POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

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MAREK JARNICKI

A method of holomorphic retractions and pseudoinverse matrices in the theory of continuation of δ-tempered functions

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PAŃSTWOWE WYDAWNICTWO NAUKOWE

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WROCŁAWSKA DRUKARNIA NAUKOWA

CONTENTS

§1. Introduction

In the paper we investigate some problems of the theory of continuation of holomorphic functions with restricted growth.

Let X be a complex analytic space countable at infinity and let M be an analytic submanifold of X ($\dim M < \dim X$). Let $\mathcal{O}(X)$ (resp. $\mathcal{O}(M)$) denote the space of all holomorphic functions on X (resp. on M).

One can pose the following general continuation problem:

Given $f \in \mathcal{O}(M)$, does f admit an extension $\hat{f} \in \mathcal{O}(X)$ such that the growth of $\hat{f}$ on X is in some sense similar to the growth of f on M?

One of the most useful definitions of the growth of holomorphic functions is by estimates of the form $\delta^k |f| \leqslant c$, where δ is a given function; more precisely:

Let $\delta\colon X \to (0, +\infty)$ be a bounded function. For $k \geqslant 0$ let $\mathcal{O}^{(k)}(X, \delta)$ (resp. $\mathcal{O}^{(k)}(M, \delta)$) denote the vector space of all functions f holomorphic on X (resp. on M) such that the function $\delta^k f$ is bounded. The space $\mathcal{O}^{(k)}(X, \delta)$ (resp. $\mathcal{O}^{(k)}(M, \delta)$) has the natural structure of a normed space with the norm given by the formula: $f \to \|\delta^k f\|_\infty$, where $\| \; \|_\infty$ denotes the supremum norm. Put $\mathcal{O}(X, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(X, \delta)$ and analogously $\mathcal{O}(M, \delta) = \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(M, \delta)$. $\mathcal{O}(X, \delta)$ (resp. $\mathcal{O}(M, \delta)$) is a complex algebra with a unit element (cf. §2).

Let $R = R^X_M$ denote the restriction operator

$$\mathcal{O}(X) \ni f \to f|_M \in \mathcal{O}(M).$$

Note that R maps $\mathcal{O}^{(k)}(X, \delta)$ continuously into $\mathcal{O}^{(k)}(M, \delta)$, $k \geqslant 0$.

Now our problem of continuation of holomorphic functions with restricted growth may be formulated as follows:

Given a triple (X, M, δ), when is

$$\text{(C)} \qquad \mathcal{O}(M, \delta) = R^X_M\big(\mathcal{O}(X, \delta)\big)$$

satisfied?

Many classical problems concerning holomorphic continuation (or interpolation) with controlled growth may easily be translated into this language. For example:

Let $X = C^n$ and let M be an analytic subset of C^n. We ask whether every function $f \in \mathcal{O}(M)$ with polynomial growth on M extends to a polynomial $\hat{f}$ of n-complex variables. Take $\delta = \delta_0 = (1+\|z\|^2)^{-1/2}$. By the Liouville theorem, the space $\mathcal{O}^{(k)}(C^n, \delta_0)$ is identical with the space of all polynomials of degree $\leqslant [k]$, and so the above question may be equivalently formulated as follows:

When does the triple (C^n, M, δ_0) satisfy (C)?

Similarly, putting $\delta = e^{-\|z\|^\tau}$, we get a problem of continuation of functions with exponential growth (see § 3).

Taking into considerations the algebraical and topological structures of $\mathcal{O}(X, \delta)$ and $\mathcal{O}(M, \delta)$, one may consider some stronger versions of (C), for instance:

(H) There exists an algebra homomorphism

$$T\colon\ \mathcal{O}(M, \delta) \to \mathcal{O}(X, \delta)$$

such that $R \circ T = \mathrm{id}$.

(L) $\exists \sigma \geqslant 0$: $\forall \eta > 1$ $\exists c = c(\eta) > 0$: $\forall k \geqslant 0$: there exists a linear continuous extension operator

$$L_k\colon\ \mathcal{O}^{(k)}(M, \delta) \to \mathcal{O}^{(k+\sigma)}(X, \delta)$$

such that $\|L_k\| \leqslant c\eta^k$.

The simplest case is that in which X is a domain of holomorphy in C^n – in this case, for some special triples (X, M, δ), problem (C) was studied, for instance, in [1], [2], [21], [22].

On the other hand, the most interesting case is that in which X is a Stein domain spread over C^n – in particular, by the passage to the envelope of holomorphy, this permits us to study (C) for all open sets in C^n. In the case of Stein domains over C^n, some results related to (C), (H), (L) were proved by the author in [1], [12], [13], [14]. In a more general context, the problems (C), (H), (L) will be studied in the present paper.

The main result of the paper is the following:

THEOREM 5.7. *Let X be a Stein domain over C^n, let M be an analytic submanifold of X and let δ be a regular weight function on X* (see Def. 2.11; *if X is a domain of holomorphy in C^n, then we can take, for instance, $\delta = \delta_X = \min\{\varrho_X, \delta_0\}$, where ϱ_X denotes the distance to the boundary of X*). *Assume that there exists a $G \in [\mathcal{O}(X, \delta)]^m$ such that*

$$M \subset G^{-1}(0),$$

$$\operatorname{rank}(d_x G) = \operatorname{codim}_x M =: r(x), \qquad x \in M,$$

$$\|\underbrace{(d_x G) \wedge \ldots \wedge (d_x G)}_{r(x)\ \text{times}}\| \geqslant b\delta^\beta(x), \qquad x \in M \qquad (b > 0,\ \beta \geqslant 0\ \textit{constants}).$$

Then (X, M, δ) satisfies (L).

The above result is a simultaneous generalization of some results of [1], [2], [11], [13], [21], [22] (for details see §5).

The paper is organized as follows:

§2 is of preparatory nature. We collect in it some basic properties of algebras of type $\mathcal{O}(X, \delta)$. Most of the results presented in that section are taken from [3], [8], [9], [10] and [14].

Some general remarks relating to (C), (H) and (L) are presented in §3. The main result of that section is a characterization of the solvability of (H) contained in Corol. 3.9. Namely we have proved that if X is a Stein domain, δ is a regular weight function and M is determined by functions from $\mathcal{O}(X, \delta)$, then each bounded homomorphic extension operator (as in (H)) is given by the formula $Tf = f \circ \pi$, $f \in \mathcal{O}(M, \delta)$, where $\pi: X \to M$ is a suitably chosen holomorphic retraction. Note that the existence of a holomorphic retraction $\pi: X \to M$ implies that the analytic subset M must be a submanifold (cf. Remark 3.10). In the second part of § 3 we present some examples which illustrate the relations between the classical theory of interpolation for holomorphic functions and (C). These examples are also studied in § 5. § 3 is based on [5], [11], [12], [13], [14], [15], [19] and [20].

In § 4 a generalized version of Nullstellensatz for holomorphic functions with restricted growth on Riemann domains is presented (Th. 4.1). As a consequence of this result we get the fundamental theorem on holomorphic continuation from some special ("regular") neighbourhoods of M (Th. 4.3). Theorem 4.3 is a particular case of Th. 1 from [11]. In the case where $X \in \operatorname{top} \mathbf{C}^n$ analogous results were proved in [2] and [16].

The main result of the paper (Th. 5.7) is formulated in § 5. In the same section we also present some of well-known results, which are special cases of Th. 5.7. The proof of Th. 5.7. is given in § 6.

The proof is based on a method of holomorphic retractions implied by Lemma 3.11. By Lemma 3.11, a triple (X, M, δ) satisfies (L) if one can find a neighbourhood U of M and a holomorphic retraction $\pi: U \to M$ such that for every holomorphic function f with controlled growth on U (specified in the lemma) there exists a holomorphic function $\hat{f}$ with controlled growth on X with $\hat{f} = f$ on M. In the case $X \in \operatorname{top} \mathbf{C}^n$, an analogous method was used in [2]. Our approach to the construction of holomorphic retractions is different from that in [2]. The main concept is based on a method of pseudoinverse matrices (cf. Lemmas 6.2, 6.3). The idea of the method is taken from [23]. This method of proof permits us to make it elementary.

Most of the results presented in this paper were announced in the preprint [15].

The author wishes to express his deep gratitude to Professor J. Siciak for his suggestions, which were helpful in the improvement of an earlier version of this paper.

§ 2. Basic properties of δ-tempered holomorphic functions

Let X be a complex analytic space countable at infinity and let $\delta: X \mapsto (0, +\infty)$ be a fixed function. For $k \geqslant 0$ let

$$\mathcal{O}^{(k)}(X, \delta) := \{f \in \mathcal{O}(X): \exists c = c(f) \geqslant 0: \delta^k |f| \leqslant c\}$$

= the space of all *δ-tempered holomorphic functions on X of degree $\leqslant k$.*

Put

$$\mathcal{O}(X, \delta) := \bigcup_{k \geqslant 0} \mathcal{O}^{(k)}(X, \delta)$$

= the set of all δ-tempered holomorphic functions on X.

2.1. It can be seen that:

(a) *The space $\mathcal{O}^{(k)}(X, \delta)$ endowed with the norm*

$$(*) \qquad \mathcal{O}^{(k)}(X, \delta) \ni f \to \|\delta^k f\|_\infty \in \boldsymbol{R}_+$$

is a complex normed space.

(b) $\mathcal{O}^{(0)}(X, \delta) = H^\infty(X) =$ *the Banach algebra of all bounded holomorphic functions on X.*

(c) $\mathcal{O}^{(k)}(X, \delta) \cdot \mathcal{O}^{(k')}(X, \delta) \subset \mathcal{O}^{(k+k')}(X, \delta), \quad k, k' \geqslant 0.$

(d) *If δ is bounded, then*

$$\mathcal{O}^{(k)}(X, \delta) \subset \mathcal{O}^{(k')}(X, \delta), \quad k \leqslant k'.$$

In particular, if δ is bounded then $\mathcal{O}(X, \delta)$ is a complex algebra with a unit element and

$$(**) \qquad \mathcal{O}(X, \delta) = \bigcup_{k \in \boldsymbol{N}} \mathcal{O}^{(k)}(X, \delta).$$

(e) *If $1/\delta$ is locally bounded* (e.g., δ is lower semi-continuous) *then for every compact $K \subset X$:*

$$({*}{*}{*}) \qquad \sup_K |f| \leqslant [\sup_K (1/\delta)]^k \|\delta^k f\|_\infty, \quad f \in \mathcal{O}^{(k)}(X, \delta).$$

Consequently, if $1/\delta$ is locally bounded then the topology induced by the norm $(*)$ is stronger than the topology of uniform convergence on compact subsets of X; in particular, $\mathcal{O}^{(k)}(X, \delta)$ is a complex Banach space.

2.2. Note that:

(a) *If δ is unbounded then $\mathcal{O}(X, \delta)$ need not be a vector space and* $(**)$ *need not be true.* For example:

Let $X = \boldsymbol{C}$, $\delta = |e^g|$, where $g \in \mathcal{O}(\boldsymbol{C})$ is such that $g(-n) = -\ln n$, $g(n) = \ln n$, $n \in \boldsymbol{N}$. For $k \geqslant 0$, let $f_k := e^{-kg}$. Obviously $f_k \in \mathcal{O}^{(k)}(\boldsymbol{C}, \delta)$, $k \geqslant 0$, but $f_1 + f_2 \notin \mathcal{O}(\boldsymbol{C}, \delta)$ and $f_{1/2} \notin \bigcup_{k \in \boldsymbol{N}} \mathcal{O}^{(k)}(\boldsymbol{C}, \delta)$.

(b) *If $1/\delta$ is not locally bounded then the space $\mathcal{O}^{(k)}(X, \delta)$ need not be complete* ($k > 0$). For example:

Let $X = \{x+iy \in C\colon 0 < x, y < 1\}$. By the Runge approximation theorem one can construct a sequence $(f_n)_{n=1}^{\infty}$ of complex polynomials and a bounded *non-holomorphic* function $f_0\colon X \to C$ such that:

$$\forall z \in X \ \exists n(z)\colon \forall n \geqslant n(z)\colon |f_n(z) - f_0(z)| \leqslant 1/n \qquad \text{(see [18], p. 382).}$$

Put $\delta := \inf\{1, (\sqrt{n}|f_n - f_0|)^{-1}, n \in N\}$. It can be seen that $\delta\colon X \to (0, 1]$ and $\|\delta(f_n - f_0)\|_\infty \leqslant 1/\sqrt{n}, n \geqslant 1$. Thus $(f_n|_X)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{O}^{(1)}(X, \delta)$ which has no limit in this space.

2.3 (cf. [8], Prop. 1). *Assume that* $\dim \mathcal{O}(X) = \infty$. *If $1/\delta$ is locally bounded then $\mathcal{O}^{(k)}(X, \delta)$ is of the first Baire category in $\mathcal{O}(X)$ in the topology of uniform convergence on compact subsets of X. Consequently, if δ is bounded then, in view of* (**), *$\mathcal{O}(X, \delta)$ is of the first Baire category in $\mathcal{O}(X)$.*

Let X_1, X_2 be two complex analytic spaces countable at infinity and let $\delta_j\colon X_j \to (0, +\infty)$ be a bounded function, $j = 1, 2$.

2.4. A linear mapping

$$L\colon \mathcal{O}(X_1, \delta_1) \to \mathcal{O}(X_2, \delta_2)$$

is said to be *bounded* if for every $k_1 \geqslant 0$ there exists a $k_2 \geqslant 0$ such that the operator $L|_{\mathcal{O}^{(k)}(X,\delta)}$ maps continuously $\mathcal{O}^{(k_1)}(X_1, \delta_1)$ into $\mathcal{O}^{(k_2)}(X_2, \delta_2)$.

A linear isomorphism $L\colon \mathcal{O}(X_1, \delta_1) \to \mathcal{O}(X_2, \delta_2)$ is called a *bounded isomorphism* if the operators L and L^{-1} are bounded (cf. [3], §§ 2.1, 2.2).

Bounded algebra homomorphisms play the role of morphisms in the category of algebras of type $\mathcal{O}(X, \delta)$; if $L\colon \mathcal{O}(X_1, \delta_1) \to \mathcal{O}(X_2, \delta_2)$ is a bounded algebra isomorphism, then the algebras $\mathcal{O}(X_1, \delta_1)$ and $\mathcal{O}(X_2, \delta_2)$ can be identified.

2.5. Observe that *if $\varphi\colon X_2 \to X_1$ is a holomorphic mapping such that*

$$\delta_2^{\gamma} \leqslant c\delta_1 \circ \varphi \qquad (\gamma, c > 0 \textit{ constants}),$$

then the operator $\varphi^*|_{\mathcal{O}(X_1,\delta_1)}$ is a bounded algebra homomorphism of $\mathcal{O}(X_1, \delta_1)$ into $\mathcal{O}(X_2, \delta_2)$ ($\varphi^* f := f \circ \varphi$) (see also 3.6).

2.6. We say that two functions $\delta_1, \delta_2\colon X \to (0, +\infty)$ are *equivalent* ($\delta_1 \sim \delta_2$) if there exist constants $\gamma_j, c_j > 0$, $j = 1, 2$, such that:

$$\delta_1^{\gamma_1} \leqslant c_1 \delta_2, \quad \delta_2^{\gamma_2} \leqslant c_2 \delta_1 \qquad \text{(cf. [3], § 1.1).}$$

It is clear that if $\delta_1 \sim \delta_2$ then $\mathcal{O}(X, \delta_1) = \mathcal{O}(X, \delta_2)$, and if, moreover, δ_1, δ_2 are bounded then the identity mapping is a bounded algebra isomorphism of $\mathcal{O}(X, \delta_1)$ onto $\mathcal{O}(X, \delta_2)$.

More generally:

2.7. *If* $\varphi\colon X_2 \to X_1$ *is a biholomorphic mapping for which* $\delta_1 \circ \varphi \sim \delta_2$, *then* $\varphi^*|_{\mathcal{O}(X_1,\delta_1)}$ *is a bounded algebra isomomorphism of* $\mathcal{O}(X_1, \delta_1)$ *onto* $\mathcal{O}(X_2, \delta_2)$ (cf. 2.5, see also 3.8).

2.8. Let $\delta\colon X \to (0, +\infty)$ be a bounded function. We shall denote by $S(X, \delta)$ the set of all *characters* on $\mathcal{O}(X, \delta)$, that is, the set of all non-zero linear and multiplicative functionals $\xi\colon \mathcal{O}(X, \delta) \to \boldsymbol{C}$. Let $S_b(X, \delta)$ denote the set of all *bounded* characters on $\mathcal{O}(X, \delta)$; a character $\xi\colon \mathcal{O}(X, \delta) \to \boldsymbol{C}$ is said to be *bounded* if for every $k \geqslant 0$ the operator $\xi|_{\mathcal{O}^{(k)}(X,\delta)}$ maps continuously $\mathcal{O}^{(k)}(X, \delta)$ into $\boldsymbol{C}$. Further, let $E(X,\delta)$ denote the set of all *evaluations on* $\mathcal{O}(X, \delta)$, that is, the set of all characters of the form

$$\mathcal{O}(X, \delta) \ni f \to f(x) \in \boldsymbol{C},$$

where x is a point of X. Note that $E(X, \delta) \subset S_b(X, \delta) \subset S(X, \delta)$.

Now we pass to the case where X is a Riemann domain over $\boldsymbol{C}^n$.

Let (X, p) be a Riemann domain over $\boldsymbol{C}^n$ countable at infinity, i.e. X is a complex n-dimensional manifold countable at infinity and $p\colon X \to \boldsymbol{C}^n$ is a local biholomorphism.

We say that (X, p) is *finitely sheeted* if for every $x \in X$ the stalk $p^{-1}(p(x))$ is a finite set.

A set $C \subset X$ is said to be *univalent* if the mapping $p|_C$ is injective.

An open set $X \subset \boldsymbol{C}^n$ will always be identified with the domain (X, id_X).

A domain (X, p) is said to be a *Stein domain* if X is a Stein manifold.

We shall frequently write X instead of (X, p).

Let $\|z\| = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}$, $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$, denote the Euclidean norm in $\boldsymbol{C}^n$.

For $x \in X$ and $r > 0$ let $\hat{B}(x, r) = \hat{B}_X(x, r)$ denote an open univalent neighbourhood of x which is mapped (biholomorphically) by p onto the Euclidean ball $B(p(x), r) \subset \boldsymbol{C}^n$.

Put $\varrho(x) = \varrho_X(x) := \sup\{r > 0\colon \hat{B}(x, r) \text{ exists}\} =$ *the Euclidean distance to the boundary of* X *of the point* x (if $X \in \operatorname{top} \boldsymbol{C}^n$ then ϱ_X coincides with the standard Euclidean distance to ∂X; $\varrho_{\boldsymbol{C}^n} \equiv +\infty$).

Let $\hat{B}(x) = \hat{B}_X(x) := \bigcup_{0 < r < \varrho(x)} \hat{B}(x, r) =$ *the maximal "ball" centred at* x,

$p_x := p|_{\hat{B}(x)}$,

$X^*\boldsymbol{C}^n := \{(x, z) \in X \times \boldsymbol{C}^n\colon \|z\| < \varrho(x)\}$ (note that $(X^*\boldsymbol{C}^n, p \times \mathrm{id}_{\boldsymbol{C}^n})$ is a Riemann domain over $\boldsymbol{C}^{2n}$),

$x \oplus z := p_x^{-1}(p(x) + z)$, $(x, z) \in X^*\boldsymbol{C}^n$.

2.9. Observe that:

(a) *The mapping*

$$\oplus\colon X^*\boldsymbol{C}^n \ni (x, z) \to x \oplus z \in X$$

is holomorphic.

