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Quasihomographies in the theory of Teichmüller spaces

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Introduction

The extended complex plane $\overline{\mathbb{C}}$, endowed with the conformal structure defined by the local coordinates $z \to z$ and $z \to 1/z$, is called the *Riemann sphere*. The stereographic projection maps the Riemann sphere conformally onto the unit sphere

$$B := \{(x, y, u) : x^2 + y^2 + u^2 - u = 0\}. \tag{1}$$

A *Jordan curve* (Jc) Γ on $\overline{\mathbb{C}}$ is the image of the unit circle $T := \{z : |z| = 1\}$ under a homeomorphism of $\overline{\mathbb{C}}$. A domain on $\overline{\mathbb{C}}$ whose boundary is a Jordan curve is called a *Jordan domain.*

The geometric approach to the notion of K-quasiconformality on the Riemann sphere $\overline{\mathbb{C}}$ implies certain easily comprehensible rules. Given two topologically equivalent domains D and D' on $\overline{\mathbb{C}}$, let $\mathcal{F}_{D,D'}$ be the family of all sense-preserving homeomorphisms mapping D onto D'. We pick up one of the *four possible configurations* that are conformally characterized by one real parameter, and associate with it a suitable conformal invariant. The simplest and most natural configuration seems to be the so-called *quadrilateral*, i.e., a Jordan domain Q, with a distinguished quadruple of points z_1, z_2, z_3, z_4 on the boundary ∂Q, ordered according to the positive orientation of ∂Q with respect to Q. The arcs $\langle z_1, z_2\rangle$ and $\langle z_3, z_4\rangle$ are called the *a-sides* and the other two arcs the *b-sides* of the quadrilateral. The quadrilateral $Q(z_1, z_2, z_3, z_4)$ carries a conformal invariant known as the *modulus of the quadrilateral*, denoted by $M(Q(z_1, z_2, z_3, z_4))$. Unfortunately, this is not a direct generalization of the real-valued cross-ratio; see Chapter IV.

The configuration consisting of a Jordan domain Q with one interior point z and two ordered and distinguished boundary points, i.e., $Q(z; z_1, z_2)$, carries a conformal invariant, the *harmonic measure* $\omega(z, \langle z_1, z_2\rangle; Q)$ of the boundary arc $\langle z_1, z_2\rangle$ of Q as seen from the point z. The $\omega(z, \langle z_1, z_2\rangle; Q)$ is a harmonic function of variable z and a probability measure of the arc variable for any fixed $z \in Q$.

Another configuration is a domain Q bounded by two disjoint Jc's. Its characteristic conformal invariant is known as the *modulus* $M(Q)$ *of the ring domain* Q.

The fourth configuration is made up of a Jordan domain Q and a pair of distinct points $z_1, z_2 \in Q$, and denoted by $Q(z_1, z_2)$. With this configuration there are associated two well-known conformal invariants, the *hyperbolic distance* $\mathbf{h}$ and the Green's function $\mathbf{g}$ related by the identity

$$\tanh(\mathbf{h}(z_1, z_2)) = \exp(-\mathbf{g}(z_1, z_2)), \qquad z_1, z_2 \in G. \tag{2}$$

Hence, given $K \geq 1$, we may state the following definition.

DEFINITION 1. A mapping $F \in \mathcal{F}_{D,D'}$ is said to be K-*quasiconformal* (K-qc) if

$$K^{-1} m(G) \le m(F(G)) \le K m(G) \tag{3}$$

for every quadrilateral $Q := Q(z_1, z_2, z_3, z_4)$ such that $\overline{Q} \subset D$.

We denote by $\mathcal{F}_{D,D'}(K)$ the class of all K-qc mappings $F \in \mathcal{F}_{D,D'}$ with a given $K \ge 1$. Clearly, $\mathcal{F}_{D,D'}(K_1) \subset \mathcal{F}_{D,D'}(K_2)$ if and only if $K_1 \le K_2$. By (3), the class $\mathcal{F}_{D,D'}(1)$ is formed by all conformal mappings $F : D \to D'$. To avoid some difficulties in an adequate formulation of the results, we put

$$\mathcal{F}^{\infty}_{D,D'} := \bigcup_{K \ge 1} \mathcal{F}_{D,D'}(K) \tag{4}$$

and call $\mathcal{F}^{\infty}_{D,D'}$ the family of *quasiconformal* (*qc*) *mappings* of D onto D'.

Given $F \in \mathcal{F}^{\infty}_{D,D'}$, the number

$$K(F) := \inf\{K \ge 1 : F \in \mathcal{F}_{D,D'}(K)\} \tag{5}$$

is called the *maximal dilatation* of F. Obviously,

$$\mathcal{F}_{D',D}(K) = \{F \in \mathcal{F}_{D',D} : F^{-1} \in \mathcal{F}_{D,D'}(K)\} \tag{6}$$

for every $K \ge 1$. Moreover, $K(F^{-1}) = K(F)$ for every $F \in \mathcal{F}^{\infty}_{D,D'}$. It is easily seen by (3) that for every $F \in \mathcal{F}_{D,D'}(K)$ and $G \in \mathcal{F}_{D',D''}(L)$ the mapping $G \circ F$ belongs to $\mathcal{F}_{D,D''}(KL)$. This also shows that

$$K(G \circ F) \le K(G)K(F) \tag{7}$$

for every $F \in \mathcal{F}^{\infty}_{D,D'}$ and $G \in \mathcal{F}^{\infty}_{D',D''}$.

The definition of quasiconformality could equally well be given in terms of other conformal invariants; see [LV]. *Analytic characterizations* of quasiconformality can be found in [LV] and [Le].

Quasiconformal mappings with prescribed angle function were considered by S. Agard and F. W. Gehring [AG] as well as by T. Sorvali [So2]. This problem is equivalent to the following problem in geodesy: How to map a given surface in $\mathbb{R}^3$ conformally onto a plane domain. In this form the existence problem was considered by C. F. Gauss already in 1822; cf. [Ga].

A *quasicircle* in $\overline{\mathbb{C}}$ is the image of the unit circle under a quasiconformal mapping of $\overline{\mathbb{C}}$. If the mapping is K-qc, the image curve is called a K-quasicircle. Clearly, a quasicircle is a Jc on $\overline{\mathbb{C}}$. The property of being a quasicircle has an obvious geometrical meaning; see [Ge3] and [Le]. Moreover, K-quasicircles can be considered *fractals*; see [As] and [BP].

The classical Schwarz Lemma for analytic functions was generalized in 1952 by J. Hersch and A. Pfluger [HP] to the class of qc mappings of the unit disc $\varDelta := \{z : |z| < 1\}$. They proved that there exists a strictly increasing *distortion function* $\varPhi_K : [0,1] \to [0,1]$ such that $|F(z)| \le \varPhi_K(|z|)$ holds for every K-qc mapping F of $\varDelta$ into itself with $F(0) = 0$ and every $z \in \varDelta$. This distortion function will be of special interest in the sequel. The classical Schwarz Lemma follows if $K = 1$ since $\varPhi_1(t) = t$, $0 \le t \le 1$.

If D and D' are simply connected domains of hyperbolic type on $\overline{\mathbb{C}}$, then each mapping $F \in \mathcal{F}^{\infty}_{D,D'}$ has a homeomorphic extension to $\overline{D}$ if and only if D and D' are Jordan domains. It then induces a sense-preserving homeomorphism $f = F|_{\partial D}$ of the oriented Jc $\varGamma =$

∂D onto $\Gamma' = \partial D'$. Given two oriented Jc's Γ and Γ' on $\overline{\mathbb{C}}$, let $A_{\Gamma,\Gamma'}$ be the family of all sense-preserving homeomorphisms of Γ onto Γ'. The *boundary value problem* is to characterize $f \in A_{\Gamma,\Gamma'}$ which are the boundary functions of K-qc mappings $F : D \to D'$, $K \geq 1$.

By the Riemann mapping theorem and the composition property of K-qc mappings one may assume, without loss of generality, that $D = D'$, provided that the obtained characterization is conformally invariant.

For arbitrary Jordan domains $D, D' \subset \mathbb{R}^n$, $n \geq 3$, a quasiconformal mapping $F : D \to D'$ does not always have a boundary extension; see [Ku]. J. Väisälä [Vä2] proved the existence of boundary extension for all qc mappings between n-dimensional Jordan domains D, $D' \subset \mathbb{R}^n$ quasiconformally equivalent to the unit ball in $\mathbb{R}^n$, $n \geq 2$.

Given a Jordan domain D on $\overline{\mathbb{C}}$ and an automorphism $F \in \mathcal{F}_D(K) := \mathcal{F}_{D,D}(K)$, $K \geq 1$, let z_1, z_2, z_3, z_4 be a quadruple of distinct points on $\Gamma = \partial D$, ordered according to the orientation of Γ. Consider the homeomorphic extension of F to the closure of D. It follows from (3) that

$$\frac{1}{K}M(D(z_1,z_2,z_3,z_4)) \leq M(D(f(z_1),f(z_2),f(z_3),f(z_4))) \leq KM(D(z_1,z_2,z_3,z_4)) \tag{8}$$

for $f = F|_\Gamma$ and every ordered quadruple of distinct points z_1, z_2, z_3, z_4 of Γ.

In the case of $D = U := \{z : \operatorname{Im} z > 0\}$ and an automorphism $F \in \mathcal{F}_U(K)$ that fixes the point at infinity, the induced automorphism $f = F|_\mathbb{R}$ of the real line $\mathbb{R}$ is a *ϱ-quasisymmetric* (ϱ-qs) function in the sense of A. Beurling and L. V. Ahlfors (see [BA]), i.e., the condition

$$\frac{1}{\varrho} \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leq \varrho \tag{BA}$$

holds for all $x \in \mathbb{R}$ and $t > 0$, with a constant $\varrho \geq 1$. In fact, setting $z_1 = x - t$, $z_2 = x$, $z_3 = x + t$ and $z_4 = \infty$, and substituting these values into (8), we see that the boundary automorphism $f = F|_\mathbb{R}$ satisfies the condition (BA) with the constant $\lambda(K) = \Phi_K\big(1/\sqrt{2}\big)^2/\Phi_{1/K}\big(1/\sqrt{2}\big)^2$ (see [LV]).

The class of all increasing homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying the (BA) condition with a given constant $\varrho \geq 1$ is denoted by $Q_\mathbb{R}(\varrho)$. On the other hand, for any $f \in Q_\mathbb{R}(\varrho)$ there is a qc automorphism F_f of U which has the boundary values given by f, and whose maximal dilatation $K^* = K^*(\varrho)$ is a function of ϱ; cf. the well-known estimates of the qc constant by Beurling–Ahlfors ([BA], [Ea], [Ln], [Le] and [PZ1]), P. Tukia [T4], Douady–Earle ([DE], [LP], [SZ] and [P1]), and E. Reich [R].

A characterization of the boundary values of K-qc automorphisms $F \in \mathcal{F}^\circ_\Delta(K)$ with the fixed point zero was given by J. Krzyż [Kr1]. Using the configuration connected with harmonic measure, he defined a class of ϱ-qs functions of $T = \partial\Delta$, representing boundary automorphisms $f = F|_T$ such that

$$\frac{1}{\varrho} \leq \frac{|f(\alpha_1)|}{|f(\alpha_2)|} \leq \varrho \tag{K}$$

for each pair of disjoint adjacent open subarcs α_1, α_2 of T, with equal harmonic measure, where $|\alpha| = \omega(0,\alpha;\Delta)$ and a constant $\varrho \geq 1$. The relations between K and ϱ remain the same as in the previous case.

The class of all sense-preserving automorphisms of T satisfying the K-condition with a given constant $\varrho \geq 1$ is denoted by $Q_T(\varrho)$.

This class is invariant under composition with increasing linear functions. Denote by $Q^\circ_{\mathbb{R}}(\varrho)$ the subclass of $Q_{\mathbb{R}}(\varrho)$ consisting of all ϱ-qs functions f normalized by $f(0) = 0$ and $f(1)=1$. The family $Q_T(\varrho)$ is invariant under composition with rotations of T. Hence, we denote by $Q^\circ_T(\varrho)$ the subset of $Q_T(\varrho)$ consisting of all automorphisms f normalized by $f(1) = 1$.

Let $Q_{\mathbb{R}} = \bigcup_{\varrho\geq 1} Q_{\mathbb{R}}(\varrho)$ and $Q_T = \bigcup_{\varrho\geq 1} Q_T(\varrho)$. A function from $Q_{\mathbb{R}}$ or from Q_T is called *quasisymmetric* (qs). Both are groups under composition.

J. A. Kelingos ([Ke]) was one of the pioneers of the rigorous study of qs functions on $\mathbb{R}$, and he also introduced the name "qs functions". Further developments, including equivalent characterizations (cf. [Go], [AK], [Hi1] and [Hi2]), cluster around Beurling–Ahlfors extension; see [BA]. It is a remarkable fact that the (BA) condition is formally independent of complex analysis and, from different points of view, should be classified to the real analysis. It gives rise to research of $Q_{\mathbb{R}}$ from a real analytic point of view. This idea is discernible in [Ke], [HH], and some other papers.

