POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE
(ROZPRAWY MATEMATYCZNE)

CXCVIII

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Regularity theorems for solutions of partial differential equations for quasiconformal mappings in several dimensions

WARSZAWA 1982
PAŃSTWOWE WYDAWNICTWO NAUKOWE

PRINTED IN POLAND

ISBN 83-01-01614-0 ISSN 0012-3862

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CONTENTS

Preliminaries

In classical definitions of conformal mappings one usually assumes suitable regularity of the mappings. In the twenties several authors studied the question whether regularity assumptions (in the two-dimensional case) are actually necessary. The answer is given by the Menshov–Looman theorem. We shall state the conclusion of this theorem (in a version valid for an arbitrary dimension) and we shall consider it as the definition of conformal mappings.

DEFINITION 0.1. A homeomorphism f of a domain $\Omega \subset \boldsymbol{R}^n$ is called *conformal* if for any $x_0 \in \Omega$ the following holds:

$$\limsup_{r\to 0} \frac{\max\limits_{|x-x_0|=r} |f(x)-f(x_0)|}{\min\limits_{|x-x_0|=r} |f(x)-f(x_0)|} = 1. \tag{0.1}$$

Geometrically this means that f takes infinitesimal spheres into infinitesimal spheres.

This definition can be generalized as follows.

DEFINITION 0.2 (Gehring's metric definition [8]). A homeomorphism f of a domain $\Omega \subset \boldsymbol{R}^n$ is said to be *K-quasiconformal* ($1 \leqslant K < \infty$) in Ω if

$$\limsup_{r\to 0} \frac{\max\limits_{|x-x_0|=r} |f(x)-f(x_0)|}{\min\limits_{|x-x_0|=r} |f(x)-f(x_0)|} \leqslant K \quad \text{for all } x_0 \in \Omega. \tag{0.2}$$

Inequality (0.2) is an analytical characterization of the fact that infinitesimal spheres are mapped into such infinitesimal ellipsoids that the ratio of the biggest and the smallest semiaxes is bounded by the constant K.

Studies of fundamental problems of the theory of quasiconformal mappings in the plane have proved to be strictly related to the theory of elliptic systems of partial differential equations, which can be considered as generalizations of the Cauchy–Riemann system (defining conformal mappings).

Studies in non-linear hydrodynamics, due mainly to M. A. Lavrentiev [14], [15], have led to systems of equations of the form

$$(0.3)\qquad \begin{cases} \varphi_1\left(\dfrac{\partial f^i}{\partial x_j}, x, f\right) = 0 \\ \varphi_2\left(\dfrac{\partial f^i}{\partial x_j}, x, f\right) = 0 \end{cases} \qquad (i, j = 1, 2).$$

These solutions $f = f(x)$ which are homeomorphisms are quasiconformal. As an effect of these studies (esspecially the studies of *linear* cases of (0.3), e.g. the Beltrami equation) one has obtained answers to several problems concerning quasiconformal mappings and differential equations.

There have been efforts to find similar *linear* systems for quasiconformal mappings in higher dimensions. These efforts, however, have failed. Even in the class of non-linear systems there is no natural subclass having, for instance, local homeomorphic solutions (being quasiconformal) unless we make some assumptions on integrability of these systems. Some results (for example Liouville's theorem discussed below) suggest that there is no boundary value problem which can be correctly posed for any system of differential equations admitting conformal mappings in several dimensions. Recently some relations have been found between quasiconformal mappings in several dimensions and differential equations. They are not as clear as in the two-dimensional case, but certainly not less important for a further development of the theory of quasiconformal mappings. These studies have their origin in the papers of L. Ahlfors [1], J. G. Rešetniak [19], [21], B. Bojarski and T. Iwaniec [5], and others. We should like to point out some much more difficult problems which have arisen. One of the main reasons for the difficulties is the fact that equations for many-dimensional quasiconformal mappings are not linear and not always uniformly elliptic, and systems of equations are overdetermined. Therefore one almost immediately encounters fundamental problems, such as existence of solutions or regularity of generalized solutions. Such problems appear in the present paper. Each of the results requires special methods, often not sufficiently developed by the general theory of differential equations.

The systems of differential equations that we shall make use of should be considered as analogoues of Beltrami's equations for quasiconformal mappings between two-dimensional domains. We shall derive them from the geometric interpretation of the definition of quasiconformal mappings.

Let us remark (F. Gehring [9]) that generalized first derivatives of a quasiconformal mapping of a domain in $\boldsymbol{R}^n$ are locally integrable with the nth power, and inequality (0.2) can be transformed into a condition on these derivatives. This observation has led us to a more general concept of quasiregular mappings.

DEFINITION 0.3. A mapping $f\colon \Omega \to \Omega'$, $f = (f^1(x), f^2(x), \ldots, f^n(x))$, $x = (x_1, x_2, \ldots, x_n)$, of a domain $\Omega \subset \boldsymbol{R}^n$ into $\Omega' \subset \boldsymbol{R}^n$ is called *quasiregular* if the following conditions are satisfied:

(a) $f \in W^1_{n,\mathrm{loc}}(\Omega)$; $W^1_{n,\mathrm{loc}}(\Omega)$ is the Sobolev space of vector-valued functions defined on Ω whose first generalized partial derivatives exist in $L_{n,\mathrm{loc}}(\Omega)$.

(b) For almost all $x \in \Omega$,

$$|Df(x)|^n \leqslant n^{n/2} \tilde{K}^n J_f(x), \tag{0.4}$$

where $Df(x)$ is the Jacobi matrix of f, $\left(\frac{\partial f^i}{\partial x_j}\right) = Df$, $J_f(x)$ is its jacobian; $\tilde{K}$— is a constant, related to the constant K. For every matrix $A = (A_{ij})$ we denote by $|A|$ its norm:

$$|A| = \Big(\sum_{ij} |A_{ij}|^2\Big)^{1/2}.$$

Remark. For every $n \times n$-matrix A we have $|A|^n \geqslant n^{n/2} \det A$ and the equality holds if and only if A is a multiple of an orthogonal matrix. Thus in fundamental inequality (0.4) the case $\tilde{K} = 1$ implies that (0.4) becomes an equality and f is conformal.

A quasiregular one-to-one mapping in Ω is called *quasiconformal*; the terms *$\tilde{K}$-quasiregular in Ω* or *$\tilde{K}$-quasiconformal* are also used.

The case of mappings of two-dimensional domains is especially interesting because of its connection with functions of a complex variable. In this case a mapping f is regarded as a function of a complex variable $z = x_1 + ix_2$; we introduce the operators

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - i\frac{\partial f}{\partial x_2}\right), \quad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\right),$$

and the fundamental inequality (0.4) takes the form

$$|f_{\bar{z}}| \leqslant q(z)|f_z|, \quad q(z) \leqslant \frac{K-1}{K+1} < 1. \tag{0.5}$$

Thus we can consider the theory of K-quasiconformal or K-quasiregular mappings as a study of solutions of differential inequality (0.5) with a parameter K. For $K = 1$ we get the Cauchy–Riemann system

$$f_{\bar{z}} = 0. \tag{0.6}$$

The geometric interpretation of (0.4) is of fundamental importance in understanding several basic constructions in the theory of quasiconformal mappings.

Let $f\colon \Omega \to \Omega'$ be a map. Consider the tangent mapping

(0.7) $$T_{x_0}\Omega \to T_{f(x_0)}\Omega';$$

some necessary regularity assumptions will be described later. This is a linear mapping between spaces tangent to Ω at x_0 and to Ω' at $f(x_0)$. In the natural bases of $T_{x_0}\Omega$ and $T_{f(x_0)}\Omega'$, the map is given by the Jacobi matrix $Df(x_0)$. If x_0 varies over Ω, then $Df(x_0)$ defines a morphism of the tangent bundles

$$T\Omega \overset{Df}{\to} T\Omega'.$$

We shall now define infinitesimal ellipsoids. An *infinitesimal ellipsoid* $\mathscr{E}(x_0)$ at $x_0 \in \Omega$ is a family $\{\mathscr{E}_h(x_0)\}$ of ellipsoids in $T_{x_0}\Omega$, centred at $0 \in T_{x_0}\Omega$ and such that each of them can be obtained from any other by a similarity transformation of $T_{x_0}\Omega$. The ellipsoids of this family are parametrized by a parameter $h > 0$; for h we may take for instance the length of their shortest axis. Let p be the ratio of the length of the longest axis of an ellipsoid $\mathscr{E}_h(x_0)$ to the length of the shortest one. Thus the ratio measures the flatness of ellipsoids from $\{\mathscr{E}_h(x_0)\}$.

An infinitesimal ellipsoid $\{\mathscr{E}_h(x_0)\}$ can be described by a quadratic equation

$$\langle G(x_0)\xi, \xi\rangle = ph^2,$$

where $\xi \in T_{x_0}\Omega$, $\langle\ ,\ \rangle$ is the scalar product in $T_{x_0}\Omega$, and $G(x_0)$ is a positive symmetric matrix such that $\det G(x_0) = 1$. The matrix $G(x_0)$ is called the *characteristic of the infinitesimal ellipsoid* $\{\mathscr{E}_h(x_0)\}$. The distribution of characteristics in Ω is a matrix-valued function $\Omega \ni x \to G(x)$.

We want to relate infinitesimal ellipsoids with mappings. Suppose (as above) that $f\colon \Omega \to \Omega'$ is a mapping, $x_0 \in \Omega$, and $Df\colon T_{x_0}\Omega \to T_{f(x_0)}\Omega$ is its tangent mapping. Suppose that Df preserves orientation. Consider an infinitesimal ellipsoid at x_0 with characteristic $G(x_0)$ and an infinitesimal ellipsoid at $f_0 = f(x_0) \in \Omega'$ with characteristic $H(f_0)$. We may ask when Df transforms one infinitesimal ellipsoid into another. This means that

$$D^*f(x_0)H(f_0)Df(x_0) = J_f(x_0)^{2/n}G(x_0),$$

where $D^*f(x_0)$ is the transposed matrix to $Df(x_0)$.

Generalizing the above situation, suppose that there are given two distributions of characteristics $G(x,f)$, $H(x,f)$, defined on the product $\Omega \times \Omega'$. Thus $G(x,f)$ and $H(x,f)$ are matrix-valued functions defined on $\Omega \times \Omega'$ and satisfying the following conditions:

(i) G and H are real, positive, symmetric $n \times n$-matrices, i.e.

$$\nu|\xi|^2 \leqslant G_{ij}(x,f)\xi^i\xi^j \leqslant \mu|\xi|^2,$$

$$\nu|\eta|^2 \leqslant H_{ij}(x,f)\eta^i\eta^j \leqslant \mu|\eta|^2.$$

(ii) $\det G(x,f) = \det H(x,f) = 1$.

(iii) Entries of the matrices G and H are measurable and uniformly bounded on $\Omega \times \Omega'$.

Then f transforms the infinitesimal ellipsoid at x with the characteristic $G(x,f)$ into the infinitesimal ellipsoid at $f = f(x)$ with the characteristic $H(x,f)$ if and only if

$$D^*f(x)\, H(x,f)\, Df(x) = J_f(x)^{2/n} G(x,f). \tag{0.8}$$

When $H(x_0, f_0) = G(x_0, f_0) = E$ — the identity matrix, then f is conformal at x_0.

The case $G = G(x)$, $H = H(f)$ includes the problem of conformal equivalence of Riemannian spaces Ω_G and Ω'_H, where Ω_G (resp. Ω'_H) denotes a manifold with the scalar product on $T\Omega$ (resp. on $T\Omega'$) defined by G (resp. H).

Relation (0.8) is a system of non-linear homogeneous equations for the vector-function $f = (f^1, f^2, \ldots, f^n)$. The number of independent equations is $(n-1)(n+2)/2$. For $n > 2$ the system is overdetermined. For $n = 2$ the system reduces to the *quasilinear* system

$$f_{\bar z}(z) = q_1(z,f) f_z(z) + q_2(z,f)\overline{f_z(z)}, \tag{0.9}$$

where

$$q_1 = \frac{G_{11} - G_{22} + 2i G_{12}}{G_{11} + G_{22} + H_{11} + H_{22}}, \quad q_2 = -\frac{H_{11} - H_{22} + 2i H_{12}}{G_{11} + G_{22} + H_{11} + H_{22}}.$$

Conditions (i), (ii), (iii) imply that q_1 and q_2 uniformly satisfy the estimate

$$|q_1(z,f)| + |q_2(z,f)| \leqslant q_0 < 1.$$

This estimate is called *uniform ellipticity* (in the terminology of the theory of systems of partial differential equations of the first order) [3].

If $H(x,f) = E$ and $G = G(x)$, then the function $f = f(z)$, $z = x_1 + ix_2$ satisfies almost everywhere Beltrami's equation

$$f_{\bar z} = q_1(z) f_z, \quad |q_1(z)| \leqslant q_0 < 1. \tag{0.10}$$

The system

$$D^* f(x)\, Df(x) = J_f(x)^{2/n} G(x) \tag{0.11}$$

may be regarded as an analogue of Beltrami's equation for quasiregular mappings of n-dimensional domains. The existence of a solution of (0.11) may be interpreted in geometric language as a conformal equivalence of the Riemannian space Ω_G and a domain in $\boldsymbol{R}^n$.

Conditions for the existence of a local homeomorphism $f = f(x)$ satisfying (0.11) are given by the theorem of Weyl–Schouten.

We shall denote the class of solutions of (0.8) by $\Lambda(G, H)$ in honour of

M. A. Lavrentiev, who was the first to consider natural influences of partial differential equations upon quasiconformal mappings.

By generalized solution of (0.8) we shall mean a vector-function f belonging to the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$ and satisfying (0.8) almost everywhere in Ω. The set of all generalized solutions of (0.8) will be denoted by $\Lambda^{k+\alpha}(G, H)$, $k = 0, 1, 2, \ldots, 0 < \alpha \leqslant 1$, if the characteristics G and H belong to $C^{k+\alpha}(\Omega \times \Omega')$.

We shall analyse system (0.8) with a variety of different smoothness conditions for the characteristics $G(x,f)$ and $H(x,f)$.

In the case of system (0.10) there is a known theorem of B. V. Šabat which says that if the coefficient $q_1 = q_1(z)$ is of the class $C^{k+\alpha}(\Omega)$, $0 < \alpha < 1$, $k = 0, 1, \ldots$, then any solution $f = f(z)$ of system (0.10) is of the class $C^{k+\alpha+1}(\Omega)$. For $\alpha = 0$ this theorem is not true.

Example (B. V. Šabat [22]). The quasiconformal mapping

$$f(z) = z(1-\ln|z|^2),$$

which is defined in some neighbourhood of $z_0 = 0$, has the continuous characteristic $q_1(z) = f_{\bar z}/f_z$, $q_1(z_0) = 0$ but f is not of the class C^1; moreover

$$\lim_{z\to 0}\frac{|f(z)|}{|z|} = \infty .$$

In the present paper we shall prove regularity theorems in higher dimensions, i.e. $\Lambda^{k+\alpha}(G,H) \subset C^{k+\alpha+1}(\Omega)$ (Theorem 8.2).