(b) $|\varrho(x\oplus z)-\varrho(x)| \leqslant \|z\|$, $(x, z)\in X^* C^n$, $x\notin X_\infty$, *where* $X_\infty := \{x\in X: \varrho(x) = +\infty\}$ = *the sum of all connected components of* X *which are mapped biholomorphically by* p *onto* C^n.

Our approach to the theory of δ-tempered holomorphic functions on Riemann domains will be based on Hörmander's L^2-estimates for the $\bar{\partial}$-problem (cf. §4) and therefore we have to restrict our considerations to the case where X is Stein and δ satisfies some additional regularity conditions.

DEFINITION 2.10. A function δ: $X\to(0, 1]$ is said to be a *Lipschitz function* on $X(\delta\in\mathscr{L}(X))$ if

(l_1) $\delta\leqslant\varrho$,

(l_2) $|\delta(x\oplus z)-\delta(x)| \leqslant \|z\|$, $(x, z)\in X^* C^n$.

Some characterizations of Lipschitz functions will be given below. Now let us only observe that the function $\min\{\varrho, 1\}$ is the maximal Lipschitz function on X (cf. 2.9 (b)).

Let $\lambda = \lambda_X$ denote the Lebesgue measure on X (λ is locally "transported" by p from the space C^n). Let $L^2(X)$ denote the space of all λ-square integrable functions on X and let $\| \|_2$ denote the norm of $L^2(X)$.

Further, let PSH(X) denote the set of all plurisubharmonic (psh.) functions on X.

For any function δ: $X\to R_+$ let us consider the following conditions:

(w_1) $\delta\leqslant\delta_0\circ p$,

where $\delta_0(z) = (1+\|z\|^2)^{-1/2}$, $z\in C^n$.

(w_2) $\exists\alpha_0\geqslant 0$: $\delta^{\alpha_0}\in L^2(X)$.

(w_3) $-\log\delta\in$ PSH(X).

(w_4) $\forall\tau>0$: $\{x\in X: \delta(x)>\tau\}\subset\subset X$.

DEFINITION 2.11. Let $\mathscr{L}_{i_1,\dots,i_k}(X)$ denote the set of all Lipschitz functions on X which satisfy the conditions (w_{i_1}), ..., (w_{i_k}), $1\leqslant i_1<\dots<i_k\leqslant 4$, $1\leqslant k\leqslant 4$. Additionally, let

$\mathscr{L}_0(X) := \mathscr{L}(X)$,

$\mathscr{L}(X) := \mathscr{L}_1(X)$ = the set of all *weight* functions on X,

$\mathscr{W}_r(X) := \mathscr{L}_{1,2,3}(X)$ = the set of all *regular* weight functions on X.

In the sequel we shall show that, generally speaking, every space of holomorphic functions with restricted growth on a Riemann (resp. Stein) domain X may be realized as a subspace of an algebra $\mathcal{O}(X, \delta)$ where $\delta\in\mathscr{L}_{1,2,4}(X)$ (resp. $\delta\in\mathscr{L}_{1,2,3,4}(X)$) (see 2.28, 2.29). We shall also show that the passage to the envelope of holomorphy always permits us to reduce the problem to the case where X is Stein (see 2.31).

2.12. Let η: $X\to[0,1]$ be such that the set $Y := \{\eta>0\}$ is open. One can easily prove that the following conditions are equivalent:

(i) $\eta|_Y\in\mathscr{L}(Y)$,

(ii) *η satisfies* (l_1) *and* (l_2) *on* X.

In particular, if $X = C^n$ then $\eta|_Y \in \mathscr{L}(Y)$ iff

$$|\eta(z) - \eta(z')| \leqslant \|z - z'\|, \quad z, z' \in C^n.$$

This shows that in the case of C^n our definition of Lipschitz functions is equivalent to that in [3], § 1.2.

2.13. The function

$$\delta_X := \min\{\varrho, \delta_0 \circ p\}$$

is the maximal weight function on X. Observe that $\delta_{C^n} = \delta_0$.

Recall that $\mathcal{O}^{(k)}(C^n, \delta_0) =$ the space of all complex polynomials of n-complex variables of degree $\leqslant [k]$ (cf. § 1). By analogy, in the general case functions from $\mathcal{O}^{(k)}(X, \delta_X)$ are called *holomorphic functions with polynomial growth on X of degree $\leqslant k$*. Observe that the mapping $p^*|_{\mathcal{O}(C^n, \delta_0)}$ is a bounded algebra monomorphism of $\mathcal{O}(C^n, \delta_0)$ into $\mathcal{O}(X, \delta_X)$ (p^* maps $\mathcal{O}^{(k)}(C^n, \delta_0)$ into $\mathcal{O}^{(k)}(X, \delta_X)$, $k \geqslant 0$).

For every analytic subset $M \subset X$, functions from $\mathcal{O}^{(k)}(M, \delta_X)$ are called *holomorphic functions with polynomial growth on M of degree $\leqslant k$.*

2.14. *For every* $0 \leqslant i \leqslant 4$:

(a) $\delta_1, \delta_2 \in \mathscr{L}_i(X) \Rightarrow \min\{\delta_1, \delta_2\} \in \mathscr{L}_1(X)$.

(b) $\delta_1, \ldots, \delta_m \in \mathscr{L}_i(X) \Rightarrow \frac{1}{m}\delta_1 \cdot \ldots \cdot \delta_m \in \mathscr{L}_i(X)$.

(c) $\delta \in \mathscr{L}_i(X)$, $\alpha \geqslant 1 \Rightarrow \frac{1}{\alpha}\delta^\alpha \in \mathscr{L}_i(X)$.

(d) *If* $\delta_1 \in \mathscr{L}_i(X_1), \ldots, \delta_m \in \mathscr{L}_i(X_m)$ *then the function* δ *given by the formula*

$$\delta(x_1, \ldots, x_m) = \frac{1}{\sqrt{m}} \min\{\delta_1(x_1), \ldots, \delta_m(x_m)\}, \quad (x_1, \ldots, x_m) \in X_1 \times \ldots \times X_m,$$

belongs to $\mathscr{L}_i(X_1 \times \ldots \times X_m)$.

Let Ψ denote the set of all C^1-functions ψ: $R_+ \to R_+$ such that:

(i) ψ is increasing and convex,

(ii) $\psi(t) \geqslant t$, $t \in R_+$,

(iii) $\psi'(t) e^t \leqslant e^{\psi(t)}, t \in R_+$.

One can prove (cf. [10], Lemma 7) that:

2.15. *For every increasing function* ψ_0: $R_+ \to R_+$ *there exists a* C^∞-*function* $\psi \in \Psi$ *such that* $\psi \geqslant \psi_0$.

For δ: $X \to (0, 1]$ and $\psi \in \Psi$ put

$$\delta_{(\psi)} := e^{-\psi(-\log\delta)}.$$

Note that $\delta_{(\psi)} \leqslant \delta$. One can easily prove (cf. [3], § 1.5, [10], the proof of Prop. 2) that:

2.16. $\delta \in \mathscr{L}_i(X)$, $\psi \in \Psi \Rightarrow \delta_{(\psi)} \in \mathscr{L}_i(X)$, $i = 0, \ldots, 4$, *whence* $\delta_{(\psi)}$ *has at least the same regularity as* δ.

The above result, as well as 2.14, give an insight into the construction of new Lipschitz functions, which satisfy some fixed regularity conditions. Let us consider the following example:

2.17. It is clear that for every $\tau > 0$ there exists a constant $c(\tau) > 0$ such that the function

$$\psi_\tau(t) := c(\tau) + e^{\tau t}, \qquad t \in \boldsymbol{R}_+,$$

belongs to Ψ. Note that $\delta_{(\psi_\tau)}$ is equivalent (in the sense of 2.6) to $e^{-\delta^{-\tau}}$. Put $\delta_\tau := (\delta_0)_{(\psi_\tau)}$ and observe that δ_τ is equivalent to $e^{-\|z\|^\tau}$. Hence

$$\bigcap_{\tau > \mu} \mathcal{O}(\boldsymbol{C}^n, \delta_\tau) = E_\mu(\boldsymbol{C}^n) =: \text{the algebra of all } \textit{entire functions}$$
$$\textit{of order} \leqslant \mu \ (\mu \geqslant 0).$$

By analogy, in the general case, if M is an analytic subset of X (including the case where $M = X$), we put:

$$E_\mu(M) := \bigcap_{\tau > \mu} \mathcal{O}(M, (\delta_X)_{(\psi_\tau)}) = \text{the algebra of all } \textit{holomorphic}$$
$$\textit{functions with exponential growth on } X \textit{ of order} \leqslant \mu$$
$$= \{f \in \mathcal{O}(M)\colon \ \forall \tau > \mu \ \exists A_\tau, B_\tau > 0\colon \ |f(x)| \leqslant A_\tau e^{B_\tau[\delta_X(x)]^{-\tau}}, \ x \in M\}.$$

More generally:

Let $\psi \in \Psi$ *be fixed. Then for every Riemann domain* X *and for every analytic subset* M *of* X *the function* ψ *generates an algebra* $\mathcal{O}(M, (\delta_X)_{(\psi)})$ *which may be considered as a generalization to* M *of the algebra* $\mathcal{O}(\boldsymbol{C}^n, (\delta_0)_{(\psi)})$.

Let $\eta\colon X \to (0, 1]$ be a function satisfying (1_1) and such that $1/\eta$ is locally bounded. Put

$$\tilde{\eta}(x) := \inf\{\eta(x \oplus z) + \|z\|\colon \ \|z\| < \varrho(x)\}, \qquad x \in X.$$

The function $\tilde{\eta}$ will be called the *Lipschitz regularization of* η (it may easily be proved that $\tilde{\eta}$ coincides with the formal convolution of η (cf. [9])).

Observe that:

2.18. (a) $\tilde{\eta} \leqslant \eta$.

(b) $\tilde{\eta} \in \mathscr{L}(X)$.

(c) $\tilde{\eta} = \sup\{\delta \in \mathscr{L}(X)\colon \ \delta \leqslant \eta\}$.

Consequently, we get the following criterion for δ to be a Lipschitz function on X:

2.19. Let $\delta\colon X \to (0, 1]$ be such that $\delta \leqslant \varrho$ and $1/\delta$ is locally bounded. Then the following conditions are equivalent:

(i) $\delta \in \mathscr{L}(X)$.

(ii) $\delta = \tilde{\delta}$.

(iii) *For every univalent set $C \subset X$ for which $p(C)$ is convex the function $\delta \circ (p|_C)^{-1}$ satisfies the standard Lipschitz condition with the constant* 1 *in* $p(C)$ (cf. [9]).

2.20 (cf. [9], Th. 3). *Let X be a Stein domain and let $\eta\colon X \to (0, 1]$ be such that $\eta \leqslant \varrho$ and $-\log \eta \in \mathrm{PSH}(X)$. Then $-\log \tilde{\eta} \in \mathrm{PSH}(X)$.*

2.21. *Let* $N(X) := \sup\{\# p^{-1}(p(x))\colon x \in X\}$. *If* $N(X) < +\infty$ (e.g. $X \in \mathrm{top}\,\mathbf{C}^n$), *then for every $\delta \in \mathscr{W}(X)$ and for every $\varepsilon > 0$: $\delta^{n+\varepsilon} \in \mathscr{L}^2(X)$. Consequently, if $N(X) < +\infty$ then $\mathscr{W}(X) = \mathscr{L}_1(X) \subset \mathscr{L}_2(X)$.*

Proof. By the Fubini theorem

$$\int_X \delta^{2(n+\varepsilon)} d\lambda_X \leqslant \int_X (\delta_0 \circ p)^{2(n+\varepsilon)} d\lambda_X \leqslant N(X) \int_{\mathbf{C}^n} \delta_0^{2(n+\varepsilon)} d\lambda_{\mathbf{C}^n} \leqslant N(X)\, c(n)/\varepsilon. \ \blacksquare$$

Note that if $N(X) = +\infty$ then δ_X need not satisfy (w_2).

2.22. *If X is finitely sheeted* (e.g. $N(X) < +\infty$) *then $\mathscr{W}(X) = \mathscr{L}_1(X) \subset \mathscr{L}_4(X)$.*

Proof. We only need to observe that a closed set K of a finitely sheeted Riemann domain X is compact iff the set $p(K)$ is bounded and $\inf_K \varrho > 0$ (cf. [4], p. 48, the proof of Th. 8). $\blacksquare$

Note that if X is not finitely sheeted then ϱ_X need not satisfy (w_4).

2.23. *For every Riemann domain X: $\mathscr{L}_{1,2,4}(X) \neq \emptyset$.*

Proof. If $N(X) < +\infty$ then the result follows from 2.21 and 2.22. In the general case, let $\eta_0\colon X \to (0, 1]$ be an arbitrarily fixed continuous function satisfying (w_4). Put

$$\eta_1 = (\min\{\eta_0, \delta_X\})^{\sim}.$$

Then η_1 satisfies (w_4) and belongs to $\mathscr{L}_{1,4}(X)$ (cf. 2.18). Now one can choose $\psi \in \Psi$ in such a way that

$$\eta := (\eta_1)_{(\psi)} \in \mathscr{L}_{1,2,4}(X) \quad \text{(cf. 2.15 and 2.16).} \ \blacksquare$$

2.24. It is clear that *if X is a Stein domain then $\delta_X \in \mathscr{L}_3(X)$.*

2.25. *If $\mathscr{L}_{3,4}(X) \neq \emptyset$ then X is a Stein domain.*

Proof. For the proof it suffices to observe that if $\delta \in \mathscr{L}_{3,4}(X)$ then $-\log \delta$ is a continuous psh. exhaustion function. $\blacksquare$

2.26. *For every Stein domain X: $\mathscr{L}_{1,2,3,4}(X) \neq \emptyset$.*

Proof. If $N(X) < +\infty$ then the result follows from 2.21, 2.22 and 2.24. In the general case the proof is analogous with that of 2.23. It is enough to observe that if X is Stein then the function η_0 may be chosen in such a way that $-\log\eta_0 \in \mathrm{PSH}(X)$ (cf. 2.20). ■

2.27 (cf. [3], § 1.5). *Let $\eta \in \mathscr{L}_4(X)$ and let G be a locally bounded family of functions $X \to \mathbf{C}$. Then there exists a $\psi \in \Psi$ such that*

$$\eta_{(\psi)}|g| \leqslant 1, \quad g \in G.$$

Proof. It suffices to take ψ such that

$$\psi(t) \geqslant \sup_{g\in G}\{\sup_{K_{e^{-t}}}\{\log^+|g|\}\}, \quad t \in \mathbf{R}_+,$$

where $K_\tau := \{\eta \geqslant \tau\}$, $\tau > 0$ (cf. 2.15). ■

The above result implies the following two important corollaries:

2.28. *Let X be a Riemann* (resp. *Stein*) *domain, let M be an analytic subset of X (including the case where $M = X$) and let $F \subset \mathcal{O}(M)$ be a locally bounded family of functions. Then there exists $\delta \in \mathscr{L}_{1,2,4}(X)$* (resp. *$\delta \in \mathscr{L}_{1,2,3,4}(X)$*) *such that*

$$F \subset \{f \in \mathcal{O}^{(1)}(X, \delta) \colon \|\delta f\|_\infty \leqslant 1\}.$$

Proof. The result is a consequence of 2.23 (resp. 2.26) and 2.27 with $G := \{f \cup 0_{X\setminus M} \colon f \in F\}$. ■

2.29. *Let X be a Riemann* (resp. *Stein*) *domain, let M be an analytic subset of X and let $\vartheta \colon M \to (0, +\infty)$ be such that $1/\vartheta$ is locally bounded. Then there exists a $\delta \in \mathscr{L}_{1,2,4}(X)$* (resp. *$\delta \in \mathscr{L}_{1,2,3,4}(X)$*) *such that*

$$\mathcal{O}^{(k)}(M, \vartheta) \subset \mathcal{O}^{(k)}(M, \delta)$$

and

$$\|\delta^k f\|_\infty \leqslant \|\vartheta^k f\|_\infty, \quad f \in \mathcal{O}^{(k)}(M, \vartheta),\ k \geqslant 0.$$

In particular, if ϑ is also bounded then the identity mapping is a bounded algebra monomorphism of $\mathcal{O}(M, \vartheta)$ into $\mathcal{O}(M, \delta)$.

Proof. The result is a consequence of 2.1 ($*_*{*}$) and 2.27 with

$$G := \{(|f|/\|\vartheta^k f\|_\infty)^{1/k} \cup 0_{X\setminus M} \colon k > 0,\ f \in \mathcal{O}^{(k)}(M, \delta),\ f \neq 0\}. \ ■$$

Let $(\hat{X}, \hat{p})$ denote the envelope of holomorphy of (X, p) and let $\varphi \colon X \to \hat{X}$ be the embedding of X into $\hat{X}$ (φ is locally biholomorphic and $\hat{p} \circ \varphi = p$).

Let $\delta \colon X \to (0, 1]$ be a lower semi-continuous function. Define

$$\hat{\delta} = e^{-\Phi^*},$$

where $\Phi := \sup\{u \in \mathrm{PSH}(\hat{X}) \colon u \circ \varphi \leqslant -\log\delta\}$, Φ^* denotes the upper regularization of Φ. The function $\hat{\delta}$ is called the *plurisubharmonic regularization of δ* (cf. [3], § 4.4).

2.30. (a) $\hat{\delta} \leqslant 1$,

(b) $\delta \leqslant \hat{\delta} \circ \varphi$,

(c) $\hat{\delta}$: $\hat{X} \to (0, 1]$ *and* $-\log \hat{\delta} \in \mathrm{PSH}(\hat{X})$.

Proof. (a), (b) follow directly from the definition of $\hat{\delta}$. We pass to the proof of (c). It is known that for every compact $L \subset \hat{X}$ there exists a compact $K \subset X$ such that

$$L \subset [\varphi(K)]^{\sim}_{\mathrm{PSH}(X)} = \{\hat{x} \in \hat{X}: \ \forall u \in \mathrm{PSH}(\hat{X}): \ u(\hat{x}) \leqslant \sup_K u \circ \varphi\}.$$

In particular,

$$\sup_L \Phi \leqslant \sup_K(-\log \delta) < +\infty .$$

This proves that the function Φ is locally upper bounded, whence $-\log \hat{\delta} \in \mathrm{PSH}(\hat{X})$ (in particular $\hat{\delta}$: $X \to (0, 1]$). ∎

Analogously with [3], § 4.4 one can easily prove that:

2.31. $\varphi^*(\mathcal{O}^{(k)}(\hat{X}, \hat{\delta})) = \mathcal{O}^{(k)}(X, \delta)$ *and*

$$\|\hat{\delta}^k f\|_\infty = \|\delta^k (f \circ \varphi)\|_\infty, \quad f \in \mathcal{O}^{(k)}(\hat{X}, \hat{\delta}), k \geqslant 0.$$

In particular, $\varphi^*|_{\mathcal{O}(\hat{X}, \hat{\delta})}$ *is a bounded algebra isomorphism of* $\mathcal{O}(\hat{X}, \hat{\delta})$ *onto* $\mathcal{O}(X, \delta)$.

2.32. $\delta \in \mathcal{L}_i(X) \Rightarrow \hat{\delta} \in \mathcal{L}_i(X)$, $i = 0, 1$.
(The author does not know whether the above implication is true for $i = 2, 4$.)