A closer look at these characterizations rises the following arguments:

* The point at infinity plays a special role in the (BA) condition which is not justified when dealing with the boundary value problem for all K-qc automorphisms of U. Since this particular characterization is invariant under linear functions only, it cannot be used to characterize all qc automorphisms of U. It thus gives a solution of the boundary value problem for $F \in \mathcal{F}_U(K)$, $K \geq 1$, such that $F(\infty) = \infty$.
* The K-condition characterizes *not uniformly* the boundary values of all qc automorphisms of Δ. This is because the qs constant does not, in general, depend on K only but also on the particular automorphism. Since every $F \in \mathcal{F}_\Delta(K)$ has a fixed point in $\bar{\Delta}$, we have $\mathcal{F}_\Delta(K) = \bigcup_{z_0\in\bar{\Delta}} \mathcal{F}^{z_0}_\Delta(K)$, but
$$\sup_{z_0\in\Delta} \varrho(K, z_0) = \infty$$
for any $K \geq 1$. Taking $K = 1$, we see that
$$\mathcal{F}_\Delta(1)|_T \not\subset Q_T(\varrho) \tag{9}$$
for any finite $\varrho \geq 1$; see Example 2.1, p. 48. Rotation invariant ϱ-qs automorphisms of T cannot, in substance, be considered 1-dimensional K-qc mappings.
* The characterizations describing $Q_{\mathbb{R}}(\varrho)$ and $Q_T(\varrho)$ involve *two real parameters*, whereas the general case includes *four parameters*. This restricts the flexibility of these characterizations. Although one may easily calculate, for instance, the qs constant of a given automorphism of $\mathbb{R}$ or T, it is quite complicated to obtain an asymptotically sharp distortion theorem for ϱ-qs functions; cf. [Ke], [HH], [Kr2] and others.
* Obviously, quasisymmetry and quasiconformality are defined by *incompatible deformations*. This explains why the constants ϱ and K do not behave similarly under the group action. Indeed, it is not in general true that $\varrho(f) = \varrho(f^{-1})$ and $\varrho(f \circ g) \leq \varrho(f)\varrho(g)$; see [Ke].

* Particular questions concerning boundary values of qc automorphisms of a Jordan domain $D \subset \overline{\mathbb{C}}$ can be reduced to equivalent problems for ϱ-qs automorphisms of $\mathbb{R}$ or T; cf [BA], [FS] and [T3]. Unfortunately, general problems require a characterization which is both conformally invariant and suitable for all K-qc automorphisms of D, i.e. *uniform.*
* One cannot introduce the *Teichmüller metric* in the class of normalized qs functions of $\mathbb{R}$, directly by using the qs constant in the same manner as the qc constant; cf. [Le]. Therefore, the standard model of the *universal Teichmüller space* (UTS), i.e., the group of normalized qs functions of $\mathbb{R}$, is equipped with the metric obtained from qc extensions. This is not an intrinsic metric for normalized qs functions.

These arguments show that we should search for a *general* and *uniform* characterization of the boundary behaviour of qc mappings.

A few years ago the author initiated a rigorous study of the *boundary value problem* for K-qc mappings by stating and then solving the *uniform boundary value problem* for qc automorphisms of a Jordan domain D on $\overline{\mathbb{C}}$; see [Z1]–[Z8]. The required characterization is described by deformation of *cross-ratios* controlled by the Hersch–Pfluger distortion function Φ_K. A continuation and certain applications of this research can be found in [CZ4], [KZ], [RZ1], [RZ2], [SZ], [Z9]–[Z17]. That gives a quite satisfactory characterization provided D is bounded by a *circle* Γ on $\overline{\mathbb{C}}$. Recall that a circle on $\overline{\mathbb{C}}$ means the stereographic projection of a circle on the unit sphere B, which is a circle or a line in $\mathbb{C}$.

By defining the concept of *harmonic cross-ratio* one may extend this idea to the most general case where D is a Jordan domain on $\overline{\mathbb{C}}$, without any restriction whatsoever, omitting the obstacles typical for qs automorphisms. The harmonic cross-ratio is a direct generalization of the real-valued cross-ratio and an alternative *conformal invariant* with respect to the modulus of a quadrilateral. Moreover, it is defined without any use of special functions and keeps the properties of the real-valued cross-ratio, expressed conveniently in the form of equalities.

Automorphisms of an oriented Jc Γ on $\overline{\mathbb{C}}$, characterized in this way, are called *K-quasihomographies* (K-qh) and can be considered without constraints the 1-*dimensional K-qc automorphisms* of Γ.

The study of how different properties of K-qc mappings behave with respect to dimension seems to be one of the most interesting topics, particularly when the function space is formed by K-qc mappings of a domain in $\overline{\mathbb{R}}^n$, $n = 1, 2, 3, \ldots$

The study of $\Phi_{K,n}$, $n \geq 1$, and other special functions of quasiconformal theory, is motivated also by some other reasons, as will be apparent from what follows. It is worth noting that some of the results give solutions for questions on special functions; cf. [Z7].

Certain properties of these special functions result from a number of distortion theorems for K-qh of a circle Γ on $\overline{\mathbb{C}}$, including the best estimates. Therefore, one may pursue rather rigorous development of this research on K-qh automorphisms, including the general case of an arbitrary Jc Γ on $\overline{\mathbb{C}}$.

We present characterizations of *normal* and *compact families* of K-qh automorphisms showing the true *compatibility* of K-qh and K-qc automorphisms and suggesting additional interesting topics in qc theory.

Our method involves a new metric which makes the family of all normalized qh automorphisms of an arbitrary circle Γ in $\overline{\mathbb{C}}$ a metric space. This metric is defined without the use of qc extensions to complementary domains. Moreover, it is fairly justified to call that metric space the *universal Teichmüller metric space* (UTMS) of a given circle Γ on $\overline{\mathbb{C}}$; see Chapter V.

There is a natural gap between the *harmonic cross-ratios* of a given ordered quadruple of distinct points of a Jc Γ in $\overline{\mathbb{C}}$. This gap measures the deviation of Γ from a circle in $\overline{\mathbb{C}}$, and is used here to characterize quasicircles. In these circumstances one can define a norm for the family of all normalized Jordan curves in $\overline{\mathbb{C}}$, and a metric in the space of certain equivalence classes of quasicircles.

Given an oriented Jc Γ on $\overline{\mathbb{C}}$, one may associate uniquely with Γ the complementary domains D and D^*, by calling them the *left-hand* and the *right-hand complementary domains*, respectively. The unique correspondence $\Gamma \leftrightarrow (D, D^*)$ is one of the convincing arguments that the UTMS is naturally related to an oriented Jc Γ on the Riemann sphere, without any reference to conformal or quasiconformal mappings of the complementary domains.

At the end of the paper, we outline some general considerations regarding normalization, extremal normalization and linearization aspects of the universal Teichmüller space of an oriented Jc Γ on the Riemann sphere.

I. Special functions of quasiconformal theory

1. Introduction. A number of distortion theorems for K-qc and K-qr quasiregular mappings follow from the properties of the distortion function $\Phi_{K,n}$ $(n \geq 2)$ of the generalized Schwarz Lemma; cf. [HP], [MRV], [AVV4] and [Vo]. Hence, information on the foregoing and other special functions such as $\mathcal{K}$, μ, M_n $(M_2 = \mu)$, τ_n and γ_n (cf. [Vo]), in the form of functional identities, inequalities and differential equations is crucial for proving distortion theorems. The special functions $\mathcal{K}$ and μ have found various applications in mathematics as well as in physics; cf. [BB].

Elliptic integrals and other special functions often provide a connection between extremal length and conformal invariants in the plane. The general properties of elliptic integrals can be found in standard books such as [WW].

It seems fair to credit the rigorous and efficient study of these special functions to G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen; see [An], [AVV1],..., and [VV].

In this part of our presentation one may find a series of new results on the distortion function Φ_K including, to some extent, the n-dimensional case, i.e., $\Phi_{K,n}$, M_n, τ_n and γ_n, $n \geq 2$ where $\Phi_K = \Phi_{K,2}$. Some of the results give solutions of the well-known problems (cf. [Vo]), and, in particular, a very fast numerical method approximating uniformly $\Phi_{K,n}$ with the help of elementary functions to any prescribed precision.

The approximation may have been transferred back in order to approximate μ, μ^{-1} and the elliptic integral $\mathcal{K}$ (see [P3] and [Z11]), still with the help of elementary functions.

2. The distortion function Φ_K. In the quasiconformal version of the Schwarz Lemma by J. Hersch and A. Pfluger [HP], it is proved that there exists a strictly increasing distortion function $\Phi_K : (0,1) \to (0,1)$ such that $|F(z)| \le \Phi_K(|z|)$ for every $z \in \Delta$ and for each K-qc mapping F of Δ into itself with $F(0) = 0$. The function Φ_K is defined by

$$\Phi_K(t) = \mu^{-1}\left(\frac{1}{K}\mu(t)\right), \tag{2.1}$$

where $(\pi/2)\mu$ is the conformal modulus of Δ slit along the real line from 0 to t, $0 < t < 1$, and is strictly decreasing with limits ∞ and 0 at 0 and 1, respectively. It can be expressed by the formula

$$\mu(t) = \frac{\mathcal{K}\big(\sqrt{1-t^2}\,\big)}{\mathcal{K}(t)}, \tag{2.2}$$

where

$$\mathcal{K}(t) = \int_0^{\pi/2} (1 - t^2 \sin^2\phi)^{-1/2}\, d\phi \tag{2.3}$$

is the complete elliptic integral of the first kind. Multiplying μ by a positive constant, we do not alter Φ_K. Thus, for convenience we normalize μ with 1 instead of the usual $\pi/2$.

The function $\mathcal{K}$ satisfies the following identities due to Landen, sometimes called *Landen transformations*:

$$\mathcal{K}\left(\frac{2\sqrt{t}}{1+t}\right) = (1+t)\mathcal{K}(t), \qquad \mathcal{K}\left(\frac{1-t}{1+t}\right) = \frac{1+t}{2}\mathcal{K}\big(\sqrt{1-t^2}\,\big). \tag{2.4}$$

From (2.2) and (2.4) it follows that for $t \in (0,1)$,

$$\mu(t)\mu\big(\sqrt{1-t^2}\,\big) = 1, \qquad \mu(t) = 2\mu\left(\frac{2\sqrt{t}}{1+t}\right), \qquad \mu(t)\mu\left(\frac{1-t}{1+t}\right) = 2. \tag{2.5}$$

The first identity in (2.5) yields, for $x > 0$,

$$\mu^{-1}(x)^2 + \mu^{-1}(1/x)^2 = 1. \tag{2.6}$$

For $K \in (0,1)$ we extend Φ_K by (2.1). We also extend the domain of Φ_K to the closed interval $[0,1]$ by setting $\Phi_K(0) = 0$ and $\Phi_K(1) = 1$ for each $K > 0$. Since μ decreases in $[0,1]$ it follows that $\Phi_K(t) \ge t$ for $K \ge 1$, and $\Phi_K(t) \le t$ for $0 < K \le 1$, with equality in each case if and only if $K = 1$. It is well known (see [LV, p. 64]) that

$$\Phi_{K_1} \circ \Phi_{K_2} = \Phi_{K_1 K_2}, \qquad \Phi_K^{-1} = \Phi_{1/K}, \qquad \Phi_2(t) = \frac{2\sqrt{t}}{1+t}, \quad 0 \le t \le 1. \tag{i}$$

The explicit estimate

$$t^{1/K} \le \Phi_K(t) \le 4^{1-1/K} t^{1/K}, \qquad 0 \le t \le 1,\ K \ge 1, \tag{ii}$$

was obtained by C. F. Wang [W] and O. Hübner [Hü].

Using (2.1), (2.5) and (2.6) one obtains

$$\Phi_K(t)^2 + \Phi_{1/K}\big(\sqrt{1-t^2}\,\big)^2 = 1, \qquad 0 \le t \le 1,\ K > 0, \tag{iii}$$

as shown in [AVV1]. We call (iii) the *circular property* of Φ_K.

In the same way we obtain

$$\text{(iv)}\qquad \Phi_{1/K}\left(\frac{1-t}{1+t}\right)=\frac{1-\Phi_K(t)}{1+\Phi_K(t)},\qquad 0\le t\le 1,\ K>0,$$

which we call the *hyperbolic property* of Φ_K; cf. [AVV1] and [Gh].

Moreover, it is shown in [AVV1] that

$$\text{(v)}\qquad \Phi_K(tu)\ge \max\{u^{1/K}\Phi_K(t),t^{1/K}\Phi_K(u)\},\qquad 0<t,\ u<1,\ K\ge 1,$$

and that the function $\Phi_K(t)/t^{1/K}$ is strictly decreasing. Furthermore, the function

$$f(t)=\frac{\Phi_K(at)}{\Phi_K(t)},\qquad 0<a<1,\ 0<t\le 1,$$

is strictly increasing and, in particular,

$$\text{(vi)}\qquad f(t)<f(1)=\Phi_K(a).$$

The first and simplest of our results reads

$$\text{(vii)}\qquad \max\left\{t^{1/K},\frac{(1+t)^K-(1-t)^K}{(1+t)^K+(1-t)^K}\right\}\le\Phi_K(t)\le\min\left\{4^{1-1/K}t^{1/K},\frac{(1+t)^K-4^{1-K}(1-t)^K}{(1+t)^K+4^{1-K}(1-t)^K}\right\}$$

for $K\ge 1$ and $0\le t\le 1$.