In principle it can be said that (in the two-dimensional case) the strongest and most general theorems on quasiconformal mappings (in particular existence theorems) have been obtained by using the analytic approach, i.e. by using properties of Beltrami's equations and its generalizations, as for instance Λ-system

$$f_{\bar z}(z) = h(z,f,f_z), \tag{0.12}$$

where $h(z,f,\zeta)$ is a Lipschitz function with respect to ζ:

$$|h(z,f,\zeta_1)-h(z,f,\zeta_2)| \leqslant q_0|\zeta_1-\zeta_2|, \quad h(z,f,0) = 0, \quad q_0 < 1 .$$

Such systems were studied in papers by B. Bojarski and T. Iwaniec [6], [4], [11]; they contain so-called strong elliptic systems in the geometric sense, the last have been studied by M. Lavrentiev [14], [15]. We shall quote the following fundamental theorem:

Theorem 0.1. *Let f be a K-quasiregular mapping defined on $\Omega \subset \mathbf{R}^n$. Then for any compact subset Ω_0 of Ω there is a constant C_{Ω_0} such that*

$$|f(x_1)-f(x_2)| \leqslant C_{\Omega_0}|x_1-x_2|^{\alpha}. \tag{0.13}$$

This theorem was proved in its full generality by F. Gehring with the best possible exponent $\alpha = 1/K^{n-1}$.

According to our definition, a quasiregular mapping is given by a function in the Sobolev class $W^1_{n,\mathrm{loc}}(\Omega)$. Recall Sobolev's imbedding theorem: if $f \in W^1_{p,\mathrm{loc}}(\Omega)$, then $f \in C^\alpha_{\mathrm{loc}}(\Omega)$, where $\alpha = 1-n/p$. Inequality (0.4) implies $f \in W^1_{n+\varepsilon,\mathrm{loc}}(\Omega)$, see [7], so we get (0.13) with some exponent $\alpha > 0$. Unfortunately, it is not proved yet that

$$f \in W^1_{nK^{n-1}/(K^{n-1}-1),\mathrm{loc}}(\Omega);$$

this result would imply $\alpha = 1/K^{n-1}$, i.e. F. Gehring theorem.

The theorem stating that any quasiconformal mapping is of the class $W^1_{p,\mathrm{loc}}(\Omega)$, $p > n$, is much stronger than Theorem 0.1.

To end these preliminaries, we shall discuss some topics in quasiconformal mappings related to the study of the non-linear system (0.8). For $n \geqslant 3$ the solutions of (0.8) are of a very special nature. This fact was observed for the first time by Liouville (in the middle of the nineteenth century) in the special case of $G = H = E$ – the identity matrix, i.e. for conformal mappings. He proved his famous theorem which says that any C^3 conformal mapping in $\mathbf{R}^n$, $n \geqslant 3$, is a Möbius transformation. This means that all C^3-conformal mappings are generated by rotations, inversions and similarity mappings. Over one hundred years later F. Gehring [8] and J. Rešetniak [20] proved Liouville's theorem without any regularity assumptions (i.e. the assumption that the mapping $f \in C^3$ is not necessary; see also [26]).

Again, their proofs were based on very deep results on generalized solutions of non-linear equations of the second order or of equations with measurable coefficients. As we proved in [12], [5], the third derivatives of the classical solutions of (0.8) can be expressed as functions of their first and second derivatives. Namely, we have the formulae

$$\frac{\partial^3 f^l}{\partial x_i \partial x_j \partial x_k} = F^l_{ijk}\left(x, f, \frac{\partial f^\alpha}{\partial x^\beta}, \operatorname{grad} J_f(x)\right), \tag{0.14}$$

$$\alpha, \beta, i, j, k, l = 1, \ldots, n,$$

where F^l_{ijk} is algebraic in $\dfrac{\partial f^\alpha}{\partial x^\beta}$, $\operatorname{grad} J_f(x)$. The proof of this fact is obtained by using only the formal structure of (0.8). A further study of (0.14) leads in particular to the following uniqueness theorem; two solutions of (0.8) which take the same value at a given point x_0 together with their first and second derivatives coincide in a neighbourhood of x_0.

This fact was first proved in [5] under strong regularity assumptions; the results of this papers are further extended to arbitrary generalized solutions of (0.8) whose characteristics are of the class $C^3(\Omega\times\Omega')$. Thus system (0.8) has the basic property of ordinary differential equations: its solution is determined by its Cauchy data of order $\leqslant 2$ at any point. This is a proper generalization of the classical theorem of Liouville.

From system (0.8) one can derive a certain number of other quasilinear equations of the second order. The are important in the study of several local and global properties of (0.8) (regularity of the solutions). To give an idea how these equations are formed let us note that the function $v(x) = J_f(x)^{(n-2)/2n}$ satisfies *linear* equations of the function second order; the coefficients of this equation are given by simple universal formulae involving coefficients of the metric tensor $G(x)$

$$G^{ij}(x)v_{x_ix_j} - G^{ij}(x)\Gamma^t_{ij}(x)v_{x_t} + ((n-2)/4(n-1))R(x)v = 0, \tag{0.15}$$

where $\Gamma^t_{ij}(x)$ are the Christoffel symbols and $R(x)$ is the scalar curvature (see [5]). It can also be written in the weak form

$$\int\limits_{\Omega}\left(\frac{\partial^2\varphi G^{ij}}{\partial x_i\partial x_j} + \frac{\partial\varphi G^{ij}\Gamma^t_{ij}}{\partial x_t} + \frac{n-2}{4(n-1)}\varphi R\right)\cdot v\,dx = 0 \tag{0.16}$$

for every test function $\varphi\in C_0^\infty(\Omega)$.

Equation (0.16) is uniformly elliptic and, in consequence, the function $v(x)$ has rather special properties, and so it is a convenient tool for a discussion of properties of the Jacobian of a quasiconformal mapping.

The presented proof of regularity Theorem 8.2 requires several new ideas. Some lemmas used in the proof and concerning nonlinear partial differential equations give extensions of earlier results, for example some of the results of O. A. Ladyzenskaya and N. Uraltseva [13]. The real difficulty is to prove that the Jacobian $J_f(x)$ is bounded. It follows from very strong versions of the results of A. Elcart–G. Meyers [7] on the L_p-estimates $(p > n)$ of first partial derivatives of quasiregular mappings.

At the begining of the paper we derive some equations of the second order satisfied by mappings from the class $\Lambda(G, H)$ with measurable characteristics. Next we discuss some classes of quasiregular mappings assuming higher regularity of characteristics, culminating in the case $\Lambda^{k+\alpha}(G, H)$, $k \geqslant 1$, $0 < \alpha < 1$, where we prove the inclusion $\Lambda^{k+\alpha}(G, H) \subset C^{k+1+\alpha}(\Omega)$. Also an integral version of Hölder's condition for the first derivatives of a mapping $f\in\Lambda^\alpha(G, H)$, $(0 < \alpha \leqslant 1)$ will be given (see Theorem 4.1 and Theorem 4.2).

The author would like to thank to Professor B. Bojarski for many valuable discussions during the preparation of this paper.

1. Auxiliary results

Before discussing the regularity theorem, we give a series of fundamental properties of system (0.8) and its solutions.

Let $x_0 \in \Omega$ be a differentiability point of a mapping f satisfying system (0.8) and such that $J_f(x_0) \neq 0$. On the basis of the theory of quasiregular mappings we know that either the set of such points is dense in Ω or f is constant.

Let $f_0 = f(x_0)$. The tangent map $Df(x_0)$: $T_{x_0}\Omega \to T_{f_0}\Omega'$ transforms the family of ellipsoids

$$\{\xi \in T_{x_0}\Omega;\ \langle G(x_0, f_0)\xi, \xi\rangle = \text{const}\}$$

into the family of ellipsoids

$$\{\eta \in T_{f_0}\Omega';\ \langle H(x_0, f_0)\eta, \eta\rangle = \text{const}\},$$

where $\eta = Df(x_0)\xi$.

Proposition 1.1 *System* (0.8) *can be reduced by linear transformations of the variables x and f to the case where G and H are equal to the identity matrix at a given point* $(x_0, f_0) \in \Omega \times \Omega'$.

The proposition allows to assume (in studying the local behaviour of the mapping f) without loss of generality that $G(x_0, f_0) = H(x_0, f_0) = E$.

Let us introduce a coefficient

$$q(x, f) = |G(x, f) - E| + |H(x, f) - E|. \tag{1.1}$$

A map f is *conformal at a point* x_0 if $q(x_0, f_0) = 0$. In general, the coefficient $q(x, f(x))$ measures the deviation from conformality of f at the point $x \in \Omega$. There is a continuous relation between $\sup_{\Omega \times \Omega'} q(x, f)$ and the constant K. Moreover, $K = 1$ if and only if $q(x, f) \equiv 0$.

Suppose that the characteristics G and H are continuous matrix-valued functions on $\Omega \times \Omega'$. In view of Proposition 1.1, the mapping f can be expressed as a composition of linear maps and a map, whose constant K is arbitrarily close to 1 in a sufficiently small neighbourhood of a given point $x_0 \in \Omega$.

According to Theorem 1 we obtain

Theorem 1.1. *Suppose that the characteristics G and H of f are continuous on $\Omega \times \Omega'$. Let α be a positive number less than 1. Then for every compact subset $\Omega^0 \subset \Omega$ there exists a constant $C_{\Omega^0}(\alpha)$ such that*

$$|f(x_1) - f(x_2)| \leqslant C_{\Omega^0}(\alpha)|x_1 - x_2|^\alpha \quad \textit{for} \quad x_1, x_2 \in \Omega^0. \tag{1.2}$$

A well-known fact of the theory of n-dimensional quasiregular mappings with $n \geqslant 3$ states that any non-constant K-quasiregular mapping is a local homeomorphism if K is sufficiently close to 1. This result is due to O. Martio [16] and V. M. Goldstein [10]. Now by Proposition 1.1 we observe that any mapping f with continuous characteristics is either a local homeomorphism or a constant map.

This does not exclude that the Jacobian of f can vanish at some points of Ω. Furthermore one can show that the inverse map $x = f^{-1}(y)$ satisfies a system of the same type as the map f, namely

$$D^* x(y) G(x, y) Dx(y) = J_x(y)^{2/n} H(x, y).$$

2. The second order equations

We collect here some partial differential equations which are satisfied by general solutions of (0.8). In what follows we extend some results of J. G. Rešetniak [21] on the subject.

We begin from the 2-dimensional case. As is well known, the components f^1 and f^2 of a conformal mapping $f = (f^1, f^2)$ satisfy the Laplace equation $\operatorname{div}(\operatorname{grad} f^l(x)) = 0$, $l = 1, 2$. In general, as we shall show later, the components f^1, f^2 of a solution of the Beltrami system satisfy uniformly elliptic equations

(2.1) $$\operatorname{div}\{G^{-1}(x) \operatorname{grad} f^l(x)\} = 0, \quad l = 1, 2.$$

Since actually $f = (f^1, f^2)$ need not be twice differentiable, we understand (2.1) as the integral equality

$$\int_\Omega \langle G^{-1}(x) \operatorname{grad} f^l(x), \operatorname{grad} \varphi(x)\rangle \, dx = 0$$

satisfied by all smooth test functions φ with compact support in Ω.

Equations of type (2.1) are particular cases of equations in higher dimensions, which we are going to derive now. First we recall the following preparatory lemma (compare Ch. Morrey [17], p. 122 and J. G. Rešetniak [21]).

Lemma 2.1. *Suppose that $f\colon \Omega \to \mathbf{R}^n$ is of the class $W^1_{n,\mathrm{loc}}(\Omega)$, where $\Omega \subset \mathbf{R}^n$. Then, for every constant vector $v \in \mathbf{R}^n$ the following identity holds:*

(2.2) $$\operatorname{div}\{J_f(x) D^{-1} f(x) v\} = 0.$$

We should mention that (2.2) is understood in the distributional sense.

Remark. Let S be a field on Ω, and suppose that S is differentiable at x. The divergence $\operatorname{div} S(x)$ of S at x is the vector with the property

$$\langle v \operatorname{div} S(x)\rangle = \operatorname{div}\{S^*(x) v\}$$

for every constant vector v, or equivalently

$$[\operatorname{div} S(x)]_i = \sum_{j=1}^{n} \frac{\partial S_{ij}(x)}{\partial x_j}.$$

Identity (2.2) means that

$$\operatorname{div}\{J_f(x)(D^* f)^{-1}(x)\} \equiv 0.$$

Proof. Obviously, it is sufficient to verify the lemma for infinitely differentiable mappings. Let us observe that the left-hand side of (2.2) is a polynomial in the variables $f^i_{x_j}, f^\alpha_{x_\beta x_\gamma}$, $1 \leqslant i, j, \alpha, \beta, \gamma \leqslant n$, while in the space of these variables the manifold defined by $\det(f^i_{x_j}) = 0$ is nowhere dense. Therefore, we can assume that the Jacobian $J_f(x)$ does not vanish at the point x to be examined.

In a neighbourhood of x, say D, f is a diffeomorphic mapping. Let φ be any smooth function with compact support contained in the neighbourhood. We can express φ by $\varphi(x) = \psi(f(x))$, where $\psi \in C_0^\infty(f(D))$. From the obvious identity

$$\int_{f(D)} \langle v, \operatorname{grad}\psi(f)\rangle\, df = 0$$

by changing variables

$$\int_D \langle v, \operatorname{grad}\psi(f(x))\rangle \cdot J_f(x)\, dx = 0.$$

On the other hand, one can express the vector function grad ψ by means of grad $\varphi(x)$ as follows:

$$\operatorname{grad}\varphi(x) = D^* f(x) \operatorname{grad}\psi \quad \text{or} \quad \operatorname{grad}\psi = (D^* f(x))^{-1} \operatorname{grad}\varphi(x).$$

Introducing this into the last identity, we obtain

$$\int_\Omega \langle v, (D^{-1} f(x))^* \operatorname{grad}\varphi(x)\rangle \cdot J_f(x)\, dx = 0.$$

Hence

$$\int_\Omega \langle J_f(x) D^{-1} f(x) v, \operatorname{grad}\varphi(x)\rangle\, dx = 0.$$

This is a weak form of (2.2). The result is proved. If f is a solution of (0.8), then $J_f D^{-1} f = J_f^{(n-2)/n} G^{-1}(D^* f) H$, and we see that every column of the matrix $J_f^{(n-2)/n} G^{-1}(D^* f) H$ is a solenoidal vector field or, in other terms, that, for every constant vector $v \in \mathbf{R}^n$, f is a weak solution of the quasilinear second order system

(2.3) $$\operatorname{div}\{J_f(x)^{(n-2)/n} G^{-1}(x, f) D^* f(x) H(x, f) v\} = 0.$$

Fixing any point $(x_0, f_0) \in \Omega \times \Omega'$ and choosing $v = H^{-1}(x_0, f_0)e^l$, where e^l is the lth basic vector of $\boldsymbol{R}^n$, we obtain the following equation for the lth component f^l of the map f.