Proof (cf. [14]). Observe that

$$\varphi(\hat{B}_X(x, r)) = \hat{B}_{\hat{X}}(\varphi(x), r), \quad 0 < r < \varrho_X(x);$$

hence $\varrho_X \leqslant \varrho_{\hat{X}} \circ \varphi$ and $\delta_X \leqslant \delta_{\hat{X}} \circ \varphi$. Obviously $-\log \varrho_{\hat{X}}$, $-\log \delta_{\hat{X}} \in \mathrm{PSH}(\hat{X})$. Consequently, if $\delta \leqslant \varrho_X$ (resp. $\delta \leqslant \delta_X$) then $\hat{\delta} \leqslant \varrho_{\hat{X}}$ (resp. $\hat{\delta}_{\hat{X}} \leqslant \delta_{\hat{X}}$). It remains to show that if $\delta \in \mathcal{L}(X)$ then $(\hat{\delta})^{\sim} \geqslant \hat{\delta}$ (cf. 2.19). In view of 2.20, it is enough to prove that $\delta \leqslant (\hat{\delta})^{\sim} \circ \varphi$. Fix $x \in X$. Since $\varphi(x \oplus z) = \varphi(x) \oplus z$, $(x, z) \in X * \boldsymbol{C}^n$, we get

$$\begin{aligned}(\hat{\delta})^{\sim}(\varphi(x)) = \min \big\{ &\inf \{(\hat{\delta} \circ \varphi)(x \oplus z) + \|z\|: \ \|z\| < \varrho_X(x)\}, \\ &\inf \{\hat{\delta}(\varphi(x) \oplus z) + \|z\|: \ \varrho_X(x) \leqslant \|z\| < \varrho_{\hat{X}}(\varphi(x))\} \big\} \\ \geqslant \min &\{\tilde{\delta}(x), \varrho_X(x)\} = \delta(x). \ \blacksquare\end{aligned}$$

2.33. Let X be a Stein domain. Then we can take $\hat{X} = X$, $\varphi = \mathrm{id}_X$. Let δ: $X \to (0, 1]$ be a lower semi-continuous function and let $\hat{\delta}$ denote its plurisubharmonic regularization. We already know that $\mathcal{O}(X, \hat{\delta}) = \mathcal{O}(X, \delta)$ (cf. 2.31). Let M be an analytic subset of X, $\dim M \leqslant n-1$. Then obviously $\mathcal{O}^{(k)}(M, \hat{\delta}) \subset \mathcal{O}^{(k)}(M, \delta)$, $k \geqslant 0$.

Note that in general $\mathcal{O}(M,\hat{\delta})$ need not be equal to $\mathcal{O}(M,\delta)$ (even if X, M, δ are very regular). For example:

Let $X = \boldsymbol{C}^2$, $M = \{z_2 = 0\}$. Set

$$\eta(z_1,z_2) = \begin{cases} \delta_0(z_1,z_2) & \text{if } z_2 \neq 0, \\ e^{-|z_1|} & \text{if } z_2 = 0. \end{cases}$$

Note that $\eta \leqslant \delta_0$ and $1/\eta$ is locally bounded. Let $\delta := \tilde{\eta}$. Then $e^{z_1} \in \mathcal{O}^{(1)}(M,\delta)$. We shall show that $e^{z_1} \notin \mathcal{O}(M,\hat{\delta})$.

Note that

$$\delta(z_1, z_2) \geqslant \min\{\delta_0(z_1, z_2), \inf\{e^{-|w_1|} + \sqrt{|z_1 - w_1|^2 + |z_2|^2}:\ w_1 \in \boldsymbol{C}\}\},$$

so $\delta(z_1, z_2) = \delta_0(z_1, z_2)$, $|z_2| \geqslant 1$. Hence, since $-\log\hat{\delta} \in \mathrm{PSH}(X)$,

$$-\log\hat{\delta}(z_1, 0) \leqslant \max_{|z_2|=1}\{-\log\hat{\delta}(z_1, z_2)\} = -\log\delta_0(z_1, 1),$$

so $\hat{\delta}(z_1, 0) \geqslant \delta_0(z_1, 1)$, $z_1 \in \boldsymbol{C}$.

Consequently, for every $k \geqslant 0$:

$$\sup_M \{\hat{\delta}^k |e^{z_1}|\} \geqslant \sup\{(2+x^2)^{-k/2} e^{x}:\ x \in \boldsymbol{R}\} = +\infty.$$

2.34. In the case where $X \in \operatorname{top} \boldsymbol{C}^n$ the class of weight functions may be extended as follows (cf. [2], [6]):

A function η: $X \to (0, +\infty)$ is said to be a *generalized weight function on* X ($\eta \in \mathscr{W}^g(X)$) if there exist constants $c_0, c_1 > 0$, $0 < c_2 < 1$, $\alpha_1 > 0$, $\alpha_2 \geqslant 1$ such that

$$\eta \leqslant c_0 \delta_0,$$

$$[x \in X,\ y \in \boldsymbol{C}^n,\ \|x-y\| \leqslant c_1 \eta^{\alpha_1}(x)] \Rightarrow [y \in X,\ \eta(y) \geqslant c_2 \eta^{\alpha_2}(x)].$$

If moreover $-\log\eta \in \mathrm{PSH}(X)$ then we say that η is *regular* ($\eta \in \mathscr{W}^g_r(X)$).

Note that $\mathscr{W}(X) \not\subseteq \mathscr{W}^g(X)$ and $\mathscr{W}_r(X) \not\subseteq \mathscr{W}^g_r(X)$, but from the point of view of the theory of δ-tempered functions the classes $\mathscr{W}(X)$ and $\mathscr{W}^g(X)$ are equivalent, namely (cf. [3], §§ 1.4, 4.4) for every $\eta \in \mathscr{W}^g(X)$ (resp. $\eta \in \mathscr{W}^g_r(X)$) there exist constants $c, \gamma > 0$ such that the function $\delta := (c\eta^{\gamma})^{\sim}$ is equivalent to η and belongs to $\mathscr{W}(X)$ (resp. $\mathscr{W}_r(X)$).

We have shown how the study of δ-tempered holomorphic functions on a Riemann domain X may be reduced to the case where X is a Stein domain and δ is a Lipschitz function satisfying some of the conditions $(\mathrm{w}_1), \ldots, (\mathrm{w}_4)$.

Now we would like to present the special properties of $\mathcal{O}(X,\delta)$ in the case where X and δ are sufficiently regular.

We need some auxiliary notations.

Let δ: $X \to (0, +\infty)$ be a λ-measurable function. For $k \geqslant 0$ define

$$H^{(k)}(X,\delta) = \{f \in \mathcal{O}(X):\ \delta^k f \in L^2(X)\}$$

and let

$$H(X, \delta) = \bigcup_{k \geqslant 0} H^{(k)}(X, \delta).$$

2.35 (cf. 2.1). (a) The space $H^{(k)}(X, \delta)$ with the scalar product

$$(f, g) \to \int_X f \bar{g} \delta^{2k} \, d\lambda_X$$

is a complex unitary space ($k \geqslant 0$).

(b) If δ is bounded then

$$H^{(k)}(X, \delta) \subset H^{(k')}(X, \delta), \qquad k' \geqslant k;$$

in particular, $H(X, \delta)$ is a complex space and

$$H(X, \delta) = \bigcup_{k \in N} H^{(k)}(X, \delta).$$

(c) If $1/\delta$ is locally bounded then for every compact $K \subset X$ there exists a constant $c(K) > 0$ such that

$$\sup_K |f| \leqslant [c(K)]^k \|\delta^k f\|_2, \qquad f \in H^{(k)}(X, \delta), \; k \geqslant 0.$$

Consequently, $H^{(k)}(X, \delta)$ is a complex Hilbert space which topology is stronger than the topology of uniform convergence on compact subsets of X.

(d) $1 \in H(X, \delta)$ iff δ satisfies (w_2).

(e) If $\delta^{\alpha_0} \in L^2(X)$ then

$$\mathcal{O}^{(k)}(X, \delta) \subset H^{(k+\alpha_0)}(X, \delta)$$

and

$$\|\delta^{k+\alpha_0} f\|_2 \leqslant \|\delta^{\alpha_0}\|_2 \|\delta^k f\|_\infty, \qquad f \in \mathcal{O}^{(k)}(X, \delta), \; k \geqslant 0.$$

2.36. One may also easily prove (cf. [10], Th. I) that *if* $\dim \mathcal{O}(X) = \infty$ *and* $1/\delta$ *is locally bounded then* $H^{(k)}(X, \delta)$ *is of the first Baire category in* $\mathcal{O}(X)$ ($k \geqslant 0$). *Consequently, if* δ *is bounded then* $H(X, \delta)$ *is of the first Baire category in* $\mathcal{O}(X)$.

2.37 ([3], § 1.3, [8], Prop. 3). *If* $\delta \in \mathcal{L}(X)$ *then*

$$H^{(k)}(X, \delta) \subset \mathcal{O}^{(k+n)}(X, \delta)$$

and

$$\|\delta^{k+n} f\|_\infty \leqslant c(n, k) \|\delta^k f\|_2, \qquad f \in H^{(k)}(X, \delta), \; k \geqslant 0,$$

where $c(n, k) = [\tau_n \max\{\varepsilon^n (1-\varepsilon)^k \colon 0 < \varepsilon < 1\}]^{-1}$, τ_n = *the volume of the unit ball in* $\mathbf{C}^n$.

In view of 2.35 (e) and 2.37, if $\delta \in \mathcal{L}_2(X)$ then $\mathcal{O}(X, \delta) = H(X, \delta)$ and the structures of $\mathcal{O}(X, \delta)$ and $H(X, \delta)$ are isomorphic.

2.38 ([3], § 1.3, [8], Prop. 2). *If* $\delta \in \mathscr{L}(X)$ *then*

$$\|\delta^{k+|\nu|}(\partial^\nu f)\|_\infty \leqslant \nu! \sqrt{n}^{|\nu|} 2^{k+|\nu|} \|\delta^k f\|_\infty, \quad f \in \mathcal{O}^{(k)}(X, \delta),\ k \geqslant 0,\ \nu \in \boldsymbol{Z}_+^n,$$

where

$$\frac{\partial f}{\partial p_j}(x) := \frac{\partial (f \circ p_x^{-1})}{\partial z_j}(p(x)), \quad j = 1, \ldots, n,$$

$$\partial^\nu := \left(\frac{\partial}{\partial p_1}\right)^{\nu_1} \circ \ldots \circ \left(\frac{\partial}{\partial p_n}\right)^{\nu_n}, \quad \nu = (\nu_1, \ldots, \nu_n) \in \boldsymbol{Z}_+^n.$$

Consequently, for every $\nu \in \boldsymbol{Z}_+^n$, ∂^ν is a bounded linear operator of $\mathcal{O}(X, \delta)$ into $\mathcal{O}(X, \delta)$.

2.39. Let $\delta \in \mathscr{L}(X)$ and let $G = (G_1, \ldots, G_m)$: $X \to \boldsymbol{C}^n$ be such that $G_j \in \mathcal{O}^{(k)}(X, \delta)$, $j = 1, \ldots, m$. Put

$$\|G\| := (|G_1|^2 + \ldots + |G_m|^2)^{1/2} \quad \text{and} \quad \|\delta^k G\|_\infty := \sup_X \{\delta^k \|G\|\}.$$

Let $d_x^{(s)} G := d_{p(x)}^{(s)}(G \circ p_x^{-1})$ = the sth differential of G at x ($s \in \boldsymbol{N}$). $d_x^{(s)} G$ is a homogeneous polynomial of degree s of $\boldsymbol{C}^n$ into $\boldsymbol{C}^m$.

By the Cauchy inequalities

$$(\begin{smallmatrix}*&*\\ *&*\end{smallmatrix}) \qquad \|d_x^{(s)} G\| \leqslant s! \left[\frac{2}{\delta(x)}\right]^s \max\{\|G(x \oplus z)\|: \|z\| = \tfrac{1}{2}\delta(x)\}$$

$$\leqslant s! \left[\frac{2}{\delta(x)}\right]^{k+s} \|\delta^k G\|_\infty, \quad x \in X,\ s \in \boldsymbol{N}.$$

In particular,

$$\delta^{k+1} \left\|\frac{\partial G}{\partial p_j}\right\| \leqslant 2^{k+1} \|\delta^k G\|_\infty, \quad j = 1, \ldots, n. \tag{2.40}$$

Note that

$$G(x \oplus z) - G(x) = \int_0^1 (d_{x \oplus tz}^{(1)} G)(z)\, dt$$

$$= (d_x^{(1)} G)(z) + \int_0^1 (1 - t)(d_{x \oplus tz}^{(2)} G)(z)\, dt, \quad (x, z) \in X^* \boldsymbol{C}^n.$$

Hence in view of $(\begin{smallmatrix}*&*\\ *&*\end{smallmatrix})$ we get the following two very useful estimates:

$$\|G(x \oplus z) - G(x)\| \leqslant \left[\frac{4}{\delta(x)}\right]^{k+1} \|\delta^k G\|_\infty \|z\|, \quad \|z\| \leqslant \tfrac{1}{2}\delta(x). \tag{2.41}$$

$$\|G(x \oplus z) - G(x)\| \geqslant \|d_x G(z)\| - 2\left[\frac{4}{\delta(x)}\right]^{k+2} \|\delta^k G\|_\infty \|z\|^2, \quad \|z\| \leqslant \tfrac{1}{2}\delta(x). \tag{2.42}$$

2.43 ([12], the proof of Lemma 3, [14], 2.3). *Let X be a Stein domain, $\delta \in \mathscr{L}_{1,3}(X)$. Then, for every $k \geqslant 0$, $a \in X$, there exists a $u_a \in \mathcal{O}^{(k+4n)}(X, \delta)$ such that*

(i) $u_a(a) = 1$,

(ii) $u_a(x) = 0, \ x \in p^{-1}(p(x)), \ x \neq a$,

(iii) $\|\delta^{k+4n} u_a\|_X \leqslant c(n, k)\delta^{k-2n}(a)$, *where $c(n, k)$ depends only on n and k.*

In particular, $\mathcal{O}^{(4n)}(X, \delta)$ separates points in X.

2.44 ([8], Th. 4). *Let X be a Stein domain, $\delta \in \mathscr{L}_{1,3}(X)$. Then*

(a) *$\mathcal{O}(X, \delta)$ is dense in $\mathcal{O}(X)$ in the topology of uniform convergence on compact subsets of X;*

(b) *For every $k > 6n$ there exists an $f \in \mathcal{O}^{(k)}(X, \delta)$ such that X is the maximal domain of existence of f.*

In particular, in view of 2.31 and 2.32 we get

2.45. *For every Riemann domain X and for every $\delta \in \mathscr{W}(X)$:*

(a) *If $\mathcal{O}(X)$ separates points then so does $\mathcal{O}^{(4n)}(X, \delta)$;*

(b) *$\mathcal{O}(X, \delta)$ is dense in $\mathcal{O}(X)$.*

2.46 ([14], Th. 2). *Let X be a Stein domain, $\delta \in \mathscr{W}_r(X)$. Then $S_b(X, \delta) = E(X, \delta)$. If, moreover, X is finitely sheeted then $S(X, \delta) = S_b(X, \delta) = E(X, \delta)$.*

(The author does not know any example of an infinitely sheeted Stein domain and $\delta \in \mathscr{W}(X)$ for which $E(X, \delta) \nsubseteq S(X, \delta)$.)

§3. Holomorphic continuation and holomorphic retractions

Let X be a complex analytic space countable at infinity, let M be a closed subspace of X and let $R = R_M^X$ denote the restriction operator

$$\mathcal{O}(X) \ni f \to f|_M \in \mathcal{O}(M).$$

We shall always assume that every connected component of X intersects M.

Let $\delta\colon X \to (0, +\infty)$ be a fixed bounded function such that $1/\delta$ is locally bounded.

We shall study the following problem of holomorphic continuation. Given X, M, δ, find when

(C) $\mathcal{O}(M, \delta) = R_M^X(\mathcal{O}(X, \delta))$, i.e., $\forall f \in \mathcal{O}(M, \delta) \ \exists \hat{f} \in \mathcal{O}(X, \delta)\colon \hat{f}|_M = f$.

The basic problem (C) has many stronger versions – for example:

(C_d) (Continuation with controlled degree)

$$\forall k \ \exists \hat{k}\colon \ \mathcal{O}^{(k)}(M, \delta) \subset R(\mathcal{O}^{(\hat{k})}(X, \delta)).$$

($C_{d,n}$) (Continuation with controlled degree and norm)

$$\forall k \ \exists \hat{k}, \exists b > 0\colon \ \mathcal{O}^{(k)}(M, \delta) \subset R(\mathcal{O}^{(\hat{k})}(X, \delta))$$

and

$$\forall f \in \mathcal{O}^{(k)}(M, \delta) \ \exists \hat{f} \in \mathcal{O}^{(\hat{k})}(X, \delta)\colon \hat{f}|_M = f, \quad \|\delta^{\hat{k}} \hat{f}\|_\infty \leqslant b \|\delta^k f\|_\infty .$$

($C_{d,n,l}$) (Linear continuation with controlled degree and norm)

$\forall k \ \exists \hat{k}, \exists L_k\colon \ \mathcal{O}^{(k)}(M, \delta) \to \mathcal{O}^{(\hat{k})}(X, \delta)\colon \ L_k$ is a linear continuous extension operator.

(L) (Linear continuation with uniform estimate of degree and geometrical estimate of norm)

$\exists \sigma \geqslant 0\colon \ \forall \eta > 1 \ \exists c = c(\eta)\colon$

$\forall k \ \exists L_k\colon \ \mathcal{O}^{(k)}(M, \delta) \to \mathcal{O}^{(k+\sigma)}(X, \delta)\colon \ L_k$ is a linear continuous extension operator with $\|L_k\| \leqslant c\eta^k$.

(L_0) (A limit version of (L))

$\exists c > 0\colon \ \forall k \ \exists L_k\colon \ \mathcal{O}^{(k)}(M, \delta) \to \mathcal{O}^{(k)}(X, \delta)\colon \ L_k$ is a linear continuous extension operator with $\|L_k\| \leqslant c$.

(H) (Homomorphic continuation)

$\exists T\colon \ \mathcal{O}(M, \delta) \to \mathcal{O}(X, \delta)\colon \ T$ is a homomorphic extension operator.

(H_b) (Bounded homomorphic continuation)

$\exists T\colon \ \mathcal{O}(M, \delta) \to \mathcal{O}(X, \delta)\colon \ T$ is a bounded homomorphic extension operator.

It can be seen that:

$$\begin{array}{ccccccccccc} (L_0) & \Rightarrow & (L) & \Rightarrow & (C_{d,n,l}) & \Rightarrow & (C_{d,n}) & \Rightarrow & (C_d) & \Rightarrow & (C) \\ & & & & \Uparrow & & & & & & \Uparrow \\ & & & & H_b & & \Longrightarrow & & & & H \end{array}$$

Below we shall show (cf. 3.5, 3.9 and 4.5) that if X is a finitely sheeted Stein domain over C^n (e.g., X is a domain of holomorphy in C^n), $\delta \in \mathcal{W}_r(X)$ and M is an analytic subset of X determined by functions from $\mathcal{O}(X, \delta)$ then:

$$\begin{array}{ccccccccccc} (L_0) & \Rightarrow & (L) & \Rightarrow & (C_{d,n,l}) & \Leftrightarrow & (C_{d,n}) & \Leftrightarrow & (C_d) & \Leftrightarrow & (C) \\ & & \Uparrow & & & & & & & & \\ (H) & \Leftrightarrow & (H_b) & & & & & & & & \end{array}$$

Let us start with a few general remarks relating to bounded linear operators between algebras of δ-tempered functions.

Let X_1, X_2 be two complex analytic spaces countable at infinity and let $\delta_j\colon X_j \to (0, +\infty)$ be a bounded function such that $1/\delta_j$ is locally bounded, $j = 1, 2$. Let $L\colon \mathcal{O}(X_1, \delta_1) \to \mathcal{O}(X_2, \delta_2)$ be a linear operator.

LEMMA 3.1. *The following conditions are equivalent:*
(i) *L is bounded.*
(ii) *For every $k_1 \geqslant 0$ and for every $x_2 \in X_2$ the operator*

$$\mathcal{O}^{(k_1)}(X_1, \delta_1) \ni f \to (Lf)(x_2) \in C$$

is continuous.