Here especially the right-hand estimate essentially has control of Φ_K within the closed interval $[0,1]$, whereas the majorant $4^{1-1/K}t^{1/K}$ (see (ii)) does not, i.e., at $t=1$ we have $4^{1-1/K}\to 4$ as $K\to\infty$.

To prove (vii) note that it is a fairly simple consequence of the Wang–Hübner inequality (ii) for $\Phi_{1/K}$, which has the form

$$\text{(ii}'\text{)}\qquad 4^{1-K}t^K\le\Phi_{1/K}(t)\le t^K,\qquad 0\le t\le 1,\ K\ge 1,$$

and of the hyperbolic property (iv).

The *multiplicative property* of Φ_K is presented in

THEOREM 1.1. *For each t, $u\in[0,1]$ and an every $K\ge 1$, there exists a constant K' such that*

$$\text{(viii)}\quad \Phi_K(tu)\le\Phi_K(t)\Phi_K(u)\le\Phi_{K'}(tu),\qquad \Phi_{1/K'}(tu)\le\Phi_{1/K}(t)\Phi_{1/K}(u)\le\Phi_{1/K}(tu),$$

where

$$K'\le\chi(K)=\begin{cases}K\left(1+\dfrac{K-1}{\log_4\frac{31}{33}}\right)^{-1}, & 1\le K\le K_0,\\ 2K, & K>K_0,\end{cases}\tag{2.7}$$

and $K_0=1+\log_{16}\frac{33}{31}$.

P r o o f. The left-hand inequality of the first line in (viii) is presented in [AVV1]. Now we prove the right-hand inequality. Suppose first that $1\le K\le K_0$, and let $\delta=2^{-5}$, $\delta_0=(1-\delta)/(1+\delta)=31/33$ and $K''=7K-6$.

Because $2\cdot 2^{21(K-1)}\le 1+2^{42(K-1)}$, we have

$$2^{9K-8}-2^{12(1-K)}\le 2^{30(K-1)},$$

and

$$\delta = 2^{-5} \le \left(\frac{2^{7K-6} - 4^{7(1-K)}}{4^{1-K}}\right)^{1/(6(1-K))} = \left(\frac{2^{K''} - 4^{1-K''}}{4^{1-K}}\right)^{1/(K-K'')}.$$

Hence, for $0 \le t \le \delta$, we have

$$\begin{aligned} 4^{1-K}t^K + 4^{1-K''}t^{K''} &\ge (2t)^{K''} = \left(\frac{2t}{1+t^2}\right)^{K''}(1+t^2)^{K''} \\ &\ge \left(\frac{2t}{1+t^2}\right)^{K''}(1 + 4^{2-K-K''}t^{K''+K}), \end{aligned}$$

and therefore,

$$\frac{1 - \left(\dfrac{2t}{1+t^2}\right)^{K''}}{1 + \left(\dfrac{2t}{1+t^2}\right)^{K''}} \ge \frac{1 - 4^{1-K}t^K}{1 + 4^{1-K}t^K}\,\frac{1 - 4^{1-K''}t^{K''}}{1 + 4^{1-K''}t^{K''}}. \tag{2.8}$$

Substituting $a = (1-t)/(1+t)$, we have

$$t = \frac{1-a}{1+a} \quad \text{and} \quad \frac{2t}{1+t^2} = \frac{1-a^2}{1+a^2}.$$

Now, by (vii) and (2.8), we have

$$\Phi_{K''}(a)\Phi_K(a) \le \Phi_{K''}(a^2) \qquad \text{for } a > \delta_0. \tag{2.9}$$

Using (vi) we see that the function $f(t) = \Phi_{K''}(at)/\Phi_{K''}(t)$ is strictly increasing for $0 < t \le 1$ with a fixed $0 < a < 1$. Hence (2.9) shows that, for $\delta_0 \le a \le b \le 1$,

$$\Phi_{K''}(ab) \ge \Phi_{K''}(b)\frac{\Phi_{K''}(a^2)}{\Phi_{K''}(a)} \ge \Phi_K(b)\Phi_K(a).$$

Since $K'' \le \chi(K)$, we have

$$\Phi_K(a)\Phi_K(b) \le \Phi_{\chi(K)}(ab) \tag{2.10}$$

for all $a, b \in [\delta_0, 1]$.

By the definition of $\chi(K)$ for $1 \le K \le K_0$ we have, for all $0 \le t \le \delta_0$,

$$4^{1-1/K}t^{1/K} \le t^{1/\chi(K)}. \tag{2.11}$$

Hence, by (ii) and (v) for $0 \le a \le \delta_0$, and $0 \le b \le 1$, we arrive at

$$\Phi_K(a)\Phi_K(b) \le 4^{1-1/K}a^{1/K}\Phi_{\chi(K)}(b) \le a^{1/\chi(K)}\Phi_{\chi(K)}(b) \le \Phi_{\chi(K)}(ab). \tag{2.12}$$

In a similar way we prove (2.12) for $0 \le a \le 1$ and $0 \le b \le \delta_0$.

The last inequality in (2.12), together with (2.10), gives us the desired result for $1 \le K \le K_0$, where K_0 is the root of the equation

$$2K = K\left(\frac{K-1}{\log_4 \delta_0} + 1\right)^{-1}. \tag{2.13}$$

Suppose now that $K > K_0$. Without any loss of generality we assume that $t \le u$. Using the basic properties in (i) we see that

$$\Phi_{2K}(tu) = \Phi_K(\Phi_2(tu)) = \Phi_K\left(\frac{2\sqrt{tu}}{1+tu}\right) \ge \Phi_K\left(\sqrt{tu}\right) \ge \Phi_K(t) \ge \Phi_K(t)\Phi_K(u). \tag{2.14}$$

Hence,

$$\chi(K) = 2K \quad \text{for } K > K_0. \tag{2.15}$$

To obtain the second line of (viii) note that, by (i) and the first line of (viii),

$$\Phi_{K'}(\Phi_{1/K}(t)\Phi_{1/K}(u)) \ge \Phi_K(\Phi_{1/K}(t))\Phi_K(\Phi_{1/K}(u)) = tu = \Phi_{K'}(\Phi_{1/K'}(tu)). \tag{2.16}$$

Composing both sides of (2.16) with $\Phi_{K'}^{-1}$, we obtain the left-hand side of the second row in (viii). The right-hand inequality can be obtained in a similar way: because

$$\Phi_K(\Phi_{1/K}(t)\Phi_{1/K}(u)) \le \Phi_K(\Phi_{1/K}(t))\Phi_K(\Phi_{1/K}(u)) = tu = \Phi_K(\Phi_{1/K}(tu)), \tag{2.17}$$

we obtain the assertion by composing both sides of (2.17) with Φ_K^{-1}. The proof is complete.

Remark 1.1. According to information obtained from M. K. Vamanamurthy and M. Vuorinen [VV], they were able to lower the bound on K' to $K' \le K^2$. Very recently, another improvement of this result was obtained by S. L. Qiu and M. Vuorinen [QV], who showed that the bound on K' can be lowered to $K' < K^c$, where $c = 2/m$ and $m \approx 1.5324$. Combining these we can see that the last result can be formulated as

$$K' \le \min\{K^c, 2K\}.$$

A sharp estimate for one of the most natural functionals on Φ_K, considered in [Be] and [Vo], is presented in

THEOREM 1.2. *For each* $K \ge 1$, *the function*

$$\varphi_K(t) := \Phi_K(t) - t$$

is concave in $[0,1]$ *and*

$$\varphi(K) := \max_{0\le t\le 1} \varphi_K(t) \le \Psi(K), \tag{ix}$$

where

$$\Psi(K) = \begin{cases} 1 - \dfrac{1+4^{1-K}}{2K}, & 1 \le K \le K_0, \\ \dfrac{1-4^{1-K}}{1+4^{1-K}}, & K > K_0, \end{cases}$$

and K_0 *satisfies the equation* $(1+4^{1-K})^2 = 4^{2-K}K$, $2.481 < K_0 < 2.482$.

Figure 2 represents the graph of $\Psi(K)$, whereas in Figure 1 we see the graph of the estimate obtained by M. K. Vamanamurthy and M. Vuorinen [VV].

Figure 3 shows both of these bounds in one picture, where the thin line means our Ψ. For K close to 1 the graph of the bound from [VV] is better than our Ψ. Using $B_2[K,4]$, defined in Section 4 of this chapter, one can easily obtain other bounds for $\Phi_K(t)$.

Proof of Theorem 1.2. Notice first that $\varphi_K(0) = \varphi_K(1) = 0$ for every $K \ge 1$ and because $\varphi_K''(t) = \Phi_K''(t) < 0$ in $(0,1)$ for every $K > 0$ we conclude that φ_K is concave and then it attains its maximum at a unique point $t_K \in (0,1)$.

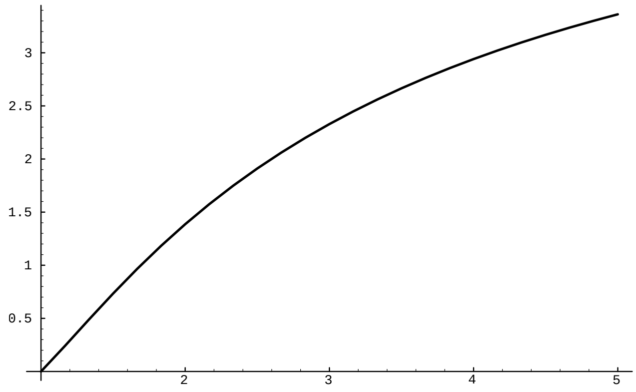

Fig. 1

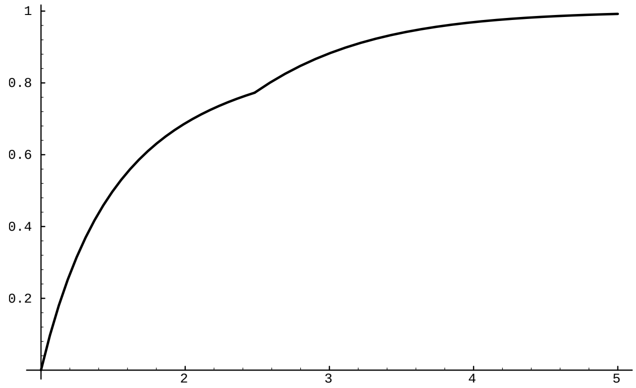

Fig. 2

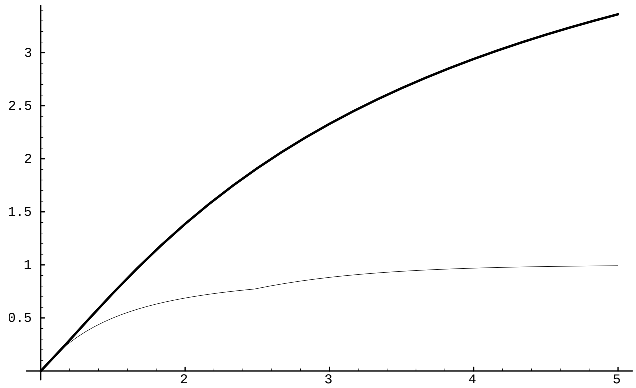

Fig. 3

Let

$$G_K(t) = \frac{(1+t)^K - 4^{1-K}(1-t)^K}{(1+t)^K + 4^{1-K}(1-t)^K}, \quad 0 \le t \le 1,\ K \ge 1. \tag{2.18}$$

Then, because of (vii), $\Phi_K(t) \le G_K(t)$ for $0 \le t \le 1$ and $K \ge 1$. Differentiating G_K we see that

$$G'_K(t) = 4^{2-K}K\frac{(1-t^2)^{K-1}}{[(1+t)^K + 4^{1-K}(1-t)^K]^2} \tag{2.19}$$

is a strictly decreasing function in $[0,1]$, hence G_K is a concave function in this interval. Then

$$G_K(t) \le tG'_K(0) + G_K(0) = t\frac{4^{2-K}K}{(1+4^{1-K})^2} + \frac{1-4^{1-K}}{1+4^{1-K}}. \tag{2.20}$$

Using (2.18) and (2.20) we see that

$$\Phi_K(t) \le \max_{0\le t\le 1}\{\min\{tG'_K(0) + G_K(0), 1\}\}. \tag{2.21}$$

Hence,

$$\max_{0\le t\le 1}[\Phi_K(t) - t] \le \max\{G_K(0), 1-r_0\} = \begin{cases} 1 - \dfrac{1+4^{1-K}}{2K}, & 1 \le K \le K_0, \\ \dfrac{1-4^{1-K}}{1+4^{1-K}}, & K > K_0, \end{cases} \tag{2.22}$$

where $G'_{K_0}(0) = 1$ and $r_0 G'_K(0) + G_K(0) = 1$. This completes the proof.