(2.4) $$\operatorname{div}\{J_f^{(n-2)/n} G^{-1}(x,f)[\operatorname{grad} f^l(x) + D^* f(x)(H(x,f)H^{-1}(x_0,f_0) - E)e^l]\} = 0.$$

For $H \equiv E$ it reduces to the case considered by J. G. Rešetniak [21], i.e. for system (0.11) we have

(2.5) $$\operatorname{div}\{J_f^{(n-2)/n} G^{-1}(x) \operatorname{grad} f^l(x)\} = 0, \quad l = 1, 2, \ldots, n.$$

Notice that the second term in the parenthesis of (2.4) in a neighbourhood of (x_0, f_0) for $H(x,f)$ continuous at (x_0, f_0) is majorated by a quantity arbitrarily small with respect to $\operatorname{grad} f^l(x)$. This remark will play an important role in some perturbation arguments in studying the regularity problem of solutions. The „principal term" in (2.4) contains differentials of the components $f^i(x)$ for $i \neq l$, only in the Jacobian $J_f(x)$.

We notice, however, that for every index l, the Jacobi determinant $J_f(x)$ can be expressed, up to „zero order" terms, as a function of $\operatorname{grad} f^l(x)$.

In fact, if H^{il} denotes the (i, l) entry of the inverse matrix H^{-1}, then system (0.8) implies the relation

$$\langle G^{-1} \operatorname{grad} f^l(x), \operatorname{grad} f^i(x)\rangle = J_f^{2/n} H^{il}, \quad i, l = 1, 2, \ldots, n,$$

which, for $i = l$ permits to express J_f in terms of $\operatorname{grad} f^l(x)$. Inserting the obtained formula for $J_f(x)$ into (2.4) we get the equations

(2.6) $$\operatorname{div}\left\{\left(\frac{\langle G^{-1} \operatorname{grad} f^l \operatorname{grad} f^l\rangle}{H^{ll}(x,f)}\right)^{(n-2)/2} G^{-1}(x,f) \operatorname{grad} f^l(x) + J_f D^{-1} f(H^{-1}(x_0,f_0) - H^{-1}(x,f))e^l\right\} = 0$$

for $l = 1, 2, \ldots, n,$

which is a system with the unknowns $f^l(x)$ essentially „separated" in the highest order terms. In the case (0.11), i.e. $H \equiv$ const $G \equiv G(x)$, this reduces to the following single equation for every component $u = f^l(x)$

(2.7) $$\operatorname{div}\{\langle G^{-1} \operatorname{grad} u, \operatorname{grad} u\rangle^{(n-2)/2} G^{-1}(x) \operatorname{grad} u(x)\} = 0.$$

In particular for the component $u = f^l(x)$ of a conformal mapping f, that is when $G(x) \equiv E$, we obtain the equation

(2.8) $$\operatorname{div}\{|\nabla u|^{n-2} \cdot \nabla u\} = 0.$$

The above two equations can be considered as Euler equations for the integral functionals

$$\int_{\Omega} |\nabla u|^n \, dx \quad \text{for} \quad (2.8),$$

and

$$\int_{\Omega} \langle G^{-1}(x) \nabla u, \nabla u \rangle^{n/2} \, dx \quad \text{for} \quad (2.7).$$

Solutions of a single equation of this type, as for example (2.7) or (2.8), were investigated from a variety of view points by J. Serrin [24], A. Elcrat and N. G. Meyers [7], O. A. Ladyzenskaya and N. N. Uraltseva [13], and others. Although the equations (2.6) or (2.7) are not equivalent to our basic system (0.8) or (0.11), they yield some properties of quasiregular mappings. As it was proved by J. Serrin [24], every solution of equation (2.7) with measurable coefficients satisfies Hölder's condition. A more general result has been obtained by A. Elcrat and N. G. Meyers [7], who proved that there exists a number $p > n$ such that solutions of (2.7) belong to the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$.

The regularity problem for a single equation of type (2.8) has been investigated by N. N. Uraltseva and O. A. Ladyzenskaya in [13]. Unfortunately, it is not clear how to extend their proofs to degenerated elliptic *systems* of type (2.6).

In the next sections we shall give complete proofs of regularity of the generalized solutions $f = f(x)$ of system (2.6). In the process we shall have to use some additional properties of the map f, which result from the general theory of quasiconformal mappings. Therefore we shall prove only the regularity of f satisfying our basic system (0.8). For this purpose we now rewrite system (2.4) in a more suitable form. First we observe that the function $J_f(x)^{2/n}$ can be expressed as a homogeneous quadratic polynomial with respect to the variables Df. From (0.8) we get

$$J_f(x)^{2/n} = \frac{\operatorname{Tr}\left(D^* f H(x, f) Df\right)}{\operatorname{Tr} G(x, f)}.$$

Let us suppose for a moment that the former system (0.8) is sufficiently close to the system for conformal mappings, i.e. that the number $\sup q(x, f)$ is sufficiently small. The matrices $G(x, f)$ and $H(x, f)$ are of the form

$$(2.9) \qquad G(x, f) = E + g(x, f) \quad \text{and} \quad H(x, f) = E + h(x, f),$$

where

$$|g(x, f)| + |h(x, f)| \leqslant q(x, f).$$

Inserting the resulting formula for $J_f(x)$ into (2.4) and then using relations (2.9), we obtain a system of the following general form:

$$(2.10) \qquad \operatorname{div}\{|Df|^{n-2} \operatorname{grad} f^l(x) + a^l(x, f, Df)\} = 0, \qquad l = 1, 2, \ldots, n.$$

The vectors $a^l(x,f,Df)$, $l = 1,2,\ldots,n$, may be interpreted as a perturbation of the system

$$\operatorname{div}\{|Df|^{n-2}\operatorname{grad} f^l(x)\} = 0, \quad l = 1,2,\ldots,n, \tag{2.11}$$

which appears in the case of conformal mappings, i.e. when $G \equiv H \equiv E$.

The vector-functions $a^l(x,f,\mathscr{A})$ are positively homogeneous of order $n-1$ with respect to $\mathscr{A} \in \mathbf{R}^{n^2}$, i.e.

$$a^l(x,f,t\mathscr{A}) = t^{n-1}a^l(x,f,\mathscr{A}) \quad \text{for} \quad t \geqslant 0,$$

and they satisfy the inequalities

$$\begin{aligned} |a^l(x,f,\mathscr{A})| &\leqslant \text{const}\, q(x,f)|\mathscr{A}|^{n-1}, \\ \left|\frac{\partial a^l(x,f,\mathscr{A})}{\partial \mathscr{A}_{ij}}\right| &\leqslant \text{const}\, q(x,f)|\mathscr{A}|^{n-2}. \end{aligned} \tag{2.12}$$

Moreover, the vector functions $a^l(x,f,\mathscr{A})$ belong to the same class of smoothness with respect to the variables (x,f) as the characteristics $G(x,f)$, and $H(x,f)$, and they are infinitely smooth on the unit sphere with respect to the variables $\mathscr{A} \in S^{n^2-1}$.

If the characteristics G and H are continuous, then the coefficient $q(x,f)$ may by arbitrarily small if we consider system (2.10) in a sufficiently small neighbourhood of the point $(x_0,f_0) \in \Omega \times \Omega'$, where $G(x_0,f_0) = H(x_0,f_0) = E$. Correct boundary value problems do not exist for the overdetermined system (0.8). They are specific for the second order system of type (2.10). As a confirmation of this hypothesis we prove the following proposition:

PROPOSITION 2.1. *Suppose that the vector functions f and g are of the class $W_n^1(\Omega)$ and $f-g \in \mathring{W}_n^1(\Omega)$ ($\mathring{W}_n^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_n^1(\Omega)$).*

Suppose also that f and g satisfy the systems

$$\begin{aligned} \operatorname{div}\{|Df|^{n-2}\operatorname{grad} f^l(x) + a^l(x,Df)\} &= 0, \\ \operatorname{div}\{|Dg|^{n-2}\operatorname{grad} g^l(x) + a^l(x,Dg)\} &= 0. \end{aligned} \tag{i}$$

Here we assume that the vector-functions $a^l(x,\mathscr{A})$ are positively homogeneous of order $n-1$ and of the class C^1 with respect to $\mathscr{A} \in \mathbf{R}^{n^2}$. Moreover, we assume that

$$|a^l(x,\mathscr{A})| + \left|\frac{\partial a^l(x,\mathscr{A})}{\partial \mathscr{A}_{ij}}\right| \cdot |\mathscr{A}| \leqslant \varepsilon |\mathscr{A}|^{n-1}.$$

If ε is sufficiently small, then $f \equiv g$.

Proof. It is easy to see that the functions

$$|Df|^{n-2}\operatorname{grad} f^l(x) + a^l(x,Df) \quad \text{and} \quad |Dg|^{n-2}\operatorname{grad} g^l(x) + a^l(x,Dg)$$

belong to the space $L_{n/(n-1),\mathrm{loc}}(\Omega)$. Since $\varphi(x) = f(x)-g(x) \in \mathring{W}_n^1(\Omega)$ and $1/n+(n-1)/n = 1$, the weak form of (i) may be written as the following system of integral identities:

$$\int_\Omega \langle |Df|^{n-2} \operatorname{grad} f^l(x) + a^l(x, Df), \operatorname{grad} \varphi^l(x)\rangle\, dx = 0,$$

$$\int_\Omega \langle |Dg|^{n-2} \operatorname{grad} g^l(x) + a^l(x, Dg), \operatorname{grad} \varphi^l(x)\rangle\, dx = 0.$$

Summing with respect to the index $l = 1, 2, \ldots, n$ and subtracting both integrals, we get

$$\int_\Omega \langle |Df|^{n-2} Df - |Dg|^{n-2} Dg, Df-Dg\rangle\, dx = \int_\Omega \langle a(x, Dg)-a(x, Df), Df-Dg\rangle\, dx.$$

Here $a(x, \mathscr{A})$ denotes the matrix $\begin{bmatrix} a^1(x, \mathscr{A}) \\ \cdots\cdots \\ a^n(x, \mathscr{A}) \end{bmatrix}$. We have used the symbol $\langle \mathscr{A}, \mathscr{B}\rangle$ for the scalar product of matrices, i.e. $\langle \mathscr{A}, \mathscr{B}\rangle = \sum_{ij}^n \mathscr{A}_{ij}\mathscr{B}_{ij}$.

To the first integral of the last equality we apply the following lemma:

LEMMA 2.1. *The inequality*

$$\langle |\mathscr{A}|^{n-2}\mathscr{A} - |\mathscr{B}|^{n-2}\mathscr{B}, \mathscr{A}-\mathscr{B}\rangle \geqslant \tfrac{1}{2}(|\mathscr{A}|^{n-2} + |\mathscr{B}|^{n-2})|\mathscr{A}-\mathscr{B}|^2 \tag{2.13}$$

holds $(n \geqslant 2)$.

Proof. This follows from the fact that

$$(|\mathscr{A}|^{n-2} - |\mathscr{B}|^{n-2})(|\mathscr{A}|^2 - |\mathscr{B}|^2) \geqslant 0.$$

The difference $a(x, Dg)-a(x, Df)$ can be estimated in the following way:

$$\begin{aligned} |a(x, Dg)-a(x, Df)| &= \left| \int_0^1 \frac{d}{d\tau} a\big(x, \tau Df + (1-\tau)Dg\big)\, d\tau \right| \\ &\leqslant |Df-Dg| \int_0^1 \sum_{ij,l} \left| \frac{\partial a^l(x, \ldots)}{\partial \mathscr{A}_{ij}} \right| d\tau \\ &\leqslant \mathrm{const}\cdot\varepsilon |Df-Dg| \int_0^1 [\tau|Df| + (1-\tau)|Dg|]^{n-2}\, d\tau \\ &\leqslant \mathrm{const}\cdot\varepsilon (|Df|^{n-2} + |Dg|^{n-2})|Df-Dg|. \end{aligned}$$

Finally we obtain the inequality

$$\int_{\Omega} (1-\text{const}\,\varepsilon)(|Df|^{n-2}+|Dg|^{n-2})\cdot|Df-Dg|^2\,dx \leqslant 0.$$

The number $(1-\text{const}\cdot\varepsilon)$ is positive if we assume ε to be sufficiently small. Hence $Df = Dg$ almost everywhere in Ω, that is, $f = g$. The result follows.

3. Some properties of Sobolev and Besov spaces

In this section we list the fundamental properties of functions from Sobolev and Besov spaces, which we shall need later.

Let f be a function of the space $L_p(\boldsymbol{R}^n)$, $1 \leqslant p \leqslant \infty$. Let us put $\omega_p(h) = \|f(x+h)-f(x)\|_p$ for $h \in \boldsymbol{R}^n$.

LEMMA 3.1. *The function f belongs to the Sobolev space $W_p^1(\boldsymbol{R}^n)$, $1 < p < \infty$, iff $f \in L_p(\boldsymbol{R}^n)$ and $\omega_p(h) = O(|h|)$. The norm*

$$\|f\|_p+\sup_{h\neq 0}\frac{\|f(x+h)-f(x)\|_p}{|h|} \tag{3.1}$$

is equivalent to the standard norm of the space $W_p^1(\boldsymbol{R}^n)$.

For the proof see [25].

Remark. It is easy to formulate corresponding results for an arbitrary domain $\Omega \subset \boldsymbol{R}^n$. We leave their formulation and proof to the reader.

LEMMA 3.2 (Sobolev's embedding lemma, see [13]). *Assume $n > 2$, $1 \leqslant p \leqslant 2n/(n-2)$, $u \in \mathring{W}_2^1(\Omega)$; then the following inequality holds:*

$$\|u\|_{L_p(\Omega)} \leqslant \frac{2(n-1)}{n-2}(\text{mes}\,\Omega_0)^{1/p+1/n-1/2}\|\text{grad}\,u\|_{L_2(\Omega)}, \tag{3.2}$$

where

$$\Omega_0 = \{x \in \Omega\,;\, u(x) \neq 0\}.$$

In particular, we get $W_{2,\text{loc}}^1(\Omega) \subset L_{2n/(n-2),\text{loc}}(\Omega)$.

Lemma (3.2) can be extended to the case of Besov spaces.