Proof (the method of the proof is taken from [20]). The implication (i)$\Rightarrow$(ii) is obvious. For the proof of (ii)$\Rightarrow$(i), let us fix $k_1 \geqslant 0$ and observe that it is enough to prove that there exist $r_1, r_2 > 0$ and $k_2 \geqslant 0$ such that:

$$(*) \qquad [f \in \mathcal{O}^{(k_1)}(X_1, \delta_1), \|\delta_1^{k_1} f\|_\infty \leqslant r_1] \Rightarrow [\|\delta_2^{k_2}(Lf)\|_\infty \leqslant r_2].$$

For $\lambda, \mu, \nu \in N$ let

$$C_{\lambda,\mu,\nu} := \{f \in \mathcal{O}^{(k_1)}(X_1, \delta_1) \colon \|\delta_1^{k_1} f\|_\infty \leqslant \lambda, \|\delta_2^{\mu}(Lf)\|_\infty \leqslant \nu\}.$$

The set $C_{\lambda,\mu,\nu}$ is absolutely convex and

$$\bigcup_{\lambda,\mu,\nu \in N} C_{\lambda,\mu,\nu} = \mathcal{O}^{(k_1)}(X_1, \delta_1).$$

The space $\mathcal{O}^{(k_1)}(X_1, \delta_1)$ has the Baire property, whence for some λ_0, μ_0, $\nu_0 \in N$, int $\mathrm{cl}(C_{\lambda_0,\mu_0,\nu_0}) \neq \emptyset$ (the interior and the closure are taken in the sense of $\mathcal{O}^{(k_1)}(X_1, \delta_1)$). In consequence there exists an $r_1 > 0$ such that

$$B_{r_1} := \{f \in \mathcal{O}^{(k_1)}(X_1, \delta_1) \colon \|\delta_1^{k_1} f\|_\infty \leqslant r_1\} \subset \mathrm{cl}(C_{\lambda_0,\mu_0,\nu_0}).$$

We shall prove that

$$\|\delta_2^{\mu_0}(Lf)\|_\infty \leqslant 2\nu_0, \quad f \in B_{r_1} \text{ (cf. } (*)).$$

Fix $f_0 \in B_{r_1}$. Since

$$\mathrm{cl}(C_{\lambda_0,\mu_0,\nu_0}) \subset C_{\lambda_0,\mu_0,\nu_0} + \tfrac{1}{2} B_{r_1},$$

there exist sequences $(g_s)_{s=0}^{\infty} \subset C_{\lambda_0,\mu_0,\nu_0}$ and $(f_s)_{s=1}^{\infty} \subset B_{r_1}$ such that

$$f_s = g_s + \tfrac{1}{2} f_{s+1}, \quad s \geqslant 0.$$

Hence

$$f_0 = \Big(\sum_{s=0}^{t} 2^{-s} g_s\Big) + 2^{-t-1} f_{t+1}, \quad t \geqslant 1.$$

The sequence $(f_t)_{t=0}^{\infty}$ is bounded in $\mathcal{O}^{(k_1)}(X_1, \delta_1)$, and so the series $\sum_{s=0}^{\infty} 2^{-s} g_s$ is convergent in $\mathcal{O}^{(k_1)}(X_1, \delta_1)$ to f_0.

On the other hand, since $g_s \in C_{\lambda_0,\mu_0,\nu_0}$, $s \geqslant 0$, the series $\sum_{s=0}^{\infty} 2^{-s} Lg_s$ is absolutely convergent in $\mathcal{O}^{(\mu_0)}(X_2, \delta_2)$ to an element with the norm $\leqslant 2\nu_0$.

In view of (ii)

$$Lf_0 = \sum_{s=0}^{\infty} 2^{-s} Lg_s,$$

which finishes the proof.

LEMMA 3.2. *Assume that L is bounded. Then the following conditions are equivalent*:

(i) L *is surjective.*

(ii) $\forall k_2 \ \exists k_1$: $\mathcal{O}^{(k_2)}(X_2, \delta_2) \subset L(\mathcal{O}^{(k_1)}(X_1, \delta_1))$.

(iii) $\forall k_2 \ \exists k_1, \exists b > 0$: $\mathcal{O}^{(k_2)}(X_2, \delta_2) \subset L(\mathcal{O}^{(k_1)}(X_1, \delta_1))$ *and*

$$\forall f_2 \in \mathcal{O}^{(k_2)}(X_2, \delta_2) \ \exists f_1 \in \mathcal{O}^{(k_1)}(X_1, \delta_1): \ Lf_1 = f_2, \ \|\delta_1^{k_1} f_1\|_\infty \leqslant b \|\delta_2^{k_2} f_2\|_\infty.$$

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. The proof of (i) $\Rightarrow$ (iii) is analogous with the proof of Lemma 3.1. Fix k_2. It suffices to prove that there exist $r_1, r_2 > 0$, $k_1 \geqslant 0$ such that

$$B_{r_2} := \{f_2 \in \mathcal{O}^{(k_2)}(X_2, \delta_2): \|\delta_2^{k_2} f_2\|_\infty \leqslant r_2\} \subset L(B_{r_1}),$$

where

$$B_{r_1} := \{f_1 \in \mathcal{O}^{(k_1)}(X_1, \delta_1): \|\delta_1^{k_1} f_1\|_\infty \leqslant r_1\}.$$

Let

$$C_{\lambda,\mu,\nu} = \{f_2 \in \mathcal{O}^{(k_2)}(X_2, \delta_2): \|\delta_2^{k_2} f_2\|_\infty \leqslant \lambda, \exists f_1 \in \mathcal{O}^{(\mu)}(X_1, \delta_1): \\ Lf_1 = f_2, \ \|\delta_1^{\mu} f_1\|_\infty \leqslant \nu\}.$$

There exist λ_0, μ_0, ν_0 and $r_2 > 0$ such that

$$B_{r_2} \subset \mathrm{cl}(C_{\lambda_0,\mu_0,\nu_0}) \quad \text{(cf. the proof of Lemma 3.1).}$$

Fix $f_0 \in B_{r_2}$. As in Lemma 3.1, one can construct a sequence $(g_s)_{s=0}^{\infty} \subset C_{\lambda_0,\mu_0,\nu_0}$ such that

$$f_0 = \sum_{s=0}^{\infty} 2^{-s} g_s,$$

where the series is convergent in $\mathcal{O}^{(k_2)}(X_2, \delta_2)$.

In view of the definition of $C_{\lambda_0,\mu_0,\nu_0}$ there exists a sequence $(h_s)_{s=0}^{\infty} \subset \mathcal{O}^{(\mu_0)}(X_1, \delta_1)$ such that

$$Lh_s = g_s, \quad s \geqslant 0$$

and the series $\sum_{s=0}^{\infty} 2^{-s} h_s$ is absolutely convergent in $\mathcal{O}^{(\mu_0)}(X_1, \delta_1)$ to an element $\hat{f}_0$ with the norm $\leqslant 2\nu_0$. Since L is bounded, we get

$$L\hat{f}_0 = f_0,$$

which gives the required result with $r_1 = 2\nu_0$, $k_1 = \mu_0$.

COROLLARY 3.3. *The conditions* (C), (C_d) *and* ($C_{d,n}$) *are equivalent.*

Proof. The result is a consequence of Lemma 3.2 with $X_1 = X$, $X_2 = M$, $\delta_1 = \delta$, $\delta_2 = \delta|_M$, $L = R^X_M$.

We shall need the following lemma on the solvability of a system of linear equations.

LEMMA 3.4. *Let E be a normed space, H a Hilbert space, D a linear subspace of H, and let T_α: $D \to E$, $\alpha \in A$, be a family of linear operators such that*

$$S := \{v \in D: \ T_\alpha v = 0, \ \alpha \in A\}$$

is a closed subspace of H.

Assume that

$$\exists c > 0: \ \forall u \in E \ \exists v \in D: \ \forall \alpha \in A: \ T_\alpha v = u \ \text{and} \ \|v\|_H \leqslant c\|u\|_E.$$

Then there exists a linear continuous operator

$$L: \ E \to D$$

such that

$$T_\alpha \circ L = \mathrm{id}, \quad \alpha \in A, \quad \text{and} \quad \|L\| \leqslant 2c.$$

Proof (cf. the proof of Lemma 1 in [11]). Let P: $H \to S$ denote the orthogonal projection. We can put

$$Lu := v - Pv, \quad u \in E,$$

where $v = v(u)$ is an element of D such that $T_\alpha v = u$, $\alpha \in A$. ∎

COROLLARY 3.5. *Let X be a Riemann domain, $\delta \in \mathscr{L}_2(X)$. Then the conditions* (C), (C_d), ($C_{d,n}$) *and* ($C_{d,n,l}$) *are equivalent.*

Proof. In view of Corol. 3.3, it is enough to prove that ($C_{d,n}$) $\Rightarrow$ ($C_{d,n,l}$). Fix k and let $\hat{k}$ and b be as in ($C_{d,n}$). In view of 2.35 (e),

$$\mathcal{O}^{(\hat{k})}(X, \delta) \subset H^{(\hat{k}+\alpha_0)}(X, \delta)$$

and

$$\|\delta^{\hat{k}+\alpha_0} g\|_2 \leqslant \|\delta^{\alpha_0}\|_2 \|\delta^{\hat{k}} g\|_{\mathscr{X}}, \quad g \in \mathcal{O}^{(\hat{k})}(X, \delta).$$

Let $E := \mathcal{O}^{(k)}(M, \delta)$, $H := H^{(\hat{k}+\alpha_0)}(X, \delta)$, $D := \{g \in H: \ g|_M \in E\}$, $T_\alpha = R^X_M$. Note that the set $S = \{g \in H: \ g|_M = 0\}$ is closed (2.35 (c)). Hence, by Lemma 3.4, there exists a linear continuous operator

$$L_k: \ E \to D$$

such that $R \circ L_k = \mathrm{id}$ and $\|L_k\| \leqslant 2b\|\delta^{\alpha_0}\|_2$.

According to 2.37, L_k may be regarded as an operator into $\mathcal{O}^{(\hat{k}+\alpha_0+n)}(X, \delta)$ with the norm $\leqslant 2b\|\delta^{\alpha_0}\|_2 c(n, \hat{k}+\alpha_0)$. ∎

LEMMA 3.6. *Assume that (X_1, p) is a Stein domain over C^n, $\delta_1 \in \mathcal{W}_r(X_1)$. Then for every bounded algebra homomorphism*

$$T\colon \mathcal{O}(X_1, \delta_1) \to \mathcal{O}(X_2, \delta_2)$$

with $T1 \equiv 1$ there exists exactly one holomorphic mapping

$$\varphi\colon X_2 \to X_1$$

such that

$$(**) \qquad \delta_2^\gamma \leqslant c\delta_1 \circ \varphi \ (\gamma, c > 0 \text{ constants}) \quad \text{and} \quad T = \varphi^*|_{\mathcal{O}(X_1, \delta_1)}.$$

Moreover, if X_1 is finitely sheeted then the same is true for every algebra homomorphism T with $T1 \equiv 1$ (in consequence T has to be bounded).

Note that, in view of 2.5, the above result gives a full characterization of all bounded homomorphisms T with $T1 \equiv 1$.

Proof (the proof is based on the methods of [12]). Fix an algebra homomorphism (resp. bounded algebra homomorphism) T with $T1 \equiv 1$. For $x_2 \in X_2$ let $\xi\colon \mathcal{O}(X_1, \delta_1) \to C$ be given by the formula

$$\xi f = (Tf)(x_2), \quad f \in \mathcal{O}(X_1, \delta_1).$$

Then $\xi \in S(X, \delta)$ (resp. $\xi \in S_b(X, \delta)$). By 2.46, there exists exactly one point $x_1 \in X_1$ such that

$$\xi f = f(x_1), \quad f \in \mathcal{O}(X_1, \delta_1).$$

Put $\varphi(x_2) := x_1$. Then $\varphi\colon X_2 \to X_1$ and $T = \varphi^*|_{\mathcal{O}(X_1, \delta_1)}$. In particular T satisfies condition (ii) of Lemma 3.1, and so T is bounded. It remains to prove $(**)$ and show that φ is holomorphic.

Due to 2.43, there exists a family $(v_a)_{a \in X_1} \subset \mathcal{O}^{(6n+1)}(X_1, \delta_1)$ such that

$$\delta_1(a) v_a(a) = 1,$$

$$\|\delta_1^{6n+1} v_a\|_\infty \leqslant c(n), \quad a \in X_1,$$

where $c(n)$ depends only on n.

Since T is bounded, there exist γ, $c > 0$ such that

$$\|\delta_2^\gamma (v_a \circ \varphi)\|_\infty \leqslant c, \quad a \in X_1.$$

In particular,

$$\delta_2^\gamma(x_2) \left| v_{\varphi(x_2)}\big(\varphi(x_2)\big) \right| \leqslant c, \quad x_2 \in X_2,$$

which proves $(**)$.

We pass to the proof that φ is holomorphic. Note that

$$f \circ \varphi \in \mathcal{O}(X_2), \quad f \in \mathcal{O}(X_1, \delta_1);$$

in particular, $p\circ\varphi\in[\mathcal{O}(X_2)]^n$, and so it suffices to show that φ is continuous.

Let $X_2\ni x_s\to x_0\in X_2$, $y_s:=\varphi(x_s)$, $z_s:=p(y_s)$, $s\geqslant 0$. Observe that $z_s\to z_0$. $1/\delta_2$ is locally bounded, and so there exists an $\varepsilon>0$ such that

$$\delta_2(x_s)\geqslant\varepsilon,\qquad s\geqslant 0;$$

thus

$$\delta_1(y_s)\geqslant\frac{1}{c}\delta_2^{\gamma}(x_s)\geqslant\frac{1}{c}\varepsilon^{\gamma}=:2r,\qquad s\geqslant 0.$$

We may assume that $\|z_s-z_0\|<r$, $s\geqslant 1$. Then there exists a sequence $(w_s)_{s=1}^{\infty}\subset p^{-1}(z_0)$ such that $w_s\in\hat{B}(y_s,r)$, $s\geqslant 1$.

Now we have two possibilities:

(a) There exists an s_0 such that $w_s=y_0$, $s\geqslant s_0$. In this case $y_s\in\hat{B}(y_0)$, $s\geqslant s_0$, and so $y_s\to y_0$.

(b) There exists a subsequence $(w_{s_t})_{t=1}^{\infty}$ such that $w_{s_t}\neq y_0$, $t\geqslant 1$. We may assume that this subsequence coincides with the initial sequence $(w_s)_{s=1}^{\infty}$. In this case, in view of 2.43, there exists a function $f_0\in\mathcal{C}^{(4n)}(X_1,\delta_1)$ such that $f_0(w_s)=0$, $s\geqslant 1$, $f_0(y_0)=1$. The function $f_0\circ\varphi$ is holomorphic; in particular, $f_0(y_s)\to f_0(y_0)=1$. On the other hand, in view of (2.41),

$$|f_0(y_s)|=|f_0(y_s)-f_0(w_s)|\leqslant\left[\frac{4}{\delta_1(y_s)}\right]^{4n+1}\|\delta_1^{4n}f_0\|_{\infty}\|z_s-z_0\|$$

$$\leqslant(2/r)^{4n+1}\|\delta_1^{4n}f_0\|_{\infty}\|z_s-z_0\|\to 0\qquad\text{as}\qquad s\to+\infty,$$

which gives a contradiction. ∎

Remark 3.7. If X_1 is not Stein then the assertion of Lemma 3.6 need not be true. For example:

Let X, $\hat{X}$, φ, δ, $\hat{\delta}$ be as in 2.31. Assume that $\delta\in\mathcal{W}(X)$ and $\varphi(X)\neq\hat{X}$. Then $(\varphi^*|_{\mathcal{O}(\hat{X},\hat{\delta})})^{-1}$ is a bounded algebra isomorphism of $\mathcal{O}(X,\delta)$ onto $\mathcal{O}(\hat{X},\hat{\delta})$ which is not given by any mapping of $\hat{X}$ into X.

COROLLARY 3.8. *Let X_1, X_2 be Stein domains, $\delta_j\in\mathcal{W}_r(X_j)$, $j=1,2$. Then for every bounded algebra isomorphism T: $\mathcal{O}(X_1,\delta_1)\to\mathcal{O}(X_2,\delta_2)$ there exists a biholomorphic mapping φ: $X_2\to X_1$ which satisfies (**) (of Lemma 3.6) and is such that $T=\varphi^*|_{\mathcal{O}(X_1,\delta_1)}$.*

If X_1, X_2 are finitely sheeted (e.g., *X_1, X_2 are domains of holomorphy in $\mathbb{C}^n$*) *then the same is true for every algebra isomorphism of $\mathcal{O}(X_1,\delta_1)$ onto $\mathcal{O}(X_2,\delta_2)$.*

COROLLARY 3.9 (A characterization of solvability of (H) and ($\mathrm{H_b}$)). *Let X be a Stein domain, $\delta\in\mathcal{W}_r(X)$ and suppose that $M=\bigcap_{f\in F}f^{-1}(0)$, where F is a family of functions from $\mathcal{O}(X,\delta)$ (we always assume that every connected component of X intersects M). Then for every bounded homomorphic extension* *operator* $T\colon \mathcal{O}(M, \delta) \to \mathcal{O}(X, \delta)$ *there exists exactly one holomorphic retraction* $\pi\colon X \to M$ *such that*

$$({}_*{}^*{}_*) \qquad \delta^\gamma \leqslant c\delta \circ \pi \ (\gamma, c > 0 \text{ constants}) \quad \text{and} \quad T = \pi^*|_{\mathcal{O}(M,\delta)}.$$

If X is finitely sheeted then the same is true for every homomorphic extension operator T (which, in consequence, has to be bounded).

Proof. The operator $T \circ R\colon \mathcal{O}(X, \delta) \to \mathcal{O}(X, \delta)$ is a bounded algebra homomorphism (resp. algebra homomorphism) and $(T \circ R)(1) \equiv 1$. Hence, by Lemma 3.6, there exists exactly one holomorphic mapping $\pi\colon X \to X$ such that $({}_*{}^*{}_*)$ holds true and $T \circ R = \pi^*|_{\mathcal{O}(X,\delta)}$. In particular, $f \circ \pi = T(f|_M) = T(0) = 0$, $f \in F$, and so $\pi\colon X \to M$. If $f \in \mathcal{O}(M, \delta)$ then we have $f \circ \pi = (Tf) \circ \pi = (T \circ R)(Tf) = Tf$. Hence $T = \pi^*|_{\mathcal{O}(M,\delta)}$. The space $\mathcal{O}(M, \delta)$ separates points and thus π is a retraction. ∎

Remark 3.10. Let X be a complex manifold and let M be an analytic subset of X. It is known that if $\pi\colon X \to M$ is a holomorphic retraction then M has to be a submanifold (if X is disconnected then connected components of M may have different dimensions). In particular, under the assumptions of Corol. 3.9, if the triple (X, M, δ) satisfies $(\mathrm{H_b})$ then M has to be a submanifold.

The existence of global holomorphic retractions $\pi\colon X \to M$ with $({}_*{}^*{}_*)$ as in Corol. 3.9 seems to be very rare (even without the condition $({}_*{}^*{}_*)$). On the other hand, it is known that if M is a submanifold of X then there exists a neighbourhood U of M and a holomorphic retraction $\pi\colon U \to M$. In the case of problem (L) this leads to the following idea (a different approach was presented, for instance, in [13], [21], [22]):

LEMMA 3.11. *Let X be Riemann domain over* $\mathbb{C}^n$, *let* $\delta \in \mathcal{L}_2(X)$. *Suppose that there exists a* $\sigma_0 \geqslant 0$ *such that for every* $0 < \varepsilon < 1$ *there exists an open neighbourhood U of M, a holomorphic retraction* $\pi\colon U \to M$ *and a constant* $c > 0$ *such that*

(i) $\forall x \in U\colon\ x \in \hat{B}(\pi(x), \varepsilon\delta(\pi(x)))$,

(ii) $\forall l\ \forall f \in H^{(l)}(U, \delta)\ \exists \hat{f} \in H^{(l+\sigma_0)}(X, \delta)$:

$$\hat{f}|_M = f \quad \text{and} \quad \|\delta^{l+\sigma_0}\hat{f}\|_2 \leqslant c\|\delta^l f\|_2.$$

Then the triple (X, M, δ) *satisfies* (L) *with* $\sigma = \sigma_0 + \alpha_0 + n$.