Next we prove a special case:

LEMMA 1.1. *Suppose that*

$$f(x) = \left(\frac{1-x}{1+x}\right)^2, \quad g_K(x) = 4^{1-K}x^K, \quad h(x) = \frac{1-x}{(1+x)^3}$$

and that $y = y(x) = m_K(x - x_K)$ *is a line such that* $y(x) \le g_K(x)$ *for each* $0 \le x \le 1$ *and* $K \ge 1$. *Moreover, let*

$$h(x_K) = \frac{1-x_K}{(1+x_K)^3} \le m_K \le \frac{1}{1+x_K}. \tag{2.23}$$

Then

$$\max_{0\le x\le 1}[f(g_K(x)) - f(x)] \le 1 - f(x_K). \tag{2.24}$$

Proof. If

$$\alpha(x) = \frac{h(x)}{h(y)} \quad \text{for } x_K \le x \le 1, \tag{2.25}$$

then

$$\alpha'(x) = \frac{h'(x)h(y) - m_K h(x)h'(y)}{h(y)^2} \le 0 \tag{2.26}$$

if and only if

$$\frac{h'(x)}{h(x)} \le m_K\frac{h'(y)}{h(y)}, \tag{2.27}$$

which is equivalent to

$$\frac{d}{dx}\log h(x) \le m_K\frac{d}{dy}\log h(y) \tag{2.28}$$

and hence equivalent to

$$\frac{1}{1-x} - \frac{m_K}{1-y} \geq 3\left(\frac{m_K}{1+y} - \frac{1}{1+x}\right) \tag{2.29}$$

and also to

$$\frac{1-m_K(1-x_K)}{(1-x)(1-y)} \geq 3\frac{m_K(1+x_K)-1}{(1+x)(1+y)}, \tag{2.30}$$

which is satisfied because of (2.23) and of the fact that $m_K(1-x_K) = y(1) \leq 1$.

Hence, α is a decreasing function in $[x_K, 1]$. Because of (2.23), we now have the inequality

$$\frac{h(x)}{h(y)} = \frac{h(x)}{h(y(x))} \leq \frac{h(x_K)}{h(y(x_K))} = \frac{h(x_K)}{h(0)} = h(x_K) \leq m_K \quad \text{for } x_K \leq x \leq 1. \tag{2.31}$$

We then see that

$$\frac{d}{dx}\left[f(x) - f(y) + 1 - f(x_K)\right] = f'(x) - m_K f'(y) = -4h(x) + 4m_K h(y) \geq 0 \tag{2.32}$$

for $x_K \leq x \leq 1$, and hence,

$$f(x) - f(y(x)) + 1 - f(x_K) \geq f(x_K) - 1 + 1 - f(x_K) = 0 \quad \text{for } x_K \leq x \leq 1. \tag{2.33}$$

This implies that

$$1 - f(x_K) \geq f(y(x)) - f(x) \tag{2.34}$$

for $x_K \leq x \leq 1$. This inequality, together with the assumption $y(x) \leq g_K(x)$ for $0 \leq x \leq 1$ of Lemma 1.1, gives

$$f(g_K(x)) - f(x) \leq f(y(x)) - f(x) \leq 1 - f(x_K) \quad \text{for } x_K \leq x \leq 1. \tag{2.35}$$

Yet for $0 \leq x \leq x_K$,

$$f(g_K(x)) - f(x) \leq f(0) - f(x) \leq 1 - f(x_K), \tag{2.36}$$

which together with (2.35) gives the desired inequality (2.24). The proof is complete.

Using Lemma 1.1 we get a sharp estimate for the most useful special function in this research:

Theorem 1.3. *For each* $K \geq 1$, *the function*

$$M_K(t) := \Phi_K\big(\sqrt{t}\big)^2 - t$$

is concave on $[0, 1]$ *and*

$$M(K) := \max_{0 \leq t \leq 1} M_K(t) \leq \Lambda(K), \tag{x}$$

where

$$\Lambda(K) = \begin{cases} 1 - \left(\dfrac{K+1}{3K-1}\right)^2 & \textit{for } 1 \leq K \leq 3/2, \\ 1 - (2K-1)^{-2} & \textit{for } 3/2 < K \leq 4, \\ 1 - 4^{1-K} & \textit{for } K > 4. \end{cases} \tag{2.37}$$

The graph of Λ is shown in Figure 4.

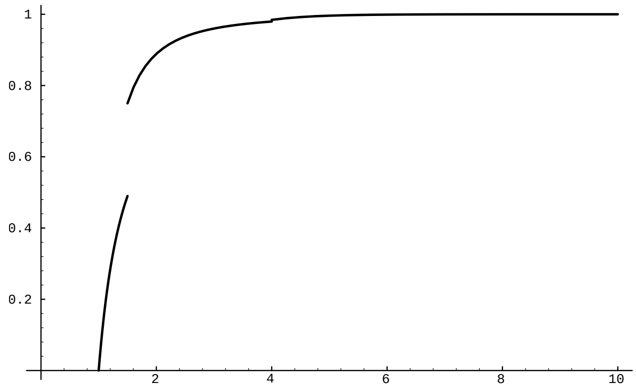

Fig. 4

P r o o f o f T h e o r e m 1.3. Since $M'_K(t) = \Phi_K(\sqrt{t})\Phi'_K(\sqrt{t})(1/\sqrt{t}) - 1$, using (v) and the remark that Φ'_K is strictly decreasing, we conclude that M_K is concave on $[0,1]$, and then it attains its maximum at a unique point $t_K \in (0,1)$. Using (vii) we can see that

$$\begin{aligned} \max_{0\le t\le 1}[\Phi_K(\sqrt{t})^2 - t] &= \max_{0\le x\le 1}[\Phi_K(x)^2 - x^2] \\ &\le \max_{0\le x\le 1}\left[\left(\frac{1-4^{1-K}(1-x)/(1+x)^K}{1+4^{1-K}(1-x)/(1+x)^K}\right)^2 - x^2\right] \\ &= \max_{0\le x\le 1}[f(g_K(x)) - f(x)], \end{aligned} \tag{2.38}$$

where f and g_K are as described in Lemma 1.1, and $0 \le x \le 1$.

Let $0 \le r \le 1$. With the notation of Lemma 1.1 we see that

$$m_K(x - x_K) = g'_K(r)(x - r) + g_K(r) \le g_K(x) \quad \text{for } 0 \le x \le 1. \tag{2.39}$$

Hence

$$m_K = g'_K(r) = 4^{1-K}Kr^{K-1} \tag{2.40}$$

and

$$x_K = r - \frac{g_K(r)}{g'_K(r)} = r\left(1 - \frac{1}{K}\right). \tag{2.41}$$

Suppose that $r = 1/2$. For each $K \ge 1$,

$$2 \cdot 8^{K-1} \ge 3K - 1.$$

Moreover, for $1 \le K \le K_0$, we have

$$4 \cdot 8^{K-1}K(K+1) \le (3K-1)^3, \tag{2.42}$$

where $K_0 \ne 1$ is the root of

$$4 \cdot 8^{K-1}K(K+1) = (3K-1)^3.$$

The inequality (2.42) is equivalent to (2.23). Hence, by Lemma 1.1, we can see that

$$\max_{0\le x\le 1}[f(g_K(x)) - f(x)] \le 1 - f(x_K) = 1 - \left(\frac{K+1}{3K-1}\right)^2 \tag{2.43}$$

for $1 \le K \le 3/2 < K_0$.

Suppose now $r = 1$. For each $K \geq 3/2$,

$$4^{K-1} \geq 2K - 1$$

and, for $3/2 \leq K \leq K_1$,

$$4^{K-1}K \leq (2K-1)^3, \tag{2.44}$$

where $K_1 \neq 1$ is the root of

$$4^{K-1}K = (2K-1)^3.$$

Hence, (2.44) is equivalent to (2.23). Then, by Lemma 1.1, we have

$$\max_{0\leq x\leq 1}[f(g_K(x)) - f(x)] \leq 1 - f(x_K) = 1 - \frac{1}{(2K-1)^2} \tag{2.45}$$

for $3/2 \leq K \leq 4 < K_1$.

Suppose that, for $K \geq 4$, m_K and x_K satisfy

$$m_K(1 - x_K) = 4^{1-K} = g_K(1) \tag{2.46}$$

and

$$m_K = \frac{1 - x_K}{(1 + x_K)^2}. \tag{2.47}$$

Hence, by (2.46) and (2.47) one can see that

$$\frac{1 - x_K}{1 + x_K} = 2^{1-K} \tag{2.48}$$

and

$$m_K = 2^{1-K}\left(1 + \frac{1 - 2^{1-K}}{1 + 2^{1-K}}\right)^{-1} = 2^{-K}(1 + 2^{1-K}) \geq 4^{1-K}K = g'_K(1) \tag{2.49}$$

for $K \geq 4$. Then

$$m_K(x - x_K) \leq g'_K(1)(x - 1) + g_K(1) \leq g(x) \quad \text{for } 0 \leq x \leq 1 \tag{2.50}$$

and, moreover,

$$\frac{1 - x_K}{(1 + x_K)^3} \leq \frac{1 - x_K}{(1 + x_K)^2} = m_K \leq \frac{1}{1 + x_K}. \tag{2.51}$$

Hence, the assumptions of Lemma 1.1 are satisfied with m_K and x_K given by (2.48) and (2.49), and so

$$\max_{0\leq x\leq 1}[f(g_K(x)) - f(x)] \leq 1 - f(x_K) = 1 - 4^{1-K} \tag{2.52}$$

for $K \geq 4$, which completes the proof.

Remark 1.2. Recently, D. Partyka [P4] proved that $M(K) = 2\Phi_{\sqrt{K}}(1/\sqrt{2})^2 - 1$.

3. Quasisymmetric functions. Now, we turn our attention to some problems on ϱ-qs functions of $\mathbb{R}$, and prove auxiliary but indispensable theorems of elementary nature on these automorphisms. In this way one is led to have control on the growth of normalized ϱ-qs functions by the distortion function Φ_K. To visualize this, we show with a quantitative estimation that automorphisms of $Q^{\circ}_{\mathbb{R}}(\varrho)$ approach the identity as $\varrho \to 0$. Comparable results can be found in [L, p. 32], [HH], [Hi1], [Hi2], and recently in [VV]. However, this subordination is crucial in establishing the passage from ϱ-quasisymmetric functions to K-quasihomographies if we do not want to make use of qc extensions.

Recall that some Hölder-type estimates for normalized ϱ-qs functions have been obtained by J. A. Kelingos [Ke]. Unfortunately these are not asymptotically sharp for $\varrho = 1$.

Sharp Hölder-type estimates for these functions are presented in

THEOREM 1.4. *Suppose that f is a normalized ϱ-qs function of $\mathbb{R}$. For each $m \in \mathbb{N}$, we have*:

$$\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)t^{\alpha_m} \le f(t) \le \left(1+\frac{1}{(\varrho+1)^m-1}\right)t^{\beta_m} \tag{3.1}$$

for $0 \le t \le 1$ *and every* $\varrho \ge 1$;

$$\begin{aligned}\left(\frac{2}{\varrho}-1\right)\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)(t_2-t_1)^{\alpha_m} \\ \le f(t_2)-f(t_1) \le (2\varrho-1)\left(1+\frac{1}{(\varrho+1)^m-1}\right)(t_2-t_1)^{\beta_m}\end{aligned} \tag{3.2}$$

for $0 \le t_1 \le t_2 \le 1$ *and every* $\varrho \ge 1$;

$$\left(1+\frac{1}{(\varrho+1)^m-1}\right)t^{\beta_m} \le f(t) \le \left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)^{-1} t^{\alpha_m} \tag{3.3}$$

for $t \ge 1$ *and every* $\varrho \ge 1$. *Here*

$$\alpha_m = \log_{(2^m-1)/2^m}\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right) \tag{3.4}$$

and

$$\beta_m = \log_{(2^m-1)/2^m}\left(1-\left(\frac{1}{\varrho+1}\right)^m\right). \tag{3.5}$$

Proof. Let $m \in \mathbb{N}$ and $c_m = 1-2^{-m}$. Since f is ϱ-qs, the inequalities

$$\frac{\varrho}{\varrho+1}f(a)+\frac{1}{\varrho+1}f(b) \le f\left(\frac{a+b}{2}\right) \le \frac{1}{\varrho+1}f(a)+\frac{\varrho}{\varrho+1}f(b) \tag{3.6}$$

hold for every $a, b \in [0,1]$, $a < b$, if and only if f is linear or a ϱ-qs function. By induction on m one can prove the inequalities

$$\begin{aligned}\left(\frac{\varrho}{\varrho+1}\right)^m f(a)+\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)f(b) \\ \le f((1-c_m)a+c_mb) \le \left(\frac{1}{\varrho+1}\right)^m f(a)+\left(1-\left(\frac{1}{\varrho+1}\right)^m\right)f(b)\end{aligned} \tag{3.7}$$

for $a, b \in [0,1]$ and each $m \in \mathbb{N}$. Induction on n gives

$$\begin{aligned}c_m^{n\alpha_m} = c_m^{n}{}^{\log_{c_m}\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)} = \left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)^n \le f(c_m^n) \\ \le \left(1-\left(\frac{1}{\varrho+1}\right)^m\right)^n = c_m^{n\log_{c_m}(1-(\frac{1}{\varrho+1})^m)} = c_m^{n\beta_m}\end{aligned} \tag{3.8}$$

for each $n = 0, 1, 2, \ldots$

Since f is strictly increasing for every $t \in [c_m^n, c_m^{n-1}]$, $m, n = 1, 2, \ldots$, we have

$$f(t) \le f(c_m^{n-1}) \le (c_m^{n-1})^{\beta_m} \le (c_m^{-1} t)^{\beta_m} = c_m^{-\beta_m} t^{\beta_m}$$

and

$$f(t) \ge f(c_m^n) \ge (c_m^n)^{\alpha_m} \ge (c_m t)^{\alpha_m} = c_m^{\alpha_m} t^{\alpha_m}.$$

This yields (3.1) because $[0,1] = \{0\} \cup \bigcup_{n=1}^{\infty} [c_m^n, c_m^{n-1}]$ for each $m \in \mathbb{N}$.