DEFINITION 3.1. Assume that $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$. We define a norm on the Besov space $B_\theta^p(\boldsymbol{R}^n)$ by the formula

$$\|f\|_{B_\theta^p} = \|f\|_p+\sup_{h\neq 0}\frac{\|f(x+h)-f(x)\|_p}{|h|^\theta}.$$

A function f is said to be *of class* $B_\theta^p(\boldsymbol{R}^n)$ iff the norm $\|f\|_{B_\theta^p}$ is finite.

LEMMA 3.3 (embedding lemma of Besov–Nikolski [25]). *The inclusion* $B^{p_1}_{\theta_1}(R) \subset B^{p_2}_{\theta_2}(R^n)$ *holds if*

$$\theta_1 \geqslant \theta_2 > 0, \quad 1 \leqslant p_1 < p_2 \leqslant \infty, \quad \theta_1 - n/p_1 = \theta_2 - n/p_2.$$

Now we summarize Lemmata 3.2 and 3.3 as follows:

LEMMA 3.4. *Let Ω be an open subset of R^n, $n > 2$, $f \in L_2(\Omega)$ and assume that the inequality*

$$\int_{\Omega^0} \frac{|f(x+h)-f(x)|^2}{|h|^{2\theta}}\,dx \leqslant C_{\Omega^0} \tag{3.3}$$

is satisfied for every compact subset $\Omega^0 \subset \Omega$ and each sufficiently small nonzero vector $h \in R^n$. (We say that $f \in B^{2,\mathrm{loc}}_{\theta}(\Omega)$.)

If $0 < \theta < 1$, then $f \in L_{p,\mathrm{loc}}(\Omega)$ for every $p \in [2, 2+4\theta/(n-2\theta))$; if $\theta = 1$, then $f \in L_{2n/(n-2)\mathrm{loc}}(\Omega)$.

Proof. The case $\theta = 1$ is a simple consequence of Sobolev's embedding lemma. To prove the case $0 < \theta < 1$ we consider the function $\eta(x) = \varphi(x) f(x)$, where φ is an arbitrary smooth function with compact support contained in Ω. Clearly $\eta \in L_2(R^n)$ and the following inequality holds:

$$\sup_{h \neq 0} \int_{R^n} \frac{|\eta(x+h)-\eta(x)|^2}{|h|^{2\theta}}\,dx < \infty,$$

that is, $\eta \in B^2_\theta(R^n)$. By the lemma of Besov and Nikolski, we can write $\eta \in B^p_{\theta'}(R^n)$ for every $\theta' < \theta$ and $\theta - n/2 = \theta' - n/p$, i.e. for $p = 2 + 4(\theta-\theta')/(n-2(\theta-\theta'))$. In particular, we have $\eta \in L_p(R^n)$. Since φ was chosen arbitrarily, $f \in L_{p,\mathrm{loc}}(\Omega)$. Finally, when θ' runs over the interval $(0, \theta)$, p runs over $(2, 4\theta/(n-2\theta))$. This ends the proof of the lemma.

4. Classes $\Lambda^\alpha(G, H)$, $0 < \alpha \leqslant 1$

Assume that for some $0 < \alpha \leqslant 1$ the characteristics $G(x,f)$ and $H(x,f)$ (in a product domain $\Omega \times \Omega'$) satisfy the Hölder condition with exponent α. In the study of local behaviour of the solution $f(x)$ we may assume, without loss of generality, that $G(x_0, f_0) = H(x_0, f_0) = E$ ($f_0 = f(x_0)$) for a given point $x_0 \in \Omega$. Equations (2.10) are satisfied in some neighbourhood of x_0. In this case the vector functions $a^l(x, f, \mathscr{A})$ satisfy the inequality

$$|a^l(x,f,\mathscr{A}) - a^l(y,g,\mathscr{A})| \leqslant \mathrm{const}\,(|x-y|^\alpha + |f-g|^\alpha)|\mathscr{A}|^{n-1}. \tag{4.1}$$

We fix an arbitrary but sufficiently small non-zero vector $h \in \boldsymbol{R}^n$. Then the vector function $g(x) = f(x+h)$ defined in a neighbourhood of x_0 satisfies the equations

$$\operatorname{div}\{|Dg|^{n-2}\operatorname{grad} g^l(x) + b^l(x, g, Dg)\} = 0, \quad l = 1, 2, \ldots, n, \tag{4.2}$$

where $b^l(x, g, Dg) = a^l(x+h, g, Dg)$. This is obtained from equation (2.10) by a translation of the argument x by h. Substracting the weak forms of equations (2.10) and (4.2), we get the equality

$$\int_\Omega \langle |Df|^{n-2}\nabla f^l - |Dg|^{n-2}\nabla g^l, \nabla\eta^l\rangle\, dx = \int_\Omega \langle b^l - a^l, \nabla\eta^l\rangle\, dx,$$

which is satisfied for every test function $\eta^l \in \mathring{W}_n^1(\Omega)$ with sufficiently small support, containing x_0 in its interior. Here ∇ denotes the gradient operation.

Putting $\eta^l(x) = \varphi^n(x)(f^l(x) - g^l(x))$, where $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$, into the last equality, we obtain

$$\begin{aligned}\int_\Omega \varphi^n \langle |Df|^{n-2}\nabla f^l - |Dg|^{n-2}\nabla g^l, \nabla f^l - \nabla g^l\rangle\, dx \\ \leqslant \text{const} \int_\Omega \varphi^{n-1}|\nabla\varphi|\,\big||Df|^{n-2}\nabla f^l - |Dg|^{n-2}\nabla g^l\big|\,|f^l - g^l|\, dx + \\ + \text{const}\int_\Omega \varphi^{n-1}|\nabla\varphi|\cdot|b^l - a^l|\cdot|f^l - g^l|\, dx + \\ + \text{const}\int_\Omega \varphi^n\cdot|b^l - a^l|\cdot|\nabla f^l - \nabla g^l|\, dx.\end{aligned} \tag{4.3}$$

We sum over l both sides of (4.3). In the sequel we do not write the symbol of summation if the indices are repeated. We will use an easy lemma.

LEMMA 4.1. *The inequality*

$$\big||Df|^{n-2}\nabla f^l - |Dg|^{n-2}\nabla g^l\big| \leqslant C_n(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg| \tag{4.4}$$

holds with some constant C_n depending only on dimension n.

From (4.3), (4.4) and (2.13) we get

$$\begin{aligned}\tfrac{1}{2}\int_\Omega \varphi^n(|Df|^{n-2} + |Dg|^{n-2})|Df - Dg|^2\, dx \\ \leqslant \text{const}\int_\Omega \varphi^{n-1}|\nabla\varphi|(|Df|^{n-2} + |Dg|^{n-2})|Df - Dg|\,|f - g|\, dx + \\ + \text{const}\int_\Omega \varphi^n|b^l - a^l|\,|\nabla f^l - \nabla g^l|\, dx + \\ + \text{const}\int_\Omega \varphi^{n-1}|\nabla\varphi|\,|b^l - a^l|\,|f^l - g^l|\, dx.\end{aligned} \tag{4.5}$$

Now it remains to estimate the difference $b^l - a^l$. For this purpose we apply property (2.12) and assumption (4.1). We conclude that

$$\begin{aligned} |b^l - a^l| &\leqslant |a^l(x+h, g, Dg) - a^l(x, f, Dg)| + |a^l(x, f, Dg) - a^l(x, f, Df)| \\ &\leqslant \text{const}\,(|h|^\alpha + |f-g|^\alpha)\,|Dg|^{n-1} + \\ &\qquad + \text{const} \cdot q(x, f)\,(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|\,. \end{aligned}$$

Putting this into (4.5), we can write

$$\int_\Omega \varphi^n(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|^2\,dx \leqslant \text{const}\,(I_1 + I_2 + I_3 + I_4 + I_5), \tag{4.6}$$

where

$$I_1 = \int_\Omega \varphi^{n-1}|\nabla\varphi|\,(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|\,|f-g|\,dx,$$

$$I_2 = \int_\Omega \varphi^n(|h|^\alpha + |f-g|^\alpha)\,|Dg|^{n-1}\,|Df - Dg|\,dx,$$

$$I_3 = \int_\Omega \varphi^{n-1}|\nabla\varphi|\,|f-g|\,(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|\,q(x, f)\,dx,$$

$$I_4 = \int_\Omega \varphi^{n-1}|\nabla\varphi|\,|f-g|\,(|h|^\alpha + |f-g|^\alpha)\,|Dg|^{n-1}\,dx,$$

$$I_5 = \int_\Omega \varphi^n(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|^2\,q(x, f)\,dx.$$

We estimate the first three integrals by using the elementary inequality

$$ab \leqslant \varepsilon a^2 + \frac{1}{4\cdot\varepsilon}\,b^2$$

for every $\varepsilon > 0$. The number ε will be chosen later. To the integral I_4 we apply the inequality $ab \leqslant a^2 + b^2$. The integral I_5 is left unchanged:

$$\begin{aligned} I_1 &\leqslant \varepsilon \int_\Omega \varphi^n(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|^2\,dx + \\ &\qquad + \frac{1}{4\varepsilon}\int_\Omega \varphi^{n-2}|\nabla\varphi|^2\,(|Df|^{n-2} + |Dg|^{n-2})\,|f-g|^2\,dx, \end{aligned}$$

$$\begin{aligned} I_2 &\leqslant \varepsilon \int_\Omega \varphi^n(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|^2\,dx \\ &\qquad + \frac{1}{4\varepsilon}\int_\Omega \varphi^n\,|Dg|^n\,(|h|^\alpha + |f-g|^\alpha)^2\,dx, \end{aligned}$$

$$\begin{aligned} I_3 &\leqslant \varepsilon \int_\Omega \varphi^n(|Df|^{n-2} + |Dg|^{n-2})\,|Df - Dg|^2\,dx + \\ &\qquad + \frac{1}{4\varepsilon}\int_\Omega \varphi^{n-2}|\nabla\varphi|^2\,q^2(x, f)\,(|Df|^{n-2} + |Dg|^{n-2})\,|f-g|^2\,dx, \end{aligned}$$

$$\begin{aligned} I_4 &\leqslant \int_\Omega \varphi^{n-2}|\nabla\varphi|^2(|Df|^{n-2} + |Dg|^{n-2})\,|f-g|^2\,dx + \\ &\qquad + \int_\Omega \varphi^n\,|Dg|^n(|h|^\alpha + |f-g|^\alpha)^2\,dx. \end{aligned}$$

Hence

$$I_1+I_2+I_3+I_4+I_5$$
$$\leqslant \int_\Omega (3\varepsilon+q(x,f))\,\varphi^n(|Df|^{n-2}+|Dg|^{n-2})\,|Df-Dg|^2\,dx+$$
$$+C_\varepsilon \int_\Omega \varphi^{n-2}|\nabla\varphi|^2(|Df|^{n-2}+|Dg|^{n-2})\,|f-g|^2\,dx+$$
$$+C_\varepsilon \int_\Omega \varphi^n|Dg|^n(|h|^\alpha+|f-g|^\alpha)^2\,dx.$$

If we choose ε sufficiently small, then the coefficient $3\varepsilon+q(x,f)$ which appears in the first integral will be arbitrarily small in a sufficiently small neighbourhood of $(x_0,f_0)\in\Omega\times\Omega'$. Instead of restricting our considerations to some neighbourhood of x_0 it is better to assume that the support of $\varphi(x)$ is small.

Now inequality (4.6) implies

$$(4.7)\quad \int_\Omega \varphi^n(|Df|^{n-2}+|Dg|^{n-2})\,|Df-Dg|^2\,dx$$
$$\leqslant \text{const} \int_\Omega \varphi^{n-2}\,|\nabla\varphi|^2\,(|Df|^{n-2}+|Dg|^{n-2})\,|f-g|^2\,dx+$$
$$+\text{const} \int_\Omega \varphi^n\,|Dg|^n\,(|h|^\alpha+|f-g|^\alpha)^2\,dx.$$

Let θ be an arbitrary number belonging to the interval $(0,\alpha)$. We divide both sides of (4.7) by $|h|^{2\theta}$. The quotient $|f-g|^\alpha/|h|^\theta$ is uniformly bounded (see Theorem 1.1). Therefore we obtain the inequality

$$(4.8)\quad \int_\Omega \varphi^n(x)\big(|Df(x)|^{n-2}+|Df(x+h)|^{n-2}\big)\frac{|Df(x+h)-Df(x)|^2}{|h|^{2\theta}}\,dx$$
$$\leqslant \text{const}\,(\varphi,\Omega,\theta,\|Df\|_{L_{n,\mathrm{loc}}(\Omega)}),$$

which is satisfied for every smooth function $\varphi\geqslant 0$ with sufficiently small support containing the point x_0 in its interior and for any sufficiently small non-zero vector $h\in\mathbf{R}^n$. From inequality (4.8) one can obtain some information about Df.

Let $\Phi=\Phi(\mathscr{A}):\mathbf{R}^{n^2}\to\mathbf{R}$ be an arbitrary positively homogeneous form of degree $\nu\geqslant 1$, i.e. $\Phi(t\mathscr{A})=t^\nu\Phi(\mathscr{A})$ for $t\geqslant 0$, of class $C^1(S^{n^2-1})$ (when restricted to the unit sphere). We consider the function $\mu(x)=\Phi(Df(x))$. The difference $\mu(x+h)-\mu(x)$ may be estimated as follows:

$$(4.9)\quad |\mu(x+h)-\mu(x)|$$
$$\leqslant \left|\int_0^1 \frac{d}{d\tau}\,\Phi\big(\tau Df(x+h)+(1-\tau)Df(x)\big)d\tau\right|$$

$$\leqslant |Df(x+h)-Df(x)| \int_0^1 \sum_{ij} \left|\frac{\partial \Phi}{\partial \mathscr{A}_{ij}}\right| d\tau$$

$$\leqslant \text{const}\, |Df(x+h)-Df(x)| \int_0^1 |\tau Df(x+h)+(1-\tau)\, Df(x)|^{\nu-1}\, d\tau$$

$$\leqslant \text{const}\, (|Df(x+h)|+|Df(x)|)^{\nu-1}\, |Df(x+h)-Df(x)|\,.$$

Here we have used the inequality

$$\left|\frac{\partial \Phi(\mathscr{A})}{\partial \mathscr{A}_{ij}}\right| \leqslant \text{const}\, |\mathscr{A}|^{\nu-1},$$

which automatically follows from the conditions imposed on Φ. Let us consider the case $\nu = n/2$. We have

$$(4.10)\quad \int_\Omega \varphi^n(x) \frac{|\mu(x+h)-\mu(x)|^2}{|h|^{2\theta}}\, dx$$

$$\leqslant \text{const} \int_\Omega \varphi^n(x)\big(|Df(x+h)|^{n-2}+|Df(x)|^{n-2}\big) \frac{|Df(x+h)-Df(x)|^2}{|h|^{2\theta}}\, dx$$

$$\leqslant \text{const}\, (\varphi, \Omega, \theta, \|Df\|_{L_{n,\text{loc}}(\Omega)})\,.$$

Since θ can be chosen arbitrarily close to α, Lemma 3.4 gives $\mu \in L_{q,\text{loc}}(\Omega)$ for $2 \leqslant q < 2+4\alpha/(n-2\alpha)$. In particular, setting $\mu(x) = |Df(x)|^{n/2}$ or $\mu(x) = \sqrt{J_f(x)}$, we obtain the following result:

THEOREM 4.1. *Every mapping $f \in \Lambda^\alpha(G, H)$, $0 < \alpha \leqslant 1$, belongs to the space $W^1_{p,\text{loc}}(\Omega)$ for every $n \leqslant p < n^2/(n-2\alpha)$. The square root of the Jacobian $\sqrt{J_f(x)}$ belongs to the Besov space $B^{2,\text{loc}}_{\theta}(\Omega)$* for every $0 < \theta < \alpha$.