Proof. Fix $\eta > 1$ and let $0 < \varepsilon < 1$ be fixed in such a way that $\frac{1+\varepsilon}{1-\varepsilon} \leqslant \eta$. Let U, π, c be associated with ε according to the assumptions.

In view of 2.35 (e), the operator

$$L_k^{(1)}\colon \mathcal{O}^{(k)}(M, \delta) \ni f \to f \circ \pi \in H^{(k+\alpha_0)}(U, \delta)$$

is well-defined, linear, continuous and has the norm $\leqslant \|\delta^{\alpha_0}\|_2 (1+\varepsilon)^k$.

In view of (ii), by Lemma 3.4 (cf. the proof of Corol. 3.5 and 2.37) there exists a linear continuous operator

$$L^{(2)}_{k+\alpha_0}: H^{(k+\alpha_0)}(U, \delta) \to \mathcal{O}^{(k+\sigma)}(X, \delta)$$

such that

$$R^X_M \circ L^{(2)}_{k+\alpha_0} = R^U_M \quad \text{and} \quad \|L^{(2)}_{k+\alpha_0}\| \leqslant 2c\,[\tau_n \varepsilon^n (1-\varepsilon)^{k+\sigma_0+\alpha_0}]^{-1}.$$

Set

$$L_k := L^{(2)}_{k+\alpha_0} \circ L^{(1)}_k.$$

Then L_k: $\mathcal{O}^{(k)}(M, \delta) \to \mathcal{O}^{(k+\sigma)}(X, \delta)$ is a linear continuous extension operator and

$$\|L_k\| \leqslant 2c\,[\tau_n \varepsilon^n (1-\varepsilon)^{\sigma_0+\alpha_0}]^{-1} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^k \leqslant c'\eta^k,$$

where c' is independent of k. ∎

For sufficiently regular (X, M, δ) a class of neighbourhoods U satisfying condition (ii) of Lemma 3.11 will be characterized in § 4 (Th. 4.3). The problem of existence of holomorphic retractions π: $U \to M$ (as in condition (i)) will be studied in § 6 (Lemma 6.2).

Problem (L) is simpler in the case where M is a graph – more precisely:

3.12. Let (Y, q) be a Riemann domain over C^{n-m} and let $F = (F_1, \ldots, F_m) \in [\mathcal{O}^{(\alpha)}(Y, \delta_Y)]^m$ $(\alpha \geqslant 1)$. Put

$$M := \{(y, F(y)): y \in Y\} \subset Y \times C^m,$$

M is the graph of F. Let $X \in \operatorname{top}(Y \times C^m)$, $p := q \times \mathrm{id}_{C^m}$. Assume that $M \subset X$. We shall shortly say that M is a graph in X.

Put $G_j(y, z) = z_j - F_j(y)$, $(y, z) \in X$, $j = 1, \ldots, m$. Obviously $M = \{G = 0\}$. One can easily check that $\delta_X(y, z) \leqslant \delta_Y(y)$, $(y, z) \in X$. Hence for every $\delta \in \mathscr{W}(X)$: $G := (G_1, \ldots, G_m) \in [\mathcal{O}^{(\alpha)}(X, \delta)]^m$.

Let $(\hat{X}, \hat{p})$ be the envelope of holomorphy of (X, p) and let φ: $X \to \hat{X}$ denote the embedding.

Fix $\delta \in \mathscr{W}(X)$ and let $\hat{\delta}$ denote its psh. regularization. In view of 2.31, there exist $\hat{G}_1, \ldots, \hat{G}_m \in \mathcal{O}^{(\alpha)}(\hat{X}, \hat{\delta})$ such that $\hat{G}_j \circ \varphi = G_j$, $j = 1, \ldots, m$. Put $\hat{M} := \{\hat{G} = 0\}$. Clearly $\varphi(M) \subset \hat{M}$. Observe that

$$\frac{\partial G_j}{\partial z_k} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases} \qquad j, k = 1, \ldots, m,$$

whence

$$\frac{\partial \hat{G}_j}{\partial \hat{p}_{n-m+k}} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

In particular, $\operatorname{rank}(d_{\hat{x}} \hat{G}) = m$, $\hat{x} \in \hat{X}$, which implies that $\hat{M}$ is an $(n-m)$-dimensional submanifold of $\hat{X}$.

We shall prove (cf. 5.8) that if $\hat{\delta} \in \mathscr{W}_r(\hat{X})$ (e.g., $N(\hat{X}) < +\infty$, cf. 2.21) then $(\hat{X}, \hat{M}, \hat{\delta})$ always satisfies (L). Consequently we get:

3.13. *If* $\hat{\delta} \in \mathcal{W}_r(\hat{X})$ *then*

$$(X, M, \delta) \text{ satisfies } (C) \Leftrightarrow \varphi^*\big(\mathcal{O}(\hat{M}, \hat{\delta})\big) = \mathcal{O}(M, \delta).$$

Remark 3.14. Note that there exist very regular graphs for which condition (C) is not fulfilled. For example: $n = 3$, $m = 2$, $Y = C$, $F_1 = F_2 = 0$, $X = C^3 \setminus A$, where $A = \{z_2 = e^{-z_1}, z_3 = 0\}$. Clearly $M = \{z_2 = z_3 = 0\} \subset X$. Observe that $\operatorname{codim} A = 2$, so $\hat{X} = C^3$, $\varphi = \mathrm{id}_X$. Set $\delta = \delta_X$. Obviously $\hat{M} = M$ and $\hat{\delta} \in \mathcal{W}_r(C^3)$. One can easily prove (on the analogy of 2.33) that $e^{z_1} \in \mathcal{O}(M, \delta) \setminus \mathcal{O}(\hat{M}, \hat{\delta})$. Hence, in view of 3.13, (X, M, δ) does not satisfy (C).

In the case $X = C^n$ one of the most important problems of the interpolation theory is to characterize those analytic subsets M of C^n for which $E_\mu(M) = R_M^{C^n}(E_\mu(C^n))$ (cf. 2.17), where $\mu \geq 0$ is a fixed number.

The same problem may be formulated in the general case – we say that an analytic subset M of a Riemann domain X is an *interpolation set for* $E_\mu(X)$ if $E_\mu(M) = R_M^X(E_\mu(X))$.

In the case $X = C$ the interpolation sets are completely characterized by the following theorem:

3.15 ([19]). Let $M = \{z_s\colon s \in N\} \subset C$, $|z_s| \nearrow +\infty$. Fix $\mu \geq 0$. Then the following conditions are equivalent:

(i) *M is an interpolation set for $E_\mu(C)$.*

(ii) (a) *For every $\tau > \mu$ the series $\sum_{s=1}^{\infty} |z_s|^{-\tau}$ is convergent and*

(b) $1/K' \in E_\mu(M)$, *where K is the Weierstrass canonical product for* $(z_s)_{s=1}^{\infty}$ (cf. [18], p. 220).

(iii) *There exists a $G \in E_\mu(C)$ such that*

(a′) $M = G^{-1}(0)$ *and*

(b′) $\forall \tau > \mu\ \exists b \geq 1\colon |G'(z)| \geq e^{-b(1+|z|^\tau)}$, $z \in M$.

Note that the implication (ii) $\Rightarrow$ (iii) is obvious (we can take $G := K$). The implication (iii) $\Rightarrow$ (ii) is a consequence of standard properties of entire functions of order $\leq \mu$ (cf. [18], pp. 218, 224).

In the case $X = C^n$, $n \geq 2$, the situation is more complicated: generally speaking, the implication (iii) $\Rightarrow$ (i) remains true but (i) $\Rightarrow$ (iii) does not, namely:

3.16 ([22]). Let M be an $(n-1)$-dimensional submanifold of C^n. Fix $\mu \geq 0$. Suppose that there exists a $G \in E_\mu(C^n)$ such that

$$M = G^{-1}(0),$$

and there exist $\tau > \mu$, $b \geq 1$:

(i) $\|d_x G\| \geq e^{-b(1+\|x\|^\tau)}$, $x \in M$.

Then

(ii) for every $m \geq 1$ there exists an $\hat{m} \geq 1$ such that every function $f \in \mathcal{O}(M)$ satisfying

$$|f(x)| \leq e^{m(1+\|x\|^\tau)}, \qquad x \in M,$$

admits an extension to an entire function $\hat{f}$ such that

$$|\hat{f}(x)| \leqslant e^{\hat{m}(1+\|x\|^{\tau})}, \qquad x \in C^n.$$

In particular, $\mathcal{O}(M, \delta_\tau) = R(\mathcal{O}(C^n, \delta_\tau))$. Consequently, if for every $\tau > \mu$ there exists a $b \geqslant 1$ for which (i) is satisfied, then M is an interpolation set for $E_\mu(C^n)$ (this corresponds to the implication (iii) $\Rightarrow$ (i) in 3.15).

On the other hand, there exists a $G \in E_0(C^2)$ such that for every $\tau > 0$ condition (ii) is fulfilled but there are no $\tau > 0$ for which (i) is true ([22], § 4).

Theorem 3.16 was generalized to the case of Stein domains in [13] (cf. also Corol. 5.10).

We shall end this section with a few remarks relating to problem (L_0) in the case where $\dim M = 0$.

LEMMA 3.17. *Let X be a Riemann domain, let $M = \{x_s\colon s \in N\} \subset X$ be a zero-dimensional subset and let $\delta\colon X \to (0, 1]$ be such that $1/\delta$ is locally bounded. Then the following conditions are equivalent*:

(i) *(X, M, δ) satisfies* (L_0) *with a constant c.*

(ii) *For every $k \geqslant 0$ there exists a sequence $(h_t)_{t\in N} \subset \mathcal{O}^{(k)}(X, \delta)$ such that*

(a) $h_t(x_s) = \begin{cases} 1 & \textit{if } s = t, \\ 0 & \textit{if } s \neq t, \end{cases}$

(b) $\delta^k \sum_{t=1}^{\infty} \delta^{-k}(x_t)|h_t| \leqslant c.$

Proof. (ii) $\Rightarrow$ (i); define $L_k f = \sum_{t=1}^{\infty} f(x_t) h_t$, $f \in \mathcal{O}^{(k)}(M, \delta)$. In view of (b)

$$\delta^k \sum_{t=1}^{\infty} |f(x_t) h_t| \leqslant \|\delta^k f\|_\infty \delta^k \sum_{t=1}^{\infty} \delta^{-k}(x_t)|h_t| \leqslant c\|\delta^k f\|_\infty.$$

Consequently, L_k is a well-defined linear continuous operator of $\mathcal{O}^{(k)}(M, \delta)$ into $\mathcal{O}^{(k)}(X, \delta)$. In view of (a), L_k is an extension operator.

(i) $\Rightarrow$ (ii); let $f_t \in H^\infty(M)$ be given by the formula

$$f_t(x_s) = \begin{cases} 1 & \text{if} \quad s = t, \\ 0 & \text{if} \quad s \neq t. \end{cases}$$

Put $h_t := L_k(f_t)$, $t \in N$ (L_k is as in (L_0)). Then obviously (a) is fulfilled, and so it remains to verify (b).

Fix $x^0 \in X$ and let $\Theta_t \in R$ be such that $e^{i\Theta_t} h_t(x^0) = |h_t(x^0)|$, $t \in N$. Let $g_m := \sum_{t=1}^{m} \delta^{-k}(x_t) e^{i\Theta_t} f_t$, $m \in N$. Then $g_m \in \mathcal{O}^{(k)}(M, \delta)$ and $\|\delta^k g_m\|_\infty \leqslant 1$, $m \in N$. Hence we get

$$c \geqslant \|\delta^k (L_k g_m)\|_\infty \geqslant \delta^k(x^0)\,|(L_k g_m)(x^0)|$$

$$\geqslant \delta^k(x^0)\left|\sum_{t=1}^{m} \delta^{-k}(x_t)\, e^{i\Theta_t} h_t(x^0)\right|$$

$$= \delta^k(x^0) \sum_{t=1}^{m} \delta^{-k}(x_t)\, |h_t(x^0)|, \qquad m \in \boldsymbol{N}. \ \blacksquare$$

Let us consider the simplest case where $X = \boldsymbol{D} := \{z \in \boldsymbol{C}\colon |z| < 1\}$, $M = \{z_s\colon s \in N\} \subset \boldsymbol{D}$. We shall say that M is a *universal interpolation sequence* (u.i.s.) if $H^\infty(M) = R_M^{\boldsymbol{D}}(H^\infty(\boldsymbol{D}))$ (cf. [5]).

3.18 ([5], Thms. 1, 2).

(i) M is a u.i.s. iff there exists a $b > 0$ such that

$$(+) \qquad r_s := \prod_{\substack{t=1\\ t\neq s}}^{\infty} \left|\frac{z_s - z_t}{1 - \bar{z}_t z_s}\right| \geqslant b, \qquad s \in \boldsymbol{N}.$$

(ii) If

$$(++) \qquad \limsup_{s\to+\infty} \frac{1 - |z_{s+1}|}{1 - |z_s|} < 1,$$

then M is a u.i.s.; if $z_s \in (0, 1)$, $s \in N$, and $z_s \nearrow 1$ then $(++)$ is also necessary for M to be a u.i.s.

(iii) Assume that there exist $\lambda < 1$, $c > 0$ such that

$$(+++) \qquad \prod_{\substack{t=1\\ t\neq s}}^{\infty} \left[1 - \left(1 - \left|\frac{z_s - z_t}{1 - \bar{z}_t z_s}\right|\right)^{\lambda}\right] \geqslant c, \qquad s \in \boldsymbol{N}.$$

Then M is a u.i.s. and, moreover, there exist $c_1, c_2 > 0$, $f_t \in \mathcal{O}(\boldsymbol{D})$, $t \in \boldsymbol{N}$, such that

$$|f_t(z_t)| \geqslant c_1, \qquad t \in \boldsymbol{N},$$

$$\sum_{t=1}^{\infty} |f_t(z)| \leqslant c_2, \qquad z \in \boldsymbol{D}.$$

(iv) If $(++)$ is satisfied then $(+++)$ holds true with $\lambda = 1/2$.

Proposition 3.19. *Assume that $(+++)$ is satisfied. Let $\psi \in \mathcal{O}(\boldsymbol{D})$ be such that*

$$0 \notin \psi(\boldsymbol{D}),$$

$$\delta(z) := \psi(|z|) \in (0, 1], \qquad z \in \boldsymbol{D},$$

$$\psi(|z|) \leqslant |\psi(z)|, \qquad z \in \boldsymbol{D}.$$

Then $(\boldsymbol{D}, M, \delta)$ satisfies (L_0).

Proof (the concept of the proof is taken from [5]). Define

$$P_s(z) = \prod_{\substack{t=1\\ t\neq s}}^{\infty} e^{i\Theta_t} \frac{z - z_t}{1 - \bar{z}_t z}, \qquad z \in \boldsymbol{D},\ s \in \boldsymbol{N},$$

where $e^{i\theta_t} z_t = -|z_t|$, $t \in N$. In view of $(+)$, $P_s \in \mathcal{O}(D)$, $|P_s| \leqslant 1$, $|P_s(z_s)| = r_s$ and $P_s(z_t) = 0$, $t \neq s$. Let c_1, c_2, f_t, $t \in N$, be as in 3.18 (iii). Fix $k \geqslant 0$ and put

$$g_t(z) := \frac{f_t(z)\,P_t(z)}{[\psi(-e^{i\theta_t} z)]^k}, \qquad z \in D, t \in N.$$

Obviously $g_t \in \mathcal{O}(D)$, $|g_t(z)| \leqslant \dfrac{|f_t(z)|}{|\psi(-e^{i\theta_t} z)|^k} \leqslant \delta^{-k}(z)\,|f_t(z)|$, $z \in D$, $g_t(z_s) = 0$, $s \neq t$, and

$$|g_t(z_t)| = \frac{|f_t(z_t)|\, r_t}{\psi(|z_t|)} \geqslant c_1\, b\delta^{-k}(z_t), \qquad t \in N.$$

Let $h_t := \dfrac{g_t}{g_t(z_t)}$, $t \in N$. It is easily seen that the sequence $(h_t)_{t=1}^{\infty}$ satisfies condition (ii) of Lemma 3.17 (with $c = c_2/c_1 b$). Thus the result is a direct consequence of Lemma 3.17. ∎

COROLLARY 3.20. *Assume that* $(+++)$ *is fulfilled. Then*

(i) (D, M, δ_D) *satisfies* (L_0).

(ii) *For every* $\tau > 0$: $(D, M, \delta_{D,\psi_\tau})$ *satisfies* (L_0) (cf. 2.17).

(iii) *For every* $\mu \geqslant 0$: M *is an interpolation set for* $E_\mu(D)$.

Proof. (i) is a consequence of Prop. 3.19 with $\psi(z) := 1-z$, $z \in D$. (ii) follows from 3.19 with $\psi(z) = \exp[-c(\tau) - e^{-\tau \operatorname{Log}(1-z)}]$, $z \in D$. Finally (iii) is a direct consequence of (ii).

§4. Continuation from regular neighbourhoods

Let X be a Riemann domain over C^n countable at infinity, $\delta \in \mathscr{L}(X)$, $F = (F_1, \ldots, F_N) \in [\mathcal{O}^{(\alpha)}(X, \delta)]^N$, $U \in \operatorname{top} X$. Assume that $F \not\equiv 0$ on any connected component of U.

Define

$$\Delta_{r,s}^0 = \{u \in L^2_{(r,s)}(U, \text{loc}) \colon \bar{\partial} u \in L^2_{(r,s+1)}(U, \text{loc})\},$$

$$\Delta_{r,s}^t = \{u = (u_I)_I \colon u_I \in \Delta_{r,s}^0,\ I = (i_1, \ldots, i_t),\ 1 \leqslant i_1, \ldots, i_t \leqslant N,$$
$$\text{the system } (u_I)_I \text{ is skew-symmetric with respect to } I\}.$$

For $u = (u_I)_I \in \Delta_{r,s}^t$ we put $\|u\| = (\sum_I |u_I|^2)^{1/2}$. Let

$$\bar{\partial} \colon \Delta_{r,s}^t \to \Delta_{r,s+1}^t$$

and

$$P \colon \Delta_{r,s}^{t+1} \to \Delta_{r,s}^t$$

be defined by the formulae

$$(\bar{\partial}u)_I := \bar{\partial}u_I,$$

$$(Pu)_{(i_1,\ldots,i_t)} := \sum_{j=1}^{N} F_j u_{(i_1,\ldots,i_t,j)}.$$

Additionally, let $P\colon \Delta^0_{r,s} \to 0$ be defined as the zero operator.

Using the same methods as in [6], [17], one can prove the following general version of Nullstellensatz for holomorphic functions with restricted growth on Riemann domains.

