$$

It is shown in Section 6 of [14] (Theorem 6.2) that D_S is topologically unique:

$$(4.23) \quad \text{If two dendrites satisfy conditions (4.21) and (4.22) with the same set } S \subset \{3, 4, \dots, \omega\}, \text{ then they are homeomorphic.}$$

If $S = \{m\}$ for some $m \in \{3, 4, \dots, \omega\}$, then we will simply write D_m in place of $D_{\{m\}}$. The dendrite D_m is called the *standard universal dendrite of order m* . A construction of this dendrite is known from Ważewski's doctoral dissertation ([36], Chapter K, p. 187). It was simplified by K. Menger in [26], Chapter X, §6, p. 318, and recalled in [11], p. 168. Another description of these continua for finite m , which uses limits of inverse sequences of finite dendrites (i.e. dendrites having a finite number of end points only) with monotone onto bonding mappings, is given in [10], p. 491.

Observe that for each $m \in \{3, 4, \dots, \omega\}$,

$$(4.24) \quad \text{each ramification point of } D_m \text{ is of order } m,$$

and

$$(4.25) \quad \text{for every arc } A \subset D_m \text{ the set of all ramification points of } D_m \text{ which belong to } A \text{ is dense in } A.$$

According to (4.23) any dendrite satisfying (4.24) and (4.25) is homeomorphic to D_m .

The following universality properties of D_S are known (see [14], Section 6, Theorems 6.6–6.8).

$$(4.26) \quad \text{If } \omega \in S, \text{ then the dendrite } D_S \text{ is universal.}$$

$$(4.27) \quad \text{If } S \text{ is finite with } \max S = m, \text{ then } D_S \text{ is universal in the family of all dendrites having orders of ramification points at most } m.$$

$$(4.28) \quad \text{If } S \text{ is infinite and } \omega \notin S, \text{ then } D_S \text{ is universal in the family of all dendrites having finite orders of ramification points.}$$

The above universality properties of the dendrites D_S together with the uniqueness property (4.23) justify their name: given $S \subset \{3, 4, \dots, \omega\}$, the dendrite D_S will be called the *standard universal dendrite of orders in S* .

Furthermore, repeating the proof of Theorem 6.2 from [14] one can easily verify that the following stronger form of (4.23) holds true.

4.29. PROPOSITION. *Let dendrites X and Y be homeomorphic to D_S for some $S \subset \{3, 4, \dots, \omega\}$. Then, for any two end points p and q of X and Y respectively, there exists a homeomorphism $h : X \rightarrow Y$ such that $h(p) = q$.*

For further generalizations the reader is referred to [12], where it is proved that, if p and q are arbitrary points of X and Y respectively, then a homeomorphism $h : X \rightarrow Y$ such that $h(p) = q$ exists if and only if $\text{ord}(p, X) = \text{ord}(q, Y)$.

5. Monotone and confluent mappings

Now we fix \mathcal{S} to be the family \mathcal{D} of all dendrites, and for the class of mappings we take either the class \mathbb{M} of monotone mappings between dendrites or any neat class \mathbb{F} which contains \mathbb{M} . Recall that a monotone image of a dendrite is again a dendrite (see e.g. [37], Chapter 8, (6.21), p. 145 and (2.41), p. 140; see also Proposition 4.19 above). Further, since every subcontinuum of a dendrite is a dendrite (see (4.1) above), we conclude from Theorems 2.2 and 4.16 that any image of a dendrite under an r -mapping is also a dendrite.

By Theorems 2.2 and 4.16 we have

5.1. PROPOSITION. *If X and Y are dendrites, then*

$$(5.2) \quad Y \leq_{\mathbb{R}} X \quad \text{implies} \quad Y \leq_{\mathbb{M}} X.$$

5.3. Remark. The converse to (5.2) is not true, because if H is the union of two simple triods with exactly one end point in common (i.e. a dendrite which looks like capital H) and if X is a 4-od (i.e. a dendrite which looks like capital X), then shrinking the horizontal bar in H to a point is a monotone mapping from H onto X , so that $X \leq_{\mathbb{M}} H$, while H and X are \mathbb{R} -incomparable.

A very important class of mappings between compacta that contains \mathbb{M} is the class \mathbb{C} of confluent mappings. Since it plays a basic role in investigations of mapping properties of continua, we will discuss the same problem of interconnections between the relations $\leq_{\mathbb{M}}$ and $\leq_{\mathbb{F}}$ (as in Proposition 5.1) for $\mathbb{F} = \mathbb{C}$. To this end, we recall two known properties of confluent mappings between compacta.

The first property concerns confluent mappings of locally connected continua. It is known (see [8], IX, p. 215) that then these mappings coincide with quasi-monotone ones, i.e., such that for each subcontinuum Q of Y having non-empty interior, $f^{-1}(Q)$ has finitely many components each of which is mapped onto Q under f . Combining this result with Whyburn's characterization of quasi-monotone mappings of locally connected continua saying that these mappings are just compositions of monotone and of open light mappings ([37], Theorem 8.4, p. 153) we get the following result (compare also [25], (6.2), p. 51).

5.4. LEMMA. *Let a mapping $f : X \rightarrow Y$ of a locally connected continuum X onto Y be confluent. Then there is a unique factorization $f = f_2 \circ f_1$ into confluent mappings such that $f_1 : X \rightarrow f_1(X)$ is monotone and $f_2 : f_1(X) \rightarrow Y$ is open and light.*

The second property is a consequence of a more general result concerning open mappings due to Whyburn (see [37], Theorem 2.4, p. 188; for a generalization of this result to confluent mappings see [15], Theorem 1.3, p. 410).

5.5. LEMMA. *If X is a compact space and a mapping $f : X \rightarrow Y$ is open and light, then for every dendrite B in Y there exists a dendrite A in X such that $f|_A : A \rightarrow f(A) = B$ is a homeomorphism.*

5.6. PROPOSITION. *If X and Y are dendrites, then there exists a monotone surjective mapping from X onto Y if and only if there exists a confluent surjective mapping from X onto Y .*

PROOF. Since each monotone mapping is confluent, one implication is trivial. So, assume that $f : X \rightarrow Y$ is a confluent surjection. According to Lemma 5.4 there are a monotone mapping $f_1 : X \rightarrow f_1(X)$ and an open light mapping $f_2 : f_1(X) \rightarrow Y$ such that $f = f_2 \circ f_1$. Being the monotone image of a dendrite, $f_1(X)$ is a dendrite. By Lemma 5.5 there exists a dendrite Z in $f_1(X)$ such that $f_2|_Z : Z \rightarrow Y$ is a homeomorphism. Let $r : f_1(X) \rightarrow Z$ be a monotone retraction from $f_1(X)$ onto Z according to Theorem 4.16. Then $g : X \rightarrow Y$ defined by $g = (f_2|_Z) \circ r \circ f_1$ is the composition of three monotone mappings, so it is monotone. The proof is complete.

5.7. COROLLARY. *If X and Y are dendrites, then*

$$(5.8) \quad Y \leq_{\mathbb{M}} X \quad \text{is equivalent to} \quad Y \leq_{\mathbb{C}} X.$$

5.9. QUESTION. To what families \mathcal{S} containing the family \mathcal{D} of dendrites can Propositions 5.1 and 5.6 be generalized?

By definition, each confluent mapping is weakly confluent. Thus we have an obvious corollary.

5.10. COROLLARY. *If X and Y are dendrites, then*

$$(5.11) \quad Y \leq_{\mathbb{C}} X \quad \text{implies} \quad Y \leq_{\mathbb{W}} X.$$

The authors do not know whether the implication (5.11) can be reversed, i.e., whether the relations $\leq_{\mathbb{C}}$ and $\leq_{\mathbb{W}}$ are equivalent for dendrites. More precisely, we have the following question.

5.12. QUESTION. Assume there is a weakly confluent surjection from a dendrite X onto a dendrite Y . Does it follow that there is a confluent (equivalently: monotone, cf. Proposition 5.6) surjection from X onto Y ?

5.13. Remark. The assumption that Y is a dendrite is essential in the above question. Namely, the function $f : [0, 1] \rightarrow S^1$ defined by $f(t) = \exp(4\pi it)$ is

weakly confluent, while there is no monotone mapping from $[0,1]$ onto S^1 because a monotone image of an arc is an arc ([37], Chapter 9, (1.1), p. 165).

Since the arc is a monotone retract of any dendrite (compare Theorem 4.16 above), and since a monotone image of an arc is an arc we obtain the following fact.

5.14. FACT. *The arc is the unique least element with respect to \mathbb{M} in the family \mathcal{D} of dendrites.*

5.15. COROLLARY. *The arc is the least element in \mathcal{D} with respect to any neat class \mathbb{F} of mappings between compacta that contains \mathbb{M} . Moreover, if \mathbb{F} has the property that for every $f \in \mathbb{F}$ the image of an arc under f is again an arc, then the arc is the unique least element in \mathcal{D} with respect to \mathbb{F} .*

Recall that the image of an arc under a weakly confluent mapping is either an arc or a simple closed curve (see [15], Corollary II.3, p. 412). So, if the range space is assumed to be a dendrite, a weakly confluent image of an arc is an arc. Thus the class \mathbb{W} of all weakly confluent mappings between dendrites can be substituted for \mathbb{F} in Corollary 5.15.

The next corollary is an immediate consequence of the previous one.

5.16. COROLLARY. *In the family \mathcal{D} of dendrites the following conditions are equivalent:*

(5.17) X is an arc;

(5.18) X is the least element with respect to \mathbb{M} ;

(5.19) X is the least element with respect to \mathbb{C} ;

(5.20) X is the least element with respect to \mathbb{W} ;

(5.21) X is the least element with respect to \mathbb{R} .

Furthermore, the arc is the unique least element with respect to each of the above mentioned classes.

Again by Theorem 4.16, if a dendrite X is monotone equivalent (i.e. equivalent with respect to \mathbb{M}) to the standard universal dendrite D_ω , then for every dendrite Y there exists a monotone mapping from X onto Y . Therefore the following fact holds true.

5.22. FACT. *The standard universal dendrite D_ω (and every dendrite X which is monotone equivalent to D_ω) is the greatest element in \mathcal{D} with respect to \mathbb{M} .*

5.23. COROLLARY. *D_ω (and every dendrite X which is monotone equivalent to D_ω) is the greatest element in \mathcal{D} with respect to any class \mathbb{F} of mappings between compacta that contains \mathbb{M} .*

Note that $[D_\omega]_{\mathbb{F}}$ contains more than one element, thus D_ω is not the unique greatest element in \mathcal{D} with respect to \mathbb{F} .

To see what the class of dendrites which are monotone equivalent to D_ω looks like, we need an example of a dendrite L such that

$$(5.24) \quad \text{all ramification points of } L \text{ are of order 3;}$$

$$(5.25) \quad R(L) \text{ is discrete.}$$

A special example of such a dendrite, denoted by L_0 , has been defined in [11], Example 6.9, p. 182, as the closure of the union of an increasing sequence of dendrites in the plane. We recall its construction here for the reader's convenience.

Let L_1 be the unit straight line segment. Divide L_1 into three equal subsegments and in the middle one, M , locate a thrice diminished copy of the Cantor ternary set C . At the mid point of each interval K contiguous to C (i.e. of a component K of $M \setminus C$) we erect perpendicularly to L_1 a straight line segment whose length equals the length of K . Denote by L_2 the union of L_1 and of all the erected segments (there are countably many of them). We perform the same construction on each of the added segments: divide such a segment into three equal parts, locate in the middle part M a copy of the Cantor set C properly diminished, at the mid point of any component K of $M \setminus C$ construct a segment perpendicular to K and as long as K is, and denote by L_3 the union of L_2 and of all the attached segments. Continuing in this manner we get a sequence

$$L_1 \subset L_2 \subset L_3 \subset \dots \subset L_i \subset L_{i+1} \subset \dots$$

Putting

$$(5.26) \quad L_0 = \text{cl} \left(\bigcup \{L_i : i \in \mathbb{N}\} \right)$$

we see that L_0 is a dendrite.

The following characterizations of dendrites which are monotone equivalent to standard universal dendrites are known (see [11], Theorem 6.14, p. 185).

5.27. THEOREM. *The following conditions are equivalent for a dendrite X :*

$$(5.28) \quad X \text{ is monotone equivalent to } D_3;$$

$$(5.29) \quad X \text{ is monotone equivalent to } D_\omega;$$

$$(5.30) \quad X \text{ is monotone equivalent to } D_m \text{ for each } m \in \{3, 4, \dots, \omega\};$$

$$(5.31) \quad X \text{ contains a homeomorphic copy of every dendrite } L \text{ satisfying (5.24) and (5.25);}$$

$$(5.32) \quad X \text{ contains a homeomorphic copy of the dendrite } L_0 \text{ defined by (5.26).}$$

5.33. Remark. It can happen that for some neat class \mathbb{F} of mappings between compacta that contains \mathbb{M} the class $[D_\omega]_{\mathbb{F}}$ of dendrites which are \mathbb{F} -equivalent to D_ω is essentially larger than the class $[D_\omega]_{\mathbb{M}}$ of dendrites which are \mathbb{M} -equivalent to D_ω as described in Theorem 5.27. For example, if \mathbb{F} is the class of all mappings between dendrites, then obviously for any dendrite X there are mappings from

X onto D_ω and from D_ω onto X , i.e. every dendrite is \mathbb{F} -equivalent to D_ω (and to any other dendrite). Therefore the following problem can be posed.

5.34. PROBLEM. For what neat classes \mathbb{F} of mappings between dendrites such that $\mathbb{M} \subset \mathbb{F}$ does the equality $[D_\omega]_{\mathbb{F}} = [D_\omega]_{\mathbb{M}}$ hold?

Note that the class \mathbb{C} of confluent mappings is one such class according to Corollary 5.7.

Our next result generalizes Theorem 5.27.

5.35. THEOREM. *The following conditions are equivalent for a dendrite X :*

- (5.29) X is monotone equivalent to D_ω ;
- (5.36) for every dendrite Y with $R(Y)$ dense, X is monotone equivalent to Y ;
- (5.37) X is monotone equivalent to D_S for each $S \subset \{3, 4, \dots, \omega\}$;
- (5.38) there exists a dendrite Y with $R(Y)$ dense such that X is monotone equivalent to Y ;
- (5.39) X is the greatest element in \mathcal{D} with respect to \mathbb{M} (equivalently: with respect to \mathbb{C}).

PROOF. By the definition of the greatest element in \mathcal{D} with respect to \mathbb{M} it is easy to see that (5.29) and (5.39) are equivalent. Now we shall prove that (5.29) \Rightarrow (5.36) \Rightarrow (5.37) \Rightarrow (5.38) \Rightarrow (5.29). Since by (4.22), $R(D_S)$ is dense in D_S , the implications (5.36) \Rightarrow (5.37) \Rightarrow (5.38) are obvious. Thus only (5.29) \Rightarrow (5.36) and (5.38) \Rightarrow (5.29) need a proof.

Assume (5.29). Let a dendrite Y have $R(Y)$ dense. It is shown in [11], Theorem 6.7, p. 180, that if a dendrite X_0 contains a subdendrite with a dense set of ramification points, then $X_0 =_{\mathbb{M}} D_\omega$. Substituting Y for X_0 we get $Y =_{\mathbb{M}} D_\omega$. Since $X =_{\mathbb{M}} D_\omega$ by (5.29), we get $X =_{\mathbb{M}} Y$, i.e. (5.36) holds.

Assume (5.38). Again by Theorem 6.7 of [11], p. 180, we have $Y =_{\mathbb{M}} D_\omega$. Since $X =_{\mathbb{M}} Y$ by (5.38), we conclude that $X =_{\mathbb{M}} D_\omega$. Thus (5.29) is shown. The proof is complete.

5.40. REMARKS. (a) Recall that, by Corollary 5.7, “monotone equivalent” can be replaced by “confluent equivalent” in conditions (5.29), (5.36), (5.37) and (5.38) of Theorem 5.35.

(b) Since for each dendrite, the density of the set of ramification points is equivalent to the density of the set of end points (see Theorem 4.6), we can replace “ $R(Y)$ dense” by “ $E(Y)$ dense” in (5.36) and (5.38).

By Theorem 5.35, $(\mathcal{D}, \leq_{\mathbb{M}})$ has a greatest element, and therefore each chain has an upper bound. So, the following questions seem to be natural.

5.41. QUESTION. In $(\mathcal{D}, \leq_{\mathbb{M}})$, (a) does every chain have a supremum? (b) does there exist a sequence $\{X_n : n \in \mathbb{N}\}$ of dendrites satisfying $X_{n+1} <_{\mathbb{M}} X_n$ for every $n \in \mathbb{N}$?