Suppose that $f \in Q_{\mathbb{R}}(\varrho)$. For every fixed $t_1 \in [0,1]$, the function

$$g_{t_1}(t) = \frac{f(t+t_1) - f(t_1)}{f(1+t_1) - f(t_1)} \tag{3.9}$$

belongs to $Q_{\mathbb{R}}^{\circ}(\varrho)$, with the same $\varrho \ge 1$. Let $0 \le t_1 \le t_2 \le 1$. Then, by (3.1) with $t = t_2 - t_1$,

$$f(t_2) - f(t_1) \le (f(1+t_1) - f(t_1)) \left(1 + \frac{1}{(\varrho+1)^m - 1}\right) (t_2 - t_1)^{\beta_m} \tag{3.10}$$

and

$$f(t_2) - f(t_1) \ge (f(1+t_1) - f(t_1)) \left(1 + \left(\frac{1}{\varrho+1}\right)^m\right) (t_2 - t_1)^{\alpha_m} \tag{3.11}$$

for every $m \in \mathbb{N}$. By (3.9) and the definition of quasisymmetry, we see that

$$\frac{1}{\varrho} g_1(t_1) - f(t_1) + 1 \le f(t_1 + 1) - f(t_1) \le \varrho g_1(t_1) - f(t_1) + 1. \tag{3.12}$$

Since the inequality

$$|f(t) - t| \le \frac{\varrho - 1}{\varrho + 1} \tag{3.13}$$

holds for each $f \in Q_{\mathbb{R}}^{\circ}(\varrho)$, $\varrho \ge 1$ and $0 \le t \le 1$ (see [Kr2]), we have

$$t_1 - \frac{\varrho - 1}{\varrho + 1} \le g_1(t_1) \le t_1 + \frac{\varrho - 1}{\varrho + 1} \tag{3.14}$$

for every $t_1 \in [0,1]$ and $\varrho \ge 1$.

Consequently,

$$f(1+t_1) - f(t_1) \le \varrho\left(t_1 + \frac{\varrho-1}{\varrho+1}\right) - t_1 + \frac{\varrho-1}{\varrho+1} + 1 = (\varrho-1)t_1 + \varrho \le 2\varrho - 1 \tag{3.15}$$

and

$$\begin{aligned} f(1+t_1) - f(t_1) &\ge \frac{1}{\varrho}\left(t_1 - \frac{\varrho-1}{\varrho+1}\right) - t_1 - \frac{\varrho-1}{\varrho+1} + 1 \\ &= \left(\frac{1}{\varrho} - 1\right) t_1 + \frac{1}{\varrho} \ge \frac{2}{\varrho} - 1. \end{aligned} \tag{3.16}$$

By (3.16), (3.10) and (3.11), we obtain

$$\frac{2}{\varrho} - 1 \le f(1+t_1) - f(t_1) \le 2\varrho - 1. \tag{3.17}$$

The left-hand estimate of (3.17) is essential for $1 \le \varrho \le 2$, but asymptotically sharp.

Inequalities (3.3) can be derived in much the same way as (3.1). This ends the proof.

COROLLARY 1.1. *For $m = 1$ the inequalities* (3.1) *and* (3.3) *reduce to those of Kelingos* [Ke, Thm. 10], *whereas* (3.2) *is better. The sharpness is obtained as $m \to \infty$.*

Now we prove

LEMMA 1.2. *Let $f : [a, b] \to \mathbb{R}$ be strictly increasing and concave. Then*

$$\frac{f(t+s_t)-f(t)}{f(t)-f(t-s_t)} \le \frac{f(t+s)-f(t)}{f(t)-f(t-s)} = F(t,s) \le 1 \tag{3.18}$$

for every $t \in (a, b)$ and $0 < s \le s_t = \min\{b - t, t - a\}$.

Proof. Let $t \in (a, b)$ and $0 < s < s_t$ be arbitrary, and set $d = s_t - s$. By the concavity of f we have

$$f(t-s) \ge \frac{s}{s_t} f(t-s_t) + \frac{d}{s_t} f(t) \quad \text{and} \quad f(t+s) \ge \frac{d}{s_t} f(t) + \frac{s}{s_t} f(t+s_t).$$

Thus

$$f(t) - f(t-s) \le \frac{s}{s_t}(f(t) - f(t-s_t)) \quad \text{and} \quad f(t+s) - f(t) \ge \frac{s}{s_t}(f(t+s_t) - f(t)).$$

Since f is strictly increasing,

$$\frac{f(t+s)-f(t)}{f(t)-f(t-s)} \ge \frac{f(t+s_t)-f(t)}{f(t)-f(t-s_t)}. \tag{3.19}$$

Using once again the concavity of f one obtains

$$f(t) \ge \tfrac{1}{2} f(t-s) + \tfrac{1}{2} f(t+s),$$

and so

$$f(t+s) - f(t) \le f(t) - f(t-s),$$

which completes the proof.

This lemma has a very practical application. It means that searching for the qs constant ϱ of a given concave and increasing homeomorphism f on $[a, b]$ we learn that it is attained on the upper frame of the domain of F, defined by (3.18).

Another application of Lemma 1.2 yields

THEOREM 1.5. *Suppose that $f : D \to \mathbb{R}$ is strictly increasing and concave. Then f is ϱ-qs in each of the following cases*:

(i) *$D = (a, b)$ and*

$$\min\left\{\inf_{t\in(a,(a+b)/2]} \frac{f(2t-a)-f(t)}{f(t)-f(a)}, \inf_{t\in[(a+b)/2,b)} \frac{f(b)-f(t)}{f(t)-f(2t-b)}\right\} = \frac{1}{\varrho} > 0;$$

(ii) *$D = (b, \infty)$ and*

$$\inf_{t\in(b,\infty)} \frac{f(2t-b)-f(t)}{f(t)-f(b)} = \frac{1}{\varrho} > 0;$$

(iii) *$D = (-\infty, a)$ and*

$$\inf_{t\in(-\infty,a)} \frac{f(a)-f(t)}{f(t)-f(2t-a)} = \frac{1}{\varrho} > 0;$$

(iv) $D = \mathbb{R}$ *and*

$$\inf_{t\in\mathbb{R}}\left(\lim_{x\to\infty}\frac{f(t+x)-f(t)}{f(t)-f(t-x)}\right)=\frac{1}{\varrho}>0.$$

Using Theorem 1.4 we shall prove the following estimates, which are crucial in establishing the passage from quasisymmetric functions to quasihomographies.

THEOREM 1.6. *Suppose that f is a ϱ-qs function of $[0,1]$ onto $[0,1]$. Then, for each $\varrho \ge 1$, there is a constant $K = K(\varrho)$ such that*

$$\Phi_{1/K}\big(\sqrt{t}\big)^2 \le f(t) \le \Phi_K\big(\sqrt{t}\big)^2 \quad \textit{for } 0 \le t \le 1, \tag{3.20}$$

where

$$K \le \nu(\varrho) = \begin{cases} \dfrac{e^{2\sqrt{\varrho-1}}}{1-2^{-m}e^{1/m}}\cdot\dfrac{1}{1+\log_2(1-2^{-m})}, & 1\le\varrho\le 5/4,\\ 3.41\cdot\log_2(1+\varrho), & 5/4<\varrho\le 6,\\ (\log 2)\left(1-\dfrac{1}{\log_2(2/\varrho\log_2(1+\varrho))}\right)(1+\varrho), & \varrho>6,\end{cases} \tag{3.21}$$

with $m = \operatorname{Ent}\{1/\sqrt{\varrho-1}\}$ and $\nu(\varrho) \cong (\log 2)(1+\varrho)$ as $\varrho\to\infty$.

Figure 5 shows the graph of the function ν. To see better the graph of ν for $1 \le \varrho \le 5/4$, one may look at Figure 6, which is a magnification of ν within this segment.

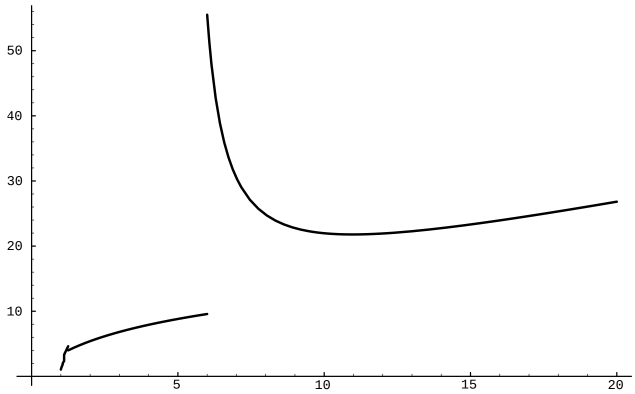

Fig. 5

M. K. Vamanamurthy and M. Vuorinen [VV] lowered the bound (3.21) for $1 < K < 4.84151$ and $6 < K < 11.4087$ by using a different technique to obtain

$$K \le \nu_1(\varrho) := \min\{\varrho^{3/2}, 2\varrho-1\}.$$

The function ν_1 can be easily recognized in Figure 7, where ν and ν_1 are in the same picture. Figure 8 shows that ν_1 is essentially better for ϱ close to 1.

P r o o f o f T h e o r e m 1.6. Note that $1-f(1-t)$ is a ϱ-qs function mapping $[0,1]$ onto $[0,1]$ as soon as f is. By Theorem 1.4,

$$f(t) \le \min\{c_m^{-\beta_m}t^{\beta_m}, 1-c_m^{\alpha_m}(1-t)^{\alpha_m}\}$$

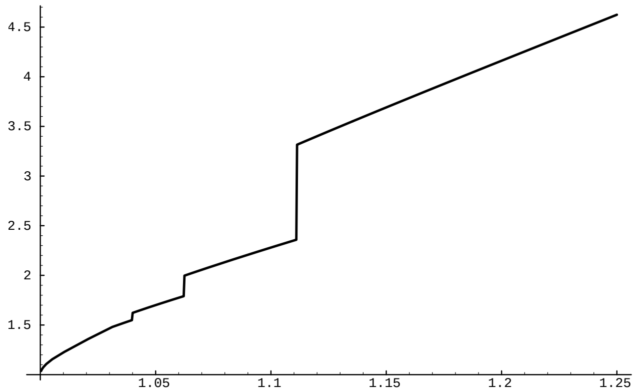

Fig. 6

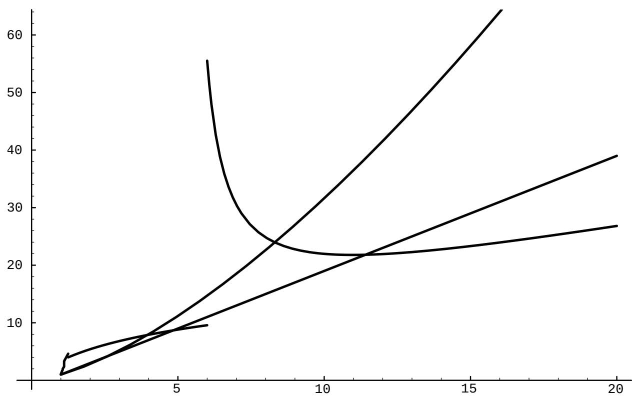

Fig. 7

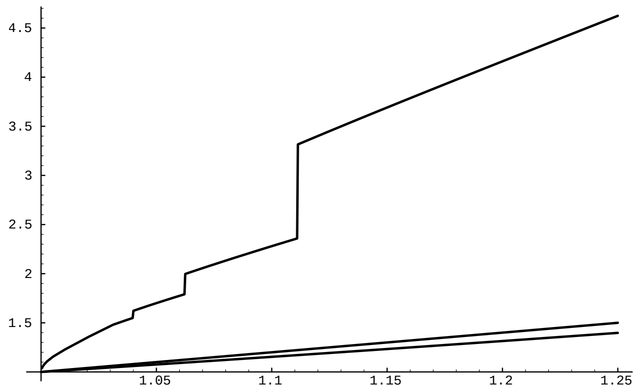

Fig. 8

for every $t \in [0,1]$ and all $m \in \mathbb{N}$. Let $\lambda \in (0, c_m)$ and

$$K_{\lambda,m} = \max\left\{\frac{1}{\beta_m}\frac{\log_{1/c_m}\lambda}{\log_{1/c_m}\lambda + 1}, \alpha_m \frac{\log_{1/c_m}(1-\lambda) - 1}{\log_{1/c_m}(1-\lambda)}\right\}. \tag{3.22}$$

Then

$$c_m^{-\beta_m} t^{\beta_m} \le t^{1/K_{\lambda,m}} \qquad \text{for } 0 \le t \le \lambda$$

and

$$(1-t)^{K_{\lambda,m}} \le c_m^{\alpha_m}(1-t)^{\alpha_m} \quad \text{for } \lambda \le t \le 1.$$

Now, by Theorem 1.5(ii) and (3.1), one has

$$f(t) \le \Phi_{K_{\lambda,m}}\big(\sqrt{t}\big)^2 \quad \text{for } 0 \le t \le \lambda, \tag{3.23}$$

and again by Theorem 1.5(ii) and (3.1),

$$\begin{aligned} f(t) &\le 1 - c_m^{\alpha_m}(1-t)^{\alpha_m} \le 1-(1-t)^{K_{\lambda,m}} \\ &\le 1 - \Phi_{1/K_{\lambda,m}}\big(\sqrt{1-t}\big)^2 = \Phi_{K_{\lambda,m}}\big(\sqrt{t}\big)^2 \quad \text{for } \lambda \le t \le 1. \end{aligned} \tag{3.24}$$