THEOREM 4.1. *Assume that U is Stein, $-\log\delta \in \mathrm{PSH}(U)$ and $\delta \leqslant \delta_0 \circ p$ on U. Then, for every $k \geqslant 0$, $0 \leqslant r, s \leqslant n$, $0 \leqslant t \leqslant N-1$, there exists a constant $c > 0$ (depending only on n, N, r, s, t, $\varkappa, \|\delta^\varkappa F\|_\infty$) such that for every $u_0 \in \Delta^t_{r,s}$ with $Pu_0 = 0$, $\bar{\partial}u_0 = 0$,*

$$J = \int_U \frac{\|u_0\|^2}{\|F\|^{2(2\mu+1)}} \delta^{2k}\, d\lambda < +\infty, \qquad \mu = \min\{n-s, N-t-1\},$$

there exists a $u \in \Delta^{t+1}_{r,s}$ such that $Pu = u_0$, $\bar{\partial}u = 0$ and

$$\int_U \|u\|^2 \delta^{2[k+\mu(2\varkappa+3)]}\, d\lambda \leqslant cJ.$$

In the proof of Th. 4.1 all L^2-estimates may be deduced from the following generalization of Hörmander's theorem (cf. [8], Th. 2).

4.2. Let X be a Stein domain, let $\delta\colon X \to (0, 1]$ be such that $-\log\delta \in \mathrm{PSH}(X)$ and $\delta \leqslant \delta_0 \circ p$ (e.g., $\delta \in \mathscr{L}_{1,3}(X)$). Then, for every $k \geqslant 0$, $0 \leqslant r$, $s \leqslant n$ and for every $\bar{\partial}$-closed form $u \in L^2_{(r,s+1)}(X, \mathrm{loc})$, there exists a $v \in L^2_{(r,s)}(X, \mathrm{loc})$ such that $\bar{\partial}v = u$ and

$$\int_X |v|^2 \delta^{2(k+2)}\, d\lambda \leqslant \int_X |u|^2 \delta^{2k}\, d\lambda.$$

Now we are able to present some examples of neighbourhoods U satisfying condition (ii) of Lemma 3.11.

THEOREM 4.3. *Let X be a Stein domain over $\mathbf{C}^n$, $\delta \in \mathscr{L}_{1,3}(X)$ and let M be an analytic subset of X for which there exists a $G \in [\mathcal{O}^{(\alpha)}(X, \delta)]^m$ such that $M \subset G^{-1}(0)$. Fix $\theta > 0$, $\gamma \geqslant 0$ and let $U = U(G, \theta, \gamma, M)$ denote the sum of all connected components of the set $V = V(G, \theta, \gamma) := \{\|G\| < \theta\delta^\gamma\}$ which intersect M (U will be called a regular neighbourhood of M). Then there exists a constant $c > 0$ (depending only on n, m, $\alpha, \|\delta^\alpha G\|_\infty$, θ, γ) such that*

$$\forall\, l \geqslant 0\ \forall f \in H^{(l)}(U, \delta)\ \exists \hat{f} \in H^{(l+\sigma_0)}(X, \delta)\colon\ \hat{f}|_M = f|_M \text{ and}$$

$$\|\delta^{l+\sigma_0} \hat{f}\|_2 \leqslant c\|\delta^l f\|_2,$$

where $\sigma_0 = q(2\alpha+2\gamma+3)$, $q := \min\{n, m\}$.

Proof (cf. [11], Th. 1). Let us fix a function $\psi \in C_0^\infty(C^m, [0, 1])$ such that $\psi = 1$ on $\bar{B}(0, 1/3)$ and $\operatorname{supp}\psi \subset \bar{B}(0, 2/3)$. Define $\chi = \psi\left(\frac{1}{\theta\delta^\gamma} G\right)$. Note that χ is locally Lipschitz (cf. 2.19), $\chi = 1$ on $\{\|G\| \leqslant \frac{1}{3}\theta\delta^\gamma\}$ and

$$\operatorname{supp}\chi \subset \{\|G\| \leqslant \tfrac{2}{3}\delta^\gamma\} \subset V.$$

Consequently $\bar{\partial}\chi \in L^2_{(0,1)}(X, \mathrm{loc})$, $\operatorname{supp}(\bar{\partial}\chi) \subset \{\frac{1}{3}\theta\delta^\gamma \leqslant \|G\| \leqslant \frac{2}{3}\theta\delta^\gamma\}$ and, in view of (2.40),

$$\delta^{\alpha+\gamma+1} |\bar{\partial}\chi| \leqslant c_1 = c_1(n, m, \alpha, \|\delta^\alpha G\|_\infty, \theta, \gamma).$$

Let us fix $f \in H^{(l)}(U, \delta)$. It suffices to consider the case where $\|\delta^l f\|_2 = 1$. Put

$$u_0 := \begin{cases} f\bar{\partial}\chi & \text{in} \quad U, \\ 0 & \text{in} \quad X\setminus U. \end{cases}$$

Clearly, $u_0 \in L^2_{(0,1)}(X, \mathrm{loc})$, $\bar{\partial}u_0 = 0$ and $\operatorname{supp}(u_0) \subset U \cap \operatorname{supp}(\bar{\partial}\chi)$. Hence

$$J = \int\limits_X \frac{|u_0|^2}{\|G\|^{2(2q-1)}} \delta^{2(l+\alpha+2q\gamma+1)} d\lambda \leqslant c_2 = c_2(n, m, \theta, \gamma, c_1).$$

According to Th. 4.1 (with $N = m$, $F = G$, $U = X$, $r = 0$, $s = 1$, $t = 0$, $k = l + \alpha + 2q\gamma + 1$) there exist $\bar{\partial}$-closed forms $u_1, \ldots, u_m \in L^2_{(0,1)}(X, \mathrm{loc})$ such that

$$u_0 = u_1 G_1 + \ldots + u_m G_m$$

and

$$\int\limits_X |u_j|^2 \delta^{2(l+\sigma_0-\alpha-2)} d\lambda \leqslant c_3 = c_3(n, m, \alpha, \|\delta^\alpha G\|_\infty, c_2), \quad j = 1, \ldots, m.$$

In view of 4.2, there exist functions $v_1, \ldots, v_m \in L^2(X, \mathrm{loc})$ such that $\bar{\partial}v_j = u_j$ and

$$\int\limits_X |v_j|^2 \delta^{2(l+\sigma_0-\alpha)} d\lambda \leqslant c_3, \quad j = 1, \ldots, m.$$

Put

$$\hat{f} := \begin{cases} f\chi - (v_1 G_1 + \ldots + v_m G_m) & \text{in} \quad U, \\ -(v_1 G_1 + \ldots + v_m G_m) & \text{in} \quad X\setminus U. \end{cases}$$

It can be seen that $\hat{f} \in L^2(X, \mathrm{loc})$, $\bar{\partial}\hat{f} = 0$ (so $\hat{f} \in \mathcal{O}(X)$), $\hat{f} = f$ on M and

$$\|\delta^{l+\sigma_0} \hat{f}\|_2 \leqslant c = c(m, \alpha, \|\delta^\alpha G\|_\infty, c_3). \ \blacksquare$$

COROLLARY 4.4 (a generalization of Th. 4 from [12]). *Let X be a Stein domain over C^n, $\delta \in \mathcal{W}_r(X)$ ($\delta^{\alpha_0} \in L^2(X)$) and let M be an analytic submanifold of X. Suppose that there exists a holomorphic retraction π: $X \to M$ such that $p \circ \pi \in [\mathcal{O}^{(\alpha)}(X, \delta)]^n$ ($\alpha \geqslant 1$). Then (X, M, δ) satisfies* (L) *with $\sigma = \alpha_0 + 2n\alpha + 6n$.*

Proof. Put $G := p - p\circ\pi$. Clearly, $G \in [\mathcal{O}^{(\alpha)}(X, \delta)]^n$ and $M \subset G^{-1}(0)$. Note that

$$x \in U\left(G, \frac{1}{1+\varepsilon}, 1, M\right) \Rightarrow x \in \hat{B}\big(\pi(x), \varepsilon\delta(\pi(x))\big).$$

Now it can be seen that the result follows from Lemma 3.11 and Th. 4.3.

COROLLARY 4.5. *Let X be a Stein domain, $\delta \in \mathcal{W}_r(X)$ and let M be an analytic submanifold of X determined by functions from $\mathcal{O}(X, \delta)$* (cf. Corol. 3.9). *If (X, M, δ) satisfies* $(\mathrm{H_b})$ *then it satisfies* (L).

Proof. The result is a consequence of Corol. 3.9 and Corol. 4.4.

COROLLARY 4.6 (a generalization of Corol. 1 from [11]). *Let M be a graph as in* 3.12. *Assume that X is Stein, $\delta \in \mathcal{W}_r(X)$ $(\delta^{\alpha_0} \in L^2(X))$. Then (X, M, δ) satisfies* (L) *with $\sigma = \alpha_0 + 2n\alpha + 6n$* (cf. 3.14).

Proof. The mapping $X \ni (y, z) \xrightarrow{\pi} (y, F(y)) \in M$ is a global holomorphic retraction such that $p\circ\pi \in [\mathcal{O}^{(\alpha)}(X, \delta)]^n$. Hence, the result is a consequence of Corol. 4.4.

Remark 4.7. Notice that, under the assumptions of Corol. 4.6, π^* need not map $\mathcal{O}(M, \delta)$ into $\mathcal{O}(X, \delta)$ (see [11]).

§ 5. Continuation from δ-regular submanifolds; Main Theorem

For $1 \leqslant r \leqslant m$ let $\mathcal{J}_r^m$ denote the set of all indices $I = (i_1, \ldots, i_r)$ such that $1 \leqslant i_1 < \ldots < i_r \leqslant m$.

Let $A = [a_{ij}]$ be an $(m \times n)$-dimensional matrix (with complex entries). For $I \in \mathcal{J}_r^m$, $J \in \mathcal{J}_r^n$ set $A_{I,J} := [a_{i_\mu, j_\nu}]_{\mu,\nu = 1,\ldots,r}$. Put

$$\Delta_r(A) := \|\underbrace{A \wedge \ldots \wedge A}_{r\,\text{times}}\| = \Big(\sum_{I,J}{}' |\det(A_{I,J})|^2\Big)^{1/2},$$

where the sum is taken over all $I \in \mathcal{J}_r^m$, $J \in \mathcal{J}_r^n$.

Throughout this section M will be an analytic submanifold of a Riemann domain over C^n countable at infinity. We denote by $M_1, \ldots, M_s$ purely dimensional components of M, $M = M_1 \cup \ldots \cup M_s$. Let $d_j := \dim M_j$, $r_j := n - d_j$, $j = 1, \ldots, s$, and assume that $0 \leqslant d_1 < \ldots < d_s \leqslant n-1$.

DEFINITION 5.1. Let $\delta\colon X \to (0, 1]$; we shall say that M is a *δ-regular submanifold of X* if there exist $m \in N$, $\alpha \geqslant 0$, $b > 0$, $\beta \geqslant 0$ and $G \in [\mathcal{O}^{(\alpha)}(X, \delta)]^m$ such that

(a) $M \subset G^{-1}(0)$,

(b) $\operatorname{rank}(d_x G) = r_j$, $x \in M_j$, $j = 1, \ldots, s$,

(c) $\Delta_{r_j}(d_x G) \geqslant b\delta^\beta(x)$, $x \in M_j$, $j = 1, \ldots, s$.

PROPOSITION 5.2. *Let M be an algebraic submanifold of C^n. Then M is δ_0-regular. In consequence, for every δ: $C^n \to (0, 1]$ with $\delta \leqslant \delta_0$, M is δ-regular.*

Proof. It is known that there exist polynomials $G_1, \ldots, G_m$ such that

$$M = G^{-1}(0) \quad (\text{where } G := (G_1, \ldots, G_m))$$

and

$$\operatorname{Ker}(d_x G) = T_x M \ (= \text{the tangent space at } x), \quad x \in M.$$

Thus it remains to verify condition (c) of Def. 5.1.

Let $G_{j,1}, \ldots, G_{j,m_j}$ be polynomials such that

$$M_j = \{G_{j,1} = \ldots = G_{j,m_j} = 0\}, \quad j = 1, \ldots, s.$$

Fix j $(1 \leqslant j \leqslant s)$. The polynomials

$$G_{j,1}, \ldots, G_{j,m_j}, \quad \det(d_x G)_{I,J}, \quad I \in \mathscr{I}^m_{r_j}, \quad J \in \mathscr{I}^n_{r_j},$$

have no common zeros in C^n; hence, by the Hilbert Nullstellensatz, there exist polynomials $P_{j,1}, \ldots, P_{j,m_j}, Q_{j,I,J}$, $I \in \mathscr{I}^m_{r_j}$, $J \in \mathscr{I}^n_{r_j}$, such that

$$\sum_{\iota=1}^{m_j} P_{j,\iota} G_{j,\iota} + \sum_{I,J}{}' Q_{j,I,J} \det(d_x G)_{I,J} = 1.$$

Set $\beta := \max\{\deg Q_{j,I,J} \colon I \in \mathscr{I}^m_{r_j}, J \in \mathscr{I}^n_{r_j}, j = 1, \ldots, s\}$ and let $b > 0$ be such that

$$\|\delta_0^\beta Q_j\|_\infty \leqslant 1/b, \quad \text{where } Q_j := (Q_{j,I,J})_{I,J}, \ j = 1, \ldots, s.$$

Then, for $x \in M_j$, we get

$$1 = \sum_{I,J}{}' Q_{j,I,J}(x) \det(d_x G)_{I,J} \leqslant \|Q_j(x)\| \Delta_{r_j}(d_x G) \leqslant [b\delta_0^\beta(x)]^{-1} \Delta_{r_j}(d_x G). \ \blacksquare$$

PROPOSITION 5.3. *Let X be a bounded domain of holomorphy in C^n such that $\bar{X}$ has a fundamental system of neighbourhoods which are domains of holomorphy. Let N be an analytic submanifold of an open neighbourhood U of $\bar{X}$. Put $M := N \cap X$. Then M is 1-regular. Consequently, for every δ: $X \to (0, 1]$, M is δ-regular.*

Proof. We may assume that U is a domain of holomorphy. Let $\mathscr{V}$ denote the sheaf of ideals of the subvariety N. $\mathscr{V}$ is a coherent sheaf (cf. [4], p. 138, Th. 2). This implies that there exist an open neighbourhood U_0 of $\bar{X}$ and functions $G_1, \ldots, G_m \in \mathcal{O}(U_0)$ such that, for every $x \in U_0$, the germs $(G_1)_x, \ldots, (G_m)_x$ generate $\mathscr{V}_x$ (cf. [4], p. 244, Th. 17). Put $G = (G_1, \ldots, G_m)$. It is clear that $N \cap U_0 = G^{-1}(0)$ and $\operatorname{Ker}(d_x G) = T_x N$, $x \in N \cap U_0$. Consequently, since $\bar{X}$ is compact, we get the required result. $\blacksquare$

PROPOSITION 5.4. *Let M be a graph as in 3.12. Then $\hat{M}$ is $\hat{\delta}$-regular (in $\hat{X}$).*

Proof (the notation is the same as in 3.12). For the proof we only need to observe that $\Delta_{n_l}(d_{\hat{x}}\hat{G}) \geqslant 1$, $\hat{x} \in \hat{X}$.

PROPOSITION 5.5. *Let V_j be a δ_j-regular submanifold of X_j, where X_j is a Riemann domain over C^{n_j}, $j = 1, \ldots, t$. Put*

$$\delta(x_1, \ldots, x_t) = \min\{\delta_1(x_1), \ldots, \delta_t(x_t)\}, \quad (x_1, \ldots, x_t) \in X_1 \times \ldots \times X_t.$$

Then $V_1 \times \ldots \times V_t$ is a δ-regular submanifold of $X_1 \times \ldots \times X_t$.

Proof. It suffices to consider the case where $t = 2$. Let $m_j, \alpha_j, b_j, \beta_j, G_j$ be associated with V_j according to Def. 5.1, $j = 1, 2$. Put $n = n_1 + n_2$, $m = m_1 + m_2$, $\alpha = \max\{\alpha_1, \alpha_2\}$, $b = b_1 b_2$, $\beta = \beta_1 + \beta_2$ and define

$$G(x_1, x_2) = (G_1(x_1), G_2(x_2)), \quad (x_1, x_2) \in X_1 \times X_2.$$

Clearly, $G \in [\mathcal{O}^{(\alpha)}(X_1 \times X_2, \delta)]^m$ and $V_1 \times V_2 \subset G^{-1}(0)$. Fix $(x_1^0, x_2^0) \in V_1 \times V_2$. Let $d = \dim_{(x_1^0, x_2^0)}(V_1 \times V_2)$, $r = n - d$, $d_j = \dim_{x_j^0} V_j$, $r_j = n_j - d_j$, $j = 1, 2$. Then

$$\operatorname{rank}(d_{(x_1^0, x_2^0)} G) = \operatorname{rank}(d_{x_1^0} G_1) + \operatorname{rank}(d_{x_2^0} G_2) = r_1 + r_2 = r$$

and

$$\Delta_r(d_{(x_1^0, x_2^0)} G) \geqslant \Delta_{r_1}(d_{x_1^0} G_1) \cdot \Delta_{r_2}(d_{x_2^0} G_2)$$
$$\geqslant b_1 \delta_1^{\beta_1}(x_1^0) b_2 \delta_2^{\beta_2}(x_2^0) \geqslant b \delta^{\beta}(x_1^0, x_2^0). \ \blacksquare$$

PROPOSITION 5.6. *Let $V_1, \ldots, V_t$ be disjoint δ-regular submanifolds of X and let $m_j, \alpha_j, b_j, \beta_j, G_j$ be associated with V_j according to* Def. 5.1, *$j = 1, \ldots, t$. Suppose that there exist constants $c > 0$, $\gamma \geqslant 0$ such that*

$$\|G_j(x)\| \geqslant c\delta^{\gamma}(x), \quad x \in V_1 \cup \ldots \cup V_{j-1} \cup V_{j+1} \cup \ldots \cup V_t, \quad j = 1, \ldots, t.$$

Then $V_1 \cup \ldots \cup V_t$ is a δ-regular submanifold of X.

Proof. We shall show that the submanifolds $V_1 \cup V_2, V_3, \ldots, V_t$ satisfy all the assumptions of Prop. 5.6 – this will permit us to apply the finite induction over t.

Let $G_j = (G_{j,1}, \ldots, G_{j,m_j})$, $j = 1, 2$, and let $c_0 \in (0, 1)$ be such that

$$\max\{|G_{j,\mu}(x)|: \mu = 1, \ldots, m_j\} \geqslant c_0 \delta^{\gamma}(x), \quad x \in (V_1 \cup \ldots \cup V_t) \setminus V_j, \quad j = 1, 2.$$

Put $m = m_1 m_2$, $\alpha = \alpha_1 + \alpha_2$, $b = c_0^n \min\{b_1, b_2\}$, $\beta = n\gamma + \max\{\beta_1, \beta_2\}$. Define $G\colon X \to C^m$, $G = (G_{\mu,\nu})_{\mu = 1, \ldots, m_1, \nu = 1, \ldots, m_2}$, by the formula

$$G_{\mu,\nu}(x) = G_{1,\mu}(x) G_{2,\nu}(x), \quad x \in X.$$

Obviously, $G \in [\mathcal{O}^{(\alpha)}(X, \delta)]^m$ and $V_1 \cup V_2 \subset G^{-1}(0)$. Fix $x^0 \in V_1 \cup V_2$. Let, for instance, $x^0 \in V_1$. Put $r = n - \dim_{x^0} V_1$. Fix $1 \leqslant \nu_0 \leqslant m_2$ in such a way that $|G_{2,\nu_0}(x^0)| \geqslant c_0 \delta^{\gamma}(x^0)$. Note that

$$\frac{\partial G_{\mu,\nu}}{\partial p_k}(x^0) = G_{2,\nu}(x^0) \frac{\partial G_{1,\mu}}{\partial p_k}(x^0).$$

Hence

$$\operatorname{rank}(d_{x^0} G) = \operatorname{rank}(d_{x^0} G_1) = r$$

and

$$\Delta_r(d_{x^0} G) \geqslant |G_{2,\nu_0}(x^0)|^r \Delta_r(d_{x^0} G_1) \geqslant [c_0 \delta^\gamma(x^0)]^r b_1 \delta^{\beta_1}(x^0) \geqslant b\delta^\beta(x^0);$$

thus $V_1 \cup V_2$ is δ-regular.