K. Sieklucki, investigating the structure of $(\mathcal{D}, \leq_{\mathbb{R}})$, constructed a family of cardinality \mathfrak{c} consisting of r -incomparable dendrites (see [34]) and a family of dendrites r -ordered similarly to the segment (see [35]). Unfortunately, these constructions cannot be directly adapted to the case of monotone mappings. Sieklucki uses the existence of a countable antichain of dendrites with respect to $\leq_{\mathbb{R}}$. The authors do not know any construction of a countable antichain of dendrites for $\leq_{\mathbb{M}}$. So, we have the following questions.

5.42. QUESTION. Does there exist a countably infinite (uncountable) antichain in $(\mathcal{D}, \leq_{\mathbb{M}})$?

5.43. QUESTION. Does there exist in $(\mathcal{D}, \leq_{\mathbb{M}})$ any chain with order structure similar to that of a segment?

5.44. QUESTION. Is every chain well-ordered in $(\mathcal{D}, \leq_{\mathbb{M}})$?

We now show that $(\mathcal{D}, \leq_{\mathbb{M}})$ is not a lattice. More precisely, we give an example of two dendrites X and Y such that the pair $\{X, Y\}$ has neither an infimum nor a supremum in $(\mathcal{D}, \leq_{\mathbb{M}})$. We use some techniques of the theory of linearly ordered sets. The reader is referred to Chapters 6 and 7 of [21] for relevant information. We start with some auxiliary notation.

If α is an order type, we denote by α^* the inverse order type. For any linearly ordered set A we denote by $\tau(A)$ the order type of A . If two sets A and B are linearly ordered by \leq_A and \leq_B respectively, then we write $\tau(A) \prec \tau(B)$ if A is order embeddable in B , i.e., there exists a one-to-one function $h : A \rightarrow B$ such that for all $x, y \in A$, if $x \leq_A y$ then $h(x) \leq_B h(y)$. Observe that if α and β are ordinal numbers (i.e. order types of well-ordered sets), then $\alpha \prec \beta$ holds if and only if $\alpha \leq \beta$.

5.45. PROPOSITION. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective monotone mapping. If X contains an arc ab such that $R(X) \subset ab$, then there exists an arc $cd \subset Y$ such that $R(Y) \subset cd$ and $R(Y)$ can be order embedded in $R(X)$, i.e., $\tau(R(Y)) \prec \tau(R(X))$.*

Proof. By Proposition 4.19, Y is a dendrite. Since f is monotone, it has the ramification point covering property, that is, $R(Y) \subset f(R(X))$ (see [15], Theorem I.1, p. 410). Hence $R(X) \subset ab$ implies $R(Y) \subset f(ab)$. Further, since f is hereditarily monotone (see Theorem 4.15), $f|_{ab} : ab \rightarrow f(ab)$ is monotone, and thus $f(ab)$ is either a point or an arc. If $f(ab)$ is a point, then $R(Y)$ is either empty or a one-point set, so there is nothing to prove. If $f(ab)$ is an arc, we put $cd = f(ab)$ and we note that $f|_{ab} : ab \rightarrow cd$ preserves the natural ordering of points. This completes the proof.

5.46. EXAMPLE. *There exist two dendrites X and Y such that $\{X, Y\}$ has no infimum and no supremum with respect to the class \mathbb{M} of monotone mappings.*

Proof. In the Euclidean plane consider a straight line segment ab ordered by \leq from a to b . Take a well-ordered discrete subset A of ab such that $\tau(A) = \omega^\omega$.

Let X be a dendrite containing the segment ab , having the points of A , and only these points, as its ramification points, all of them being of order 3. Thus the dendrite X is determined by the conditions

$$A \subset ab \subset X, \quad \tau(A) = \omega^\omega,$$

$$\text{ord}(p, X) = 3 \text{ for } p \in A \quad \text{and} \quad \text{ord}(p, X) < 3 \text{ for } p \in X \setminus A.$$

Note that the set $R(X) = A \subset ab$ is well-ordered.

To construct the dendrite Y take in the segment ab (ordered from a to b as previously) a sequence of sets B_1, B_2, \dots such that

- 1) if $i < j$, and if $x \in B_i$ and $y \in B_j$, then $x < y$;
- 2) $\tau(B_i) = (\omega^i)^*$ for every $i \in \mathbb{N}$;
- 3) the union $B = \bigcup \{B_i : i \in \mathbb{N}\}$ is a discrete set.

Put

$$\beta = \omega^* + (\omega^2)^* + (\omega^3)^* + \dots$$

and observe that $\tau(B) = \beta$.

Let Y be a dendrite containing the segment ab , having the points of B , and only these points, as its ramification points, all of them being of order 3. Thus the dendrite Y is determined by the conditions

$$B \subset ab \subset X, \quad \tau(B) = \beta,$$

$$\text{ord}(p, Y) = 3 \quad \text{for } p \in B \quad \text{and} \quad \text{ord}(p, Y) < 3 \quad \text{for } p \in Y \setminus B.$$

In particular, we have $R(Y) = B$, and thus $\tau(R(Y)) = \beta$.

Suppose on the contrary that there exists a dendrite Z which is the infimum of $\{X, Y\}$ with respect to $\leq_{\mathbb{M}}$. Then Z is a monotone image of X . By Proposition 5.45, there is an arc in Z containing $R(Z)$, and $\tau(R(Z)) \prec \tau(R(X)) = \omega^\omega$. The latter statement means that $R(Z)$ is order embeddable in $R(X)$, which is well-ordered. Thus $R(Z)$ is also well-ordered. Put $\zeta = \tau(R(Z))$. Since it is not true that $\omega^\omega \prec \beta$, while $\zeta \leq \omega^\omega$ and $\zeta \prec \beta$, we have $\zeta < \omega^\omega$.

For every $i \in \mathbb{N}$ let A_i be the smallest segment contained in ab and containing B_i , and let Y_i stand for the smallest dendrite contained in Y , containing A_i and all maximal free arcs which have one of their end points in B_i . Note that $R(Y_i)$ is contained in an arc, and that $\tau(R(Y_i)) = \omega^i$.

Take $i \in \mathbb{N}$ such that $\zeta < \omega^i$. Then $Y_i \leq_{\mathbb{M}} X$ and $Y_i \leq_{\mathbb{M}} Y$, hence $Y_i \leq_{\mathbb{M}} Z$, a contradiction to Proposition 5.45.

Suppose now that there exists a dendrite W which is the supremum of $\{X, Y\}$ for $\leq_{\mathbb{M}}$. We construct two auxiliary dendrites U and V in the plane. Take a straight line segment ab' linearly ordered by the natural relation \leq from a to b' , and choose two distinct points b and a' of this segment such that $a < b < a' < b'$. Let X' stand for a copy of the dendrite X such that the segment $a'b'$ plays the same role in X' as does ab in X , and assume that $ba' \cap X' = \{a'\}$. Next take

a copy of Y that contains the segment $ab \subset ab'$ by its definition and such that $Y \cap ba' = \{b\}$ and $Y \cap X' = \emptyset$. Then

$$U = Y \cup ba' \cup X'$$

is a dendrite. To define V we interchange the roles of X and Y in the construction of U . More precisely, we take a copy Y' of Y such that $a'b'$ plays the same role in Y' as ab in Y , and we put

$$V = X \cup ba' \cup Y'$$

assuming that $X \cap ba' = \{b\}$, $Y' \cap ba' = \{a'\}$ and $X \cap Y' = \emptyset$.

It follows from the construction that

$$\begin{aligned} \tau(R(U)) &= \tau(R(Y)) + \tau(R(X')) = \tau(R(Y)) + \tau(R(X)) = \beta + \omega^\omega, \\ \tau(R(V)) &= \tau(R(X)) + \tau(R(Y')) = \tau(R(X)) + \tau(R(Y)) = \omega^\omega + \beta. \end{aligned}$$

Since U and V contain copies of X and Y by their construction, Theorem 4.16 implies that $X \leq_{\mathbb{M}} U$ and $Y \leq_{\mathbb{M}} U$, as well as $X \leq_{\mathbb{M}} V$ and $Y \leq_{\mathbb{M}} V$, whence $W \leq_{\mathbb{M}} U$ and $W \leq_{\mathbb{M}} V$, which means that there are monotone mappings from U onto W and from V onto W . Applying Proposition 5.45 we see that $R(W)$ lies in an arc contained in W , so it is linearly ordered, and that

$$(5.47) \quad \tau(R(W)) \prec \tau(R(U)) = \beta + \omega^\omega,$$

and

$$(5.48) \quad \tau(R(W)) \prec \tau(R(V)) = \omega^\omega + \beta.$$

Since W is the supremum of $\{X, Y\}$, we have $X \leq_{\mathbb{M}} W$ and $Y \leq_{\mathbb{M}} W$, and thus there are monotone mappings of W onto X and onto Y respectively. Applying Proposition 5.45 once more, we conclude that

$$\tau(R(X)) \prec \tau(R(W)) \quad \text{and} \quad \tau(R(Y)) \prec \tau(R(W)),$$

or, equivalently, that

$$(5.49) \quad \omega^\omega \prec \tau(R(W)) \quad \text{and} \quad \beta \prec \tau(R(W)).$$

Thus, by (5.48), we have $\omega^\omega \prec \tau(R(W)) \prec \omega^\omega + \beta$. We claim that

$$(5.50) \quad \tau(R(W)) = \omega^\omega.$$

Indeed, the greatest ordinal α satisfying $\alpha \prec \tau(R(Y)) + \tau(R(X)) = \beta + \omega^\omega$ is $\omega + \omega^\omega = \omega^\omega$. Thus putting

$$\xi = \sup\{\gamma : \gamma \text{ is an ordinal, and } \gamma \prec \tau(R(W))\},$$

we conclude that $\xi \leq \omega^\omega$. If $\tau(R(W))$ were of the form

$$\tau(R(W)) = \omega^\omega + \mu$$

for some nonzero order type μ , then $\xi > \omega^\omega$, a contradiction. Thus (5.50) is established.

However, (5.50) contradicts (5.49), because the order type β is not order embeddable in ω^ω . So, the proof is complete.

5.51. COROLLARY. $(\mathcal{D}, \leq_{\mathbb{M}})$ is not a lattice.

5.52. Remark. In the same way it can be shown that $(\mathcal{D}, \leq_{\mathbb{R}})$ is not a lattice. In fact, a proposition analogous to 5.45 holds true for retractions. Therefore for the same two dendrites X and Y constructed in Example 5.46 one can prove, using similar arguments, that $\{X, Y\}$ has no infimum and no supremum for $\leq_{\mathbb{R}}$.

Recall that the *Gehman dendrite* G is a dendrite having the Cantor ternary set C in $[0,1]$ for the set $E(G)$ of its end points, such that all ramification points of G are of order 3 and

$$E(G) = \text{cl } R(G) \setminus R(G)$$

(see [17], the example on p. 42; see also [31], pp. 422–423 for a detailed description, and [32], Fig. 1 on p. 203 for a picture).

We now construct a family \mathcal{C} of dendrites G_α indexed by ordinals $\alpha < \omega_1$ such that for every $\alpha, \beta < \omega_1$ we have

$$(5.53) \quad G_\alpha \subset G_\beta,$$

and

$$(5.54) \quad \text{if } \alpha < \beta, \text{ then } G_\alpha \text{ is embeddable in } G_\beta.$$

The family \mathcal{C} is needed to describe some order phenomena in $(\mathcal{D}, \leq_{\mathbb{M}})$ and $(\mathcal{D}, \leq_{\mathbb{O}})$.

Each G_α will be uniquely determined by the set E_α of its end points. The latter sets will be defined by transfinite induction as closed subsets of the Cantor set C .

We define $E_1 = \{0\} \cup \{1/3^n : n \in \{0, 1, 2, \dots\}\} \subset C = E(G)$. Assume closed subsets E_α of C are already defined for all ordinals α less than an ordinal β such that $1 \leq \beta < \omega_1$. To define E_β consider two cases. First, assume that $\beta = \alpha + 1$ for some ordinal α . For each $n \in \{0, 1, 2, \dots\}$ we locate in $[2/3^{n+1}, 1/3^n]$ a copy of E_α diminished 3^{n+1} times. Then we define $E_{\alpha+1}$ as the union of all these copies together with the singleton $\{0\}$. Second, assume β is a limit ordinal, and let $\{\beta_n\}$ be the sequence of all ordinals less than β . Then, for each $n \in \{0, 1, 2, \dots\}$, we locate in $[2/3^{n+1}, 1/3^n]$ a copy of E_{β_n} diminished 3^{n+1} times, and define E_β as previously. The inductive procedure is thus finished.

The following is a consequence of the definition.

$$(5.55) \quad \text{For every } \alpha < \omega_1 \text{ the set } (E_\alpha)^{(\alpha)} \text{ is a singleton, and thus } (E_\alpha)^{(\alpha+1)} = \emptyset.$$

Now, we define G_α to be the subcontinuum of G irreducible with respect to containing E_α . By the hereditary unicoherence of G such a subcontinuum is unique (see e.g. [7], T1, p. 187). Therefore the dendrites G_α are defined for all ordinals $\alpha < \omega_1$. Properties (5.53) and (5.54) are consequences of this definition.

We put

$$(5.56) \quad \mathcal{C} = \{G_\alpha : \alpha < \omega_1\}.$$

5.57. THEOREM. *The family $\mathcal{C} = \{G_\alpha : \alpha < \omega_1\}$ forms a chain of dendrites in $(\mathcal{D}, \leq_{\mathbb{M}})$ such that*

(5.58) *if $\alpha < \beta$, then $G_\alpha <_{\mathbb{M}} G_\beta$.*

Hence the chain is not embeddable in a segment.

Proof. Property (5.54) and Theorem 4.16 imply that

(5.59) *for any $\alpha < \beta < \omega_1$ there is a monotone r -mapping from G_β onto G_α .*

Now we prove that

(5.60) *there is no monotone mapping from G_α onto G_β .*

Suppose on the contrary that $f : G_\alpha \rightarrow G_\beta$ is a monotone surjection. Then, taking $X = G_\alpha$ and $Y = G_\beta$, we see from Proposition 4.20, (4.11) and (4.12) that for every ordinal $\gamma < \omega_1$ we have

$$(E(Y))^{(\gamma)} \subset (f(E(X)))^{(\gamma)} \subset f((E(X))^{(\gamma)}).$$

Since $E(Y) = E_\beta$ and $E(X) = E_\alpha$, taking $\gamma = \alpha + 1 < \beta + 1$ in the above inclusions, we deduce from (5.55) that $(E(Y))^{(\gamma)} \neq \emptyset$, while $(E(X))^{(\gamma)} = \emptyset$, a contradiction.

Now (5.59) and (5.60) imply (5.58). The proof is then complete.

5.61. **Remarks.** (a) According to (5.53) and (5.54) the family $\{G_\alpha : \alpha < \omega_1\}$ with both quasi-orders $\leq_{\mathbb{R}}$ and $\leq_{\mathbb{M}}$ forms a chain isomorphic to ω_1 . It will be shown later (see Theorem 6.54) that it is also a chain with respect to $\leq_{\mathbb{O}}$. Note that there is no family of embeddings $i_\alpha : G_\alpha \rightarrow G$ (for $\alpha < \omega_1$) such that $\alpha < \beta$ implies $i_\alpha(G_\alpha) \subset i_\beta(G_\beta)$. Indeed, otherwise there would exist an ω_1 -sequence of subcontinua of G ordered by inclusion. This is impossible because the hyperspace of all subcontinua of G (equipped with the Hausdorff metric, see [20], §42, II, p. 47), being a (metric) continuum, cannot contain any such ω_1 -sequence.

(b) It can be shown that the Gehman dendrite G is the supremum of the chain $\mathcal{C} = \{G_\alpha : \alpha < \omega_1\}$ (see (5.56)) for $\leq_{\mathbb{M}}$, $\leq_{\mathbb{R}}$, and $\leq_{\mathbb{O}}$, while it cannot be represented as the limit of an inverse system of G_α 's with bonding mappings belonging to \mathbb{M} , \mathbb{R} , and \mathbb{O} , respectively.

6. Open mappings

Still keeping the family \mathcal{D} of all dendrites as the family \mathcal{S} of spaces under consideration, we now fix the class \mathbb{F} to be the class \mathbb{O} of open mappings between dendrites. Recall the following result (see e.g. [37], Chapter 8, (7.7), p. 148 and Chapter 10, p. 185; for a more general result see [25], (7.36), p. 68; compare Proposition 4.19 above):

(6.1) *An open image of a dendrite is again a dendrite.*

This resembles of course the corresponding property for monotone mappings of dendrites. However, as we will see in this chapter, the order structure of $(\mathcal{D}, \leq_{\mathbb{O}})$ differs much from that of $(\mathcal{D}, \leq_{\mathbb{M}})$.

An important difference concerns arcs. Though the property of being an arc is invariant under open mappings ([37], Chapter 10, (1.3), p. 184), just as for monotone ones (see [37], Chapter 9, (1.1), p. 165), no analog of Fact 5.14 and Corollary 5.15 holds for the class \mathcal{O} . In fact, there are dendrites X (different from an arc) such that for every open mapping defined on X the image of X is homeomorphic to X . Namely, the following is known (see [10], Theorem 1, p. 490 and Theorem 3, p. 493; [14], Corollary 6.10).