Hence,

$$f(t) \le \Phi_K\big(\sqrt{t}\big)^2 \quad \text{for } 0 \le t \le 1,$$

where

$$K = \min_{m\in\mathbb{N}} \min_{0<\lambda<c_m} K_{\lambda,m} \le \min_{m\in\mathbb{N}} K_{\lambda_m,m}$$

and λ_m is the solution of

$$\frac{\log_{1/c_m}\lambda}{1+\log_{1/c_m}\lambda} = \alpha_m\beta_m \frac{\log_{1/c_m}(1-\lambda)-1}{\log_{1/c_m}(1-\lambda)}. \tag{3.25}$$

Consider first the case $1 \le \varrho \le 5/4$. We have the following estimates:

$$\begin{aligned} \alpha_m &= \frac{\log(1-(\varrho/(\varrho+1))^m)}{\log(1-2^{-m})} \le \left(\frac{2\varrho}{\varrho+1}\right)^m \frac{1}{1-(\varrho/(\varrho+1))^m} \\ &\le \varrho^m \frac{1}{1-(\varrho/(\varrho+1))^m} \le \left(1+\frac{1}{m^2}\right)^m \frac{1}{1-2^{-m}e^{1/m}} \\ &\le \frac{e^{1/m}}{1-2^{-m}e^{1/m}} \quad \text{for } 1 \le \varrho \le 1+\tfrac{1}{m^2}. \end{aligned} \tag{3.26}$$

Similarly we obtain the estimate

$$\beta_m \ge (1-2^{-m})e^{-1/(2m)} \quad \text{for } 1 \le \varrho \le 1+1/m^2. \tag{3.27}$$

Suppose that $m \ge 2$ is the smallest possible number for which (3.26) and (3.27) are satisfied with $\lambda = 1/2$. Then

$$\begin{aligned} K \le K_{1/2,m} &= \max\left\{\frac{1}{\beta_m}\cdot\frac{1}{1+\log_2(1-2^{-m})}, \alpha_m(1-\log_2(1-2^{-m}))\right\} \\ &\le \max\left\{\frac{e^{1/(2m)}}{(1-2^{-m})(1+\log_2(1-2^{-m}))}, \frac{e^{1/m}}{1-2^{-m}e^{1/m}}(1-\log_2(1-2^{-m}))\right\} \\ &\le \max\left\{\frac{e^{1/(2m)}}{(1-2^{-m})(1+\log_2(1-2^{-m}))}, \frac{e^{1/m}(1-\log_2^2(1-2^{-m}))}{(1-2^{-m}e^{1/m})(1+\log_2(1-2^{-m}))}\right\} \\ &\le \max\left\{\frac{e^{1/(2m)}}{(1-2^{-m})(1-\log_2(1-2^{-m}))}, \frac{e^{1/m}}{(1-2^{-m}e^{1/m})(1+\log_2(1-2^{-m}))}\right\} \end{aligned}$$

$$\le \frac{e^{1/m}}{1-2^{-m}e^{1/m}}\cdot\frac{1}{1+\log_2(1-2^{-m})} \le \frac{e^{1/m}}{1-2^{-m}e^{1/m}}\cdot\frac{1}{1+\log_2\frac{1}{2}},$$

where $m = \operatorname{Ent}\left\{1/\sqrt{\varrho-1}\right\}$. Since

$$\frac{1}{m} < \frac{\sqrt{\varrho-1}}{1-\sqrt{\varrho-1}} \le 2\sqrt{\varrho-1},$$

we obtain

$$K \le \nu(\varrho) = \frac{e^{2\sqrt{\varrho-1}}}{1-2^{-\operatorname{Ent}\{1/\sqrt{\varrho-1}\}}e^{1/m}}\cdot\frac{1}{1+\log_2(1-2^{-m})}. \tag{3.28}$$

Note that $\nu(\varrho)\to 1$ as $\varrho \to 1$.

Consider now the case $1\le\varrho\le 6$. By setting $m=1$ and $\lambda = 1/4$, we have

$$\begin{aligned} K &\le \min_{0<\lambda<c_1} K_{\lambda,1} \le K_{1/4,1} \\ &= \max\left\{\frac{1}{\beta_1}\frac{\log_{1/c_1}(1/4)}{\log_{1/c_1}(1/4)+1}, \alpha_1\frac{\log_{1/c_1}(3/4)-1}{\log_{1/c_1}(3/4)}\right\} \\ &= \max\left\{\frac{2}{\log_2(1+1/\varrho)}, \log_2(1+\varrho)\frac{\log_2 3-3}{\log_2 3-2}\right\} \\ &\le \frac{\log_2(3/8)}{\log_2(3/4)}\log_2(1+\varrho) < 3.41\cdot\log_2(1+\varrho) = \nu(\varrho) \end{aligned} \tag{3.29}$$

for $5/4<\varrho\le 6$.

To obtain the last case we set $m=1$, $\alpha_1=\alpha$, $\beta_1=\beta$ and $\varrho\ge 6$. Then we have

$$\alpha\beta\log 2 = \log_2(1+\varrho)\log_2\left(1+\frac{1}{\varrho}\right)\log 2 < \frac{1}{\varrho}\log_2(1+\varrho) < \frac{1}{2} < \frac{\log^3 2}{2(1-\log 2)}.$$

Hence

$$2^{1+1/(\alpha\beta)} \ge 2\left(\frac{\log 2}{\alpha\beta}+\frac{\log^2 2}{2(\alpha\beta)^2}\right) \ge \frac{1}{\alpha\beta\log 2}$$

and so $\alpha\beta < 1/(r-1)$ with $r=\log_2(\alpha\beta\log 2)$. By setting $\lambda = 2^{-r}$ we arrive at

$$\begin{aligned} K \le K_{\lambda,1} &= \max\left\{\frac{1}{\beta}\cdot\frac{r}{r-1}, \alpha\left(1-\frac{1}{\log_2(1-2^{-r})}\right)\right\} \\ &\le \max\left\{\frac{1}{\beta}\cdot\frac{r}{r-1}, \alpha(1+(\log 2)2^r)\right\} \\ &\le \max\left\{\frac{1}{\beta}\cdot\frac{r}{r-1}, \alpha+\frac{1}{\beta}\right\} \le \frac{1}{\beta}\frac{r}{r-1}. \end{aligned} \tag{3.30}$$

Then

$$\begin{aligned} K &\le \frac{1}{\log_2(1+1/\varrho)}\left(1-\frac{1}{\log_2(\alpha\beta\log 4)}\right) \\ &\le (\log 2)(\varrho+1)\left(1-\frac{1}{\log_2(\alpha\beta\log 4)}\right) \\ &\le (\log 2)\left(1-\frac{1}{\log_2(2/\varrho)\log_2(1+\varrho)}\right)(\varrho+1) = \nu(\varrho) \end{aligned} \tag{3.31}$$

for $\varrho > 6$.

We have $\nu(\varrho) \cong (\log 2)(\varrho + 1)$ as $\varrho \to \infty$. To obtain the left-hand inequality in (3.20) we notice that $g(t) = 1 - f(1-t)$ is a ϱ-qs function if f is. Substituting $x = 1 - t$ we have

$$f(x) \ge 1 - \Phi_K\big(\sqrt{1-x}\,\big)^2 = \Phi_{1/K}\big(\sqrt{x}\,\big)^2, \tag{3.32}$$

which ends the proof.

The following result is in some sense opposite:

THEOREM 1.7. *For every $K \ge 1$, there exists $\varrho \ge 1$ such that Φ_K is ϱ-qs on $[0,1]$ with*

$$\varrho < \max\{2^{5K-3}, 2^{2-3/K}(1 - \Phi_K(1/2))^{-1}\} = 2^{5K-3}. \tag{3.33}$$

Proof. Note that Φ_K is concave for each $K > 1$. Let $t \in (0, 1/2]$. Then, by Lemma 1.2 and (ii), we have

$$\begin{aligned}\frac{\Phi_K(2t) - \Phi_K(t)}{\Phi_K(t)} &= \frac{\Phi_K(2t) - \Phi_K\big(2t\cdot\frac12\big)}{\Phi_K(t)} \ge \frac{\Phi_K(2t)}{\Phi_K(t)}\big(1 - \Phi_K\big(\tfrac12\big)\big)\\ &\ge \frac{(2t)^{1/K}}{4^{1-1/K}t^{1/K}}\big(1 - \Phi_K\big(\tfrac12\big)\big) = \frac{8^{1/K}}{4}\big(1 - \Phi_K\big(\tfrac12\big)\big).\end{aligned}$$

For $t \in [1/2, 1)$ we have, since $\Phi_{1/K} = \Phi_K^{-1}$,

$$\begin{aligned}\frac{\Phi_K(1) - \Phi_K(t)}{\Phi_K(t) - \Phi_K(2t-1)} &\ge \frac{1 - \Phi_K(t)}{1 - \Phi_K(2t-1)} = \frac{1 - \Phi_K(t)^2}{1 - \Phi_K(2t-1)^2}\,\frac{1 + \Phi_K(2t-1)}{1 + \Phi_K(t)}\\ &\ge \frac{\Phi_{1/K}\big(\sqrt{1-t^2}\,\big)^2}{\Phi_{1/K}\big(\sqrt{1-(2t-1)^2}\,\big)^2}\cdot\frac12 \ge \frac{\big(4^{1-K}\big(\sqrt{1-t^2}\,\big)^K\big)^2}{\big(\sqrt{1-(2t-1)^2}\,\big)^{2K}}\cdot\frac12\\ &= \frac{16^{1-K}(1-t^2)^K}{2(4t-4t^2)^K} = 8\cdot 4^{-3K}\left(1 + \frac1t\right)^K \ge 8\cdot 2^{-5K}.\end{aligned}$$

The last equality is an immediate consequence of a new estimate on the distortion function Φ_K obtained by M. K. Vamanamurthy and M. Vuorinen [VV, Thm. 1.4]. The proof is complete.

Hence, the following open problem arises: *Find an asymptotically sharp bound for the qs constant of the distortion function $\Phi_{K,n}$, $n \ge 3$.*

4. Functional identities for special functions. Concerning the theory of quasiconformal mappings in $\mathbb{R}^n$ with $n = 2, 3, \ldots$ (see [Cm], [Vä3] and [BI]) we are particularly interested in two rings having extremal properties. The first is the Grötzsch ring $R_{\mathrm{G},n}(s)$, $s > 1$, whose complementary components are the closed unit ball $\overline{B}^n$ and the ray $[se_1, \infty]$, where e_1 is the first unit vector of the rectangular coordinate axes in $\mathbb{R}^n$. The second is the Teichmüller ring $R_{\mathrm{T},n}(t)$, $t > 0$, whose complementary components are the segment $[-e_1, 0]$ and the ray $[te_1, \infty]$. The conformal capacities $\gamma_n(s)$ and $\tau_n(t)$ of $R_{\mathrm{G},n}(s)$ and $R_{\mathrm{T},n}(t)$, respectively, are decreasing functions related by the functional identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1) \quad \text{for } s > 1.$$

The work of F. W. Gehring (see [Ge1] and [Ge2]) suggests that the role of special functions in the plane case is taken by these capacities. It seems that this idea can be modified by introducing the n-dimensional counterpart of the elliptic integral, denoted by $\mathcal{K}_n$, where $\mathcal{K}_2 = \mathcal{K}$. Unfortunately, for $n \geq 3$ this is not a special case of the hypergeometric function; cf. [Z11].

Our main idea for studying special functions is a generalization of problems that were intensively studied by several authors; cf. [AVV4], [Vo] and other papers. This gives immediately a number of final results on these functions, both on the plane and in space; see [Z7].

The distortion function $\Phi_{K,n} : [0,1] \to [0,1]$ in the n-dimensional ($n \geq 2$) quasiregular version of the Schwarz Lemma (see [MRV] and [Vo]) is defined for $K > 0$ and $n = 2,3,4,\ldots$ by $\Phi_{K,n}(0) = 0$, $\Phi_{K,n}(1) = 1$, and

$$\Phi_{K,n}(t) = M_n^{-1}\left(\frac{1}{K_n} M_n(t)\right) \tag{4.1}$$

for $0 < t < 1$, where $K_n = K^{1/(n-1)}$ and M_n is given by

$$\gamma_n(1/t) = \omega_{n-1}^{\circ} M_n^{1-n}(t). \tag{4.2}$$

Here $\omega_{n-1}^{\circ} = (2/\pi)^{n-1}\omega_{n-1}$, with ω_{n-1} the $(n-1)$-dimensional surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$. Introducing any positive constant multiplier to the formula (4.2) that defines M_n, we do not alter $\Phi_{K,n}$. Thus, for convenience, we normalize ω_{n-1} as above. For $n = 2$ we have $M_2 = \mu$, where μ is given by (2.2) and $\Phi_{K,2} = \Phi_K$. This results in several functional identities; see [AVV1] and [Vo].

For the higher-dimensional case $n \geq 3$ neither such explicit expressions nor functional identities were known; cf. [Vo].