Now let $x^0 \in V_3 \cup \ldots \cup V_t$. There exist $1 \leqslant \mu_0 \leqslant m_1$, $1 \leqslant \nu_0 \leqslant m_2$ such that

$$|G_{1,\mu_0}(x^0)| \geqslant c_0 \delta^\gamma(x^0), \qquad |G_{2,\nu_0}(x^0)| \leqslant c_0 \delta^\gamma(x^0).$$

Consequently, $\|G(x^0)\| \geqslant |G_{1,\mu_0}(x^0)| |G_{2,\nu_0}(x^0)| \geqslant c_0^2 \delta^{2\gamma}(x^0)$. ∎

The main result of the paper is the following:

THEOREM 5.7. *Let X be a Stein domain over C^n, $\delta \in \mathscr{W}_r(X)$ ($\delta^{\alpha_0} \in L^2(X)$) and let M be a δ-regular submanifold of X. Let m, α, b, β and G be as in* Def. 5.1. *Then (X, M, δ) satisfies* (L) *with the constant σ of the form $\sigma = P\alpha_0 + Q\alpha + R\beta + S$, where P, Q, R, S depend polynomially on $n, m, d_1, \ldots, d_s$ (P, Q, R, S, as polynomials of the variables $n, m, d_1, \ldots, d_s$, are of degree $\leqslant 3$) and $P, Q, R, S \leqslant 96n^3$ (P, Q, R, S may be effectively calculated!).*

The proof of Th. 5.7 will be given in § 6. Th. 5.7 is a simultaneous generalization of some results of [1], [2], [11], [13], [21], [22]; more exactly:

COROLLARY 5.8 (a generalization of Corol. 4.6 and, in consequence, of the results of [1] and [11]). *Let M be a graph as in* 3.12. *If $\hat{\delta} \in \mathscr{W}_r(\hat{X})$ then $(\hat{X}, \hat{M}, \hat{\delta})$ satisfies* (L).

Proof. The result is a consequence of Prop. 5.4 and Th. 5.7.

COROLLARY 5.9 (a generalization of Corol. 2 from [2]). *Let X be a domain of holomorphy in C^n. Let $\delta \in \mathscr{W}_r^q(X)$* (cf. 2.34) *and let M be a δ-regular submanifold of X. Then $\mathcal{O}(M, \delta) = R_M^X(\mathcal{O}(X, \delta))$.*

Proof. The result follows from 2.34 and Th. 5.7.

COROLLARY 5.10 (a generalization of Th. 1 from [13] and, in consequence, of [22]). *Let X be a Stein domain, $\delta \in \mathscr{W}_r(X)$ and let M be an $(n-1)$-dimensional submanifold of X such that there exist $\alpha \geqslant 0$, $b > 0$, $\beta \geqslant 0$ and $G \in \mathcal{O}^{(\alpha)}(X, \delta)$ with $M \subset G^{-1}(0)$, $\|d_x G\| \geqslant b\delta^\beta(x)$, $x \in M$. Then (X, M, δ) satisfies* (L).

COROLLARY 5.11. *Let M be an algebraic submanifold of C^n. Then*

(i) (a generalization of Th. 7 from [21]) *for every $\delta \in \mathscr{W}_r(C^n)$ the triple (C^n, M, δ) satisfies* (L),

(ii) *M is an interpolation set for every $E_\mu(C^n)$.*

Proof. The result is a consequence of Prop. 5.2 and Th. 5.7.

Analogously, Prop. 5.3 and Th. 5.7 imply:

COROLLARY 5.12. *Let* X, M *be as in* Prop. 5.3. *Then*

(i) (a generalization of Th. 6 from [21]) *for every* $\delta \in \mathscr{W}_r(X)$ *the triple* (X, M, δ) *satisfies* (L),

(ii) M *is an interpolation set for every* $E_\mu(X)$.

The following result is a consequence of 3.15, Prop. 5.5 and Th. 5.7:

COROLLARY 5.13. *Let* V_j *be an interpolation set for* $E_\mu(\mathbf{C})$, $j = 1, \ldots, t$. *Then* $V_1 \times \ldots \times V_t$ *is an interpolation set for* $E_\mu(\mathbf{C}^n)$.

Remark 5.14. Let X be a Riemann (resp. Stein) domain. Note that if $G \in [\mathcal{O}(X)]^m$ is such that $M \subset G^{-1}(0)$ and $\operatorname{rank}(d_x G) = r_j$, $x \in M_j$, $j = 1, \ldots, s$, then, in view of 2.27, there exists a $\delta \in \mathscr{L}_{1,2,4}(X)$ (resp. $\delta \in \mathscr{L}_{1,2,3,4}(X)$) such that M is δ-regular.

COROLLARY 5.15. *Let* X *be a Stein domain and let* $G \in [\mathcal{O}(X)]^m$ *be such that* $M \subset G^{-1}(0)$ *and* $\operatorname{rank}(d_x G) = r_j$, $x \in M_j$, $j = 1, \ldots, s$. *Then for every locally bounded family* $F \subset \mathcal{O}(M)$ *there exists a locally bounded family* $\hat{F} \subset \mathcal{O}(X)$ *such that* $F = R_M^X(\hat{F})$.

Proof. In view of 2.28 and Remark 5.14, there exists a $\delta \in \mathscr{W}_r(X)$ such that M is δ-regular and $F \subset \{f \in \mathcal{O}^{(1)}(M, \delta) \colon \|\delta f\|_\infty \leqslant 1\}$. In virtue of Th. 5.7, (X, M, δ) satisfies (L). Fix $\eta = 2$ and let $c = c(2)$ and $(L_k)_{k \geqslant 0}$ be as in (L). Define $\hat{F} = L_1(F)$. Then $F = R(\hat{F})$ and $\|\delta^{\sigma+1} \hat{f}\|_x \leqslant 2c$, $\hat{f} \in \hat{F}$, and so, in view of 2.1 (e), $\hat{F}$ is locally bounded. ∎

§ 6. Holomorphic retractions and pseudoinverse matrices; Proof of Main Theorem

At first we shall show how to reduce the proof of Th. 5.7 to the case where M is purely dimensional.

LEMMA 6.1. *Under the assumptions of* Th. 5.7, *there exists a* $0 < \theta_0 < 1$ *such that the sets*

$$\bigcup_{x \in M_j} \hat{B}(x, \theta_0 \delta^{\gamma_0}(x)), \quad j = 1, \ldots, s,$$

are disjoint, where $\gamma_0 := r_1(\alpha + 1) + \beta + 1$.

Proof. Let $\theta_0 := 2^{-\gamma_0 - 1}\theta_1$, where $\theta_1 \in (0, \frac{1}{2})$. Fix $1 \leqslant j < k \leqslant s$, $x_1 \in M_j$, $x_2 \in M_k$ and suppose that

$$\hat{B}(x_1, \theta_0 \delta^{\gamma_0}(x_1)) \cap \hat{B}(x_2, \theta_0 \delta^{\gamma_0}(x_2)) \neq \emptyset.$$

Then $x_2 = x_1 \oplus z$ and $\|z\| < \theta_1 \delta^{\gamma_0}(x_1)$.

The functions $f_{I,J} := \det(d_x G)_{I,J}$, $I \in \mathscr{I}^m_{r_j}$, $J \in \mathscr{I}^n_{r_j}$, are of the class $\mathcal{O}^{(r_j(\alpha+1))}(X, \delta) \subset \mathcal{O}^{(r_1(\alpha+1))}(X, \delta)$ (cf. (2.40)) and, moreover, if $f := (f_{I,J})_{I,J}$ then $\|\delta^{r_1(\alpha+1)} f\|_\infty \leqslant c = c(n, m, d_1, \ldots, d_s, \alpha, \|\delta^\alpha G\|_\infty)$.

In view of (2.41):

$$\Delta_{r_j}(d_{x_2}G) = \|f(x_2)\| \geqslant \|f(x_1)\| - \|f(x_1 \oplus z) - f(x_1)\|$$
$$\geqslant \Delta_{r_j}(d_{x_1}G) - \left[\frac{4}{\delta(x_1)}\right]^{r_1(\alpha+1)+1} c\|z\| \geqslant \delta^{\beta}(x_1)\,[b - 4^{r_1(\alpha+1)+1} c\theta_1].$$

Hence if $\theta_1 < b\,[4^{r_1(\alpha+1)+1} c]^{-1}$ then $\Delta_{r_j}(d_{x_2}G) > 0$, and so $r_k = \operatorname{rank}(d_{x_2}G) \geqslant r_j$, which is a contradiction. ∎

In view of the above Lemma (and of Lemma 3.11 and Th. 4.3), for the proof of Th. 5.7 it suffices to prove the following two lemmas:

LEMMA 6.2 (on the existence of holomorphic retractions). *Let X be a Riemann domain over C^n, $\delta \in \mathscr{L}(X)$ and let M be an analytic submanifold of X of pure dimension $d \leqslant n-1$. Assume that $G \in [\mathcal{O}^{(\alpha)}(X, \delta)]^m$ is such that*

$$M \subset G^{-1}(0),$$
$$\operatorname{rank}(d_x G) = r = n-d, \qquad x \in M.$$

Let Q: $X \to C^{n\times m}$ be a matrix-valued function with entries in $\mathcal{O}^{(\tau)}(X, \delta)$ such that, for every $x \in M$, $Q(x)$ is pseudoinverse to $d_x G$, i.e., $(d_x G)\cdot Q(x)\cdot(d_x G) = d_x G$ *and* $Q(x)\cdot(d_x G)\cdot Q(x) = Q(x)$. *Then, for every $0 < \varepsilon_0 < 1$, $\gamma_0 \geqslant 1$, there exist $\theta > 0$ and a holomorphic retraction*

$$\pi\colon U \to M,$$

where $U = U(G, \theta, \gamma, M)$ (cf. Th. 4.3), $\gamma = \max\{\tau+\gamma_0, 2\alpha+3\tau+3\}$, *such that*

$$x \in U \Rightarrow x \in \hat{B}\big(\pi(x), \varepsilon_0 \delta^{\gamma_0}(\pi(x))\big).$$

LEMMA 6.3 (on the existence of pseudoinverse matrices). *Let X be a Stein domain over C^n, $\delta \in \mathscr{W}_r(X)$ ($\delta^{\alpha_0} \in L^2(X)$) and let M be a δ-regular submanifold of X of pure dimension $d \leqslant n-1$. Let m, α, b, β, G be as in* Def. 5.1. *Then there exist $\tau = \tau(n, m, d, \alpha_0, \alpha, \beta)$ and a matrix-valued function*

$$Q\colon X \to C^{n\times m}$$

with entries in $\mathcal{O}^{(\tau)}(X, \delta)$ such that, for every $x \in M$, $Q(x)$ is pseudoinverse to $d_x G$ (see also Prop. 6.20).

Proof of Lemma 6.2. The proof will be divided into six steps.

Step 1. Fix a, $t \geqslant 1$ such that $\|\delta^{\alpha} G\|_\infty \leqslant a$, $\|\delta^{\tau} Q\|_\infty \leqslant t$. Let N: $X \to C^{n\times n}$ be a matrix-valued function given by the formula

$$N(x) = Q(x)\cdot(d_x G), \qquad x \in X.$$

In view of (2.40), the entries of N belong to $\mathcal{O}^{(\alpha+\tau+1)}(X, \delta)$ and

(6.4) $$\|\delta^{\alpha+\tau+1}N\|_{\infty} \leqslant 2^{\alpha+1}\,at.$$

Note that $\operatorname{Ker} N(x) = \operatorname{Ker}(d_x G)$, $x \in M$, whence

(6.5) $$\operatorname{rank} N(x) = n-d, \qquad x \in M.$$

Observe that

(6.6) $$N(x)\cdot N(x) = N(x), \qquad x \in M.$$

Let S: $X \to C^{n\times n}$ be a matrix-valued function defined by the formula

(6.7) $$S(x) = I_n - N(x), \qquad x \in X.$$

In view of (6.6) we get

(6.8) $$\operatorname{Ker} S(x) = \operatorname{Im} N(x), \qquad x \in M,$$

and by (6.5),

(6.9) $$\operatorname{rank} S(x) = d, \qquad x \in M.$$

Put
$$Y := \{(x, z) \in M \times C^n\colon\ S(x)z = 0\},$$
and note that Y is a closed subset of $X \times C^n$.

Step 2. Y is an n-dimensional submanifold of $X \times C^n$.

Proof. The case $d = 0$ is trivial, and so assume that $d \geqslant 1$. The analyticity of Y is a local property, whence, without loss of generality, we may assume that X is an open neighbourhood of 0 in C^n and $M = \{(x_1, \ldots, x_n) \in X\colon\ x_{d+1} = \ldots = x_n = 0\}$.

Fix $x^0 \in M$ and let $S_0(x^0)$ denote a $(d \times n)$-dimensional submatrix of $S(x^0)$ such that rank $S_0(x^0) = d$ (cf. (6.9)). Let S_0: $X \to C^{d\times n}$ be a matrix-valued function such that $S_0(x)$ is constructed by deleting the same rows as in $S(x^0)$. Let U be an open neighbourhood of x^0 such that rank $S(x) = d$, $x \in U$. In particular,

(6.10) $$\operatorname{Ker} S_0(x) = \operatorname{Ker} S(x), \qquad x \in U \cap M.$$

Let ψ: $X \times C^n \to C^n$ be defined by the formula
$$\psi(x, z) = \big(x_{d+1}, \ldots, x_n, S_0(x)z\big).$$

In view of (6.10), $Y \cap (U \times C^n) = \{(x, z) \in U \times C^n\colon\ \psi(x, z) = 0\}$. Note that,

$$d_{(x,z)}\psi = \left[\begin{array}{c|c|c} \overbrace{\ 0\ }^{d} & \overbrace{\ I_{n-d}\ }^{n-d} & \overbrace{\ 0\ }^{n} \\ \hline & & S_0(x) \end{array}\right]\begin{array}{l}\}\,n-d\\ \}\,d\end{array}$$

$(x, z) \in X \times C^n$. Consequently, $\operatorname{rank} d_{(x,z)}\psi = n$, $(x, z) \in U \times C^n$, which proves that $Y \cap (U \times C^n)$ is an n-dimensional manifold.

Step 3. Put $c_1 = (4^{\alpha+3}at)^{-1}$, $\gamma_1 = \alpha+\tau+2$. Then

$$\|G(x\oplus z)\| \geqslant \frac{1}{2t}\delta^{\tau}(x)\|z\|, \qquad (x, z)\in Y,\ \|z\| \leqslant c_1\delta^{\gamma_1}(x). \tag{6.11}$$

Proof. Note that $c_1\delta^{\gamma_1}(x) < \frac{1}{2}\delta(x)$, so by (2.42),

$$\|G(x\oplus z)\| \geqslant \|(d_x G)z\| - 2\left[\frac{4}{\delta(x)}\right]^{\alpha+2} a\|z\|^2.$$

On the other hand, in view of (6.8),

$$\|z\| = \|N(x)z\| = \|Q(x)(d_x G)z\| \leqslant \|Q(x)\|\,\|(d_x G)z\| \leqslant \frac{t}{\delta^{\tau}(x)}\|(d_x G)z\|.$$

Finally,

$$\|G(x\oplus z)\| \geqslant \frac{1}{2t}\delta^{\tau}(x)\|z\|\left[2 - \frac{\|z\|}{c_1\delta^{\gamma_1}(x)}\right],$$

which proves the required estimate.

Step 4. Put $c_2 = (4^{3\alpha+\tau+5}a^2t^2)^{-1}$, $\gamma_2 = 2\alpha+2\tau+3$, $Y_0 = \{(x, z)\in Y: \|z\| < c_2\delta^{\gamma_2}(x)\}$. Note that $Y_0 \in \operatorname{top} Y$ and $Y_0 \subset X^* C^n$. We shall prove that the mapping

$$Y_0 \ni (x, z) \overset{\Phi}{\rightarrow} x\oplus z \in X$$

is injective.

Proof. Let us fix (x_1, z_1), $(x_2, z_2)\in Y_0$, $x_1 \neq x_2$, and suppose that $x_1\oplus z_1 = x_2\oplus z_2$. Let, for instance, $\delta(x_2) \leqslant \delta(x_1)$. Put $z_0 = z_1 - z_2$. Then $x_2 = x_1\oplus z_0$ (in particular $z_0 \neq 0$). Define

$$w_1 = [N(x_1) - N(x_2)]z_2, \qquad w_2 = N(x_1)(z_1 - z_2).$$

Note that $(x_1, w_2)\in Y$ and (in view of (6.6)) $w_1 - w_2 = z_0$.

Since $\|z_0\| < 2c_2\delta^{\gamma_2}(x_1) < \frac{1}{2}\delta(x_1)$ and $\|z_2\| < c_2\delta^{\gamma_2}(x_1)$, it follows that, by (2.41) and (6.4), we have

$$\begin{aligned}\|w_1\| &\leqslant \|N(x_1\oplus z_0) - N(x_1)\|\,\|z_2\| \\ &< \left[\frac{4}{\delta(x_1)}\right]^{\alpha+\tau+2} 2^{\alpha+1}at\|z_0\|c_2\delta^{\gamma_2}(x_1),\end{aligned} \tag{6.12}$$

whence, in view of the definition of c_2 and γ_2, we get

$$\|w_1\| < \tfrac{1}{2}\|z_0\|.$$

In consequence,

$$\tfrac{1}{2}\|z_0\| < \|w_2\| < \tfrac{3}{2}\|z_0\| < 3c_2\delta^{\gamma_2}(x_1) < c_1\delta^{\gamma_1}(x_1),$$

and so, according to (6.11),

$$\|G(x_1 \oplus w_2)\| \geqslant \frac{1}{2t}\delta^{\tau}(x_1)\|w_2\| > \frac{1}{4t}\delta^{\tau}(x_1)\|z_0\|. \tag{6.13}$$

On the other hand, since $x_1 \oplus w_2 = x_2 \oplus (-w_1)$, it follows by (2.41) and (6.12) that

$$\begin{aligned}\|G(x_1 \oplus w_2)\| &\leqslant \left[\frac{4}{\delta(x_2)}\right]^{\alpha+1} a\|w_1\| \\ &< \left[\frac{8}{\delta(x_1)}\right]^{\alpha+1} a\left[\frac{4}{\delta(x_1)}\right]^{\alpha+\tau+2} 2^{\alpha+1} atc_2\,\delta^{\gamma_2}(x_1)\|z_0\| \leqslant \frac{1}{4t}\delta^{\tau}(x_1)\|z_0\|,\end{aligned}$$

which contradicts (6.13).

Step 5. Φ is an injective holomorphic mapping of an n-dimensional analytic manifold Y_0 into a Riemann domain X. This implies that $U_0 := \Phi(Y_0)$ is an open neighbourhood of M and Φ is a biholomorphism of Y_0 onto U_0.

Let us define $\pi\colon U_0 \to M$ by the formula

$$\pi = (\text{projection onto } M) \circ \Phi^{-1}, \qquad \Phi(x \oplus z) = x.$$

It is clear that π is a holomorphic retraction.