6.2. PROPOSITION. *Let S be a subset of $\{3, 4, \dots, \omega\}$ and let D_S be the standard universal dendrite of orders in S . Then each open image of D_S is homeomorphic to D_S if and only if S is a nonempty subset of $\{3, \omega\}$.*

Below we construct an uncountable family of dendrites homeomorphic to their open images (see Theorem 6.45).

Before we study the structure $(\mathcal{D}, \leq_{\mathcal{O}})$ we recall some known properties of open mappings which will be needed later. We start with the following proposition (see [37], Chapter 8, (7.31), p. 147).

6.3. PROPOSITION. *The order of a point is never increased under an open mapping.*

6.4. COROLLARY. *If $f : X \rightarrow Y$ is open, then*

$$f(E(X)) \subset E(Y).$$

The next proposition is taken from [10], Lemma, p. 489.

6.5. PROPOSITION. *Open mappings of dendrites preserve points of order ω .*

Recall that an arc ab is said to be *free* if $ab \setminus \{a, b\}$ is an open subset of the space, and that if all points of a continuum are of order two at most, then the continuum is either an arc or a simple closed curve (see e.g. [20], §51, Theorems 5 and 6, p. 293 and 294). Thus the next results are consequences of Proposition 6.3.

6.6. COROLLARY. *If $f : X \rightarrow Y$ is open, then the image under f of a free arc in X is either a free arc or a simple closed curve in Y .*

6.7. COROLLARY. *If $f : X \rightarrow Y$ is open and Y is a dendrite, then the image under f of a free arc in X is a free arc in Y .*

Let us recall that each open mapping defined on an arc or on a simple closed curve is light. This is a consequence of the characterization of light mappings given in [37], Chapter 10, (1.2) and (1.3), p. 184. Note that both an arc and a simple closed curve are examples of *locally dendritic* spaces, i.e. spaces with each point having a neighbourhood which is a dendrite. It is known that open mappings of such spaces are light provided the range space is dense in itself, i.e., has no isolated point (see [13], Theorem 5, p. 214). We repeat the argument here, for the reader's convenience. We need a lemma ([13], Lemma 4, p. 214).

6.8. LEMMA. *Let X be a metric space and $f : X \rightarrow Y$ a surjective open mapping. Let A be a compact subset of X , and B a closed connected subset of Y ,*

such that

$$f(A) \cap B \neq \emptyset \neq B \setminus f(A).$$

Then for each component Q of $A \cap f^{-1}(B)$ we have

$$Q \cap \text{bd } A \neq \emptyset.$$

Proof. Suppose on the contrary that $Q \subset \text{int } A$. Since Q is a component of a compact set $A \cap f^{-1}(B)$, it coincides with a quasi-component of this set ([20], §47, II, Theorem 2, p. 169), and thus there is an open set V such that $Q \subset V \subset \text{int } A$ and $(\text{cl } V \setminus V) \cap (f^{-1}(B) \cup \text{bd } A) = \emptyset$. So we have

$$\emptyset \neq f(V) \cap B = f(\text{cl } V) \cap B \neq B,$$

which implies that $f(V) \cap B$ is a nonempty, open and closed, proper subset of a connected set B , a contradiction.

Now we are ready to show the result ([13], Theorem 5, p. 214).

6.9. THEOREM. *Let X be a locally dendritic metric space, and suppose Y has no isolated points. Then each open surjective mapping $f : X \rightarrow Y$ is light.*

Proof. Suppose on the contrary that f is not light, i.e., there is $y \in Y$ with a nondegenerate component Q' of $f^{-1}(y)$. Take $x \in Q'$ and let a dendrite D be a neighbourhood of x . Thus there is a nondegenerate continuum K in X such that $x \in K \subset Q' \cap \text{int } D$ (compare [20], §47, III, Theorem 2, p. 172). Since D has nonempty interior and f is open and Y has no isolated point, we see that $f(D)$ is a nondegenerate locally connected subcontinuum of Y containing y . Hence there exists a nondegenerate arc yy' in Y . Let $B(K, \varepsilon)$ stand for the open ε -ball around K . Take $\varepsilon > 0$ such that

$$(6.10) \quad \text{cl } B(K, \varepsilon) \subset \text{int } D \quad \text{and} \quad y' \in Y \setminus f(B(K, \varepsilon)).$$

Now we construct, by induction, a sequence of arcs $\{x_i x'_i : i \in \mathbb{N}\}$ with the following properties (for each $i \in \mathbb{N}$):

$$(6.11) \quad x_i x'_i \subset \text{cl } B(K, \varepsilon),$$

$$(6.12) \quad x_i x'_i \cap K = \{x_i\},$$

$$(6.13) \quad x_i x'_i \setminus B(K, \varepsilon) = \{x'_i\},$$

$$(6.14) \quad x_i x'_i \cap x_j x'_j = \emptyset \quad \text{for } i \neq j.$$

Consider the component Q of $f^{-1}(yy') \cap \text{cl } B(K, \varepsilon)$ which contains K , and note that Q meets $\text{cl } B(K, \varepsilon) \setminus B(K, \varepsilon)$ by Lemma 6.8. By (6.10), Q is a subcontinuum of the dendrite D , thus it is a dendrite by (4.1), whence it is arcwise connected. Choose $a \in K$ and $b \in Q \cap (\text{bd } B(K, \varepsilon) \setminus B(K, \varepsilon))$, and note that there is a unique arc $ab \subset Q \subset D$. Order this arc from a to b , and let x_1 be the last point of ab in K and x'_1 be the first point of ab in $\text{bd } B(K, \varepsilon)$. Then the arc $x_1 x'_1 \subset ab$ satisfies (6.11)–(6.13).

Assume now there is $k \geq 2$ such that the finite sequence of $k-1$ arcs $\{x_i x'_i : i \in \{1, \dots, k-1\}\}$ has properties (6.11)–(6.14). Take $c \in K \setminus \{x_1, \dots, x_{k-1}\}$ and let U

be an arcwise connected neighbourhood of c such that $U \subset B(K, \varepsilon)$ and $U \cap x_i x'_i = \emptyset$ for $i \in \{1, \dots, k-1\}$. Since f is open, there is $z \in U$ such that $f(z) \in yy' \setminus \{y\}$. Let $f(z)y' \subset yy'$, and denote by Q_k the component of $f^{-1}(f(z)y') \cap \text{cl } B(K, \varepsilon)$ to which z belongs. Again by Lemma 6.8 we see that Q_k meets $\text{cl } B(K, \varepsilon) \setminus B(K, \varepsilon)$ at a point d . Further, $Q_k \cap K = \emptyset$, whence $Q_k \cap x_i x'_i = \emptyset$ for $i \in \{1, \dots, k-1\}$, because otherwise one can find a simple closed curve in $Q_k \cup U \cup K \cup x_i x'_i$. Finally, $Q_k \cup U$ is arcwise connected and contains $c \in K$ and $d \in \text{cl } B(K, \varepsilon) \setminus B(K, \varepsilon)$, whence we can easily find an arc $x_k x'_k \subset cd \subset Q_k \cup U$ satisfying (6.11)–(6.14). So the inductive procedure is finished.

Observe that the diameters of all the constructed arcs $x_i x'_i$ are greater than or equal to ε , so the limit L of a convergent sequence of these arcs is nondegenerate. Since X is locally dendritic, the terms of the sequence are disjoint from L , so that L is a continuum of convergence contained in the dendrite D . This contradicts (4.2). The proof is complete.

6.15. COROLLARY. *Every nonconstant open mapping defined on a dendrite is light.*

The next result gives further information on the structure of point inverses for open mappings between dendrites, and thus it extends the above corollary.

6.16. PROPOSITION. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open mapping. Then*

$$(6.17) \quad f^{-1}(y) \text{ is finite for every } y \in Y \setminus E(Y);$$

$$(6.18) \quad f^{-1}(E(Y)) \setminus E(X) \text{ is finite.}$$

PROOF. By (6.1) the continuum Y is a dendrite. We show (6.17). Suppose $f^{-1}(y)$ is infinite for some $y \in Y \setminus E(Y)$. Then $Y \setminus \{y\}$ is not connected (compare Theorem 4.4). Let P and Q be the closures of two distinct components of $Y \setminus \{y\}$. For each $x \in f^{-1}(y)$ let $P(x)$ and $Q(x)$ be the components of $f^{-1}(P)$ and of $f^{-1}(Q)$ respectively, that contain x . Since f is open, and since each open mapping between compacta is confluent (see [37], Chapter 8, Theorem 7.5, p. 148), we infer that

$$(6.19) \quad f(P(x)) = P \quad \text{and} \quad f(Q(x)) = Q.$$

It follows from the hereditary unicoherence of X that for any two points x_1 and x_2 of $f^{-1}(y)$, if $P(x_1) = P(x_2)$, then $Q(x_1) \neq Q(x_2)$, and if $Q(x_1) = Q(x_2)$, then $P(x_1) \neq P(x_2)$. Therefore we have in X either infinitely many continua of the form $P(x)$ or infinitely many continua of the form $Q(x)$. Without loss of generality we can assume that they are of the form $P(x)$. Then the continua are mutually disjoint, so by the hereditary local connectedness of X they form a null-family (see [37], Chapter 5, (2.6), p. 92). This contradicts (6.19). So (6.17) is established.

To show (6.18) suppose on the contrary that $f^{-1}(E(Y)) \setminus E(X)$ is infinite. Since f is light (see Corollary 6.15) and $E(Y)$ is zero-dimensional (see e.g. [20],

§51, V, Theorem 2, p. 292), it follows that $f^{-1}(E(Y))$ does not contain any arc in X , i.e., it is also zero-dimensional. Further, it is a simple consequence of the structure of X that each infinite zero-dimensional subset of the (connected) set $X \setminus E(X)$ disconnects X into infinitely many (arc) components. This obviously implies that

(6.20) $X \setminus f^{-1}(E(Y))$ has infinitely many (arc) components.

On the other hand, it is quite obvious that there exists $\varepsilon > 0$ such that for each $y \in Y \setminus E(Y)$ there is an arc $B \subset Y \setminus E(Y)$ with $y \in B$ and $\text{diam } B > \varepsilon$. Since f , being open, is confluent (compare [37], Theorem 7.5, p. 148) and since it is uniformly continuous, we conclude that there exists $\delta > 0$ such that for every $x \in X \setminus f^{-1}(E(Y))$ there is a subcontinuum A of $X \setminus f^{-1}(E(Y))$ with $x \in A$ and $\text{diam } A > \delta$. Thus every arc component of $X \setminus f^{-1}(E(Y))$ has diameter greater than δ . Therefore we conclude from (6.20) that X contains a nondegenerate continuum of convergence, which contradicts (4.2). The proof is complete.

As a consequence of Corollary 6.15, of (6.1) and of Lemma 5.5, we get the following result.

6.21. THEOREM. *Let X be a locally dendritic compactum and $f : X \rightarrow Y$ a surjective open mapping. Suppose Y has no isolated points. Then for each dendrite B in Y and for each $x_0 \in f^{-1}(B)$ there exists a subdendrite A of X containing x_0 and such that $f|_A : A \rightarrow B$ is a homeomorphism.*

6.22. COROLLARY. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open mapping. Then for each subcontinuum B of Y and for each $x_0 \in f^{-1}(B)$ there exists a subcontinuum A of X containing x_0 and such that $f|_A : A \rightarrow B$ is a homeomorphism. Certainly, both A and B are dendrites.*

6.23. COROLLARY. *If there exists a surjective open mapping from a dendrite X onto a dendrite Y , then there exists a surjective monotone r -mapping from X onto Y .*

PROOF. Let $f : X \rightarrow Y$ be an open surjection. According to Corollary 6.22 there exists a subdendrite A of X such that $f|_A : A \rightarrow Y$ is a homeomorphism. Then, by Theorem 4.16, there exists a monotone retraction $g : X \rightarrow A$. The composition $(f|_A) \circ g : X \rightarrow Y$ is the desired monotone r -mapping of X onto Y .

6.24. COROLLARY. *If X and Y are dendrites, then*

(6.25) $Y \leq_{\mathbb{O}} X$ implies $Y \leq_{\mathbb{R}} X$.

6.26. REMARK. The converse to (6.25) does not hold. In fact, consider an arc A , embed it in D_3 and note that, by Theorem 4.16, there is a monotone retraction from D_3 onto A , whence $A \leq_{\mathbb{R}} D_3$. On the other hand, there is no open mapping from D_3 onto A by Proposition 6.2. Thus the inequality $A \leq_{\mathbb{O}} D_3$ is not true.

Corollary 6.24, Proposition 5.1 and Corollaries 5.7 and 5.10, as well as Remarks 6.26, 5.3 and Question 5.12 are summarized below.

6.27. COROLLARY. *If X and Y are dendrites, then*

$$Y \leq_{\mathbb{O}} X \Rightarrow Y \leq_{\mathbb{R}} X \Rightarrow Y \leq_{\mathbb{M}} X \Leftrightarrow Y \leq_{\mathbb{C}} X \Rightarrow Y \leq_{\mathbb{W}} X.$$

The first two implications cannot be reversed, while the reversibility of the last one remains an open question.

Applying Proposition 6.5 and Corollary 6.22 one gets the next result.

6.28. PROPOSITION. *Let Y_1 and Y_2 be dendrites such that Y_1 contains, while Y_2 does not contain, a point of order ω . Then there is no common upper (lower) bound for Y_1 and Y_2 in $(\mathcal{D}, \leq_{\mathbb{O}})$.*

To exhibit other pairs of dendrites Y_1 and Y_2 without any common upper bound we use another argument.

6.29. PROPOSITION. *Let Y_1 and Y_2 be dendrites such that Y_1 contains a non-degenerate component of $[E(Y_1)]^{\text{d}}$, and all components of $\text{cl } E(Y_2)$ are degenerate. Then there is no common upper (lower) bound for Y_1 and Y_2 in $(\mathcal{D}, \leq_{\mathbb{O}})$.*

Proof. We show the upper bound case. The argument for the lower bound is similar. Take a dendrite X such that $Y_1 \leq_{\mathbb{O}} X$, i.e. that there is an open mapping f from X onto Y_1 . According to Corollary 6.22, X contains a homeomorphic copy A of Y_1 , and therefore there is a dendrite $A \subset X$ and a nondegenerate component Q of $[E(A)]^{\text{d}}$. By Proposition 4.14 we have $[E(A)]^{\text{d}} \subset [E(X)]^{\text{d}}$, and since $[E(X)]^{\text{d}} \subset \text{cl } E(X)$ (see [19], §9, III, (8), p. 77), we conclude that $Q \subset \text{cl } E(X)$.

Suppose there is an open surjection $g : X \rightarrow Y_2$. Then $g(\text{cl } E(X)) \subset \text{cl } g(E(X))$ by continuity of g , and since $g(E(X)) \subset E(Y_2)$ by Corollary 6.4, we get $\text{cl } g(E(X)) \subset \text{cl } E(Y_2)$, which implies $g(Q) \subset \text{cl } E(Y_2)$. Since all components of $\text{cl } E(Y_2)$ are singletons, $g(Q)$ must also be a singleton. Therefore $Q \subset g^{-1}(g(Q))$ contradicts the lightness of g , which was shown in Corollary 6.15. The proof is complete.

6.30. COROLLARY. *If Y_1 is a universal dendrite D_S for some set $S \subset \{3, 4, \dots, \omega\}$, and if Y_2 is an arc, then there is no common upper bound for Y_1 and Y_2 in $(\mathcal{D}, \leq_{\mathbb{O}})$.*

To examine the structure of chains and antichains in $(\mathcal{D}, \leq_{\mathbb{O}})$ it will be convenient to use a new concept.

Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of mutually disjoint continua (lying e.g. in the Hilbert cube), tending to a point p . For each $n \in \mathbb{N}$ choose two points a_n and b_n in X_n , and consider a sequence of mutually disjoint arcs $\{b_n a_{n+1} : n \in \mathbb{N}\}$, also having p as the only point of its topological limit, and such that

$$X_m \cap b_n a_{n+1} = \begin{cases} \emptyset & \text{if } n \neq m \neq n+1, \\ \{b_n\} & \text{if } m = n, \\ \{a_{n+1}\} & \text{if } m = n+1. \end{cases}$$

Then

$$(6.31) \quad X = \bigcup \{X_n \cup b_n a_{n+1} : n \in \mathbb{N}\} \cup \{p\}$$

is a continuum, and is called a *string* of continua X_n . Each X_n is called a *bead* of the string, a_n and b_n are the *extreme points* of the bead X_n , and p is the *final point* of X .