Remark. The existence of a number $p > n$ such that $f \in W^1_{p,\text{loc}}(\Omega)$ was known earlier for every quasiregular mapping f with a characteristic not neccessary continuous. This result is due to B. Bojarski [3] for a two-dimensional mapping, F. W. Gehring [9] and A. Elcrat, G. Meyers [7] in higher dimensions. Unfortunately, their methods do not allow us to obtain the exact value $p = p(K, n)$ as a function of the dilatation K and the dimension n. From the Sobolev embedding theorem we get the estimate

$$1-\frac{n}{p(K,\, n)} \leqslant \frac{1}{K^{n-1}},$$

because, as is well known, the number $1/K^{n-1}$ coincides with the Hölder exponent for K-quasiconformal mappings. Some examples permit us to expect that $1-n/p(K, n) = 1/K^{n-1}$. If this were true, then the number p in Theorem 4.1 could be chosen arbitrarily. Fortunately, we do not need this hypothesis: the inequality for p in the theorem is sufficient for our purpose.

The following theorem gives a little more information about the Jacobian $J_f(x)$.

THEOREM 4.2. *If* $f \in \Lambda^\alpha(G, H)$, $0 < \alpha \leqslant 1$, *then the Jacobian* $J_f(x)$ *and all homogeneous n-forms of Df belong to the Besov space* $B_\theta^{s,\mathrm{loc}}(\Omega)$ *for every* $0 < \theta < \alpha$ *and* $1 < s < n/(n-\alpha)$. *This means that*

$$\int_{\Omega^0} \frac{|J_f(x+h) - J_f(x)|^s}{|h|^{s\theta}} dx \leqslant C_{\Omega^0}$$

for sufficiently small $h \in \mathbf{R}^n \setminus \{0\}$, *where* Ω^0 *is an arbitrary compact subset of* Ω *and* C_{Ω^0} *is a constant independent of h.*

Proof. Inequality (4.9) for $\nu = n$ takes the form

$$|\mu(x+h) - \mu(x)| \leqslant \mathrm{const}\big(|Df(x+h)| + |Df(x)|\big)^{n-1} |Df(x+h) - Df(x)|.$$

Let μ denote the Jacobian $J_f(x)$. Then

$$|\mu(x+h) - \mu(x)|^s \leqslant \mathrm{const}\left\{\big(|Df(x+h)| + |Df(x)|\big)^{(n-2)/2} |Df(x+h) - Df(x)|\right\}^s \times \times \left\{\big(|Df(x+h)| + |Df(x)|\big)^{n/2}\right\}^s.$$

We divide both sides of the above inequality by $|h|^{\theta s}$. Next we use Young's inequality

$$ab \leqslant \frac{s}{2} |a|^{2/s} + \frac{2-s}{2} |b|^{2/2-s}.$$

Finally we get

$$\text{(4.11)} \quad \frac{|\mu(x+h) - \mu(x)|^s}{|h|^{\theta s}} \leqslant \mathrm{const}\big(|Df(x+h)| + |Df(x)|\big)^{ns/(2-s)} + \\ + \mathrm{const}\big(|Df(x+h)| + |Df(x)|\big)^{n-2} \frac{|Df(x+h) - Df(x)|^2}{|h|^{2\theta}}.$$

Since $s < n/(n-\alpha)$, we have $ns/(2-s) < n^2/(n-2\alpha)$ and according to Theorem 4.1 the function $(|Df(x+h)| + |Df(x)|^{ns/(2-s)}$ is integrable on $\Omega^0 \subset \Omega$. Integrating both sides of inequality (4.11) over the compact subset $\Omega^0 \subset \Omega$ we obtain our theorem.

5. The case of Lipschitz characteristics

In this section we study the case where $\alpha = 1$. We shall first show that (4.8) also holds for $\theta = 1$. For this purpose let us observe that Theorem 4.1 implies $f \in W^1_{p,\mathrm{loc}}(\Omega)$ for every $n \leqslant p < n^2/(n-2) = n+2+4/(n-2)$. In other words, if the characteristics of f are Lipschitz continuous, then

$$\text{(5.1)} \qquad f \in W^1_{n+2,\mathrm{loc}}(\Omega).$$

The estimation (4.8) for $\theta = 1$ cannot be obtained by using the same arguments, because the quotient $|f-g|/|h|$ is not known to be bounded yet. We shall use another approach to the problem. Dividing both sides of (4.7) by $|h|^2$, we easily obtain the inequality

$$\int_\Omega \varphi^n(|Df(x+h)|^{n-2}+|Df(x)|^{n-2})\frac{|Df(x+h)-Df(x)|^2}{|h|^2}\,dx$$
$$\leqslant \text{const}\int_\Omega \varphi^n|\nabla\varphi|\,(|Df(x+h)|^{n-2}+|Df(x)|^{n-2})\frac{|f(x+h)-f(x)|^2}{|h|^2}\,dx+$$
$$+\text{const}\int_\Omega \varphi^n|Df(x+h)|^n\left(1+\frac{|f(x+h)-f(x)|}{|h|^2}\right)^2 dx.$$

Now we use Young's inequalities

$$ab \leqslant \frac{n-2}{n}|a|^{n/(n-2)}+\frac{2}{n}|b|^{n/2}\,; \quad ab \leqslant \frac{n}{n+2}|a|^{n/(n+2)}+\frac{2}{n+2}|b|^{(n+2)/2}.$$

We obtain an upper bound of the right side terms by the following integrals:

$$\text{const}\int_\Omega \varphi^n|\nabla\varphi|\left(|Df(x+h)|^n+|Df(x)|^n+\frac{|f(x+h)-f(x)|^n}{|h|^n}\right)dx$$

and

$$\text{const}\int_\Omega \varphi^n\left(1+|Df(x+h)|^{n+2}+\frac{|f(x+h)-f(x)|^{n+2}}{|h|^{n+2}}\right)dx.$$

As we know, $f\in W^1_{n+2,\text{loc}}(\Omega)$; then, by Lemma 3.1, we get

$$\int_\Omega \varphi^n(x)\,(|Df(x+h)|^{n-2}+|Df(x)|^{n-2})\frac{|Df(x+h)-Df(x)|^2}{|h|^2}\,dx \tag{5.2}$$
$$\leqslant \text{const}(\varphi, \Omega, \|Df\|_{L_{n+2,\text{loc}}(\Omega)}).$$

Now we prove the following preparatory lemma, analogous to Theorem 4.2:

LEMMA 5.1. *Let $\Phi(x,f,\mathscr{A})$: $\Omega\times\Omega'\times\mathbf{R}^{n^2}\to\mathbf{R}$ be a positively homogeneous form of degree ν, $n/2\leqslant\nu\leqslant n+1$, with respect to $\mathscr{A}\in\mathbf{R}^{n^2}$ and of class* Lipsch $(\Omega\times\Omega'\times\mathbf{R}^{n^2})$. *Then the function $\mu(x)=\Phi(x,f(x),Df(x))$ belongs to $W^1_{(n+2)/(\nu+1),\text{loc}}(\Omega)$ for every f with Lipschitz continuous characteristics.*

Proof. By the Rademacher–Stepanov theorem, Φ is differentiable almost everywhere [23]. We first write

$$|\mu(x+h)-\mu(x)| = \left|\int_0^1 \frac{d}{d\tau}\Phi\big(\tau x+(1-\tau)(x+h),\ \tau f(x)+ +(1-\tau)f(x+h),\ \tau Df(x)+(1-\tau)Df(x+h)\big)d\tau\right|.$$

The obvious estimates $|D_x\Phi(x,f,\mathscr{A})|+|D_f\Phi(x,f,\mathscr{A})| \leqslant \text{const}\,|\mathscr{A}|^\nu$ and $|D_{\mathscr{A}}\Phi(x,f,\mathscr{A})| \leqslant \text{const}\,|\mathscr{A}|^{\nu-1}$ give

$$\begin{aligned}|\mu(x+h)-\mu(x)| &\leqslant \text{const}\int_0^1\{|D_x\Phi|\,|h|+\\ &\quad +|D_f\Phi|\,|f(x+h)-f(x)|+|D_{\mathscr{A}}\Phi|\,|Df(x+h)-Df(x)|\}\,d\tau\\ &\leqslant \text{const}\int_0^1\{|\tau Df(x)+(1-\tau)Df(x+h)|^\nu\cdot(|h|+|f(x+h)-f(x)|)\}\,d\tau+\\ &\quad +\text{const}\int_0^1|\tau Df(x)+(1-\tau)Df(x+h)|^{\nu-1}|Df(x+h)-Df(x)|\,d\tau.\end{aligned}$$

It is easy to see that the last inequality implies

$$\begin{aligned}\frac{|\mu(x+h)-\mu(x)|}{|h|} &\leqslant \text{const}(|Df(x)|+|Df(x+h)|)^\nu\left(1+\frac{|f(x+h)-f(x)|}{|h|}\right)+\\ &\quad +\text{const}\big(|Df(x)|+|Df(x+h)|\big)^{\nu-1}\frac{|Df(x+h)-Df(x)|}{|h|}.\end{aligned}$$

To estimate the first term we apply the inequality

$$ab \leqslant \frac{\nu}{\nu+1}|a|^{(\nu+1)/\nu}+\frac{1}{\nu+1}|b|^{\nu+1}$$

and for the second one the inequality

$$ab \leqslant \frac{n+2}{2\nu+2}|a|^{(2\nu+2)/(n+2)}+\frac{2\nu-n}{2\nu+2}|b|^{(2\nu+2)/(2\nu-n)}.$$

Before using that we decompose the factor $\big(|Df(x+h)|+|Df(x)|\big)^{\nu-1}$ by the factors $\big(|Df(x)|+|Df(x+h)|\big)^{(n-2)/2}$ and $\big(|Df(x)|+|Df(x+h)|\big)^{(2\nu-n)/2}$. This immediately leads to

$$\begin{aligned}\frac{|\mu(x+h)-\mu(x)|}{|h|} &\leqslant \text{const}\big(|Df(x+h)|+|Df(x)|\big)^{\nu+1}+\\ &\quad +\text{const}\left(1+\frac{|f(x+h)-f(x)|}{|h|}\right)^{\nu+1}+\end{aligned}$$

$$+\text{const}\left\{\left(|Df(x)|+|Df(x+h)|\right)^{(2\nu-n)/2}\right\}^{(2\nu+2)/(2\nu-n)}+$$

$$+\text{const}\left\{\left(|Df(x)|+|Df(x+h)|\right)^{(n-2)/2}\frac{|Df(x+h)-Df(x)|}{|h|}\right\}^{(2\nu+2)/(n+2)},$$

or equivalently

$$\left(\frac{|\mu(x+h)-\mu(x)|}{|h|}\right)^{(n+2)/(\nu+1)} \leqslant \text{const}\left(|Df(x)|+|Df(x+h)|\right)^{n+2}+$$

$$+\text{const}\left(1+\frac{|f(x+h)-f(x)|}{|h|}\right)^{n+2}+$$

$$+\text{const}\left(|Df(x)|^{n-2}+|Df(x+h)|^{n-2}\right)\frac{|Df(x+h)-Df(x)|^2}{|h|^2}.$$

In view of (5.1), (5.2) and Lemma 3.1 we conclude that all terms of the right side are integrable on $\Omega^0 \subset \Omega$. Thus the lemma is proved.

COROLLARY 5.1. *Let f be a mapping with characteristics satisfying the Lipschitz condition. Then*

$$|Df|^{n+1} \in W^1_{1,\mathrm{loc}}(\Omega), \qquad J_f(x) \in W^1_{n+2/n+1,\mathrm{loc}}(\Omega),$$

$$|Df|^{n/2}, \qquad \sqrt{J_f(x)} \in W^1_{2,\mathrm{loc}}(\Omega), \qquad J_f \cdot f^l_{x_j} \in W^1_{1,\mathrm{loc}}(\Omega).$$

This means that distributional derivatives of the above functions are locally integrable with exponents 1, $1+1/(n+1)$, 2, and 1, respectively.

Remark. Actually, as we shall prove in the following the mapping f with Lipschitz characteristics belongs to the space $W^1_{\infty,\mathrm{loc}}(\Omega)$. Some other properties of f will also be given.

6. Existence of second partial derivatives and its consequences

In what follows we assume that the characteristics of f satisfy the Lipschitz condition.

By Corollary 5.1 one can obtain the following differentiability properties of f.

LEMMA 6.1. *There exist functions $f^l_j(x)$, $l, j = 1, 2, \ldots, n$, which coincide with $f^l_{x_j}(x)$ almost everywhere in Ω and are such that the partial derivatives*

$f^l_{jk}(x) = \dfrac{\partial f^l_j(x)}{\partial x_k}$ *exist for almost all* $x \in \Omega$. *Moreover*, $f^l_{jk}(x) = f^l_{kj}(x)$ *almost everywhere for* $l, k, j = 1, 2, \ldots, n$.

Proof. Using the fundamental properties of Sobolev spaces, one can construct a representative $f^l_j(x)$ of the function $f^l_{x_j}(x)$ as follows:

Let $\widetilde{J_f(x)}$ be a representative of the class of function $J_f(x)$ which has all first order partial derivatives almost everywhere in Ω. This is possible by Corollary 5.1 (see Ch. Morrey [17], Theorem 3.1.8). Also by Corollary 5.1 there exists a representative $\widetilde{J_f f^l_{x_j}}$ of the class of function $J_f(x) f^l_{x_j}(x)$ (it may be considered as an $(n+1)$-homogeneous form of Df) admitting differentiation in the classical sense almost everywhere in Ω.

On the other hand, we know that the Jacobian $J_f(x)$ does not vanish almost everywhere if f is a non-constant quasiregular mapping. Setting

$$(6.1)\qquad f^l_j(x) = \frac{\widetilde{J_f(x) f^l_{x_j}(x)}}{\widetilde{J_f(x)}}, \qquad l, j = 1, 2, \ldots, n,$$

we observe that the function f^l_j has first order partial derivatives almost everywhere in Ω.