Step 6. Fix $0 < \varepsilon_0 < 1$, $\gamma_0 \geqslant 1$ (cf. Lemma 6.2) and let

$$c_3 = \min\{\varepsilon_0, c_2/2\}, \qquad \gamma_3 = \max\{\gamma_0, \gamma_2\}, \qquad Y_1 = \{(x, z) \in Y_0\colon \|z\| < c_3\delta^{\gamma_3}(x)\}.$$

Obviously, $\bar{Y}_1 \subset Y_0$. Put $U_1 := \Phi(Y_1)$. It is clear that U_1 is an open neighbourhood of M, $\bar{U}_1 \subset U_0$ and, in view of the definition of π,

$$x \in U_1 \Rightarrow x \in \hat{B}\big(\pi(x), \varepsilon_0\delta^{\gamma_0}(\pi(x))\big).$$

Put $\gamma = \gamma_3 + \tau$, $\theta = \frac{1}{2t}(\frac{2}{3})^{\gamma} c_3$ and let $U := U(G, \theta, \gamma, M)$ (recall that U is the sum of all connected components of $\{\|G\| < \theta\delta^{\gamma}\}$ which intersect M).

It remains to show that $U \subset U_1$.

Proof. Suppose that $U \not\subset U_1$. Then $U \cap \partial U_1 \neq \emptyset$, and so (since $\partial U_1 \subset U_0$) there exists $(x, z) \in Y_0 \setminus Y_1$ such that $x \oplus z \in U$. Note that $c_3\delta^{\gamma_3} < c_1\delta^{\gamma_1}$, whence by (6.11)

$$\|G(x \oplus z)\| \geqslant \frac{1}{2t}\delta^{\tau}(x)\|z\|.$$

On the other hand, in view of the definition of U,

$$\|G(x \oplus z)\| < \theta\delta^{\gamma}(x \oplus z) \leqslant \theta(\tfrac{3}{2})^{\gamma}\delta^{\gamma}(x).$$

Hence $\|z\| < c_3 \delta^{\gamma_3}(x)$, which implies that $(x, z) \in Y_1$. We get a contradiction.
The proof of Lemma 6.2 is completed.

Proof of Lemma 6.3. We start with the following two auxiliary results:

PROPOSITION 6.14. *Let X be a Stein domain over C^n, $\delta \in \mathscr{W}_r(X)$ $(\delta^{\alpha_0} \in L^2(X))$, $G \in [\mathcal{O}^{(\alpha)}(X, \delta)]^m$, $F \in [\mathcal{O}^{(\bar{\alpha})}(X, \delta)]^N$, $\bar{b}, \bar{\theta} > 0$, $\bar{\beta}, \bar{\gamma} \geqslant 0$ and let M be an analytic subset of X such that $M \subset G^{-1}(0)$. Suppose that*

$$\|F\| \geqslant \bar{b}\delta^{\bar{\beta}} \quad on\ U = U(G, \bar{\theta}, \bar{\gamma}, M).$$

Then there exist $f_1, \ldots, f_N \in \mathcal{O}^{(\nu)}(X, \delta)$ such that

$$f_1 F_1 + \ldots + f_N F_N = 1 \quad on\ M,$$

where $\nu = \alpha_0 + \mu(2\bar{\alpha} + 3) + (2\mu + 1)\bar{\beta} + q(2\alpha + 2\bar{\gamma} + 3) + n$, $\mu = \min\{n, N-1\}$, $q = \min\{n, m\}$.

Proof. Since $-\log\delta \in \mathrm{PSH}(X)$, it follows that $\{\|G\| < \bar{\theta}\delta^{\bar{\gamma}}\}$ is a Stein domain and therefore U is also Stein. Observe that

$$J = \int\limits_U \frac{1}{\|F\|^{2(2\mu+1)}} \delta^{2k} d\lambda < +\infty,$$

where $k := \alpha_0 + (2\mu + 1)\bar{\beta}$. Hence, in virtue of Th. 4.1 (with $r = s = t = 0$, $u_0 = 1$), there exist $u_1, \ldots, u_N \in H^{(l)}(U, \delta)$, $l = k + \mu(2\bar{\alpha} + 3)$, such that

$$u_1 F_1 + \ldots + u_N F_N = 1 \quad \text{on}\ U.$$

Now, by Th. 4.3 (and 2.37), there exist $f_1, \ldots, f_N \in \mathcal{O}^{(\nu)}(X, \delta)$, $\nu = l + q(2\alpha + 2\bar{\gamma} + 3) + n$, such that

$$f_j = u_j \quad \text{on}\ M, \quad j = 1, \ldots, N. \ \blacksquare$$

A thorough analysis of the proof of Lemma 4.2 in [23] leads to the following

PROPOSITION 6.15 (an algebraical criterion of existence of pseudoinverse matrices). *Let P be a commutative ring with a unit element. Assume that $P = \bigcup_{k \geqslant 0} P_k$, where P_k, $k \geqslant 0$, are subgroups of P and $P_k P_{k'} \subset P_{k+k'}$, $k, k' \geqslant 0$. Let A be an $(m \times n)$-dimensional matrix with entries in P_k (k is fixed). Suppose that there exists $1 \leqslant r \leqslant m, n$ such that:*

(i) *there exists a system $(f_{I,J})_{I \in \mathscr{I}_r^m, J \in \mathscr{I}_r^n}$ of elements of P_ν with*

$$\sum_{I,J}' f_{I,J} \det(A_{I,J}) = 1,$$

(ii) *if $r < \min\{m, n\}$ then, for every $I \in \mathscr{I}_{r+1}^m$, $J \in \mathscr{I}_{r+1}^n$,* $\det(A_{I,J}) = 0$.

Then there exists an $(n\times m)$-dimensional matrix B with entries in $P_{2\nu+(2r-1)k}$ which is pseudoinverse to A, i.e., $ABA = A$ *and* $BAB = B$.

In view of the above two propositions, for the proof of Lemma 6.3 we only need to prove the following

LEMMA 6.16. *Let X be a Riemann domain over C^n, $\delta\in\mathscr{L}(X)$, let M be a δ-regular submanifold of X of pure dimension d and let m, α, b, β, G, be as in* Def. 5.1. *Then there exist $\bar{b}$, $\bar{\theta}>0$ (depending only on n, m, d, α, $\|\delta^\alpha G\|_\infty$, b, β) such that*

$$\Delta_r(d_x G)\geqslant \bar{b}\delta^\beta(x),\qquad x\in U=U(G,\bar{\theta},\bar{\gamma},M),$$

where $\bar{\gamma}=(3r-1)\alpha+3\beta+3r$.

Proof. The proof will be divided into six steps.

Step 1°. Let $b_0\in(0,1]$ be such that

$$\max\{|\det(d_x G)_{I,J}|:\ I\in\mathscr{I}_r^m,\ J\in\mathscr{I}_r^n\}>b_0\delta^\beta(x),\qquad x\in M.$$

Put $t:=\frac{1}{2}(\frac{2}{3})^\beta$ and let

$$M_{I,J}^j:=\{x\in M:\ |\det(d_x G)_{I,J}|>t^{j-1}b_0\delta^\beta(x)\},\qquad I\in\mathscr{I}_r^m,\ J\in\mathscr{I}_r^n,\ j=1,2,3.$$

Note that $\overline{M_{I,J}^j}\subset M_{I,J}^{j+1}$ and $M=\bigcup_{I,J}M_{I,J}^1$.

Step 2°. Fix $a\geqslant 1$ such that $\|\delta^\alpha G\|_\infty\leqslant a$. Put

$$b_1=b_0t[2r!4^{r(\alpha+1)+1}(2^{\alpha+1}a)^r]^{-1},\qquad \beta_1=r(\alpha+1)+\beta+1.$$

Then

$$(6.17)\qquad |\det(d_{x\oplus z}G)_{I,J}|>t^jb_0\delta^\beta(x\oplus z),\qquad x\in M_{I,J}^j,\ \|z\|\leqslant b_1\delta^{\beta_1}(x),\ j=1,2.$$

Proof (cf. the proof of Lemma 6.1). The function $f_{I,J}:=\det(d_xG)_{I,J}$ is of the class $\mathcal{O}^{(r(\alpha+1))}(X,\delta)$ and $\|\delta^{r(\alpha+1)}f_{I,J}\|_\infty\leqslant r!(2^{\alpha+1})^r$. Fix $x\in M_{I,J}^j$, $\|z\|\leqslant b_1\delta^{\beta_1}(x)$. In view of (2.41):

$$\begin{aligned}|\det(d_{x\oplus z}G)_{I,J}|&=|f_{I,J}(x\oplus z)|\geqslant|f_{I,J}(x)|-|f_{I,J}(x\oplus z)-f_{I,J}(x)|\\&>t^{j-1}b_0\delta^\beta(x)-\left[\frac{4}{\delta(x)}\right]^{r(\alpha+1)+1}r!(2^{\alpha+1}a)^r\|z\|\\&\geqslant\frac{b_0}{2}t^{j-1}\delta^\beta(x)\geqslant t^jb_0\delta^\beta(x\oplus z),\end{aligned}$$

which proves (6.17).

Observe that, in view of (6.17), for the proof of the lemma it suffices to construct $\bar{\theta}$ such that

$$U(G,\bar{\theta},\bar{\gamma},M)\subset\bigcup_{I,J}\ \bigcup_{x\in M_{I,J}^2}\hat{B}(x,b_1\delta^{\beta_1}(x))$$

(then $\bar{b}=b_0t^2$).

Step 3°. For $J=(j_1,\dots,j_r)\in\mathscr{J}_r^n$, let

$$E_J=\{z=(z_1,\dots,z_n)\in C^n\colon j\notin\{j_1,\dots,j_r\}\Rightarrow z_j=0\}.$$

Put $t_1=b_0t^2[2r!(2^{\alpha+1}a)^{r-1}]^{-1}$, $\tau_1=(r-1)(\alpha+1)+\beta$, $b_2=t_1(2\cdot 4^{\alpha+2}a)^{-1}$. Then

$$\|G(x\oplus z)\|\geqslant t_1\delta^{\tau_1}(x)\|z\|,\qquad (x,z)\in M_{I,J}^3\times E_J,\ \|z\|\leqslant b_2\delta^{\beta_1}(x). \tag{6.18}$$

Proof (cf. the proof of (6.11)). Put $G_I=(G_{i_1},\dots,G_{i_r})$. In view of (2.42):

$$\|G(x\oplus z)\|\geqslant\|G_I(x\oplus z)\|\geqslant\|(d_xG_I)z\|-2\left[\frac{4}{\delta(x)}\right]^{\alpha+2}a\|z\|^2.$$

Let $z^*:=(z_{j_1},\dots,z_{j_r})\in C^r$. According to the definition of E_J, we get

$$\|z\|=\|z^*\|=\|[(d_xG)_{I,J}]^{-1}(d_xG)_{I,J}z^*\|\leqslant\|[(d_xG)_{I,J}]^{-1}\|\,\|(d_xG_I)z\|$$
$$\leqslant\frac{1}{2t_1\delta^{\tau_1}(x)}\|(d_xG_I)z\|.$$

Hence

$$\|G(x\oplus z)\|\geqslant t_1\delta^{\tau_1}(x)\|z\|\left[2-\frac{\|z\|}{b_2\delta^{\beta_1}(x)}\right]\geqslant t_1\delta^{\tau_1}(x)\|z\|.$$

Step 4°. Put $b_3=\min\left\{\frac{b_1}{3^{\beta_1}+1},\frac{b_2}{2}\right\}$ and let

$$Y_{I,J}^j:=\{(x,z)\in M_{I,J}^j\times E_J\colon \|z\|<\tfrac{1}{3}jb_3\delta^{\beta_1}(x)\},\qquad I\in\mathscr{J}_r^m,\ J\in\mathscr{J}_r^n,\ j=1,2,3.$$

Note that $\overline{Y_{I,J}^j}\subset Y_{I,J}^{j+1}$ and $Y_{I,J}^3\subset X^*C^n$. We shall prove that the mapping

$$Y_{I,J}^3\ni(x,z)\xrightarrow{\Phi_{I,J}}x\oplus z\in X$$

is injective (cf. Step 4 of the proof of Lemma 6.2).

Proof. Suppose that $x_1\oplus z_1=x_2\oplus z_2$ and $\delta(x_2)\leqslant\delta(x_1)$. Put $z_0=z_1-z_2$. Then $x_2=x_1\oplus z_0$ and $\|z_0\|<2b_3\delta^{\beta_1}(x_1)\leqslant b_2\delta^{\beta_1}(x_1)$. In view of (6.18) we get

$$0=\|G(x_2)\|\geqslant t_1\delta^{\tau_1}(x_1)\|z_0\|,$$

which implies that $z_0=0$, and consequently $(x_1,z_1)=(x_2,z_2)$.

Step 5°. Observe that $Y_{I,J}^3$ is an n-dimensional analytic manifold, whence as in Step 5 of the proof of Lemma 6.2, the set $U_{I,J}^j:=\Phi_{I,J}(Y_{I,J}^j)$ is an open neighbourhood of $M_{I,J}^j$ and $\overline{U_{I,J}^j}\subset U_{I,J}^{j+1}$.

Put $b_4=2b_3t_1[3^{\beta_1+\tau_1+1}4^{\alpha+1}a]^{-1}$, $\beta_2=\beta_1+\tau_1+\alpha+1$.

Then, for every $x \in M^1_{I,J}$, the ball $\hat{B}(x, b_4 \delta^{\beta 2}(x))$ is contained in $U^2_{I,J}$.

Proof. Suppose $\hat{B}(x, b_4 \delta^{\beta 2}(x)) \not\subset U^2_{I,J}$. Then there exists a point $(x_1, z_1) \in Y^3_{I,J} \setminus Y^2_{I,J}$ such that $x_1 \oplus z_1 \in \hat{B}(x, b_4 \delta^{\beta 2}(x))$. Note that $\delta(x_1) \leqslant 3\delta(x)$ and $\delta(x) \leqslant 3\delta(x_1)$; in particular:

$$\|p(x_1) - p(x)\| < b_3 \delta^{\beta 1}(x_1) + b_4 \delta^{\beta 2}(x) \leqslant b_1 \delta^{\beta 1}(x).$$

Hence, in view of (6.17), $x_1 \in M^2_{I,J}$.

On the other hand, in view of (6.18) (and (2.41)):

$$t_1 \delta^{\tau 1}(x_1) \|z_1\| \leqslant \|G(x_1 \oplus z_1)\| = \|G(x_1 \oplus z_1) - G(x)\|$$
$$\leqslant \left[\frac{4}{\delta(x)}\right]^{\alpha+1} a \|p(x_1 \oplus z_1) - p(x)\| < \frac{2}{3} b_3 \delta^{\beta 1}(x_1) t_1 \delta^{\tau 1}(x_1).$$

Thus $\|z_1\| < \frac{2}{3} b_3 \delta^{\beta 1}(x_1)$ and therefore $(x_1, z_1) \in Y^2_{I,J}$, which is a contradiction.

Step 6°. Put $\bar{\theta} = b_4 t_1 (\frac{2}{3})^{\bar{\gamma}}$ $(\bar{\gamma} = \tau_1 + \beta_2 = (3r-1)\alpha + 3\beta + 3r)$. It remains to show that $U = U(G, \bar{\theta}, \bar{\gamma}, M) \subset W := \bigcup_{I,J} U^2_{I,J}$ (cf. the remark after Step 2°).

Proof (cf. Step 6). Suppose $U \not\subset W$. Then there exist I, J, $(x, z) \in Y^3_{I,J}$ such that $x \oplus z \in U \cap \partial W$. Since $M = \bigcup_{I,J} M^1_{I,J}$, there exist I', J' such that $x \in M^1_{I',J'}$. The ball $\hat{B}(x, b_4 \delta^{\beta 2}(x))$ is contained in $U^2_{I',J'}$ (Step 5°), whence $\|z\| \geqslant b_4 \delta^{\beta 2}(x)$. Now, by (6.18),

$$\|G(x \oplus z)\| \geqslant t_1 \delta^{\tau 1}(x) \|z\| \geqslant \bar{\theta} \delta^{\bar{\gamma}}(x \oplus z),$$

which contradicts the definition of U.

The proof of Lemma 6.3 is completed.

Remark 6.19. In the case where $\dim M = 0$ the proof of the existence of holomorphic retraction $\pi\colon U \to M$ may be simplified, namely:

Let X be a Riemann domain over C^n, $\delta \in \mathscr{L}(X)$ and let M be a 0-dimensional δ-regular submanifold of X. Then (as in Step 3°) one can prove that there exist $t_1 > 0$, $0 < b_1 < 1/2$ such that

$$\|G(x \oplus z)\| \geqslant t_1 \delta^{\tau 1}(x) \|z\|, \qquad x \in M,\ z \in C^n,\ \|z\| \leqslant b_1 \delta^{\beta 1}(x),$$

where $\tau_1 = (n-1)(\alpha+1) + \beta$, $\beta_1 = n(\alpha+1) + \beta + 1$.

Consequently, if $0 < b_2 < b_1/2$, $\beta_2 \geqslant \beta_1$, then, for every $x_1, x_2 \in M$, $x_1 \neq x_2$:

$$\hat{B}(x_1, b_2 \delta^{\beta 2}(x_1)) \cap \hat{B}(x_2, b_2 \delta^{\beta 2}(x_2)) = \emptyset.$$

Put $U_0 := \bigcup_{x \in M} \hat{B}(x, b_2 \delta^{\beta 2}(x))$ and let $\pi\colon U_0 \to M$ be defined by the formula $\pi(y) = x$, $y \in \hat{B}(x, b_2 \delta^{\beta 2}(x))$ $(x \in M)$. Define $\gamma = \beta_2 + \tau_1$, $\theta = b_2 t_1 (\frac{2}{3})^{\gamma}$. The standard arguments show that $U(G, \theta, \gamma, M) \subset U_0$.

Note that the existence of a pseudoinverse matrix Q (as in Lemma 6.3) is in some sense equivalent to the δ-regularity of M; namely, we have the following:

PROPOSITION 6.20. *Let X be a Stein domain, $\delta \in \mathscr{W}_r(X)$ and let M be a d-dimensional analytic submanifold of X. Suppose that there exist $m \in N$ and $G \in [\mathcal{O}(X, \delta)]^m$ such that*

$$M \subset G^{-1}(0)$$

and

$$\operatorname{rank}(d_x G) = r = n-d, \qquad x \in M.$$

Then the following conditions are equivalent:

(i) $\exists\, b > 0,\ \beta \geqslant 0$: $\Delta_r(d_x G) \geqslant b\delta^\beta(x)$, $x \in M$ (i.e., *M is δ-regular*).

(ii) $\exists\, (f_{I,J})_{I \in \mathscr{I}_r^m, J \in \mathscr{I}_r^n} \subset \mathcal{O}(X, \delta)$:

$$\sum_{I,J}' f_{I,J}(x) \det(d_x G)_{I,J} = 1, \qquad x \in M.$$

(iii) *There exists a matrix-valued function Q: $X \to C^{n \times m}$ with entries in $\mathcal{O}(X, \delta)$ such that, for every $x \in M$, $Q(x)$ is pseudoinverse to $d_x G$.*

Proof. The implication (i) $\Rightarrow$ (ii) is a consequence of Prop. 6.14 and Lemma 6.16. The implication (ii) $\Rightarrow$ (iii) follows from Prop. 6.15. It remains to prove that (iii) $\Rightarrow$ (i).

Let A (resp. B) be an $(m \times n)$ (resp. $(n \times p)$)-dimensional matrix with complex entries. Then, for every $1 \leqslant r \leqslant m, n, p$,

$$\Delta_r(A \cdot B) \leqslant \Delta_r(A) \cdot \Delta_r(B).$$

Consequently, $\Delta_r(d_x G) \leqslant [\Delta_r(d_x G)]^2 \Delta_r(Q(x))$, $x \in M$. Since $\Delta_r(d_x G) > 0$, $x \in M$, we get: $1 \leqslant \Delta_r(d_x G)\Delta_r(Q(x))$, $x \in M$. Hence $\Delta_r(d_x G) \geqslant b\delta^{\tau r}(x)$, $x \in M$ ($b > 0$ constant) provided that the entries of Q lie in $\mathcal{O}^{(\tau)}(X, \delta)$. ∎

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