Observe that each arc $b_n a_{n+1}$ is a free arc in X , p is an end point of X , and

$$\text{Lim } X_n = \text{Lim } b_n a_{n+1} = \{p\},$$

whence $\lim \text{diam } X_n = \lim \text{diam } b_n a_{n+1} = 0$.

We shall use the concept of a string exclusively in the case when for each $n \in \mathbb{N}$ the following three conditions hold:

$$(6.32) \quad X_n \text{ is a dendrite;}$$

$$(6.33) \quad \text{cl } E(X_n) = X_n;$$

$$(6.34) \quad a_n, b_n \in E(X_n).$$

It is easy to verify that X is then a dendrite. We will say that X is a *string of dendrites*.

Observe that (by construction)

$$(6.35) \quad \text{if } X \text{ is a string of dendrites, then}$$

$$E(X) = \{a_1\} \cup \bigcup \{(E(X_n) \setminus \{a_n, b_n\}) : n \in \mathbb{N}\} \cup \{p\},$$

whence we conclude by (6.33) that

$$(6.36) \quad \text{if } X \text{ is a string of dendrites, then}$$

$$\text{cl } E(X) = \{p\} \cup \bigcup \{X_n : n \in \mathbb{N}\},$$

and each member of this union is a component of $\text{cl } E(X)$.

6.37. PROPOSITION. *The property of being a string of dendrites is invariant under open mappings.*

Proof. Let X be a string of dendrites defined by (6.31), and satisfying (6.32)–(6.34). Consider an open mapping $f : X \rightarrow Y$ onto a continuum Y . First, Y is clearly a dendrite (see (6.1)). The string structure of Y is defined by

$$Y_n = f(X_n), \quad c_n = f(a_n), \quad d_n = f(b_n), \quad q = f(p).$$

Then obviously we have

$$(6.38) \quad Y = \bigcup \{Y_n \cup d_n c_{n+1} : n \in \mathbb{N}\} \cup \{q\}.$$

By Corollary 6.7 the arcs $d_n c_{n+1}$ are free arcs in Y . Further, each Y_n is a dendrite by (4.1). By hereditary unicoherence of Y it follows that the dendrites Y_n , as well as the free arcs $d_n c_{n+1}$, are mutually disjoint. Condition (6.34) implies that the points $b_1, a_2, b_2, a_3, b_3, \dots$ are of order two in X , whence by Proposition 6.3 and by (6.38) their images $d_1, c_2, d_2, c_3, d_3, \dots$ are of order two in Y . Thus $c_n, d_n \in E(Y)$. Finally, to show that $\text{cl } E(Y_n) = Y_n$ for each n , it is enough to use (6.36) and Corollary 6.4. The proof is complete.

6.39. PROPOSITION. *Let*

$$(6.31) \quad X = \bigcup \{X_n \cup b_n a_{n+1} : n \in \mathbb{N}\} \cup \{p\}$$

and

$$(6.38) \quad Y = \bigcup \{Y_n \cup d_n c_{n+1} : n \in \mathbb{N}\} \cup \{q\}$$

be two strings of dendrites with beads X_n and Y_n , with extreme points a_n , b_n and c_n , d_n and with final points p and q , respectively. If a surjective mapping $f : X \rightarrow Y$ is open, then $f(p) = q$, and, for each $n \in \mathbb{N}$,

$$(6.40) \quad f(X_n) = Y_n, \quad f(a_n) = c_n \quad \text{for } n > 1, \quad f(b_n) = d_n.$$

PROOF. Since end points of X are mapped to end points of Y (see Corollary 6.4), the components of $\text{cl } E(X)$ are mapped onto continua with a dense set of end points, i.e.,

$$(6.41) \quad \text{for each } n \in \mathbb{N} \text{ there exists } m \in \mathbb{N} \text{ such that } f(X_n) \subset Y_m.$$

Therefore the final point p , being the only accumulation point of the beads X_n , must go to the only accumulation point of the beads Y_n , i.e., $f(p) = q$. We claim that

$$(6.42) \quad f(X_1) = Y_1 \quad \text{and} \quad f(b_1) = d_1.$$

To see this, choose $u \in b_1 a_2 \setminus \{b_1, a_2\}$, put $A = X_1 \cup b_1 u \setminus \{u\}$ and note that A is a connected open subset of X . Thus

$$f(A) = f(X_1 \setminus \{b_1\}) \cup \{f(b_1)\} \cup f(b_1 u \setminus \{b_1, u\})$$

is an open subset of Y , being the union of an open set $f(X_1 \setminus \{b_1\})$, of a free arc without its end points $f(b_1 u \setminus \{b_1, u\})$ (compare Corollary 6.7) and of a singleton $\{f(b_1)\}$. By the definition of Y the only open subsets in Y having this structure are of the form $Y_1 \cup (d_1 w \setminus \{w\})$ for some $w \in d_1 c_2$. Now (6.42) follows from (6.41) and Corollary 6.7.

Next we claim that

$$(6.43) \quad f(X_2) = Y_2, \quad f(a_2) = c_2 \quad \text{and} \quad f(b_2) = d_2.$$

Indeed, $A = b_1 a_2 \cup X_2 \cup b_2 a_3 \setminus \{b_1, a_3\}$ is an open subset of X , thus $f(A)$ is an open subset of Y . It is the union of a free arc without end points $f(b_1 a_2 \setminus \{b_1, a_2\})$ (again Corollary 6.7 is used here), of a singleton $\{f(a_2)\}$, of an open set $f(X_2 \setminus \{a_2, b_2\})$, of a singleton $\{f(b_2)\}$, and of a free arc without end points $f(b_2 a_3 \setminus \{b_2, a_3\})$. We already know by (6.42) that $f(b_1) = d_1$. Therefore the only open subsets in Y having this structure are of the form $d_1 c_2 \cup Y_2 \cup d_2 w \setminus \{d_1, w\}$ for some $w \in d_2 c_3$. Using the same argument as previously we deduce (6.43).

Continuing in this way we get (6.41) by an easy induction. The proof is complete.

Now we come to constructions of some special strings.

6.44. PROPOSITION. *Let X and Y be two strings of dendrites defined by (6.31) and (6.38) such that, for each $n \in \mathbb{N}$, the beads X_n and Y_n are homeomorphic standard universal dendrites D_{S_n} of orders in S_n for some $S_n \subset \{3, 4, \dots, \omega\}$. Then X and Y are homeomorphic.*

PROOF. By Theorem 6.2 of [14], for each $n \in \mathbb{N}$ there is a homeomorphism $h_n : X_n \rightarrow Y_n$ such that $h_n(a_n) = c_n$ and $h_n(b_n) = d_n$. Further, let $g_n : b_n a_{n+1} \rightarrow d_n c_{n+1}$ be a homeomorphism. Define $h : X \rightarrow Y$ by $h(p) = q$ and, for each $n \in \mathbb{N}$, by $h(x) = h_n(x)$ if $x \in X_n$ and $h(x) = g_n(x)$ if $x \in b_n a_{n+1}$. One can easily verify that h is a homeomorphism. The proof is complete.

Using Propositions 6.37, 6.39, 6.44 and 6.2 we get the following result.

6.45. THEOREM. *For each 0-1 sequence $\delta = \{\delta_n : n \in \mathbb{N}\}$ the string of dendrites $X(\delta) = X$ defined by*

$$(6.31) \quad X = \bigcup \{X_n \cup b_n a_{n+1} : n \in \mathbb{N}\} \cup \{p\},$$

where

$$(6.46) \quad X_n = D_3 \quad \text{if } \delta_n = 0 \quad \text{and} \quad X_n = D_\omega \quad \text{if } \delta_n = 1,$$

is homeomorphic to any of its open images, i.e., X is uniquely minimal in the family $(\mathcal{D}, \leq_{\mathbb{O}})$.

6.47. Remark. Taking in the construction of $X(\delta)$ instead of (6.46) either

$$(6.48) \quad X_n = D_3 \quad \text{if } \delta_n = 0 \quad \text{and} \quad X_n = D_{\{3, \omega\}} \quad \text{if } \delta_n = 1,$$

or

$$(6.49) \quad X_n = D_\omega \quad \text{if } \delta_n = 0 \quad \text{and} \quad X_n = D_{\{3, \omega\}} \quad \text{if } \delta_n = 1,$$

one gets another two families of cardinality \mathfrak{c} composed of dendrites with the same property.

As an immediate consequence of Theorem 6.45 and of the definition of a unique minimal element we have the following.

6.50. COROLLARY. *In $(\mathcal{D}, \leq_{\mathbb{O}})$ there are continuum many uniquely minimal dendrites.*

6.51. Remark. Since the strings $X(\delta)$ of dendrites constructed in Theorem 6.45 are minimal elements in $(\mathcal{D}, \leq_{\mathbb{O}})$, they are \mathbb{O} -incomparable, and therefore the family $\{X(\delta) : \delta \text{ is a 0-1 sequence}\}$ is an antichain in $(\mathcal{D}, \leq_{\mathbb{O}})$.

Now we pass to the structure of chains in $(\mathcal{D}, \leq_{\mathbb{O}})$. We start with the following proposition.

6.52. PROPOSITION. *Denote by $S(0)$ the string of dendrites X defined by*

$$(6.31) \quad X = \bigcup \{X_n \cup b_n a_{n+1} : n \in \mathbb{N}\} \cup \{p\},$$

with all beads X_n equal to D_3 . Further, for each positive integer k denote by $S(k)$ the string of dendrites defined by (6.31) with $X_k = D_4$ and $X_n = D_3$ for $n \neq k$. Then for any $k_1, k_2 \in \{0\} \cup \mathbb{N}$ we have

$$S(k_1) \leq_{\mathbb{O}} S(k_2) \quad \text{if and only if} \quad k_1 = 0 \quad \text{or} \quad k_1 = k_2.$$

Proof. It is known that, given two natural numbers m_1 and m_2 with $m_1 > m_2 \geq 3$, there exists an open mapping from D_{m_1} onto D_{m_2} (see [10], Theorem 2, p. 492) but not conversely (by Proposition 6.3). Thus in particular D_4 can be openly mapped onto D_3 , and there is no open mapping from D_3 onto D_4 . Now the conclusion is a straightforward consequence of Propositions 6.37 and 6.39.

6.53. THEOREM. *In $(\mathcal{D}, \leq_{\mathbb{O}})$ there exists a chain of continuum many dendrites which has the order structure of a segment.*

Proof. We apply Sieklucki's construction from [35], where a chain of dendrites has been constructed having a similar property with respect to the class \mathbb{R} .

Let $I = \{(x, 0) : x \in [0, 1]\}$ be the unit segment in the plane, and let C denote the standard Cantor ternary set lying in I . Arrange the components of $I \setminus C$ in a sequence $\{Z_k : k \in \mathbb{N}\}$ and let z_k stand for the middle point of the (open) segment Z_k . Fix $c \in C \subset I$. To each z_k we assign a copy of either $S(0)$ or $S(k)$ (where $S(0)$ and $S(k)$ are the strings of dendrites described in Proposition 6.52), according as either $c < z_k$ or $z_k < c$. These copies are diminished in such a way that the diameter of the copy assigned to z_k equals the diameter of Z_k . Now form the union $X[c]$ of the unit segment I and all the diminished copies of $S(0)$ and $S(k)$ assigned to the midpoints z_k for $k \in \mathbb{N}$ situated so that each z_k coincides with the final point p (see (6.31) in 6.52) of the corresponding copy of either $S(0)$ or $S(k)$. It is clear that all this can be done in such a way that the constructed continuum $X[c]$ is a dendrite.

It is evident from the construction that if $c_1, c_2 \in C$ and if $<$ is the standard order in I , then

$$c_1 < c_2 \quad \text{implies} \quad X[c_1] \leq_{\mathbb{O}} X[c_2].$$

We will show that the dendrites $X[c_1]$ and $X[c_2]$ are not \mathbb{O} -equivalent. Suppose on the contrary that there is an open surjective mapping $f : X[c_1] \rightarrow X[c_2]$. Then between c_1 and c_2 there is a component Z_{k_0} of $I \setminus C$. Hence, by construction, a copy of $S(k_0)$ is contained in $X[c_2]$ while no homeomorphic copy of $S(k_0)$ is contained in $X[c_1]$. Note that for each $c \in C$ the closures of components of $X[c] \setminus I$ coincide with the copies of either $S(0)$ or $S(k)$ attached to I in the construction of $X[c]$. Therefore it follows from Propositions 6.37 and 6.39 that the closures of components of $X[c_1] \setminus I$ are mapped under f onto the closures of components of $X[c_2] \setminus I$. But there is no copy of $S(k)$ in $X[c_1]$ that could be mapped under f onto $S(k_0)$. This is because of the \mathbb{O} -incomparability of $S(k_1)$ and $S(k_2)$ for $k_1 \neq k_2$ (see Proposition 6.52). The proof is complete.

An analogue of Theorem 5.57 holds for open mappings. Moreover, the same family \mathcal{C} of dendrites defined by (5.56) can be used to construct a chain that cannot be embedded into any segment with respect to the quasi-order $\leq_{\mathbb{O}}$. Namely, we have the following result.

6.54. THEOREM. *In $(\mathcal{D}, \leq_{\mathbb{O}})$, the family $\mathcal{C} = \{G_{\alpha} : \alpha < \omega_1\}$ forms a chain of dendrites such that*

$$(6.55) \quad \text{if } \alpha < \beta, \text{ then } G_{\alpha} <_{\mathbb{O}} G_{\beta}.$$

Hence the chain is not embeddable into a segment with respect to the ordering $\leq_{\mathbb{O}}$.

PROOF. First we show that

$$(6.56) \quad \text{for any } \alpha < \beta < \omega_1 \text{ there is an open mapping from } G_{\beta} \text{ onto } G_{\alpha}.$$

In fact, by (5.54) the dendrite G_{β} contains a homeomorphic copy Y of G_{α} . Moreover, it can be verified that, by construction, this copy can be embedded in G_{β} in such a way that $\text{bd}Y = Y \cap \text{cl}(G_{\beta} \setminus Y)$ is a one-point set. Denote this point by p . Choose $q \in E(Y)$ such that the arc pq is free in Y . Note that q is uniquely determined. Now define a surjection $f : G_{\beta} \rightarrow Y$ as follows: (a) $f|_Y$ is the identity; (b) for each $x \in E(G_{\beta}) \setminus E(Y)$ put $f(x) = q$, and let $f|_{px}$ be a homeomorphism from px onto pq . All this can obviously be done in such a way that f is well-defined and continuous. One can verify that f is open. Thus (6.56) is established.

Further, it follows from Corollary 6.23 and (5.60) that

$$(6.57) \quad \text{there is no open mapping from } G_{\alpha} \text{ onto } G_{\beta}.$$

Assertions (6.56) and (6.57) imply (6.55). The proof is complete.

Now we shall study equivalence classes of the relation $\simeq_{\mathbb{O}}$. We show that, unlike for some other mappings (r -mappings, monotone, confluent, weakly confluent, see Corollary 4.18) these classes are proper subfamilies of the family \mathcal{D} of all dendrites. We characterize the most important ones: the equivalence class of an arc and of the universal dendrites D_3 , D_{ω} , and $D_{\{3, \omega\}}$.

We start our study with the class of dendrites which admit an open mapping onto an arc. To describe the equivalence class of an arc with respect to the relation $\simeq_{\mathbb{O}}$ we need the following result.

6.58. PROPOSITION. *Let X be a dendrite with $E(X)$ closed. Then for each $p \in X \setminus E(X)$ there exists an open surjective mapping $f : X \rightarrow [0, 1]$ such that*

$$(6.59) \quad f^{-1}(0) = E(X) \quad \text{and} \quad f^{-1}(1) = \{p\}.$$

PROOF. Let $\{e_1, e_2, \dots\}$ be a dense subset of $E(X)$. For each $n \in \mathbb{N}$ we define X_n as the minimal subdendrite of X containing $\{p, e_1, \dots, e_n\}$. Then

$$X = \text{cl} \left(\bigcup \{X_n : n \in \mathbb{N}\} \right) \quad \text{and} \quad X \setminus \bigcup \{X_n : n \in \mathbb{N}\} \subset E(X).$$

Take a homeomorphism $f_1 : X_1 \rightarrow [0, 1]$ with $f_1(e_1) = 0$ and $f_1(p) = 1$. Assume that for a fixed $n \in \mathbb{N}$ and for each $i \in \{1, \dots, n\}$ we have defined mappings $f_i : X_i \rightarrow [0, 1]$ such that

$$(6.60) \quad \text{if } 1 \leq i \leq j \leq n, \text{ then } f_j|_{e_i p} \text{ is a homeomorphism with } f_j(e_i) = 0 \text{ and } f_j(p) = 1, \text{ and } f_j|_{X_i} = f_i.$$

Note that $X_{n+1} = X_n \cup e_{n+1}r_n$ for some $r_n \in X_n$ such that $e_{n+1}r_n \cap X_n = \{r_n\}$. Define $f_{n+1} : X_{n+1} \rightarrow [0, 1]$ by putting $f_{n+1}|_{X_n} = f_n$, and letting $f_{n+1}|_{e_{n+1}r_n} : e_{n+1}r_n \rightarrow [0, f_n(r_n)]$ be a homeomorphism. To finish the construction it is enough to put $f(x) = f_n(x)$ if $x \in X_n$ and $f(x) = 0$ if $x \in X \setminus \bigcup\{X_n : n \in \mathbb{N}\}$. Since $E(X)$ is closed, it follows that f is continuous. Finally, (6.60) implies the openness of f . The proof is complete.