Unfortunately, we are not able at present to say much about functions

$$f^l_{jk}(x) = \frac{\partial f^l_j(x)}{\partial x_k}.$$

Until get some further information about the second order partial derivatives $f^l_{jk}(x)$, we must use them with great caution.

Let φ be an infinitely smooth function with compact support contained in Ω. Let us consider the function $J_f[\varphi f^l]x_j$, which, by Lemma 5.1, is an element of the space $\mathring{W}^1_1(\Omega)$. Therefore the integral

$$\int_\Omega \frac{\partial}{\partial x_k} \{J_f(x)[\varphi(x) f^l(x)]_{x_j}\}\, dx$$

is equal to zero. Hence we obtain

$$(6.2)\qquad 0 = \int_\Omega (J_f)_{x_k} (\varphi f^l)_{x_j}\, dx + \int_\Omega J_f [(\varphi f^l)_{x_j}]_{x_k}\, dx = \int_\Omega (J_f)_{x_k} (\varphi f^l)_{x_j}\, dx + $$
$$+ \int_\Omega J_f \cdot \varphi f^l_{jk}\, dx + \int_\Omega J_f [\varphi_{x_j x_k} + \varphi_{x_j} \cdot f^l_{x_k} + \varphi_{x_k} \cdot f^l_{x_j}]\, dx.$$

The last integral is symmetric with respect to indices j and k. By an approximation method we now prove that the first integral on the right side does not change under permutation of j and k. For this purpose we observe that $J_f \in W^1_{(n+2)/(n+1),\mathrm{loc}}(\Omega)$ (see Corollary (5.1) and $\varphi f^l \in \mathring{W}^1_{n+2}(\Omega)$ (compare with formula (5.1)).

Let $(J_n(x))$ be a sequence of smooth functions converging to $J_f(x)$ in the space $W^1_{(n+2)/(n+2),\mathrm{loc}}(\Omega)$, and let $(u_n(x))$ be a sequence of smooth functions with compact support converging to φf^l in the space $\mathring{W}^1_{n+2}(\Omega)$. Since $(n+1)/(n+2)+1/(n+2) = 1$,

$$\int_\Omega (J_f)_{x_k}(\varphi f^l)_{x_j}\,dx = \lim_{n\to\infty}\int_\Omega (J_n)_{x_k}(u_n)_{x_j}\,dx$$
$$= \lim_{n\to\infty}\int_\Omega (J_n)_{x_j}(u_n)_{x_k}\,dx = \int_\Omega (J_f)_{x_j}(\varphi f^l)_{x_k}\,dx.$$

Now from (5.2) interchanging the indices j and k, we obtain

$$\int_\Omega \varphi J_f\,[f^l_{jk}-f^l_{kj}]\,dx = 0.$$

Hence, as $J_f(x) \neq 0$ a.e. and $\varphi(x)$ has been chosen arbitrarily, we conclude that $f^l_{jk}(x) = f^l_{kj}(x)$ almost everywhere in Ω. This completes the proof.

Now we shall derive other second order equations for the mapping f, which will be more suitable for the study of the regularity problem.

According to the theorem of Rademacher–Stepanov [23], the coefficients of system (0.8), when they satisfy the Lipschitz condition, are differentiable almost everywhere. Moreover, their derivatives are bounded almost everywhere by the Lipschitz constant. We start with the system

$$D^*f(x)\,H(x,f)\,Df(x) = \|Df(x)\|^2\,G(x,f), \tag{6.3}$$

where

$$\|Df\|^2 = \frac{\mathrm{Tr}\,(D^*fHDf)}{\mathrm{Tr}\,G},$$

or equivalently

$$Df(x)\,G^{-1}(x,f)\,D^*f(x) = \|Df(x)\|^2\,H^{-1}(x,f). \tag{6.4}$$

In tensor notation this reads

$$f^s_i H_{sk} f^k_j = \|Df\|^2 G_{ij}, \tag{6.3$'$}$$

$$f^i_s G^{sk} f^j_k = \|Df\|^2 H^{ij}. \tag{6.4$'$}$$

Differentiating (6.3)′ with respect to x_r, we get

$$\frac{\partial(\|Df\|^2 G_{ij})}{\partial x_r} = f^s_{ir} H_{sk} f^k_j + f^s_i H_{sk} f^k_{jr} + f^s_i f^k_j \frac{\partial H_{sk}}{\partial x_r}.$$

Analogously we get the other equations by the permutation of i, j, r. Summing up, we obtain

$$\frac{\partial(\|Df\|^2 G_{ij})}{\partial x_r}+\frac{\partial(\|Df\|^2 G_{ir})}{\partial x_j}-\frac{\partial(\|Df\|^2 G_{rj})}{\partial x_i}$$
$$= 2f_i^s H_{sk} f_{jr}^k + f_i^s \frac{\partial H_{sk}}{\partial x_r} f_j^k + f_i^s \frac{\partial H_{sk}}{\partial x_j} f_r^k - f_r^s \frac{\partial H_{sk}}{\partial x_i} f_j^k.$$

Now we multiply both sides of the above equation by $\frac{1}{2}G^{ti}f_t^l$. From (6.4)′ we deduce that

$$G^{ti}f_t^l f_i^s = \|Df\|^2 H^{ls} \quad \text{and} \quad G^{ti}f_t^l f_i^s H_{sk} = \|Df\|^2 \delta_k^l,$$

hence

$$(6.5)\quad \frac{1}{2}f_t^l G^{ti}\left[\frac{\partial(\|Df\|^2 G_{ij})}{\partial x_r}+\frac{\partial(\|Df\|^2 G_{ir})}{\partial x_j}-\frac{\partial(\|Df\|^2 G_{rj})}{\partial x_i}\right]$$
$$= \|Df\|^2\left(f_{jr}^l + \frac{1}{2}f_j^k H^{sk}\frac{\partial H_{sk}}{\partial x_r} + \frac{1}{2}f_r^k H^{ls}\frac{\partial H_{sk}}{\partial x_j}\right) -$$
$$-\frac{1}{2}G^{ti} f_t^l f_r^s f_j^k \frac{\partial H_{sk}}{\partial x_i}.$$

A chain rule of differential calculus leads to the following formulae:

$$(6.6)\quad 2\|Df\|^2 f_{rj}^l(x)$$
$$= (\|Df\|^2)_{x_r} f_j^l(x) + (\|Df\|^2)_{x_j} f_r^l(x) - (\|Df\|^2)_{x_i} G^{ti} G_{rj} f_t^l +$$
$$+ \|Df\|^2 G^{ti}[(G_{ij})_{x_r} + (G_{ir})_{x_j} - (G_{rj})_{x_i}] f_t^l +$$
$$+ \|Df\|^2 G^{ti}[(G_{ij})_{f^\alpha} f_r^\alpha + (G_{ir})_{f^\alpha} f_j^\alpha - (G_{rj})_{f^\alpha} f_i^\alpha] f_t^l +$$
$$+ \|Df\|^2 H^{sl}(H_{sk})_{x_r} f_j^k + \|Df\|^2 H^{sl}(H_{sk})_{x_j} f_r^k - G^{ti}(H_{sk})_{x_i} f_t^l f_r^s f_j^k +$$
$$+ \|Df\|^2 [H^{sl}(H_{sk})_{f^\alpha} f_r^\alpha f_j^k + H^{sl}(H_{sk})_{f^\alpha} f_j^\alpha f_r^k - H^{\alpha l}(H_{sk})_{f^\alpha} f_r^s f_j^k].$$

According to formula (2.9) we can express $\|Df\|^2$ as follows:

$$(6.7)\quad \|Df\|^2 = \frac{\operatorname{Tr}[D^*f(E+h)Df]}{\operatorname{Tr}[E+g]} = \frac{1}{n}|Df|^2 + \sum_{\substack{ij\\ \alpha\beta}} a_{\alpha\beta}^{ij}(x,f) f_i^\alpha f_j^\beta,$$

where the coefficients $a_{\alpha\beta}^{ij}(x,f)$ are of the same class of smoothness as the characteristics $G(x,f)$ and $H(x,f)$ of the map f. Moreover, we have $a_{\alpha\beta}^{ij}(x_0,f_0) = 0$. Inserting this into (6.6), we obtain a system of n^3 linearly independent equations (in a neighbourhood of (x_0,f_0)) with unknowns $f_{rj}^l(x)$.

Hence one can easily derive

(6.8) $|Df|f_{rj}^l = |Df|_{x_r}f_j^l+|Df|_{x_j}f_r^l-|Df|_{x_t}f_t^l\delta_r^j+|Df|\mathcal{R}_{rj}^l(x,f,Df,\nabla|Df|)$.

Here the remainder term $\mathcal{R}_{rj}^l(x,f,Df,\nabla|Df|)$ satisfies the following inequality:

(6.9) $$|\mathcal{R}_{rj}^l(x,f,Df,|Df|_x)| \leqslant C(\text{Lip. const. of } G, H)\big(|Df|+|Df|^2+q(x,f)\,\big||Df|_x\big|\big).$$

The symbol $|Df|_x$ denotes grad $|Df|$. This new notation is more suitable for further notations.

Finally we present the following lemma.

LEMMA 6.2. *Let f be a map with characteristics satisfying the Lipschitz condition. Then almost everywhere*

(6.10) $$|f_{rj}^l(x)| \leqslant \text{const}\,\big(|Df(x)|+|Df(x)|^2+\big||Df(x)|_x\big|\big).$$

The equations

(6.11) $$\operatorname{div}\{|Df|^{n-2}\operatorname{grad} f^l\} = \mathcal{R}_{ii}^l(x,f,Df,|Df|_x)|Df|^{n-2},\qquad l=1,2,\ldots,n,$$

hold almost everywhere in some neighbourhood of $x_0\in\Omega$.

Proof. Estimate (6.10) is obvious. We transform the left side of (6.11) using (6.8).

$$\begin{aligned}
&\operatorname{div}\{|Df|^{n-2}\operatorname{grad} f^l\}\\
&= \frac{\partial}{\partial x_i}(|Df|^{n-2}f_i^l) = |Df|^{n-2}f_{ii}^l+(n-2)|Df|^{n-3}|Df|_{x_i}f_i^l\\
&= |Df|^{n-3}\{|Df|_{x_i}f_i^l+|Df|_{x_i}f_i^l-n|Df|_{x_t}f_t^l+\mathcal{R}_{ii}^l(x,f,Df,|Df|_x)|Df|\}+\\
&\qquad +(n-2)|Df|^{n-3}|Df|_{x_i}f_i^l\\
&= \mathcal{R}_{ii}^l(x,f,Df,|Df|_x)|Df|^{n-2}.
\end{aligned}$$

Remark. System (6.11) can be deduced from equations (2.10), but it requires some information about the vector-functions $a^l(x,f,Df)$.

7. Local boundedness of the Jacobian

We continue the study of the case where the characteristics $G(x,f)$ and $H(x,f)$ satisfy the Lipschitz condition. We first recall some basic facts about the Sobolev class necessary to prove the local boundedness of the Jacobian $J_f(x)$.

DEFINITION 7.1. Let $u(x)$ be a function of the Sobolev space $W_2^1(\Omega)$, and let λ be a real parameter. Put $A_\lambda = \{x\in\Omega;\ u(x) > \lambda\}$. The function

$$u^\lambda(x) = \max\{u(x)-\lambda;\ 0\}$$

will be called a *truncation of* $u(x)$. The following lemma is well known in the theory of Sobolev spaces (see Ch. Morrey [17]).

LEMMA 7.1. *The truncation* $u^\lambda(x)$ *of the function* $u\in W_2^1(\Omega)$ *belongs to the space* $W_2^1(\Omega)$. *Moreover,* $\operatorname{grad} u^\lambda(x) = \operatorname{grad} u(x)$ *for almost all* $x\in A_\lambda$ and $\operatorname{grad} u^\lambda(x) = 0$ *for almost all* $x\in\Omega\backslash A_\lambda$.

We shall use a version of a lemma of Ladyzenskaya–Uraltseva [13]. The proof will imitate and at the same time simplify their own arguments.

The main theorem of this section is as follows:

THEOREM 7.1. *Let* f *be a non-constant quasiregular mapping with the Lipschitz continuous characteristics* $G(x,f)$ *and* $H(x,f)$. *Then its Jacobian* $J_f(x)$ *satisfies the inequality*

$$C_{\Omega^0}^{-1} \leqslant J_f(x) \leqslant C_{\Omega^0} \tag{7.1}$$

almost everywhere in Ω^0 *for every compact subset* $\Omega^0 \subset \Omega$, *and some positive constant* C_{Ω^0}.

LEMMA 7.2. *Suppose that the function* $u\in W^1_{2,\mathrm{loc}}(\Omega)$, $u \geqslant 0$ *satisfies the integral inequality*

$$\int_{A_\lambda} |\nabla u|^2\zeta^2\,dx \leqslant \text{const}\int_{A_\lambda}(u-\lambda)^2|\nabla\zeta|^2\,dx + \text{const}\int_{A_\lambda} u^{2+4/n}\zeta^2\,dx \tag{7.2}$$

with any parameter $\lambda \geqslant 1$ *and for all smooth functions* ζ *with compact support contained in* Ω. *Then* $u\in L_{\infty,\mathrm{loc}}(\Omega)$.

Proof of the lemma. Let $x_0\in\Omega$. Consider the ball $B(x_0, 2R) = B_0$ with the radius $2R$ such that

(i) $\|u\|_{L_{2n/(n-2)}(B_0)} \leqslant 1$.

(ii) $\operatorname{mes} B_0 \leqslant 1$, $B_j = B(x_0, R+R/2^j) \subset B_0$, $j = 0, 1, 2, \ldots$

(iii) For $\zeta\in C_0^\infty(B_0)$ inequality (7.2) holds.

We shall show that u is bounded in $B_R = B(0, R)$. Indeed, consider smooth functions ζ_j satisfying the following properties.

(iv) $\zeta_j\in C_0^\infty(B_j)$ $(j = 0, 1, 2, \ldots)$, $0 \leqslant \zeta_j(x) \leqslant 1$.

(v) $\zeta_j(x) \equiv 1$, for $x\in B_{j+1} \subset B_j$.

(vi) $|\nabla\zeta_j(x)| \leqslant \text{const}\cdot 2^j$, with some constant depending upon R and dimension n, the existence of ζ_j being obvious.

We shall utilize (7.2) with ζ_j and

$$\lambda_j = 2\lambda_0 - \frac{\lambda_0}{2^j}, \quad j = 0, 1, \ldots,$$

where $\lambda_0 \geqslant 1$ will be chosen later. It is easy to see that $2\lambda_0 \geqslant \lambda_j \geqslant \lambda_0$ and $\lambda_{j+1}-\lambda_j = \lambda_0/2^{j+1}$.