Remark 1.3. It is worth noting that the generalized Schwarz Lemma is valid for *quasiregular mappings*, which is wider than the class of qc mappings.

We recall that the explicit estimate

$$t^{1/K_n} \leq \Phi_{K,n}(t) \leq \lambda_n^{1-1/K_n} t^{1/K_n} \tag{4.3}$$

holds for $K \geq 1, n = 2,3,\ldots$ and $0 \leq t \leq 1$, where $K_n = K^{1/(n-1)}$. The constant λ_n is known only when $n = 2$, and in this case $\lambda_2 = 4$; see [LV, p. 62]. Generally, $2e^{0.76(n-1)} \leq \lambda_n \leq 2e^{n-1}$ for $n \geq 3$; see [Vo, p. 89].

The main purpose of this section is to show how one can obtain

* a one-parameter family of identities satisfied by $\Phi_{K,n}$, $n = 2,3,\ldots$ and $K > 0$,
* equivalent identities for M_n, γ_n, τ_n, and for $\mathcal{K}$ when $n = 2$;
* a *dynamical* convergence formula for $\Phi_{K,n}$ as an application.

Our idea can be realized since we know that M_n is differentiable in $(0,1)$ for $n = 2,3,\ldots$; see [An].

Let $\mathcal{H}$ denote the family of all differentiable automorphisms of $(0,1)$, and $\mathcal{H}_n$, $n = 2,3,\ldots$, be the set of all involutions $h \in \mathcal{H}$ such that

$$h \circ \Phi_{K,n} = \Phi_{1/K,n} \circ h \tag{4.4}$$

for each $K > 0$. The formula (4.4) is called the *involute identity.*

Moreover, let $Q = (0,1) \times (0,1)$. Then we have

THEOREM 1.8. *Let $n = 2, 3, \ldots,$ be fixed. A function h belongs to $\mathcal{H}_n$ if and only if there is a number $L > 0$ such that*

$$h(t) = M_n^{-1}(L_n/M_n(t)) \tag{4.5}$$

for $0 < t < 1$ and $L_n = L^{1/(n-1)}$. Moreover, if $(\xi, \eta) \in Q$ is an arbitrary point, then there is a number $L_n^{\xi\eta}$ such that $h(\xi) = \eta$.

Proof. Assume that $h \in \mathcal{H}_n$. Introducing $\widetilde{M}_n = M_n \circ h \circ M_n^{-1}$, we see that (4.4) with (4.1) may be written as

$$\widetilde{M}_n(t/K_n) = K_n \widetilde{M}_n(t) \tag{4.6}$$

for $0 < t < \infty$, $K_n = K^{1/(n-1)}$, $K > 0$ and $n = 2, 3, \ldots$ By the definition of $\mathcal{H}_n$ and the regularity of M_n, it follows that $\widetilde{M}_n$ is differentiable in $0 < t < \infty$. Hence, the well-known Euler identity implies that all the solutions of (4.6) can be written as

$$\widetilde{M}_n(t) = L_n/t \tag{4.7}$$

for $0 < t < \infty$, $L_n = L^{1/(n-1)}$, $L > 0$ and $n = 2, 3, \ldots$ Thus

$$h(t) = M_n^{-1}(L_n/M_n(t)). \tag{4.8}$$

Let $(\xi, \eta) \in Q$. Setting $L_n^{\xi\eta} = M_n(\xi)M_n(\eta)$ we see that $M_n^{-1}(L_n^{\xi\eta}/M_n(\xi)) = \eta$, so the second part of our theorem follows.

On the other hand, it is evident that each function of the form (4.8) belongs to $\mathcal{H}_n$ for $n = 2, 3, \ldots,$ which ends the proof.

Let

$$\mathcal{H}^\infty = \bigcup_{n=2}^{\infty} \mathcal{H}_n. \tag{4.9}$$

We may additionally assume that each function of $\mathcal{H}^\infty$ maps 0 and 1 to 1 and 0, respectively. A function from $\mathcal{H}^\infty$ is called a *conjugate distortion function*. To justify the name let us consider the family of functions defined as

$$\Psi_{K,\alpha,\beta}[\nu, t] = \nu^{-1}(K^\alpha \nu^\beta(t)) \tag{4.10}$$

for $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ and $K > 0$, where ν is a differentiable homeomorphism mapping $(0, 1)$ onto $(0, \infty)$. The family of functions defined by (4.10) forms a group of automorphisms of (0,1) under composition. With each automorphism $\Psi_{K,\alpha,\beta}[\nu, \cdot\,]$ we associate the automorphism

$$\Psi^*_{K,\alpha,\beta}[\nu, \cdot\,] = \Psi_{K,-\alpha,-\beta}[\nu, \cdot\,]. \tag{4.11}$$

The correspondence $\Psi_{K,\alpha,\beta}[\nu, \cdot\,] \to \Psi^*_{K,\alpha,\beta}[\nu, \cdot\,]$ is called the *conjugation*.

Substituting $\nu = M_n$, $\alpha = 1/(1-n)$ and $\beta = 1$ in (4.10) we obtain $\Phi_{K,n}$, whereas

$$\Phi^*_{K,n} := \Psi_{K,-1/(1-n),-1}[M_n, \cdot\,] \tag{4.12}$$

is called the *space conjugate distortion function*.

Figure 9 illustrates the family of conjugate distortion functions $\Phi^*_L := \Phi^*_{L,2}$ for $L = 0.2, 0.3, \ldots, 5$.

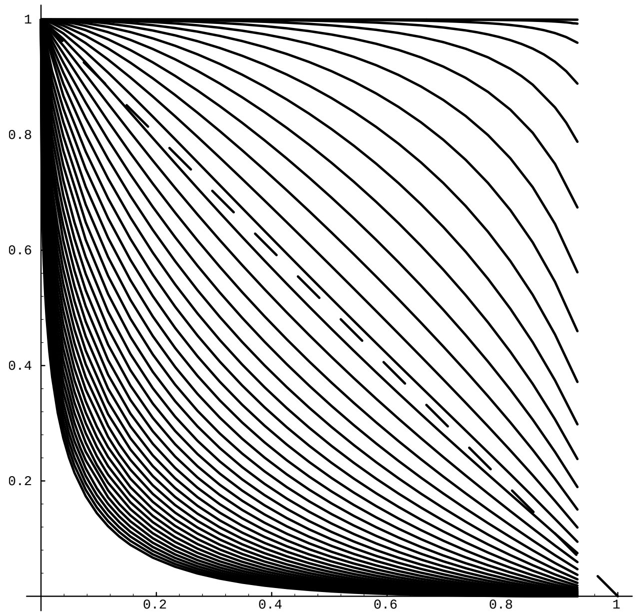

Fig. 9

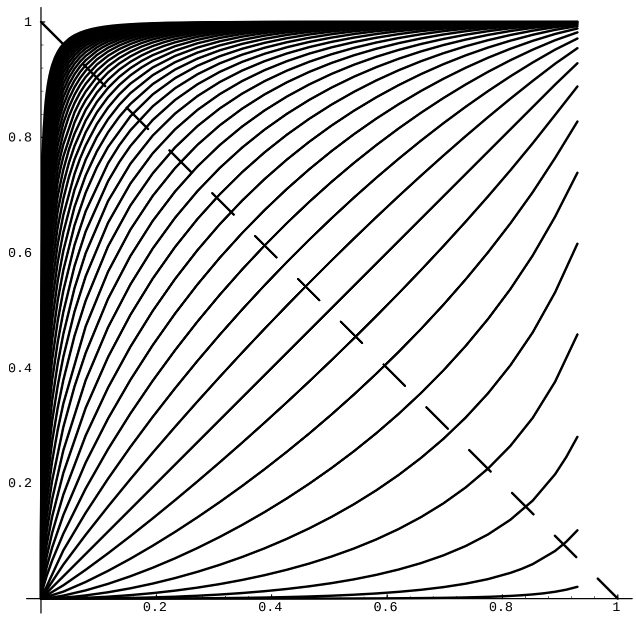

Fig. 10

Figure 10 shows the family Φ_K for $K = 0.2, 0.3, \ldots, 5$.

The graphs are obtained by identifying Φ_K with $B_2[K, 8]$, and Φ^*_L with $\Phi^*_1 \circ B_2[L, 8]$, defined by (4.22) below.

Figure 11 shows these families in one picture.

Figure 12 is a magnification of a corner of Figure 10. The line near the diagonal represents the value $L = 1.9$.

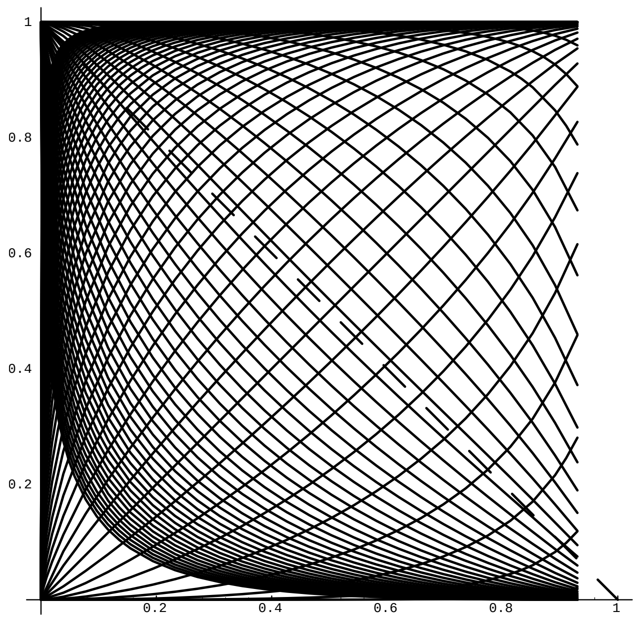

Fig. 11

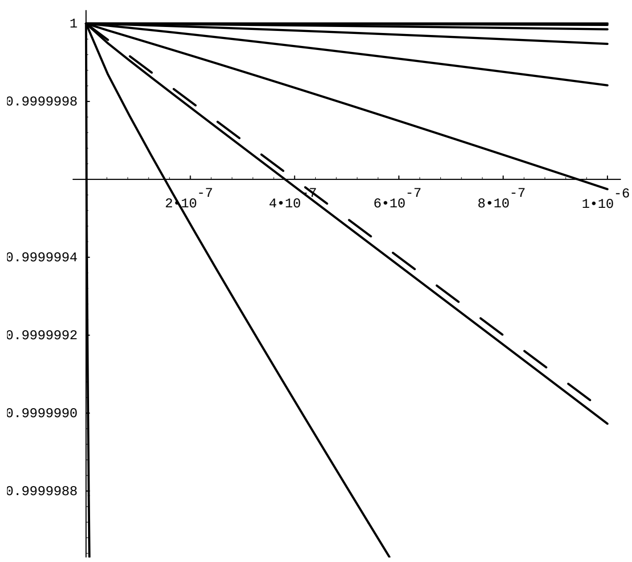

Fig. 12

For each $n = 2, 3, \dots$, let

$$\mathcal{F}_n = \bigcup_{K>0} \Phi_{K,n}. \tag{4.13}$$

Then we have

THEOREM 1.9. *Let $n = 2, 3, \dots$ be fixed. Then $\mathcal{F}_n^* = \mathcal{H}_n$ and $\mathcal{F}_n \cup \mathcal{H}_n$ is a group under composition.*

The basic properties of the conjugate distortion functions are stated in

THEOREM 1.10. *For each $K, L > 0$ and $n = 2, 3, \dots$ we have*:

(i) *for every fixed $L \in (0, \infty)$, $\Phi^*_{L,n}$ is a decreasing automorphism of $(0, 1)$, and for each $t \in (0, 1)$, $\Phi^*_{L,n}$ is a decreasing diffeomorphism of $(0, \infty)$ onto $(0, 1)$;*
(ii) $\Phi^*_{L,n} \circ \Phi^*_{K,n} = \Phi_{K/L,n}$;
(iii) $\Phi^*_{L,n} \circ \Phi_{K,n} = \Phi^*_{LK,n}$;
(iv) $\Phi_{K,n} \circ \Phi^*_{L,n} = \Phi^*_{L/K,n}$;
(v) $\Phi^*_{L,n} \circ \Phi_{K,n} \circ \Phi^*_{L,n} = \Phi_{1/K,n}$;
(vi) *with $\Phi^*_L := \Phi^*_{L,2}$ and $0 < t < 1$, we have*

$$\Phi^*_1(t) = \sqrt{1-t^2}, \quad \Phi^*_2(t) = \frac{1-t}{1+t}, \quad \Phi^*_4(t) = \left(\frac{1-\sqrt{t}}{1+\sqrt{t}}\right)^2, \quad \Phi^*_8(t) = \left(\frac{\sqrt{1+t} - \sqrt{2}\sqrt[4]{t}}{1-\sqrt{t}}\right)^4, \quad \textit{etc.}$$

Proof. The properties (i)–(v) follow from the definition of $\Phi^*_{L,n}$ and the properties of $\Phi_{K,n}$. Since $M_2(\sqrt{1-t^2}) = \mu(\sqrt{1-t^2}) = \mathcal{K}(t)/\mathcal{K}(\sqrt{1-t^2}) = 1/\mu(t)$, we have $\Phi^*_1(t) = \mu^{-1}(1/\mu(t)) = \sqrt{1-t^2}$. Making use of (iii) we obtain $\Phi^*_{2,2}$, $\Phi^*_{4,2}$, $\Phi^*_{8,2}$, etc. in explicit form. The proof is complete.