Our next result, which is a characterization of dendrites admitting an open mapping onto an arc, can be deduced from Theorem 2 of [27], p. 455. The theorem says that a local dendrite X (i.e. a continuum whose points have neighbourhoods which are dendrites) is retractable onto an arc under an open mapping if and only if X contains no point of order ω and $\text{cl } E(X) \setminus E(X)$ is finite. Since the proof given in [27] uses some methods and arguments from graph theory, we present an independent proof.

6.61. THEOREM. *For each dendrite X the following conditions are equivalent:*

$$(6.62) \quad \text{there exists an open surjective mapping from } X \text{ onto } [0, 1];$$

$$(6.63) \quad \text{the set } \text{cl } E(X) \setminus E(X) \text{ is finite, and } \text{ord}(x, X) \text{ is finite for each } x \in X.$$

Proof. Assume (6.62) and let $f : X \rightarrow [0, 1]$ be the mapping. By Corollary 6.4 we have $f(E(X)) \subset \{0, 1\}$, whence $f(\text{cl } E(X)) \subset \text{cl } f(E(X)) \subset \{0, 1\}$ by continuity of f . Thus $\text{cl } E(X) \subset f^{-1}(\{0, 1\})$. Since $f^{-1}(\{0, 1\}) \setminus E(X)$ is finite by (6.18), we infer that $\text{cl } E(X) \setminus E(X)$ is finite. Further, X does not contain any point of order ω by Proposition 6.5. Thus (6.63) is shown.

Assume (6.63). Put $A = \text{cl } E(X) \setminus E(X)$ and note that if $A = \emptyset$, then (6.62) holds by Proposition 6.58. So let $A = \{a_1, \dots, a_m\}$ for some $m \in \mathbb{N}$. Since each point of X is of finite order, it follows from Theorem 4.4 that $X \setminus A$ has finitely many, say n , components P_1, \dots, P_n for some $n \in \mathbb{N}$. For each $i \in \{1, \dots, n\}$, $X_i = \text{cl } P_i$ is a subdendrite of X with $E(X_i)$ closed, and we see that $a_j \in E(X_i)$ for each $j \in \{1, \dots, m\}$ and some $i \in \{1, \dots, n\}$. Note that

$$X = \bigcup\{X_i : i \in \{1, \dots, n\}\}.$$

Next, for each $i \in \{1, \dots, n\}$ choose $p_i \in X_i \setminus E(X_i)$, and let $f_i : X_i \rightarrow [0, 1]$ be an open surjection such that $f_i^{-1}(0) = E(X_i)$ and $f_i^{-1}(1) = \{p_i\}$; it exists by Proposition 6.58. One can verify that $f : X \rightarrow [0, 1]$ defined by $f|_{X_i} = f_i$ is the needed open mapping. So (6.62) is shown and the proof is complete.

As an immediate consequence of Proposition 6.58 (or of Theorem 6.61) we have a corollary.

6.64. COROLLARY. *Every dendrite X with $E(X)$ closed admits an open mapping onto an arc.*

Observe that, by Proposition 4.14, the property described in (6.63) is hereditary. Thus we have

6.65. COROLLARY. *If a dendrite X admits an open mapping onto an arc, then so does every nondegenerate subdendrite of X .*

6.66. COROLLARY. *If a dendrite X admits an open mapping onto an arc, then so does every nondegenerate image of X under an open mapping.*

PROOF. This is a consequence of Corollaries 6.23 and 6.65.

The converse implication to that of Corollary 6.66 is not true. This can be seen by the example below.

6.67. EXAMPLE. *There exist dendrites X and Y both admitting open mappings onto arcs such that $Y \subset X$ without any open mapping from X onto Y .*

PROOF. Let A be an arm of a simple triod Y . Fix a sequence of points $\{p_n\}$ of A converging to the center of the triod, and take a sequence of arcs A_n such that

$$Y \cap A_n = \{p_n\} \quad \text{and} \quad \lim \text{diam } A_n = 0.$$

Then $X = Y \cup \bigcup \{A_n : n \in \mathbb{N}\}$ is the needed dendrite. One can verify, e.g. using Proposition 6.3 and Corollaries 6.4 and 6.22, that there is no open mapping from X onto Y .

6.68. PROPOSITION. *The equivalence class $\{[0, 1]\}_\circ$ is the family of all dendrites admitting an open mapping onto an arc.*

PROOF. If a dendrite admits an open mapping onto an arc, then it obviously is in $\{[0, 1]\}_\circ$. Since the property of dendrites to have an open mapping onto an arc is preserved by taking open images (Corollary 6.66) and by taking open preimages (simply by composition), the family of all dendrites that admit an open mapping onto an arc is closed under the relation \simeq_\circ . This completes the proof.

The class of dendrites which admit an open mapping onto an arc is not closed under taking unions even if the union is still a dendrite. Therefore the opposite property is not hereditary. Namely, we have the following example.

6.69. EXAMPLE. *There exists a dendrite which does not admit any open mapping onto an arc and which is the union of two subdendrites admitting open mappings onto an arc.*

PROOF. Observe that each member G_α of the family \mathcal{C} (see (5.56)) admits an open mapping onto an arc by Corollary 6.64. Take the dendrite G_2 , recall that $E(G_2) = E_2$ and that $E_2^{(2)}$ is a singleton, and join to each point p_n of $E_2^{(1)} \setminus E_2^{(2)}$ an arc A_n such that

$$G_2 \cap A_n = \{p_n\} \quad \text{and} \quad \lim \text{diam } A_n = 0.$$

Then $X = G_2 \cup \bigcup\{A_n : n \in \mathbb{N}\}$ is a dendrite, and by construction $p_n \in \text{cl}E(X) \setminus E(X)$ for each $n \in \mathbb{N}$. It follows by Theorem 6.61 that X does not admit any open mapping onto an arc. Further, define the subdendrite X_1 of X to be irreducible with respect to containing the singleton $E_2^{(2)}$ and the union $\bigcup\{A_n : n \in \mathbb{N}\}$. Note that X_1 is homeomorphic to G_1 and that X is the union of two dendrites admitting open mappings onto an arc, namely of X_1 and of the copy of G_2 which is naturally embedded in X . The proof is complete.

Observe the following consequence of (6.17) and (6.18).

6.70. COROLLARY. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open mapping. Then for every $y \in \text{cl}E(Y) \setminus E(Y)$ there exists $x \in \text{cl}E(X) \setminus E(X)$ such that $f(x) = y$.*

Define

$$\mu(X) = \text{card}[\text{cl}E(X) \setminus E(X)].$$

Corollary 6.70 and Theorem 6.61 yield

6.71. COROLLARY. *If a dendrite Y is the image of a dendrite X under an open mapping, then $\mu(Y) \leq \mu(X)$.*

6.72. LEMMA. *Each element of $\{[0, 1]\}_\circ$ contains a free arc.*

Proof. Let $X \in \{[0, 1]\}_\circ$. Then X admits an open mapping onto an arc by Proposition 6.68, and thus condition (6.63) holds true by Theorem 6.61. Hence $\text{cl}E(X) \neq X$, and therefore $X \setminus \text{cl}E(X)$ is a non-empty open subset of X . By Proposition 4.13 we have $[R(X)]^d \subset \text{cl}E(X)$, thus $X \setminus \text{cl}E(X)$ contains a non-empty open set disjoint from $R(X)$, and thus it contains a free arc. The proof is complete.

6.73. PROPOSITION. *For every $Y \in \{[0, 1]\}_\circ$ there is an $X \in \{[0, 1]\}_\circ$ such that $Y <_\circ X$.*

Proof. Consider three cases depending on the structure of $E(Y)$.

Case 1: $\mu(Y) > 0$. By Lemma 6.72 one can choose a point p belonging to a free arc in Y . Let X be the one-point union of two copies of Y having p as the only common point. Then identification of the corresponding points of the copies of Y in X is an open mapping of X onto Y . On the other hand, $\mu(X) = 2\mu(Y)$, thus by Corollary 6.71 there is no open mapping of Y onto X .

Case 2: $\mu(Y) = 0$, i.e., $E(Y)$ is closed, and there is an isolated end point p of $E(Y)$. Take a free arc $A \subset Y$ ending at p . Fix a sequence $\{p_n\}$ of points of A converging to p , and take a sequence of arcs A_n such that

$$Y \cap A_n = \{p_n\} \quad \text{and} \quad \lim \text{diam } A_n = 0.$$

Then $Z = Y \cup \bigcup\{A_n : n \in \mathbb{N}\}$ is a dendrite. Let X be the one-point union of two copies of Z having p as the only common point. It can easily be verified that there is an open mapping from the dendrite X onto Z (identification of the

corresponding points) and from Z onto Y (namely, one which maps the end points of A_n 's onto p). So $Y \leq_{\mathbb{O}} X$. Further, $\mu(X) = 1$, thus there is no open mapping from Y onto X according to Corollary 6.71.

Case 3: $\mu(Y) = 0$, i.e., $E(Y)$ is closed, and there is no isolated end point p of $E(Y)$. Fix $p \in E(Y)$. Define X to be the one-point union of two copies of Y having p as the only common point. Then $Y \leq_{\mathbb{O}} X$ as previously, and $\mu(X) = 1$, so $Y <_{\mathbb{O}} X$ by the same argument. The proof is finished.

6.74. COROLLARY. *There is no maximal element in the class $\{[0, 1]\}_{\mathbb{O}}$.*

The following corollary is an immediate consequence of Corollary 6.71 and Proposition 6.73.

6.75. COROLLARY. *In $(\mathcal{D}, \leq_{\mathbb{O}})$ there exists a countable chain having no upper bound.*

We prove even more:

6.76. PROPOSITION. *For every $n \in \mathbb{N}$ there exists a dendrite X_n such that*

$$(6.77) \quad \mu(X_n) = n, \quad \text{and} \quad X_m \leq_{\mathbb{O}} X_n \quad \text{if} \quad m \leq n.$$

PROOF. Take a straight line segment A with end points a and b and with midpoint c . In the arc $ac \subset A$ fix a sequence of points p_n converging to c , and take a sequence of arcs A_n such that

$$A \cap A_n = \{p_n\} \quad \text{and} \quad \lim \text{diam } A_n = 0.$$

Put $X_1 = A \cup \bigcup \{A_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ and $n > 1$ define X_n to be the one-point union of n copies of X_1 , with b the only common point of any two of them. Observe that, for every $m, n \in \mathbb{N}$, condition (6.77) is satisfied. So, the proof is finished.

Now we pass to studying the class $\{D_3\}_{\mathbb{O}}$, i.e., the class of all dendrites X such that $X \simeq_{\mathbb{O}} D_3$. We will show that it coincides with the class of dendrites admitting open mappings onto D_3 , and we will provide a structural characterization of elements of this class. To this end we need some easy observations.

6.78. OBSERVATION. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open nonconstant mapping. Then $\text{cl } R(X) = X$ if and only if $\text{cl } R(Y) = Y$.*

PROOF. By (6.1), Y is a dendrite. For each dendrite X the condition $\text{cl } R(X) = X$ is equivalent to $\text{cl } E(X) = X$ (see Theorem 4.6). Since by Corollary 6.4 we have $f(E(X)) \subset E(Y)$, the conclusion $\text{cl } R(Y) = Y$ follows. On the other hand, if $R(X)$ is not dense, then X contains a free arc. Now it is enough to apply Corollary 6.7.

6.79. OBSERVATION. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open mapping. Then*

$$\text{ord}(x, X) < \omega \text{ for all } x \in X \quad \text{if and only if} \quad \text{ord}(y, Y) < \omega \text{ for all } y \in Y.$$

PROOF. By (6.1), Y is a dendrite. If Y contained a point of order ω , then according to Corollary 6.22 the dendrite X would contain a homeomorphic copy of Y , so it would be a point of order ω in X , a contradiction. For the opposite implication apply Proposition 6.5.

To make some further proofs shorter, we introduce the following notation. Given a dense subset $\{e_1, e_2, \dots\}$ of a dendrite X and an open set $U \subset X$, we denote by $e(U)$ the first point in the sequence e_1, e_2, \dots that belongs to U .

6.80. THEOREM. *For each dendrite X the following conditions are equivalent:*

(6.81) *there exists an open surjective mapping from X onto D_3 ;*

(6.82) $X \in \{D_3\}_\mathbb{O}$;

(6.83) $R(X)$ is dense in X , and $\text{ord}(x, X)$ is finite for each $x \in X$.

PROOF. (6.81) \Rightarrow (6.82) is obvious, and (6.82) \Rightarrow (6.83) is a consequence of Observations 6.78 and 6.79. We now prove (6.83) \Rightarrow (6.81).

Put $Y = D_3$. We will define by induction two increasing sequences of dendrites:

$$X_1 \subset X_2 \subset \dots \subset X \quad \text{and} \quad Y_1 \subset Y_2 \subset \dots \subset Y$$

such that $\bigcup\{X_n : n \in \mathbb{N}\}$ and $\bigcup\{Y_n : n \in \mathbb{N}\}$ are dense subsets of X and Y respectively. Further, for each $n \in \mathbb{N}$ we will define monotone retractions $\pi_n : X_{n+1} \rightarrow X_n$ and $\varrho_n : Y_{n+1} \rightarrow Y_n$ which are bonding mappings for inverse sequences $\{X_n, \pi_n\}$ and $\{Y_n, \varrho_n\}$ respectively, in such a way that $X = \varprojlim\{X_n, \pi_n\}$ and $Y = \varprojlim\{Y_n, \varrho_n\}$. Next we will define a surjective mapping $f : X \rightarrow Y$ so that $f_n = f|X_n : X_n \rightarrow Y_n$ is an open surjection for each $n \in \mathbb{N}$, and the diagrams

$$(6.84) \quad \begin{array}{ccc} X_n & \xleftarrow{\pi_n} & X_{n+1} \\ f_n \downarrow & & \downarrow f_{n+1} \\ Y_n & \xleftarrow{\varrho_n} & Y_{n+1} \end{array}$$

commute. Then the openness of $f = \varprojlim f_n$ will follow.

Let $\{e_1, e_2, \dots\}$ be a dense subset of $E(X)$ and $\{e'_1, e'_2, \dots\}$ be a dense subset of $E(Y)$. Put $X_1 = e_1 e_2$, $Y_1 = e'_1 e'_2$ and let $f_1 : X_1 \rightarrow Y_1$ be a homeomorphism such that

$$f_1(e_1) = e'_1, \quad f_1(e_2) = e'_2, \quad \text{and} \quad f_1(X_1 \cap R(X)) = Y_1 \cap R(Y).$$

Assume that the dendrites $X_n \subset X$, $Y_n \subset Y$, and open surjections $f_n : X_n \rightarrow Y_n$ are defined for some $n \in \mathbb{N}$ such that $f_n(X_n \cap R(X)) = Y_n \cap R(Y)$. The union of X_n and of all arcs of the form $pe(K)$, where $p \in X_n \cap R(X)$ and K is a component of $X \setminus \{p\}$ disjoint from X_n , is denoted by X_{n+1} . Similarly, the union of all arcs of the form $f_n(p)e'(L)$, where L stands for the (only) component of $Y \setminus \{f_n(p)\}$ disjoint from Y_n , is denoted by Y_{n+1} . Observe that for each $p \in X_n \cap R(X)$ we have

$$\text{ord}(p, X_{n+1}) = \text{ord}(p, X) \quad \text{and} \quad \text{ord}(f_n(p), Y_{n+1}) = \text{ord}(f_n(p), Y) = 3.$$

Define a surjection $f_{n+1} : X_{n+1} \rightarrow Y_{n+1}$ putting $f_{n+1}|_{X_n} = f_n$, and for each ramification point $p \in X_n \cap R(X)$ and for each end point $e(K)$ let $f_{n+1}|_{pe(K)} : pe(K) \rightarrow f_n(p)e'(L)$ be a homeomorphism such that $f_{n+1}(p) = f_n(p)$, $f_{n+1}(e(K)) = e'(L)$, and $f_{n+1}(pe(K) \cap R(X)) = f_{n+1}(p)e'(L) \cap R(Y)$. Thus the dendrites $X_n \subset X$, $Y_n \subset Y$, and mappings $f_n : X_n \rightarrow Y_n$ are defined for each $n \in \mathbb{N}$. Note that the mappings f_n are open.