Consider the following number sequence

$$I_j = \|u-\lambda_j\|_{L_{2n/(n-2)}(A_j\cap B_j)} \leqslant \|u\|_{L_{2n/(n-2)}(B_0)} \leqslant 1, \quad j = 0, 1, 2, \ldots,$$

where $A_j = A_{\lambda_j} = \{x\in\Omega\,;\, u(x) > \lambda_j\}$. Since $\zeta_j(u-\lambda_{j+1})\in \mathring{W}_2^1(B_j)$, by the Sobolev embedding Lemma 3.2 we get

$$\begin{aligned}
I_{j+1} &= \|\zeta_j(u-\lambda_{j+1})\|_{L_{2n/(n-2)}(A_{j+1}\cap B_{j+1})}\\
&\leqslant \|\zeta_j(u-\lambda_{j+1})\|_{L_{2n/(n-2)}(A_{j+1}\cap B_j)}\\
&\leqslant \frac{2(n-1)}{n-2}\|\nabla(u-\lambda_{j+1})\zeta_j\|_{L_2(A_{j+1}\cap B_j)}\\
&\leqslant \frac{2(n-1)}{n-2}\cdot\|(u-\lambda_{j+1})\nabla\zeta_j\|_{L_2(A_{j+1}\cap B_j)}+\frac{2(n-1)}{n-2}\cdot\|\lambda_j\cdot\nabla u\|_{L_2(A_{j+1}\cap B_j)}.
\end{aligned}$$

For the second term we use estimate (7.2), getting

$$I_{j+1} \leqslant \text{const}\cdot\|(u-\lambda_{j+1})\nabla\zeta_j\|_{L_2(A_{j+1}\cap B_j)}+\text{const}\,\Big(\int\limits_{A_{j+1}\cap B_j} u^{2+4/n}\zeta_j^2\Big)^{1/2}.$$

Now, observe that

$$\begin{aligned}
|(u-\lambda_{j+1})\nabla\zeta_j| &\leqslant |(u-\lambda_j)\nabla\zeta_j|+|(\lambda_j-\lambda_{j+1})\nabla\zeta_j|\\
&\leqslant \text{const}\cdot 2^j|u-\lambda_j|+\text{const}\,\lambda_0,\\
u^{2+4/n} &= u^{2n/(n-2)}\cdot u^{-8/n(n-2)} \leqslant (u-\lambda_j+\lambda_j)^{2n/(n-2)}\lambda_0^{-8/n(n-2)}\\
&\leqslant \text{const}\,(u-\lambda_j)^{2n/(n-2)}\lambda_0^{-8/n(n-2)}+\text{const}\cdot\lambda_0^{2+4/n} \quad \text{if } u>\lambda_j\geqslant\lambda_0.
\end{aligned}$$

The above inequalities lead to the following estimate:

$$\begin{aligned}
I_{j+1} \leqslant\ & \text{const}\cdot 2^j\|u-\lambda_j\|_{L_2(A_{j+1}\cap B_j)}+\text{const}\,\lambda_0(\text{mes}\,A_{j+1}\cap B_j)^{1/2}+\\
&+\text{const}\,\lambda_0^{-4/n(n-2)}\Big(\int\limits_{A_{j+1}\cap B_j}(u-\lambda_j)^{2n/(n-2)}\Big)^{1/2}+\\
&+\text{const}\,\lambda_0^{1+2/n}\cdot(\text{mes}\,A_{j+1}\cap B_j)^{1/2}.
\end{aligned}$$

By the Hölder inequality we get

$$\begin{aligned}
\|u-\lambda_j\|_{L_2(A_{j+1}\cap B_j)} &\leqslant \|u-\lambda_j\|_{L_{2n/(n-2)}(A_{j+1}\cap B_j)}(\text{mes}\,A_{j+1}\cap B_j)^{1/n}\\
&\leqslant I_j(\text{mes}\,A_{j+1}\cap B_j)^{1/n}.
\end{aligned}$$

Since $\lambda_0 < (\lambda_0)^{1+2/n}$, we have

$$\begin{aligned}
I_{j+1} \leqslant\ & \text{const}\,I_j(\text{mes}\,A_{j+1}\cap B_j)^{1/n}+\text{const}\,(\lambda_0)^{-4/n(n-2)}\cdot I_j^{n/(n-2)}+\\
&+\text{const}\,(\lambda_0)^{1+2/n}\cdot(\text{mes}\,A_{j+1}\cap B_j)^{1/2}.
\end{aligned}$$

To estimate the measure of $A_{j+1} \cap B_j$ we proceed as follows

$$I_j \geqslant \|u-\lambda_j\|_{L_{2n/(n-2)}(A_{j+1}\cap B_j)} \geqslant \|\lambda_{j+1}-\lambda_j\|_{L_{2n/(n-2)}(A_{j+1}\cap B_j)}$$
$$= \frac{\lambda_0}{2^{j+1}}(\operatorname{mes} A_{j+1} \cap B_j)^{(n-2)/2n},$$

thus

$$\operatorname{mes}(A_{j+1} \cap B_j) \leqslant 2^{2n(j+1)/(n-2)} \cdot (\lambda_0)^{-2n/(n-2)} I_j^{2n/(n-2)}.$$

Finally we obtain

$$I_{j+1} \leqslant \text{const}\ \lambda_0^{-2/(n-2)} 2^{2(j+1)/(n-2)} I_j^{n(n-2)} + \text{const}\,(\lambda_0)^{-4/n(n-2)} I_j^{n/(n-2)} +$$
$$+\text{const}\ 2^{n(j+1)/(n-2)}(\lambda_0)^{-4/n(n-2)} I_j^{n/(n-2)}$$
$$\leqslant \text{const}\ 2^{n(j+1)/(n-2)} \lambda_0^{-4/n(n-2)} I_j^{n/(n-2)}$$
$$\leqslant \text{const}\ 8^{j+1}(\lambda_0)^{-4/n(n-2)} I_j^{n/(n-2)}.$$

Now choose λ_0 sufficiently large so that

$$I_{j+1} \leqslant 8^{j-n} \cdot I_j^{n/(n-2)}.$$

It is easy to see that

$$8^{(n-2)(j+1)} I_{j+1} \leqslant 8^{(n-2)(j+1)} 8^{-jn} (8^{(n-2)j} I_j)^{n/(n-2)}$$
$$= 8^{-j-2}(8^{(n-2)j} I_j)^{n/(n-2)};$$

thus by induction one gets $8^{(n-2)j} I_j \leqslant 1$ $(I_0 \leqslant 1)$. Hence I_j tends to zero as $j \to \infty$. We have

$$\|u-\lambda_0\|_{L_{2n/(n-2)}(A_0\cap B_R)} \leqslant \|u-\lambda_j\|_{L_{2n/(n-2)}(A_j\cap B_j)} = I_j \to 0,$$

so that the measure of $A_0 \cap B_R = \{x \in B_R\,;\ u(x) > \lambda_0\}$ is zero. In other words, $u(x) \leqslant \lambda_0$ almost everywhere in $B(x_0, R)$. This completes the proof of the lemma.

Before we start to prove Theorem 7.1 let us make a few remarks. First we observe that the boundedness from below of the Jacobian is a simple consequence of the boundedness from above by considering the inverse map $x = f^{-1}(y)$, which satisfies an equation of the same type as the map $f = f(x)$. Furthermore, we can consider, without loss of generality, the domain Ω^0, sufficiently small, which is a neighbourhood of an arbitrary point of Ω, say x_0. Moreover, we may assume that $G(x_0, f(x_0)) = H(x_0, f(x_0)) = E$ – the identity matrix. For simplicity we denote $u(x) = |Df(x)|^{n/2}$.

By Corollary 6.1 we know that $u \in W^1_{2,\text{loc}}(\Omega)$ and by Sobolev's lemma 3.2 we have $u \in L_{2n(n-2),\text{loc}}(\Omega)$. Summarizing we see that it is enough to show that $u \in L_{\infty,\text{loc}}(\Omega)$.

Proof of Theorem 7.1. Let ζ be a smooth function with sufficiently small support containing the point $x_0 \in \Omega$ in its interior. Fix an arbitrary number $\lambda \geqslant 1$. Let us consider the function

$$\varphi(x) = f_j^l(x)\max\{1-\lambda/u(x)\,;\,0\}\,\zeta^2(x), \qquad u(x) = |Df(x)|^{n/2} \qquad (l, j = 1, 2, \ldots, n).$$

It is easy to check that $\varphi \in L_{n+2}(\Omega)$. Set $A_\lambda = \{x \in \Omega;\ u(x) > \lambda\}$. The following relations are obviously true:

$$(7.3)\qquad \begin{aligned} \nabla\varphi &= \nabla f_j^l\left(1-\frac{\lambda}{u}\right)\zeta^2 + \frac{n\lambda\nabla u}{2u^2} f_j^l\zeta^2 + 2f_j^l\left(1-\frac{\lambda}{u}\right)\zeta\nabla\zeta \quad \text{in} \quad A_\lambda, \\ \nabla\varphi &= 0 \quad \text{almost everywhere in} \quad \Omega\backslash A_\lambda. \end{aligned}$$

We start from the equality

$$(7.4)\qquad \int_\Omega (|Df|^{n-2} f_i^l)_{x_\alpha}\varphi_{x_\beta}\,dx = \int_\Omega (|Df|^{n-2} f_i^l)_{x_\beta}\varphi_{x_\alpha}\,dx, \qquad \alpha, \beta = 1, 2, \ldots, n,$$

which follows from the identity

$$\begin{aligned} &\int_\Omega (|Df|^{n-2} f_i^l)_{x_\alpha}\varphi_{x_\beta}\,dx \\ &= \frac{n-2}{2}\int_\Omega |Df|^{n-4}\{|Df|\,|Df|_{x_\alpha}\varphi_{x_\beta} + |Df|\,|Df|_{x_\beta}\varphi_{x_\alpha} - |Df|_{x_\alpha}|Df|_{x_\beta}\varphi\}\,f_i^l\,dx + \\ &\qquad + \int_\Omega (|Df|^{(n-2)/2} f_i^l)_{x_\alpha}(|Df|^{(n-2)/2}\varphi)_{x_\beta}\,dx - \\ &\qquad - \frac{n-2}{n}\int_\Omega (|Df|^{n/2})_{x_\beta}(|Df|^{(n-4)/2} f_i^l\varphi)_{x_\alpha}\,dx, \end{aligned}$$

and from the following facts:

The first integral on the right side is symmetric in α and β. The other integrals are also symmetric. Indeed, the definition of φ and Lemma 5.1 imply

$$|Df|^{(n-2)/2} f_i^l \in W_{2,\mathrm{loc}}^1(\Omega); \qquad |Df|^{(n-2)/2}\varphi \in \mathring{W}_2^1(\Omega),$$
$$|Df|^{n/2} \in W_{2,\mathrm{loc}}^1(\Omega); \qquad |Df|^{(n-4)/2} f_i^l\varphi \in \mathring{W}_2^1(\Omega).$$

Thus, these functions can be replaced by aproximating smooth ones. So, by a limiting process, we conclude that the indices α and β can be transposed. In particular, setting $\alpha = j$ and $\beta = i$, we obtain the equality

$$(7.5)\qquad \int_\Omega (|Df|^{n-2} f_i^l)_{x_j}\varphi_{x_i}\,dx = \int_\Omega (|Df|^{n-2} f_i^l)_{x_i}\varphi_{x_j}\,dx.$$

We take into account equations (6.11). Formula (7.5) gives

$$(7.6)\qquad \int_{A_\lambda} (|Df|^{n-2} f_i^l)_{x_j} \varphi_{x_i}\, dx = \int_{A_\lambda} R_{ii}^l(x, f, Df, |Df|_x)|Df|^{n-2} \varphi_{x_j}\, dx.$$

We are going to estimate both integrals. Formula (7.3) and Lemma 6.2 imply the following inequality:

$$|\nabla\varphi| \leqslant \mathrm{const}(|Df|^2 + |Df|_x)\zeta^2 + \mathrm{const}\left(1 - \frac{\lambda}{|Df|^{n/2}}\right)|Df|\,|\zeta|\,|\nabla\zeta|$$

almost everywhere in $A_\lambda = \{x \in \Omega;\ |Df|^{n/2} > \lambda \geqslant 1\}$. Hence, by (6.9) we have

$$\begin{aligned}
&\mathscr{R}_{ii}^l(x, f, Df, |Df|_x)\varphi_{x_j} \cdot |Df|^{n-2}\\
&\qquad \leqslant \mathrm{const}(|Df|^2 + q(x, f)|Df|_x)|Df|^{n-2} \times\\
&\qquad\qquad \times\left\{(|Df|^2 + |Df|_x)\zeta^2 + \left(1 - \frac{\lambda}{|Df|^{n/2}}\right)|Df|\,|\zeta|\,|\nabla\zeta|\right\}.
\end{aligned}$$

The chain of inequalities of the type $ab \leqslant \varepsilon a^2 + b^2/4\varepsilon$, where the positive number ε will be chosen later, leads to the following inequalities:

$$\begin{aligned}
(7.7)\qquad &|\mathscr{R}_{ii}^l(\ldots)\varphi_{x_j}|\,|Df|^{n-2}\\
&\qquad \leqslant C(\text{independ. of } \varepsilon)(3\varepsilon + q(x, f))|Df|^{n-2}|Df|_x^2\zeta^2 +\\
&\qquad\qquad + C_\varepsilon|Df|^{n+2}\zeta^2 + C_\varepsilon(|Df|^{n/2} - \lambda)^2|\nabla\zeta|^2
\end{aligned}$$

valid almost everywhere in A_λ.

Now we estimate the left side of (7.6). Formula (7.3) implies

$$\begin{aligned}
\int_{A_\lambda} (|Df|^{n-2} f_i^l)_{x_j}\varphi_{x_i}\, dx &= \int_{A_\lambda} |Df|\, f_{ij}^l \varphi_{x_i}\, dx + (n-2)\int_{A_\lambda} |Df|^{n-3} f_i^l |Df|_{x_j}\varphi_{x_i}\, dx\\
&= \int_{A_\lambda} |Df|^{n-2} f_{ij}^l f_{ij}^l \left(1 - \frac{\lambda}{|Df|^{n/2}}\right)^2 dx +\\
&\quad + \frac{n}{2}\int_{A_\lambda} |Df|^{n-2} f_{ij}^l f_j^l \frac{\lambda|Df|_{x_j}}{|Df|^{(n+2)/2}}\zeta^2\, dx +\\
&\quad + 2\int_{A_\lambda} |Df|^{n-2} f_{ij}^l f_j^l \left(1 - \frac{\lambda}{|Df|^{n/2}}\right)\zeta\zeta_{x_i}\, dx +\\
&\quad + (n-2)\int_{A_\lambda} |Df|^{n-3}|Df|_{x_j} f_{ij}^l f_i^l \left(1 - \frac{\lambda}{|Df|^{n/2}}\right)\zeta^2\, dx +
\end{aligned}$$

$$+\frac{n(n-2)}{2}\int_{A_\lambda}|Df|^{n-3}|Df|_{x_j}f_i^l f_j^l\frac{\lambda|Df|_{x_i}}{|Df|^{(n+2)/2}}\zeta^2\,dx+$$

$$+2(n-2)\int_{A_\lambda}|Df|^{n-3}|Df|_{x_j}f_i^l f_j^l\left(1-\frac{\lambda}{|Df|^{n/2}}\right)\zeta\zeta_{x_i}\,dx.$$

By the identity $f_i^l f_{ij}^l = |Df||Df|_{x_j}$ one can simplify some of the integrals. In particular, the fourth and the fifth ones on the right side are non-negative.