By construction, the sequences $\{X_n\}$ and $\{Y_n\}$ are increasing and $\bigcup\{X_n : n \in \mathbb{N}\}$ and $\bigcup\{Y_n : n \in \mathbb{N}\}$ are dense subsets of X and Y respectively. Thus, if for each point $x \in X_k \subset \bigcup\{X_n : n \in \mathbb{N}\}$ we put $f(x) = f_k(x)$, then the mapping f can be continuously (and uniquely) extended from $\bigcup\{X_n : n \in \mathbb{N}\}$ to X . In this way $f : X \rightarrow Y$ is defined, and it is a surjection.

Since for each $n \in \mathbb{N}$ we have $X_n \subset X_{n+1}$ and $Y_n \subset Y_{n+1}$, by Theorem 4.16 there exist monotone retractions $\pi_n : X_{n+1} \rightarrow X_n$ and $\varrho_n : Y_{n+1} \rightarrow Y_n$. We can view π_n and ϱ_n as bonding mappings for the inverse sequences $\{X_n, \pi_n\}$ and $\{Y_n, \varrho_n\}$ respectively, and we see that

$$X = \varprojlim\{X_n, \pi_n\} \quad \text{and} \quad Y = \varprojlim\{Y_n, \varrho_n\}.$$

One can verify in a routine way that for each $n \in \mathbb{N}$ the diagrams (6.84) commute. Thus the sequence $\{f_n\}$ determines a mapping from the inverse sequence $\{X_n, \pi_n\}$ to $\{Y_n, \varrho_n\}$ with $f : X \rightarrow Y$ being the induced limit mapping. The reader is referred e.g. to [16], pp. 138 and 139, for the definitions of these concepts. Since $f_n = f|_{X_n} : X_n \rightarrow Y_n$ is open for each $n \in \mathbb{N}$, the openness of f follows from Theorem 4 of [33], p. 61. The proof is complete.

6.85. COROLLARY. *The standard universal dendrite D_3 of order 3 is the unique least element in the equivalence class $\{D_3\}_\circ$.*

PROOF. It follows from the equivalence of (6.81) and (6.82) that D_3 is the least element in $\{D_3\}_\circ$. Uniqueness is a consequence of the fact that an open image of D_3 is homeomorphic to D_3 (see Proposition 6.2).

Now we are going to study the class $\{D_{\{3,\omega\}}\}_\circ$. We need a result concerning some special homeomorphisms between arcs. The result is proved in [14] as Lemma 6.1.

6.86. LEMMA. *Let*

$$A_1, A_2, \dots, A_j, \dots \quad \text{and} \quad B_1, B_2, \dots, B_j, \dots$$

be two sequences of subsets of the open unit interval $(0,1)$ such that

(6.87) *for each $j \in \mathbb{N}$ the sets A_j and B_j are either both countable dense in $(0,1)$, or are both empty, and*

(6.88) *for each $j, k \in \mathbb{N}$ with $j \neq k$ we have $A_j \cap A_k = \emptyset = B_j \cap B_k$.*

Then there is a homeomorphism $h : [0, 1] \rightarrow [0, 1]$ with $h(A_j) = B_j$ for each $j \in \mathbb{N}$.

6.89. PROPOSITION. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open mapping. Then:*

- (a) $A \cap R_\omega(X)$ is dense in A for every arc A in X if and only if $B \cap R_\omega(Y)$ is dense in B for every arc B in Y ;
- (b) $A \cap R_{\mathbb{N}}(X)$ is dense in A for every arc A in X if and only if $B \cap R_{\mathbb{N}}(Y)$ is dense in B for every arc B in Y .

PROOF. (a) “only if”. Let B be an arc in Y . Then by Corollary 6.22 there is an arc A in X such that $f|_A : A \rightarrow B$ is a homeomorphism. By the assumption, $A \cap R_\omega(X)$ is dense in A . Thus the conclusion follows by Proposition 6.5.

(a) “if”. Let A be an arc in X . Then by the assumption the continuum $f(A)$ contains points of order ω in Y . So A contains a point of order ω in X by Proposition 6.3.

(b) “only if”. Let B be an arc in Y . Then by Corollary 6.22 there is an arc A in X such that $f|_A : A \rightarrow B$ is a homeomorphism. By the assumption, $A \cap R_{\mathbb{N}}(X)$ is dense in A . Thus there is a convergent sequence of points $x_n \in A \cap R_{\mathbb{N}}(X)$. Denote by K_n a sequence of components of $X \setminus A$ such that $x_n \in \text{cl } K_n$. Then $f(K_n)$ is a null-sequence of open sets in Y with one-point boundary. Thus $f(K_n) \cap B = \emptyset$ for almost all n , and hence $f(x_n) \in B \cap R(Y)$ for almost all n . Because the order of ramification points cannot be increased under an open mapping (see Proposition 6.3), we conclude that $f(x_n) \in B \cap R_{\mathbb{N}}(Y)$, which finishes the argument.

(b) “if”. Let A be an arc in X . If A contains points of order 2 and ω only, then the continuum $f(A)$ contains points of order 2 and ω only by Propositions 6.3 and 6.5. The proof is complete.

Propositions 6.3 and 6.5 lead to the following corollary.

6.90. COROLLARY. *Let X and Y be dendrites and $f : X \rightarrow Y$ a surjective open mapping. Then*

$$f(R_{\mathbb{N}}(X)) \supset R_{\mathbb{N}}(Y) \quad \text{and} \quad f(R_\omega(X)) = R_\omega(Y).$$

6.91. THEOREM. *For each dendrite X the following conditions are equivalent:*

- (6.92) *there exists an open surjective mapping from X onto $D_{\{3,\omega\}}$;*
- (6.93) $X \in \{D_{\{3,\omega\}}\}_{\mathbb{O}}$;
- (6.94) *for every arc A in X the sets $A \cap R_{\mathbb{N}}(X)$ and $A \cap R_\omega(X)$ are both dense in A .*

PROOF. The implication (6.92) \Rightarrow (6.93) is obvious, and (6.93) \Rightarrow (6.94) is a consequence of Proposition 6.89. We now prove (6.94) \Rightarrow (6.92). We use the same method as in the proof of (6.83) \Rightarrow (6.81). However, we have to construct the required mapping f from X onto $D_{\{3,\omega\}}$ much more carefully than in the previous proof.

Put $Y = D_{\{3,\omega\}}$. Just as in the proof of (6.83) \Rightarrow (6.81) let $\{e_1, e_2, \dots\}$ be a dense subset of $E(X)$ and $\{e'_1, e'_2, \dots\}$ be a dense subset of $E(Y)$. Put $X_1 = e_1 e_2$ and $Y_1 = e'_1 e'_2$. By Lemma 6.86 applied to the sequence $X_1 \cap R_{\mathbb{N}}(X)$, $X_1 \cap R_{\omega}(X)$ of countable dense subsets of X_1 and to the sequence $Y_1 \cap R_{\mathbb{N}}(Y)$, $Y_1 \cap R_{\omega}(Y)$ of countable dense subsets of Y_1 there exists a homeomorphism $f_1 : X_1 \rightarrow Y_1$ such that $f_1(e_1) = e'_1$, $f_1(e_2) = e'_2$ and $f_1(X_1 \cap R_{\mathbb{N}}(X)) = Y_1 \cap R_{\mathbb{N}}(Y)$ as well as $f_1(X_1 \cap R_{\omega}(X)) = Y_1 \cap R_{\omega}(Y)$.

Assume that dendrites $X_n \subset X$, $Y_n \subset Y$, and open surjections $f_n : X_n \rightarrow Y_n$ are defined for some $n \in \mathbb{N}$ such that

$$(6.95) \quad f_n(X_n \cap R_{\mathbb{N}}(X)) = Y_n \cap R_{\mathbb{N}}(Y)$$

and simultaneously

$$(6.96) \quad f_n(X_n \cap R_{\omega}(X)) = Y_n \cap R_{\omega}(Y).$$

The union of X_n and of all arcs of the form $pe(K)$, where $p \in X_n \cap R(X)$ and K is a component of $X \setminus \{p\}$ disjoint from X_n , is denoted by X_{n+1} ; and the union of Y_n and of all arcs $f_n(p)e'(L)$, where L is a component of $Y \setminus \{f_n(p)\}$ disjoint from Y_n , is denoted by Y_{n+1} .

Observe that if $p \in X_n \cap R_{\mathbb{N}}(X)$, then

$$\text{ord}(p, X_{n+1}) = \text{ord}(p, X) \quad \text{and} \quad \text{ord}(f_n(p), Y_{n+1}) = \text{ord}(f_n(p), Y) = 3,$$

and if $p \in X_n \cap R_{\omega}(X)$, then

$$\text{ord}(p, X_{n+1}) = \text{ord}(p, X) = \omega \quad \text{and} \quad \text{ord}(f_n(p), Y_{n+1}) = \text{ord}(f_n(p), Y) = \omega.$$

Define a surjection $f_{n+1} : X_{n+1} \rightarrow Y_{n+1}$ by the following three conditions. First, $f_{n+1}|_{X_n} = f_n$. Second, for each ramification point $p \in X_n \cap R_{\mathbb{N}}(X)$ and for each end point $e(K)$ define $f_{n+1}|_{pe(K)} : pe(K) \rightarrow f_n(p)e'(L)$ to be a homeomorphism (whose existence follows from Lemma 6.86 properly applied) such that $f_{n+1}(p) = f_n(p)$, $f_{n+1}(e(K)) = e'(L)$, and $f_{n+1}(pe(K) \cap R_{\mathbb{N}}(X)) = f_{n+1}(p)e'(L) \cap R_{\mathbb{N}}(Y)$ as well as $f_{n+1}(pe(K) \cap R_{\omega}(X)) = f_{n+1}(p)e'(L) \cap R_{\omega}(Y)$. Third, for each $p \in X_n \cap R_{\omega}(X)$ and for each $e(K)$ define $f_{n+1}|_{pe(K)} : pe(K) \rightarrow f_n(p)e'(L)$ to be a homeomorphism such that $f_{n+1}(p) = f_n(p)$, $f_{n+1}(e(K)) = e'(L)$, and $f_{n+1}(pe(K) \cap R_{\mathbb{N}}(X)) = f_{n+1}(p)e'(L) \cap R_{\mathbb{N}}(Y)$ as well as $f_{n+1}(pe(K) \cap R_{\omega}(X)) = f_{n+1}(p)e'(L) \cap R_{\omega}(Y)$ (use Lemma 6.86 again). Observe that f_{n+1} so defined is continuous, surjective, and open.

Thus the dendrites $X_n \subset X$, $Y_n \subset Y$ and mappings $f_n : X_n \rightarrow Y_n$ are defined for each $n \in \mathbb{N}$. The rest of the proof mimicks the corresponding part of the proof of Theorem 6.80. The proof is complete.

In much the same way as Corollary 6.85 was deduced from Theorem 6.80, the following is a consequence of Theorem 6.91.

6.97. COROLLARY. *The standard universal dendrite $D_{\{3,\omega\}}$ is the unique least element in the equivalence class $\{D_{\{3,\omega\}}\}_{\circ}$.*

Now we shall characterize elements of the class $\{D_\omega\}_\mathbb{O}$. To this end we need a definition and some lemmas. We start with two propositions concerning confluent mappings onto locally connected continua.

6.98. PROPOSITION. *Let $f : X \rightarrow Y$ be a surjective confluent mapping from a continuum X onto a locally connected continuum Y , and let B be a closed subset of Y . Then the image of every component of $X \setminus f^{-1}(B)$ is a component of $Y \setminus B$.*

PROOF. Suppose there is a component K of $X \setminus f^{-1}(B)$ such that $f(K)$ is not a component of $Y \setminus B$. Then $f(K)$ is a proper subset of a component of $Y \setminus B$, and therefore by local connectedness of Y there exists a continuum $M \subset Y \setminus B$ such that $M \cap f(K) \neq \emptyset \neq M \setminus f(K)$. Let A be a component of $f^{-1}(M)$ such that $A \cap K \neq \emptyset$. Then $A \subset f^{-1}(M) \subset f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$, whence $A \subset K$ by the definition of K . This implies $f(A) \subset f(K)$. Since $f(A) = M$ by confluence of f , we get $M \subset f(K)$, a contradiction.

6.99. PROPOSITION. *Let $f : X \rightarrow Y$ be a surjective confluent mapping from a continuum X onto a locally connected continuum Y , let A be a closed subset of X , and let K be a component of $X \setminus A$. If $f(K) \setminus f(A) \neq \emptyset$, then $f(K)$ contains a component of $Y \setminus f(A)$.*

PROOF. Note that $B = f(A)$ is a closed subset of Y , and that $X \setminus f^{-1}(B) \subset X \setminus A$. Take $p \in K \setminus f^{-1}(B)$, and let L be the component of $X \setminus f^{-1}(B)$ that contains p . Then $L \subset K$, whence $f(L) \subset f(K)$. By Proposition 6.98 the set $f(L)$ is a component of $Y \setminus B$, and so we are done.

Let X be a dendrite. We define a subset $P(X)$ of $X \setminus E(X)$ to consist of those points p for which there are an arc $A \subset X$ and a sequence of points $p_n \in (A \setminus \{p\}) \cap R_\mathbb{N}(X)$ such that $p = \lim p_n$. So, a point p of X is in $P(X)$ if and only if there exists an arc $A \subset X$ such that $p \in (A \cap R_\mathbb{N}(X))^d \setminus E(X)$.

6.100. LEMMA. *Let X be a dendrite and $f : X \rightarrow Y$ a surjective open mapping. Then $P(Y) \subset f(P(X))$.*

PROOF. By (6.1), Y is a dendrite. Assume $y \in P(Y)$. Then there exist an arc $L \subset Y$ containing y not as its end point, and a sequence of points $y_n \in L$ such that $y = \lim y_n$ and $y_n \in R_\mathbb{N}(Y)$. For each $n \in \mathbb{N}$ let L_n be an arc in Y such that $L \cap L_n = \{y_n\}$. By hereditary local connectedness of Y we infer that $\{L_n\}$ is a null-sequence. Put $B = L \cup \bigcup \{L_n : n \in \mathbb{N}\}$. Take $p \in X$ with $f(p) = y$. According to Corollary 6.22 there exists in X a continuum K containing p such that $f|_K : K \rightarrow B$ is a homeomorphism. Thus $(f|_K)^{-1}(L)$ is an arc that contains a sequence of points p_n such that $f(p_n) = y_n$. By Proposition 6.5 we have $p_n \in R_\mathbb{N}(X)$ for each $n \in \mathbb{N}$. Thus $p \in P(X)$, so $y = f(p) \in f(P(X))$, and the inclusion is shown.

6.101. LEMMA. *Let $f : X \rightarrow Y$ be a surjective open mapping defined on a dendrite X such that $P(X) \cap R_\omega(X) = \emptyset$. Then $f(P(X)) \subset E(Y) \cup P(Y)$.*

Proof. Again, Y is a dendrite by (6.1). Let $p \in P(X)$. Take a sequence of points p_n as in the definition of $P(X)$. Denote by K_n a component of $X \setminus \{p_n\}$ such that $\{K_n\}$ is a null-sequence with $\{p\}$ as its limit. Then $\{f(K_n)\}$ is also a null-sequence. By Proposition 6.99 the images $f(K_n)$ contain some components of $Y \setminus \{f(p_n)\}$. Since $p_n \in R_\omega(X)$ by the definition of p_n and since $f(R_\mathbb{N}(X)) \subset Y \setminus R_\omega(Y)$ by Proposition 6.3, we conclude that either $f(p) \in E(Y)$, or $f(p_n) \in R_\mathbb{N}(Y)$ for almost all $n \in \mathbb{N}$. To finish the proof we have to show that if $f(p) \in Y \setminus E(Y)$, then there is a subsequence of $\{f(p_n)\}$ which lies on an arc in Y .

Assume the contrary. Put, for short, $q = f(p)$ and $q_n = f(p_n)$ for each $n \in \mathbb{N}$. We claim that

(6.102) there is a subsequence $\{q_{n_k}\}$ no three points of which lie on one arc in Y .

Indeed, take $q_{n_1} = q_1$. Since only finitely many points q_n lie on the arc $q_{n_1}q$, there is a q_{n_2} not in this arc. Similarly, there is a q_{n_3} outside $q_{n_1}q \cup q_{n_2}q$, and so on. Thus (6.102) is shown.

The condition $P(X) \cap R_\omega(X) = \emptyset$ implies that p is of finite order in X , whence, by Proposition 6.3, q is of finite order in Y . Therefore by Theorem 4.4 there is a component of $Y \setminus q$ containing infinitely many q_{n_k} . Without loss of generality we can assume that all q_{n_k} lie in one component of $Y \setminus q$. Further, by construction, for any two points q_{n_i} and q_{n_j} the union of the arcs $q_{n_i}q$ and $q_{n_j}q$ is a simple triod with q , q_{n_i} and q_{n_j} as end points. Put $B = \{q\} \cup \{q_{n_k} : k \in \mathbb{N}\}$ and note that B is closed. It follows from hereditary unicoherence of Y that if the boundary of a component of $Y \setminus B$ contains more than one point, then this boundary contains the whole B . Moreover, there is only one such component. Call it L .

Let us come back to the dendrite X . Since the points p_{n_k} lie on an arc, and p is their accumulation point, there exist components of $X \setminus (\{p\} \cup \{p_{n_k} : k \in \mathbb{N}\})$ with two-point boundaries and of arbitrarily small diameter. Let K be one. By Proposition 6.99, $f(K)$ contains a component of $Y \setminus B$, and the boundary of this component has at least two points. Thus the component must be just L , i.e. $L \subset f(K)$, which contradicts the diameter of K being arbitrarily small. The proof is complete.

6.103. **LEMMA.** *Let $f : X \rightarrow Y$ be a surjective open mapping defined on a dendrite X such that $P(X) \cap R_\omega(X) = \emptyset$. If $P(Y)$ is finite, then so is $P(X)$.*

Proof. By (6.1), Y is a dendrite. According to Lemma 6.101, $P(X) = P_1 \cup P_2$ with $f(P_1) \subset E(Y)$ and $f(P_2) \subset P(Y)$. Since $P(X) \subset X \setminus E(X)$ by its definition, we see that $P_1 \subset f^{-1}(E(Y)) \setminus E(X)$, and so P_1 is finite by (6.18). Since $f(P_2)$ is finite being a subset of $P(Y)$, we conclude from (6.17) that P_2 is also finite. The proof is finished.

6.104. **PROPOSITION.** *Let a dendrite X be the union of k dendrites X_i :*

$$(6.105) \quad X = X_1 \cup \dots \cup X_k$$

such that, for any two distinct $i, j \in \{1, \dots, k\}$, either $X_i \cap X_j = \{p\}$ for some $p \in E(X_i) \cap E(X_j)$ or $X_i \cap X_j = \emptyset$. If, for each $i \in \{1, \dots, k\}$, there exists an open surjective mapping $f_i : X_i \rightarrow D_\omega$, then there exists an open surjective mapping $f : X \rightarrow D_\omega$.

Proof. Define $f|X_1 = f_1$. For each $i_1 \in \{1, \dots, k\}$ such that

$$(6.106) \quad i_1 \neq 1 \quad \text{and} \quad X_{i_1} \cap X_1 \neq \emptyset$$

define p_{i_1} by $p_{i_1} \in X_{i_1} \cap X_1$. Then $p_{i_1} \in E(X_{i_1})$, whence $f_{i_1}(p_{i_1}) \in E(D_\omega)$ by Corollary 6.4. According to Proposition 4.29 there exists a homeomorphism $h_{i_1} : D_\omega \rightarrow D_\omega$ such that $h_{i_1}(f_{i_1}(p_{i_1})) = f(p_{i_1}) = f_1(p_{i_1})$. Define $f|X_{i_1} : X_{i_1} \rightarrow D_\omega$ by $f|X_{i_1} = h_{i_1} \circ f_{i_1}$. Next, for each $i_2 \in \{1, \dots, k\}$ distinct from 1 and from all indices i_1 already considered and such that $X_{i_2} \cap X_{i_1} \neq \emptyset$ for some i_1 satisfying (6.106) define p_{i_2} as the only point of $X_{i_2} \cap X_{i_1}$. Again $p_{i_2} \in E(X_{i_2})$, whence $f_{i_2}(p_{i_2}) \in E(D_\omega)$. Applying Proposition 4.29 once more we find a homeomorphism $h_{i_2} : D_\omega \rightarrow D_\omega$ with $h_{i_2}(f_{i_2}(p_{i_2})) = f(p_{i_2})$, and we define $f|X_{i_2} : X_{i_2} \rightarrow D_\omega$ by $f|X_{i_2} = h_{i_2} \circ f_{i_2}$, and so on. After a finite number of steps we exhaust all members of the union in (6.106), and we have f defined on the whole dendrite X . The reader can verify that f is an open surjection, as required.

Given two points p and q of a dendrite X , we denote by $X(p, q)$ the closure of the unique component of $X \setminus \{p, q\}$ such that $pq \subset X(p, q)$. Note that $\text{bd } X(p, q) = \{p, q\} \setminus E(X)$; in particular, if $p, q \in E(X)$, then $X(p, q) = X$.

Now we are ready to formulate and to prove the characterization of dendrites in $\{D_\omega\}_\emptyset$.

6.107. THEOREM. *For each dendrite X the following conditions are equivalent:*

$$(6.108) \quad \text{there exists an open surjective mapping from } X \text{ onto } D_\omega;$$

$$(6.109) \quad X \in \{D_\omega\}_\emptyset;$$

$$(6.110) \quad X \text{ has the following three properties:}$$

- (a) $R_\omega(X)$ is dense in X ;
- (b) $P(X)$ is finite;
- (c) $R_\omega(X) \cap P(X) = \emptyset$.

Proof. The implication (6.108) \Rightarrow (6.109) is obvious. Assume (6.109). Then $R(D_\omega) = R_\omega(D_\omega)$ by (4.24), and it follows from (4.22) that $R_\omega(D_\omega)$ is dense in D_ω . Then Observation 6.78 leads to (6.110)(a). Further, note that $P(D_\omega) = \emptyset$, and so D_ω satisfies conditions (b) and (c) of (6.110). Thus, if $X \in \{D_\omega\}_\emptyset$, then X satisfies (b) by Lemmas 6.100 and 6.103. Finally, the equality in (c) is invariant under open mappings between dendrites X and Y by Lemma 6.100 and Corollary 6.90, and it is also inverse invariant under such mappings by Lemma 6.101 and the same corollary. Therefore condition (c) follows.

Finally, we prove (6.110) \Rightarrow (6.108). First we reduce the general case to the case when $P(X) = \emptyset$. In fact, (b) and (c) imply, by Theorem 4.4, that $P(X)$ disconnects X into finitely many (say k) components. Denote by X_i their closures,

for $i \in \{1, \dots, k\}$. Then $p \in P(X)$ implies that $p \in E(X_i)$ for some $i \in \{1, \dots, k\}$, whence

$$P(X_i) = \emptyset \quad \text{for each } i \in \{1, \dots, k\}.$$

We see that all the assumptions of Proposition 6.104 concerning the structure of X and X_i are satisfied. So, if we construct, for each $i \in \{1, \dots, k\}$, an open surjection $f_i : X_i \rightarrow D_\omega$ then the required mapping f exists by Proposition 6.104.

Assume then that, besides (6.110), the condition $P(X) = \emptyset$ is satisfied. We claim that

$$(6.111) \quad A \cap R_\omega(X) \text{ is dense in } A \text{ for each arc } A \subset X.$$

In fact, otherwise some arc A in X contains a subarc A' such that $A' \cap R_\omega(X)$ is empty. Hence $A' \cap R_{\mathbb{N}}(X)$ is dense in A' (because otherwise we would have a free arc contained in A' , a contradiction to (a)), which implies that $A' \subset P(X)$ contrary to (b). So (6.111) is established.

We will construct an inverse sequence $\{X_n, f_n\}$ such that:

- 1) X is homeomorphic to $\varprojlim \{X_n, f_n\}$;
- 2) X_1 is homeomorphic to D_ω ;
- 3) $X_n \subset X_{n+1} \subset X$ for each $n \in \mathbb{N}$;
- 4) $f_n : X_{n+1} \rightarrow X_n$ are open retractions for each $n \in \mathbb{N}$;
- 5) $\text{cl}(\bigcup \{X_n : n \in \mathbb{N}\}) = X$.

Then the natural projection from X onto X_1 will be the required open mapping.

Fix a sequence of points e_1, e_2, \dots of X such that $\text{cl}\{e_1, e_2, \dots\} = X$. Given an arc $A \subset X$ with end points a and b , we define $M(A)$ by the following inductive procedure. Take $M_0 = \{a, b\}$ and $M_1 = A$. For every $n \in \mathbb{N}$ we define

$$M_{n+1} = M_n \cup \bigcup \{pe(K) : p \in (M_n \setminus M_{n-1}) \cap R_\omega(X) \text{ and } K \text{ is a component of } X \setminus \{p\} \text{ disjoint from } M_n\}.$$

Put $M(A) = \text{cl}(\bigcup \{M_n : n \in \mathbb{N}\})$. Note that $M(A)$ is a maximal (in the sense of inclusion) dendrite homeomorphic to D_ω contained in $X(a, b)$.

We put $X_0 = \{e_1\}$ and $X_1 = M(e_1e_2)$. For each $n \geq 1$ we put

$$X_{n+1} = X_n \cup \bigcup \{M(pe(K)) : p \in (X_n \setminus X_{n-1}) \cap R_{\mathbb{N}}(X) \text{ and } K \text{ is a component of } X \setminus \{p\} \text{ disjoint from } X_n\},$$

and note that each X_n is a dendrite.

Now we define $f_n : X_{n+1} \rightarrow X_n$. Let $f_1|_{X_1}$ be the identity. We will define $f_1|M(pe(K))$ separately for each p and K according to the definition of X_2 . Fix $c \in e_1e_2 \setminus \{e_1, e_2\}$ such that $\text{ord}(c, X) = 2$ and consider three cases depending on whether $p \in e_1c$, $p \in ce_2$, or $p \in X_1 \setminus e_1e_2$.

(i) If $p \in e_1c$, then for each component K of $X \setminus \{p\}$ disjoint from X_1 the dendrite $M(pe(K))$ is homeomorphic to D_ω , so define $f_1|M(pe(K))$ to be a homeomorphism onto $X_1(e_1, p)$ such that $f_1(p) = p$.

(ii) If $p \in ce_2$, then we define $f_1|M(pe(K))$ similarly, as a homeomorphism onto $X_1(p, e_2)$.

(iii) If $p \in X_1 \setminus e_1e_2$, then for each component K of $X \setminus \{p\}$ disjoint from X_1 the dendrite $M(pe(K))$ is again homeomorphic to D_ω . Since X_1 , being homeomorphic to D_ω , has ramification points of order ω only, and since $p \in R_{\mathbb{N}}(X)$, we conclude that $\text{ord}(p, X_1) = 2$, and therefore by Theorem 4.4 there is exactly one component L of $X_1 \setminus \{p\}$ disjoint from e_1e_2 . We define $f_1|M(pe(K))$ to be a homeomorphism onto $\text{cl} L$ such that $f_1(p) = p$. Thus f_1 is well-defined. Continuity of f_1 is a consequence of the fact that $P(X) = \emptyset$, and by construction f_1 is an open retraction.

Now we define $f_n : X_{n+1} \rightarrow X_n$ for $n > 1$. As previously, for each $p \in (X_n \setminus X_{n-1}) \cap R_{\mathbb{N}}(X)$ and for each component K of $X \setminus \{p\}$ disjoint from X_n , the dendrite $M(pe(K))$ is homeomorphic to D_ω . Since $\text{ord}(p, X_n) = 2$, there is exactly one component L of $X_n \setminus \{p\}$ disjoint from X_{n-1} . Since $p \notin X_{n-1}$, L contains no ramification point of finite order in L , and so $\text{cl} L$ is homeomorphic to D_ω . Define $f_n|M(pe(K))$ to be a homeomorphism onto $\text{cl} L$ such that $f_n(p) = p$. Thus f_n is well-defined, continuous since $P(X) = \emptyset$, and an open retraction simply by its definition.

For each $m \in \mathbb{N}$ let $\psi_m : \varprojlim\{X_n, f_n\} \rightarrow X_m$ denote the natural projection. Since for each $n \in \mathbb{N}$ the bonding mappings f_n are open, so are ψ_n (see [33], Theorem 5, p. 61). Further, it is evident from the construction that for each $x \in \varprojlim\{X_n, f_n\}$ the diameter of $f_n^{-1}(\psi_n(x))$ tends to zero as $n \rightarrow \infty$. Since the f_n are retractions, $\varprojlim\{X_n, f_n\}$ is homeomorphic to $X = \text{cl}(\bigcup\{X_n : n \in \mathbb{N}\})$ (see [1], Theorem I, p. 348). Recall that X_1 is homeomorphic to D_ω . Neglecting the homeomorphisms for simplicity, we see that $\psi_1 : X \rightarrow D_\omega$ is the required open mapping. The proof is finished.

In much the same way as Corollaries 6.85 and 6.97 were deduced from Theorems 6.80 and 6.91, the following is a consequence of Theorem 6.107.

6.112. COROLLARY. *The standard universal dendrite D_ω is the unique least element in the equivalence class $\{D_\omega\}_\mathbb{O}$.*

Having the above characterizations of elements in the classes $\{[0, 1]\}_\mathbb{O}$, $\{D_3\}_\mathbb{O}$, $\{D_{\{3, \omega\}}\}_\mathbb{O}$ and $\{D_\omega\}_\mathbb{O}$ (Theorem 6.61, Proposition 6.68 and Theorems 6.80, 6.91 and 6.107), and knowing the existence of unique least elements in these classes (Corollaries 6.85, 6.97 and 6.112), one could expect the following two statements to hold.

(1) If a dendrite M is a minimal element in $(\mathcal{D}, \leq_\mathbb{O})$, then M is the unique least element in the equivalence class $\{M\}_\mathbb{O}$.

(2) For any minimal element M in $(\mathcal{D}, \leq_\mathbb{O})$ the following two conditions are equivalent:

- (i) there exists an open mapping from X onto M ;
- (ii) $X \in \{M\}_\mathbb{O}$.

We now show that both (1) and (2) fail. This can be seen by the following example.

6.113. EXAMPLE. *There are two minimal elements M_1 and M_2 in $(\mathcal{D}, \leq_\mathbb{O})$ such that $M_1 \simeq_\mathbb{O} M_2$ and $M_1 \neq_\mathbb{O} M_2$.*

PROOF. Consider two strings of dendrites (defined by (6.31)–(6.34)) with D_3 and D_ω alternately:

$$M_1 = \bigcup \{X_n \cup b_n a_{n+1} : n \in \mathbb{N}\} \cup \{p\}$$

with $X_n = D_3$ if n is odd and $X_n = D_\omega$ if n is even, and

$$M_2 = \bigcup \{Y_n \cup d_n c_{n+1} : n \in \mathbb{N}\} \cup \{q\}$$

where $Y_n = D_\omega$ if n is odd and $Y_n = D_3$ if n is even. In particular, $X_1 = D_3$, while $Y_1 = D_\omega$. To show that $M_1 \simeq_\mathbb{O} M_2$ recall that a_1 is the extreme point of X_1 , and take the one-point union X of two copies of M_1 meeting in a_1 . Do the same for M_2 and c_1 , and denote by Y the resulting one-point union. Then there is an open mapping f from X onto M_1 , namely identification of the corresponding points in the two copies of M_1 . Similarly, there is an open mapping g from Y onto M_2 . Further, X and Y are homeomorphic. If h is the homeomorphism, then we have

$$M_1 \xleftarrow{f} X \xrightarrow{h} Y \xrightarrow{g} M_2$$

and therefore $M_1 \simeq_\mathbb{O} M_2$. Since M_1 and M_2 are uniquely minimal in $(\mathcal{D}, \leq_\mathbb{O})$ according to Theorem 6.45, we have $M_1 \neq_\mathbb{O} M_2$ by Proposition 6.39.

Note that, by the above example, (1) is evidently not true. Taking $X = M_2$ and $M = M_1$ we have (ii) and not (i), whence (2) is false as well.

Now we show that there is no maximal element in $(\mathcal{D}, \leq_\mathbb{O})$. We start with some lemmas.

6.114. LEMMA. *For each dendrite X there is an increasing sequence of sub-dendrites whose union is dense in X :*

$$X_1 \subset X_2 \subset \dots \subset \bigcup \{X_n : n \in \mathbb{N}\} \subset \text{cl} \left(\bigcup \{X_n : n \in \mathbb{N}\} \right) = X,$$

monotone retractions $\pi_n : X_{n+1} \rightarrow X_n$, and points $r_n \in X_n$ such that

- (i) $X = \varprojlim \{X_n, \pi_n\}$;
- (ii) $\pi_n^{-1}(x)$ is degenerate for each $n \in \mathbb{N}$ and each $x \in X_n \setminus \{r_n\}$;
- (iii) $R(\pi_n^{-1}(r_n)) \subset \{r_n\}$ for each $n \in \mathbb{N}$;
- (iv) $\text{ord}(r_n, X_{n+1}) = \text{ord}(r_n, X)$ for each $n \in \mathbb{N}$;
- (v) $R(X) \subset \{r_n : n \in \mathbb{N}\}$.