Finally

$$(7.8)\qquad \int_{A_\lambda}(|Df|^{n-2}f_i^l)_{x_j}\varphi_{x_i}\,dx \geqslant \int_{A_\lambda}|Df|^{n-2}\left\{\left(1-\frac{\lambda}{|Df|^{n/2}}\right)f_{ij}^l f_{ij}^l+\right.$$

$$\left.+\frac{n}{2}|Df|_{x_i}|Df|_{x_i}\frac{\lambda}{|Df|^{n/2}}\right\}\zeta^2\,dx+$$

$$+2\int_{A_\lambda}|Df|^{n-1}|Df|_{x_i}\left(1-\frac{\lambda}{|Df|^{n/2}}\right)\zeta\zeta_{x_i}\,dx+$$

$$+2(n-2)\int_{A_\lambda}|Df|^{n-3}f_i^l f_j^l|Df|_{x_j}\left(1-\frac{\lambda}{|Df|^{n/2}}\right)\zeta\zeta_{x_i}\,dx.$$

Now we apply the following lemma:

LEMMA 7.3. *The inequality*

$$(7.9)\qquad |\operatorname{grad} Df| \geqslant |\operatorname{grad}|Df||$$

holds. Equivalently $f_{ij}^l f_{ij}^l \geqslant |Df|_{x_i}\cdot|Df|_{x_i}$.

Proof. The lemma is a consequence of the following facts:

$$|Df||Df|_{x_i} = f_\beta^\alpha f_{\beta i}^\alpha$$

and

$$2|Df|^2(f_{ij}^l f_{ij}^l-|Df|_{x_i}|Df|_{x_i}) = 2(f_\beta^\alpha f_\beta^\alpha f_{ij}^l f_{ij}^l - f_i^l f_{ij}^l f_\beta^\alpha f_{\beta i}^\alpha)$$

$$= \sum_{\substack{\alpha,\beta\\ i,j,l}}(f_{\beta i}^\alpha f_i^l - f_\beta^\alpha f_{ji}^l)^2 \geqslant 0.$$

Now we return to (7.8)

$$\int_{A_\lambda}(|Df|^{n-2}f_i^l)_{x_i}\varphi_{x_i}\,dx \geqslant \int_{A_\lambda}|Df|^{n-2}|Df|_{x_i}|Df|_{x_i}\zeta^2\,dx-$$

$$-C\int_{A_\lambda}|Df|^{n-1}\left(1-\frac{\lambda}{|Df|^{n/2}}\right)||Df|_x||\zeta||\nabla\zeta|\,dx$$

for a positive constant C. Using the inequality $ab \leqslant a^2/2C + Cb^2/2$, we obtain

$$\int_{A_\lambda} (|Df|^{n-2} f_i^l)_{x_j} \varphi_{x_i} dx$$

$$\geqslant \tfrac{1}{2} \int_{A_\lambda} |Df|^{n-2} |Df|_x^2 \zeta^2 dx - \frac{C^2}{2} \int_{A_\lambda} |Df|^n \left(1 - \frac{\zeta}{|Df|^{n/2}}\right)^2 |\nabla\zeta|^2 dx.$$

Comparing the last inequality to (7.7) and (7.6) we conclude that

$$\tfrac{1}{2} \int_{A_\lambda} |Df|^{n-2} |Df|_x^2 \zeta^2 dx$$

$$\leqslant C(\text{independ. of } \varepsilon) \int_{A_\lambda} (3\varepsilon + q(x,f))|Df|^{n-2} |Df|_x^2 \zeta^2 dx +$$

$$+ C_\varepsilon \int_{A_\lambda} |Df|^{n+2} \zeta^2 dx + C_\varepsilon \int_{A_\lambda} (|Df|^{n/2} - \lambda)^2 |\nabla\zeta|^2 dx.$$

The coefficient $3\varepsilon + q(x, f(x))$ can be made arbitrarily small on $A_\lambda \cap \operatorname{supp}\zeta$ if we take for ζ a function with a sufficiently small support containing the point x_0 in its interior and if we choose a sufficiently small number $\varepsilon > 0$. Therefore we get

$$\int_{A_\lambda} |Df|^{n-2} |Df|_x^2 \zeta^2 dx \leqslant \text{const} \int_{A_\lambda} |Df|^{n+2} \zeta^2 dx + \text{const} \int_{A_\lambda} (|Df|^{n/2} - \lambda)^2 |\nabla\zeta|^2 dx.$$

This inequality can be expressed in terms of the function $u(x) = |Df|^{n/2}$. Obviously $u^2(x) > \lambda^2 \geqslant 1$ in A_λ, and $u \in L_{2n/(n-2),\text{loc}}(\Omega) \subset L_{2+4/n,\text{loc}}(\Omega)$. Then we have

$$\int_{A_\lambda} |\nabla u|^2 \zeta^2 dx \leqslant \text{const} \int_{A_\lambda} u^{2+4/n} \zeta^2 dx + \text{const} \int_{A_\lambda} (u - \lambda)^2 |\nabla\zeta|^2 dx.$$

Finally Lemma (7.2) implies that the function $u = |Df|^{n/2}$ has a finite upper bound in a neighbourhood of the point x_0. This concludes the proof of the theorem.

COROLLARY 7.1. *Any quasiregular map $f = f(x)$ with characteristics $G(x,f)$, $H(x,f)$ satisfying the Lipschitz condition belongs to the Sobolev class $W^2_{2,\text{loc}}(\Omega)$, and inequality* (7.1) *holds with $J_f(x)$ replaced by $|Df(x)|$.*

Proof. Quasiregularity of f means that $|Df(x)|^n \leqslant \text{const}\, J_f(x)$. So the second assertion of Corollary 7.1 immediately follows from inequality (7.1). In consequence, integral inequality (5.2) has the following simplified form:

$$\int_\Omega \varphi^n(x) \frac{|Df(x+h) - Df(x)|^2}{|h|^2} dx \leqslant \text{const}(\varphi, \Omega, \|Df\|_{L_{n+2,\text{loc}}(\Omega)}).$$

Now, in view of Lemma 3.1 we deduce that $f \in W^2_{2,\text{loc}}(\Omega)$.

8. Smoothness

Smoothness of the map f will be studied via the theory of quasilinear and linear equations of elliptic type (not systems of equations) by making use of estimate (7.1).

We begin with a simplification of system (2.6), assuming that the characteristics $G(x,f)$ and $H(x,f)$ are Lipschitz or smooth. For this purpose we prove a preparatory lemma, similar to Lemma 2.1.

LEMMA 8.1. *Let $f\colon \Omega \to \mathbf{R}^n$ be a map of the class $W^2_{2,\mathrm{loc}}(\Omega)$ and let $v(x)$ be an n-vector function of the class $C^1(\Omega)$. Then the identity*

$$\operatorname{div}\{(J_f D^{-1} f) v\} = \langle J_f D^{-1} f, \operatorname{grad} v\rangle \text{ a.e. } in\ \Omega \tag{8.1}$$

holds.

Remark. (8.1) holds also for v being Lipschitz. In particular, if v is constant, we obtain equation (2.2).

Proof. It can be seen, as in Lemma 2.1, that the vector $\operatorname{div}\{(J_f D^{-1} f)^*\}$ vanishes almost everywhere. By simple calculations we obtain

$$\begin{aligned}\operatorname{div}\{(J_f D^{-1} f) v\} &= \langle \operatorname{div}(J_f D^{-1} f)^*, v\rangle + \langle J_f D^{-1} f, \operatorname{grad} v\rangle \\ &= \langle J_f D^{-1} f, \operatorname{grad} v\rangle .\end{aligned}$$

This ends the proof.

As a result of the lemma one can write system (2.6) in a more convenient form

$$\begin{aligned}\operatorname{div}\left\{\left(\frac{\langle G^{-1}(x,f)\nabla f^l, \nabla_f^l\rangle}{H^{ll}(x,f)}\right)^{(n-2)/2} \cdot G^{-1}(x,f)\nabla f^l\right\} \\ = \langle J_f(x) D^{-1} f(x), \operatorname{grad}(H^{-1}(x,f) - H^{-1}(x_0,f_0)e^l)\rangle\end{aligned} \tag{8.2}$$

for each $l = 1, 2, \ldots, n$. This system may by viewed in two ways.

First it can be considered as a single quasilinear equation (for $u = f^l(x) \in W^2_{2,\mathrm{loc}}(\Omega)$) of the form

$$\frac{d}{dx_i} a_i(x, \operatorname{grad} u) = a(x), \tag{8.3}$$

where the ellipticity condition

$$\nu|\xi|^2 \leqslant \frac{\partial a_i(x,p)}{\partial p_j}\xi^i\xi^j \leqslant \mu|\xi|^2$$

on a solution $u = u(x)$, $p = \nabla u$ holds for $\xi \in \mathbf{R}^n$ and some positive constants ν, μ. Moreover, we have

$$\frac{\partial a_i(x, \operatorname{grad} u(x))}{\partial u_{x_j}}, \quad a(x) \in L_{\infty,\mathrm{loc}}(\Omega).$$

Using the results of Ladyzenskaya and Uraltseva [13] (see Theorem 6.1, p. 330), we conclude that $u \in C^{1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ depending only on ν/μ.

THEOREM 8.1. *Every non-constant quasiregular mapping with Lipschitz characteristics is a local diffeomorphism of the class $C^{1+\varepsilon}(\Omega)$ for some $\varepsilon > 0$ and of the class $W^2_{2,\mathrm{loc}}(\Omega)$.*

We come now to our main smoothness result.

THEOREM 8.2. *Every quasiregular mapping f with the characteristics $G(x,f)$ and $H(x,f)$ of the class $C^{k+\alpha}(\Omega\times\Omega')$, $k \geqslant 1$, $0<\alpha\leqslant 1$ is of the class $C^{k+\alpha+1}(\Omega)$.*

Before starting the proof let us present the second interpretation of system (8.2). It can be treated as a *linear* uniformly elliptic equation (for $u = f^l(x) \in C^{1+\varepsilon}(\Omega)$) of the form

$$\mathscr{A}^l_{ij}(x,f,Df)\,u_{x_ix_j} = \mathscr{A}^l(x,f,Df), \qquad l = 1,2,\ldots,n, \tag{8.4}$$

where $\mathscr{A}^l_{ij}(x,f,Df)$ and $\mathscr{A}^l(x,f,Df)$ are infinitely smooth with respect to the variables $Df \in \mathbf{R}^{n^2}\setminus\{0\}$ and of the class $C^{k+\alpha-1}$ with respect to $(x,f)\in \Omega\times\Omega'$. The coefficients $\mathscr{A}^l_{ij}$ are given by the formulae

$$\mathscr{A}^l_{ij}(x,f,Df) = \left(\frac{\sum\limits_{\alpha\beta} G^{\alpha\beta}(x,f)f^l_\alpha f^l_\beta}{H^{ll}(x,f)}\right)^{(n-2)/2} \times$$
$$\times\left[G^{ij}(x,f)+(n-2)\,\frac{\sum\limits_{\alpha\beta} G^{i\alpha}(x,f)G^{j\beta}(x,f)f^l_\alpha f^l_\beta}{\sum\limits_{\alpha\beta} G^{\alpha\beta}(x,f)f^l_\alpha f^l_\beta}\right].$$

Hence we obtain the ellipticity condition

$$\nu|\xi|^2 \leqslant \mathscr{A}^l_{ij}(x,f,Df)\xi^i\xi^j \leqslant \mu|\xi|^2 \quad \text{for} \quad \xi\in\mathbf{R}^n$$

with some positive constants ν and μ.

We shall use the following well known classical fact about the regularity of solutions of *linear* elliptic equations:

LEMMA 8.2. *Let $u \in W^2_{2,\mathrm{loc}}(\Omega)\cap L_{\infty,\mathrm{loc}}(\Omega)$ satisfies the elliptic equation*

$$\sum_{ij} a_{ij}(x)u_{x_ix_j} = a(x) \tag{8.5}$$

with coefficients $a_{ij}(x)$ and $a(x)$ of the class $C^{r+\alpha}(\Omega)$, $r\geqslant 0$, $\alpha>0$. Then $u\in C^{r+2+\alpha}(\Omega)$.

For the proof see [13], p. 235·Theorem 12.1.

The proof of Theorem 8.2 is by induction with respect to k. First we consider the case $k=1$, that is, the case where the characteristics $G(x,f)$ and $H(x,f)$ are of the class $C^{1+\alpha}(\Omega\times\Omega')$. In view of Theorem 8.1, the mapping f is of the class $W^2_{2,\mathrm{loc}}(\Omega) \cap C^{1+\varepsilon}(\Omega)$ with some positive constant ε. Then the coefficients $\mathscr{A}^l_{ij}(x) = \mathscr{A}^l_{ij}(x, f(x), Df(x))$ and $\mathscr{A}^l(x) = \mathscr{A}^l(x, f(x), Df(x))$ of equation (8.4) are of the class $C^{\min(\alpha,\varepsilon)}(\Omega)$. Lemma 8.2 gives $u \in C^{2+\min(\alpha,\varepsilon)}(\Omega)$. Now we revert to equation (8.4). Repeating the argument given above, we conclude that the coefficients $\mathscr{A}^l_{ij}(x)$ and $\mathscr{A}^l(x)$ actually belong to the class $C^\alpha(\Omega)$. Again using Lemma 8.2 we, correct the estimation of smoothness of u up to the statement that $u \in C^{2+\alpha}(\Omega)$. This estimation is the best possible for the case $k = 1$. We assume at the moment that our theorem holds for some $k \geqslant 1$ and consider the case of the characteristics $G(x,f)$, $H(x,f)$ of the class $C^{k+1+\alpha}(\Omega)$. The induction hypothesis implies that $f \in C^{k+1+\alpha}(\Omega)$. Then the coefficients $\mathscr{A}^l_{ij}(x)$ and $\mathscr{A}^l(x)$ of system (8.4) are of the class $C^{k+\alpha}(\Omega)$. Finally Lemma 8.2 implies that $f^l(x) = u(x) \in C^{k+2+\alpha}(\Omega)$. The result follows.

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