PROOF. Let $\{e_1, e_2, \dots\}$ be a dense subset of $E(X)$ (we do not require that these end points are distinct). Define X_1 to be a maximal arc in X . Assume X_n is

defined. If $e_n \in X_n$, we put $r_n = e_n$, $X_{n+1} = X_n$, and we define $\pi_n : X_{n+1} \rightarrow X_n$ to be the identity. Otherwise choose $r_n \in X_n$ such that $e_n r_n \cap X_n = \{r_n\}$. For every component K of $X \setminus X_n$ satisfying $X_n \cap \text{cl} K = \{r_n\}$ we choose $q(K) \in K \cap E(X)$, and we define

$$X_{n+1} = X_n \cup \bigcup \{r_n q(K) : K \text{ is a component of } X \setminus X_n \text{ with } X_n \cap \text{cl} K = \{r_n\}\}.$$

Finally, define $\pi_n : X_{n+1} \rightarrow X_n$ to be the natural monotone retractions. The inductive procedure is finished. The reader can verify in a routine way that the conditions (i)–(v) are satisfied. The proof is complete.

Recall that we use the symbol F_ω to denote a dendrite which is homeomorphic to the union of countably many straight line segments in the plane emanating from a common point (called the *vertex* of F_ω), disjoint off this point, and forming a null-sequence.

6.115. LEMMA. *For each dendrite X and for each sequence $\{k_n\}$ of natural numbers tending to infinity there exists a dendrite Y such that*

(6.116) *for each $p \in R_{\mathbb{N}}(Y)$ there exists an $i \in \mathbb{N}$ such that*

$$\text{ord}(p, Y) = k_i,$$

(6.117) *if $p, q \in R_{\mathbb{N}}(Y)$ and $p \neq q$, then $\text{ord}(p, Y) \neq \text{ord}(q, Y)$,*
and

(6.118) $X \leq_{\mathbb{O}} Y.$

Proof. We apply the inverse limit method known to the reader from the proof of Theorem 6.80. Let $X = \varprojlim \{X_n, \pi_n\}$, where X_n and π_n are as in Lemma 6.114. We construct Y as the inverse limit of an inverse sequence of dendrites Y_n and bonding mappings $\varrho_n : Y_{n+1} \rightarrow Y_n$ which are monotone retractions such that for each $n \in \mathbb{N}$ there are open and finite-to-one mappings $f_n : Y_n \rightarrow X_n$ having the property that the diagrams

$$(6.119) \quad \begin{array}{ccc} X_n & \xleftarrow{\pi_n} & X_{n+1} \\ f_n \uparrow & & \uparrow f_{n+1} \\ Y_n & \xleftarrow{\varrho_n} & Y_{n+1} \end{array}$$

commute. Then the openness of $f = \varprojlim f_n$ will follow.

Put $Y_1 = X_1$ and let $f_1 : Y_1 \rightarrow X_1$ be the identity. Assume there are defined dendrites Y_i and open finite-to-one mappings $f_i : Y_i \rightarrow X_i$ for $i \in \{1, \dots, n\}$, as well as monotone retractions $\varrho_i : Y_{i+1} \rightarrow Y_i$ for $i \in \{1, \dots, n-1\}$, such that the corresponding diagrams (6.119) commute. For each $r_n \in X_n$ the set $f_n^{-1}(r_n)$ is finite by the inductive hypothesis. Let $f_n^{-1}(r_n) = \{s_n^1, \dots, s_n^m\}$. To construct Y_{n+1} consider three cases.

If $X_{n+1} = X_n$, we put $Y_{n+1} = Y_n$ and define ϱ_n to be the identity mapping.

If $X_{n+1} \neq X_n$ and $\text{ord}(r_n, X_{n+1}) = \omega$, then for each $j \in \{1, \dots, m\}$ we take a homeomorphic copy $F_\omega(n, j)$ of F_ω with vertex s_n^j , and such that $F_\omega(n, j) \cap Y_n =$

$\{s_n^j\}$. Put

$$Y_{n+1} = Y_n \cup \bigcup \{F_\omega(n, j) : j \in \{1, \dots, m\}\},$$

define $\varrho_n : Y_{n+1} \rightarrow Y_n$ by the conditions

- $\varrho_n|_{Y_n}$ is the identity, and
- $\varrho_n(F_\omega(n, j)) = \{s_n^j\}$ for each $j \in \{1, \dots, m\}$,

and $f_{n+1} : Y_{n+1} \rightarrow X_{n+1}$ by the conditions

- $f_{n+1}|_{Y_n} = f_n$, and
- $f_{n+1}|_{F_\omega(n, j)}$ is a homeomorphism from $F_\omega(n, j)$ onto $\pi_{n+1}^{-1}(r_n)$ for each $j \in \{1, \dots, m\}$.

If $X_{n+1} \neq X_n$ and $\text{ord}(r_n, X_{n+1})$ is finite, for each $j \in \{1, \dots, m\}$ we choose a member α_j of the sequence $\{k_n\}$ such that:

- 1° $\alpha_j \geq \text{ord}(r_n, X_{n+1}) = \text{ord}(r_n, X)$,
- 2° $\alpha_i \neq \text{ord}(s, Y_n)$ for each $s \in R(Y_n)$ and each $i \in \{1, \dots, n\}$,
- 3° if $j_1 \neq j_2$, then $\alpha_{j_1} \neq \alpha_{j_2}$.

Now, for each point s_n^j (where $j \in \{1, \dots, m\}$) we take α_j arcs $A_n^j(u)$ for $u \in \{1, \dots, \alpha_j\}$ emanating from s_n^j , disjoint off s_n^j , and having s_n^j as the only common point with Y_n . Put

$$Y_{n+1} = Y_n \cup \bigcup \left\{ \bigcup \{A_n^j(u) : u \in \{1, \dots, \alpha_j\}\} : j \in \{1, \dots, m\} \right\},$$

define $\varrho_n : Y_{n+1} \rightarrow Y_n$ by the conditions

- $\varrho_n|_{Y_n}$ is the identity, and
- $\varrho_n(A_n^j(u)) = \{s_n^j\}$ for each $j \in \{1, \dots, m\}$,

and $f_{n+1} : Y_{n+1} \rightarrow X_{n+1}$ by the conditions

- $f_{n+1}|_{Y_n} = f_n$, and
- $f_{n+1}|_{A_n^j(u)}$ is a homeomorphism from $A_n^j(u)$ onto some arc of the form $r_n q(K) \subset X_{n+1}$, where K is a component of $X \setminus X_n$ with $X_n \cap \text{cl} K = \{r_n\}$, for each $j \in \{1, \dots, m\}$, and
- $f_{n+1}(\bigcup \{A_n^j(u) : u \in \{1, \dots, \alpha_j\}\}) = \text{cl}(X_{n+1} \setminus X_n) = \pi_{n+1}^{-1}(r_n)$.

One can verify that in all three cases considered the mapping ϱ_n is a monotone retraction, the mapping f_{n+1} is open and finite-to-one, and the diagram (6.84) commutes. Now $Y = \varprojlim \{Y_n, \varrho_n\}$ is a dendrite as the inverse limit of dendrites with monotone bonding mappings, by Nadler's theorem ([30], Theorem 4, p. 229), and the orders of ramification points of Y satisfy the required conditions (6.116) and (6.117). Moreover, $f = \varprojlim f_n$ is open since all $f_n : X_n \rightarrow Y_n$ are ([33], Theorem 4, p. 61). So, (6.118) holds, and the proof is complete.

6.120. LEMMA. *Let X be a dendrite and $f : X \rightarrow Y$ an open surjective mapping. If there are three points a, b , and c of $X \setminus E(X)$ such that $b \in ac \setminus \{a, c\}$ and $f(a) = f(b) = f(c)$, then $f(X(a, c)) = Y$.*

PROOF. Note that $X(a, c) \setminus \{a, c\} = \text{int } X(a, c)$ and $\text{bd } X(a, c) = \text{bd}(X(a, c) \setminus \{a, c\}) = \{a, c\}$. Since $b \in \text{int } X(a, c)$, and since f is open, we infer that

$$(6.121) \quad f(b) \in \text{int } f(X(a, c)).$$

Consequently, since $f(a) = f(c) = f(b)$, we conclude that

$$f(X(a, c)) = f(X(a, c) \setminus \{a, c\}) \cup f(\{a, c\}) = f(X(a, c) \setminus \{a, c\}).$$

Since $\text{bd } f(A) \subset f(\text{bd } A)$ for each open subset A of X provided f is open (see [37], Chapter 8, (7.3), (iii), p. 147), taking $A = X(a, c) \setminus \{a, c\}$ we get

$$\begin{aligned} \text{bd } f(X(a, c)) &= \text{bd } f(X(a, c) \setminus \{a, c\}) \subset f(\text{bd}(X(a, c) \setminus \{a, c\})) \\ &= f(\{a, c\}) = f(b), \end{aligned}$$

whence $\text{bd } f(X(a, c)) = \emptyset$ by (6.121). By connectedness of Y we conclude that $f(X(a, c)) = Y$, and so the proof is complete.

Another special dendrite is needed to prove the result. We construct it now. Take a straight line segment ab in the plane; let p be its midpoint, and for each $n \in \mathbb{N}$ let $p_n \in ap$ denote the point such that $p_1 = a$, and p_{n+1} is the midpoint of the segment $p_n p$. Thus $p = \lim p_n$. Take a straight line segment $p_n q_n$ perpendicular to ab with length equal to that of $p_n p$. Then

$$(6.122) \quad F = ab \cup \bigcup \{p_n q_n : n \in \mathbb{N}\}$$

is the required dendrite. We see that $P(F) = \{p\}$, with P defined just before Lemma 6.100.

6.123. THEOREM. *There is no maximal element in (\mathcal{D}, \leq_0) .*

PROOF. Assume that such a maximal element X exists. Observe that

$$(6.124) \quad \text{if there exists an open mapping } f : Y \rightarrow X \text{ from a dendrite } Y \text{ onto } X, \text{ then } X \text{ contains a homeomorphic copy of } Y.$$

Indeed, by the maximality of X there is an open surjective mapping $g : X \rightarrow Y$; thus (6.124) is a consequence of Corollary 6.22.

We shall prove that

$$(6.125) \quad X \text{ contains a homeomorphic copy of the dendrite } F \text{ defined by (6.122).}$$

To do this, fix $p \in E(X)$, take an arc $xp \subset X$ and choose a sequence of points $p_n \in xp$ such that $p = \lim p_n$ and $\text{ord}(p_n, X) = 2$. Observe that the dendrites $X(p_n, p) \subset X$ form a null-sequence with limit $\{p\}$. Let X_n be a homeomorphic copy of $X(p_n, p)$ joined to X in such a way that $X \cap X_n = \{p_n\}$. Then

$$Z' = X \cup \bigcup \{X_n : n \in \mathbb{N}\}$$

is a dendrite. Let Z'' stand for a homeomorphic copy of Z' such that $Z' \cap Z'' = \{p\}$. Then $Z = Z' \cup Z''$ is also a dendrite.

Observe that there is a natural open mapping from Z onto Z' . Further, there exists an open mapping g from Z' onto X . Indeed, define $g|_X$ to be the identity

and, for each $n \in \mathbb{N}$, take for $g|X_n : X_n \rightarrow X(p_n, p)$ the natural homeomorphism. The composition of the two mappings is an open mapping from Z onto X . According to (6.124), X contains a homeomorphic copy of Z . To simplify notation we assume that $Z \subset X$. Taking, for each $n \in \mathbb{N}$, a point $q_n \in X_n \setminus \{p_n\} \subset Z' \subset Z$ we see that Z contains $xp \cup \bigcup \{p_n q_n : n \in \mathbb{N}\} \cup px'$, where $x' \in Z'' \setminus \{p\}$. This union is homeomorphic to F . Thus (6.125) is proved.

It follows from (6.125) that $P(X) \neq \emptyset$. Now we prove more:

(6.126) $P(X)$ is infinite.

In fact, if $P(X)$ were finite, then taking the one-point union U of X and of a homeomorphic copy X' of X with the common point being an end point of both X and X' we would conclude that $P(U)$ has twice as many points as $P(X)$ has. However, since there is a natural open mapping from U onto X , by (6.124) the dendrite X contains a homeomorphic copy of U , and thus $\text{card } P(U) \leq \text{card } P(X)$, a contradiction. Hence (6.126) follows.

Choose a sequence of points p_n in $P(X)$. By the definition of $P(X)$ for each $n \in \mathbb{N}$ there exists a sequence $\{p_n(i) : i \in \mathbb{N}\}$ tending to p_n as $i \rightarrow \infty$ and such that, for all $i \in \mathbb{N}$, the points $p_n(i)$ lie on some arc in X , and $p_n(i) \in R_{\mathbb{N}}(X)$. For each $n \in \mathbb{N}$, define

$$r_n = \max\{\text{ord}(p_i(j), X) + 1 : i, j \leq n\}.$$

Then, for each fixed $n \in \mathbb{N}$, we have $r_i > \text{ord}(p_n(i), X)$ for almost all i (precisely, for all $i \geq n$). We now define a subsequence $\{r_{k_n}\}$ as follows: $k_1 = 1$, and for each $n > 1$ we put $k_n = k_{n-1} + 3n$. Now we apply Lemma 6.115 to the dendrite X and to the sequence $\{r_{k_n}\}$, which yields a dendrite Y . We shall prove that there is no open mapping from X onto Y .

Suppose on the contrary that there is an open surjection $f : X \rightarrow Y$. Since $f^{-1}(E(Y)) \setminus E(X)$ is finite by (6.18), we see that there exists an $n \in \mathbb{N}$ such that $f(p_n)$ is not an end point of Y .

We claim that (for this fixed n)

(6.127) $f(p_n(i)) \in R_{\mathbb{N}}(Y)$ for all but finitely many i .

If not, denote by V_i a component of $X \setminus \{p_n(i)\}$ such that $\{V_i : i \in \mathbb{N}\}$ is a null-sequence. Then $\{f(V_i) : i \in \mathbb{N}\}$ is a null-sequence of open sets in Y that contain components of $Y \setminus \{f(p_n(i))\}$ by Proposition 6.99. Therefore there is a null-sequence of components of $Y \setminus \{f(p_n(i))\}$, whence $f(p_n)$, being the limit point of $\{f(p_n(i)) : i \in \mathbb{N}\}$, must be an end point of Y , contrary to (6.126).

Thus (6.127) is established, and so we can assume that

$$f(p_n(i)) \in R_{\mathbb{N}}(Y) \quad \text{for all } i.$$

Observe that by the definition of r_n ,

$$\text{if } n < j \leq k_{i+1}, \text{ then } \text{ord}(p_n(j), X) < r_{k_{i+1}}.$$

By (6.116) and (6.117) we have in Y at most i points of order less than $r_{k_{i+1}}$. Consider all points $p_n(j)$ for j such that $k_i < j \leq k_{i+1}$. Because there are $3i$ such points, there are three points $a, b, c \in \{p_n(j) : k_i < j \leq k_{i+1}\}$ with $b \in ac$ and $f(a) = f(b) = f(c)$. The condition $b \in ac \subset p_n(k_i)p_n(k_{i+1})$ implies that

$$b \in X(a, c) \subset X(p_n(k_i), p_n(k_{i+1})),$$

whence, by Lemma 6.120, $f(X(a, c)) = Y$. Consequently,

$$f(X(p_n(k_i), p_n(k_{i+1}))) = Y.$$

However, $\{X(p_n(k_i), p_n(k_{i+1})) : i \in \mathbb{N}\}$ is a null-sequence, and we have a contradiction with the continuity of f . The proof is finished.

7. Problems

We end this paper with a list of unsolved problems (or questions) relating to the family \mathcal{D} and to the classes \mathbb{M} , \mathbb{O} , and \mathbb{R} . These are—in general—particular cases of questions Q1–Q8 listed in the final part of Chapter 3.

Q1(\mathbb{O}). We do not have any (structural) characterization of minimal elements in $(\mathcal{D}, \leq_{\mathbb{O}})$.

Q2(\mathbb{O}). We do not know whether all minimal elements of $(\mathcal{D}, \leq_{\mathbb{O}})$ are unique minimal. The known examples are.

Q3(a)(\mathbb{M}). Only finite antichains are known in $(\mathcal{D}, \leq_{\mathbb{M}})$. We do not know whether there is any infinite one.

Q3(b)(\mathbb{M}). We have in $(\mathcal{D}, \leq_{\mathbb{M}})$ chains of cardinality \aleph_1 , but we do not know if there are of cardinality \mathfrak{c} .

Q4(a)(\mathbb{O}). Does every chain in $(\mathcal{D}, \leq_{\mathbb{O}})$ have a lower bound?

Q4(b)($\mathbb{M}, \mathbb{O}, \mathbb{R}$). Does every bounded chain in $(\mathcal{D}, \leq_{\mathbb{M}})$, $(\mathcal{D}, \leq_{\mathbb{O}})$ or $(\mathcal{D}, \leq_{\mathbb{R}})$ have an infimum (a supremum)?

Q5(\mathbb{M}). Does there exist a chain in $(\mathcal{D}, \leq_{\mathbb{M}})$ with order structure (a) dense, (b) similar to a segment?

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