The involute identity described by (4.4) has a natural geometric interpretation. To see it, let us consider the mapping

$$\widetilde{\Phi}_{K,n}(t, x) = (\Phi_{K,n}(t), \Phi_{1/K,n}(x)), \tag{4.14}$$

which maps $\overline{Q}$ onto itself for each $K > 0$ and $n = 2, 3, \dots$ We have

$$\widetilde{\Phi}_{K,n}(t, \Phi^*_{L,n}(t)) = (\Phi_{K,n}(t), \Phi_{1/K,n} \circ \Phi^*_{L,n}(t)) = (\Phi_{K,n}(t), \Phi^*_{L,n}(\Phi_{K,n}(t))) \tag{4.15}$$

for each $0 \le t \le 1$, $K, L > 0$ and $n = 2, 3, \dots$ Thus, every curve

$$\Gamma_{L,n} = \{(t, x) : x = \Phi^*_{L,n}(t),\ 0 < t < 1\}, \qquad L > 0, \tag{4.16}$$

is invariant under each mapping from $\mathcal{F}_n$ for $n = 2, 3, \dots$ Moreover, by Theorem 1.8 we have

$$Q = \bigcup_{L>0} \Gamma_{L,n}$$

for $n = 2, 3, \dots$

THEOREM 1.11. *Let $n = 2, 3, \dots$ be fixed. The identities*

(i) $M_n(\Phi_{K,n}(t)) M_n(\Phi^*_{L,n}(t)) = L_n/K_n$;
(ii) $\gamma_n(1/\Phi_{K,n}(t)) \gamma_n(1/\Phi^*_{L,n}(t)) = (K/L)(\omega^\circ_{n-1})^2$;
(iii) $\tau_n((1/\Phi_{K,n}(t))^2 - 1) \tau_n((1/\Phi^*_{L,n}(t))^2 - 1) = (K/L)\pi^{2(1-n)}\omega^2_{n-1}$

hold for $0 < t < 1$ *and each* $K, L > 0$. *Moreover*, *these three identities and* (4.2) *are equivalent.*

Proof. By (4.12), (4.1) and (4.10) we obtain (i). The identities (ii) and (iii) follow from (4.2) and the relationship between τ_n and γ_n. This finishes the proof.

The equation

$$\frac{\mathcal{K}\circ \Phi^*_{1,2}(s)}{\mathcal{K}(s)} = p\frac{\mathcal{K}\circ \Phi^*_{1,2}(r)}{\mathcal{K}(r)} \tag{4.17}$$

is called the *modular equation* of degree p; see [BB, Ch. IV] and [AVV4].

The unique solution of (4.17) is given by $s = \Phi_{1/p}(r)$. For integer values of p, the modular equation has been studied in number theory (see [Be1], [Be2]).

In the case $n = 2$ one obtains

THEOREM 1.12. *The identity*

$$\mu = K\,\frac{\mathcal{K}\circ \Phi^*_{K,2}}{\mathcal{K}\circ \Phi_{K,2}} = L\,\frac{\mathcal{K}\circ \Phi^*_{L,2}}{\mathcal{K}\circ \Phi_{L,2}} \tag{4.18}$$

holds for each $K, L > 0$, *with* $\mathcal{K}$ *the elliptic integral.*

Proof. This observation is an immediate consequence of (2.1) and Theorem 1.11(i).

Remark 1.4. Given $n = 2, 3, \ldots$ and $L > 0$, the equation

$$\Phi^*_{L,n}(t) = t$$

has a unique solution $t_{L,n} = M_n^{-1}\big(\sqrt{L_n}\,\big)$. This observation follows directly from the definition of $\Phi^*_{L,n}$.

Remark 1.5. By Theorem 1.10 and Remark 1.3 we have the following assertions:

(i) setting $n = 2$ and then $L = 1$ or $L = 2$ in (iv), (v) of Theorem 1.10 we obtain the well-known identities (3.4)–(3.6) and (3.8)–(3.9) of [AVV1];

(ii) setting $n = 2$ and then $L = 4$ with $t = r^2$ in Theorem 1.10(iv) we obtain (3.10) of [AVV1];

(iii) the identity (1.7) of [AVV1] follows from Theorem 1.10(vi);

(iv) setting $n = 2$ and then $L = 1$ or $L = 2$ in (i) of Theorem 1.11 we obtain the identities (2.5);

(v) setting $K = 1/p$ and $L = 1$ in (4.18) we obtain (4.17). Moreover, Theorem 1.8 describes all differentiable solutions of (4.17) as well;

(vi) for $p = 2^{-n}$, $n = 1, 2, \ldots$, the solutions of (4.17) are elementary functions;

(vii) the identity (2.6) can easily be obtained from Theorem 1.11(i) by setting $n = 2$ and $K, L = 1$. Hence one can obtain the related identity satisfied by M_n^{-1}, $n \geq 2$;

(viii) by Theorem 1.12 and the first Landen identity in (2.4), one can obtain the second of these identities by taking $(K, L) = (2, 1)$. Substituting any pair of numbers $(2^n, 2^m)$, $n, m = \pm 1, \pm 2, \ldots$, in (4.18), one may easily generalize the Landen identities.

Remark 1.6. For every $K > 0$ and $0 \leq t \leq 1$, we have

(i) $\Phi^*_K(t) = \sqrt{1 - \Phi_K(t)^2} = \Phi_{1/K}\big(\sqrt{1-t^2}\,\big)$;

(ii) $\Phi^*_{2K}(t) = (1 - \Phi_K(t))/(1 + \Phi_K(t))$;

(iii) $\Phi^*_{4K}(t) = \big(1 - \sqrt{\Phi_K(t)}\,\big)^2 / \big(1 + \sqrt{\Phi_K(t)}\,\big)^2$;

(vi) $\Phi^*_{8K}(t) = \big(\sqrt{1 + \Phi_K(t)} - \sqrt[4]{4\Phi_K(t)}\,\big)^2 / \big(\sqrt{1 + \Phi_K(t)} + \sqrt[4]{4\Phi_K(t)}\,\big)^2$.

Remark 1.7. There does not exist a number $L_0 > 0$ such that

$$\Phi^*_{L_0}(t) = 1 - t.$$

Proof. Since the involute identity (4.4) must be satisfied for all K, assuming $K = 2$ and $\Phi^*_{L_0}(t) = 1 - t$ we obtain a contradiction to (4.4) at any point $t \in [0, 1]$.

Now we prove

THEOREM 1.13. *For each $L \geq 1$ and $n = 2, 3, \ldots,$ the inequalities*

$$\begin{aligned} &\lambda_n^{1-L_n} \Phi^*_{1,n}(t)^{L_n} \leq \Phi^*_{L,n}(t) \leq \Phi^*_{1,n}(t)^{L_n}, \\ &\Phi^*_{1,n}(t)^{1/L_n} \leq \Phi^*_{1/L,n}(t) \leq \lambda_n^{1-1/L_n} \Phi^*_{1,n}(t)^{1/L_n} \end{aligned} \tag{4.19}$$

hold for $0 \leq t \leq 1$ with $L_n = L^{1/(n-1)}$.

Proof. Using (iv) of Theorem 1.10 with $L = 1$ and the inequality (4.3) we obtain the second line of (4.19). Since $\Phi_{1/K,n} \circ \Phi^*_{1,n} = \Phi^*_{K,n}$, the first line of (4.19) follows by applying (4.3) to $\Phi^{-1}_{K,n} = \Phi_{1/K,n}$. This ends the proof.

In the particular case $n = 2$ we have, by Theorems 1.10(vi) and 1.13,

Remark 1.8. The inequalities

$$\begin{aligned} &4^{1-L}(1 - t^2)^{L/2} \leq \Phi^*_L(t) \leq (1 - t^2)^{L/2}, \\ &(1 - t^2)^{1/(2L)} \leq \Phi^*_{1/L}(t) \leq 4^{1-1/L}(1 - t^2)^{1/(2L)} \end{aligned} \tag{4.19$'$}$$

hold for each $0 \leq t \leq 1$ and $L \geq 1$.

From (v) of Theorem 1.10 and (4.3), applied to $\Phi_{1/K,n} = \Phi^{-1}_{K,n}$, we see that

$$\Phi^*_{L,n}(\Phi^*_{L,n}(t)^{K_n}) \leq \Phi_{K,n}(t) \leq \Phi^*_{L,n}(\lambda_n^{1-K_n} \Phi^*_{L,n}(t)^{K_n}) \tag{4.20}$$

for every $0 \leq t \leq 1$, $n = 2, 3, \ldots,$ $K \geq 1$ and $L > 0$, where $K_n = K^{1/(n-1)}$.

Let

$$b_n[K, L](t) = \Phi^*_{L,n}(\Phi^*_{L,n}(t)^{K_n}) \tag{4.21}$$

and

$$B_n[K, L](t) = \Phi^*_{L,n}(\lambda_n^{1-K_n} \Phi^*_{L,n}(t)^{K_n}) \tag{4.22}$$

for $K \geq 1$, $L > 0$, $0 \leq t \leq 1$, $n = 2, 3, \ldots,$ and $K_n = K^{1/(n-1)}$. Using these definitions one may write (4.20) as

$$\begin{aligned} &b_n[K, L](t) \leq \Phi_{K,n}(t) \leq B_n[K, L](t), \\ &\Phi^*_{1,n}(B_n[K, L](t)) \leq \Phi^*_{K,n}(t) \leq \Phi^*_{1,n}(b_n[K, L](t)). \end{aligned} \tag{4.23}$$

Setting $n = 2$ and $L = 2$, we immediately obtain (vii) of Remark 1.5. Setting $n = 2$ and $L = 4$, we see that

$$\left(\frac{\big(1 + \sqrt{t}\,\big)^K - \big(1 - \sqrt{t}\,\big)^K}{\big(1 + \sqrt{t}\,\big)^K + \big(1 - \sqrt{t}\,\big)^K} \right)^2 \leq \Phi_K(t) \leq \left(\frac{\big(1 + \sqrt{t}\,\big)^K - 2^{1-K}\big(1 - \sqrt{t}\,\big)^K}{\big(1 + \sqrt{t}\,\big)^K + 2^{1-K}\big(1 - \sqrt{t}\,\big)^K} \right)^2 \tag{4.24}$$

for $0 \leq t \leq 1$ and $K \geq 1$. The right-hand inequality is just Theorem 5.7 of [AVV4].

Figure 13 shows the graphs representing

$$B_2[K,4](t) - t, \qquad 0 \le t \le 1,$$

which are bounds for $\varphi_K(t)$ described in Theorem 1.2 and $K = 1, 1.5, 2, \ldots, 4.5, 5$.

In Figures 14 and 15 one can see the graphs which are graph bounds for $\varphi_K(t)$ with $K = 1.5, 2, \ldots, 4.5, 5$. These bounds were obtained by M. K. Vamanamurthy and M. Vuorinen [VV].

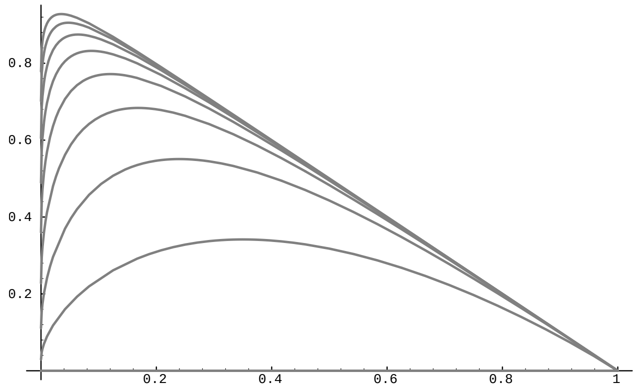

Fig. 13

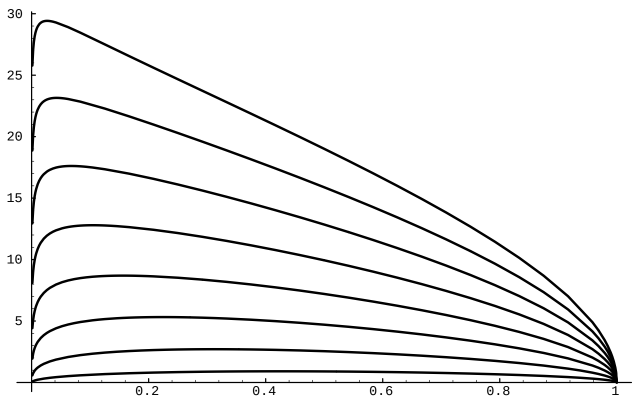

Fig. 14

In Figure 16 one may see these results in one picture, whereas Figure 17 is a magnification of Figure 16, obtained for graphs as in Figure 13 but with $K = 1, 1.01, 1.02, 1.03, 1.04, 1.09$, for graphs as in Figure 14 but with $K = 1.04, 1.09$, and for graphs as in Figure 15 but with $K = 1.04, 1.09$.

Also the graphs $4^{1-1/K}\sqrt{1-x^2}(K-1)x^{1/(4K)}\log 4$ for $K = 1.01, 1.02, 1.03$ can be found in Figure 17.

By Corollary 1.2 and (4.23), one can improve the estimates of (4.24) by taking $L = 2^i$, $i \ge 3$ and $n = 2$. This can be easily seen by the use of a computer, which is also useful in illustrating (4.21), (4.22) and (4.23).

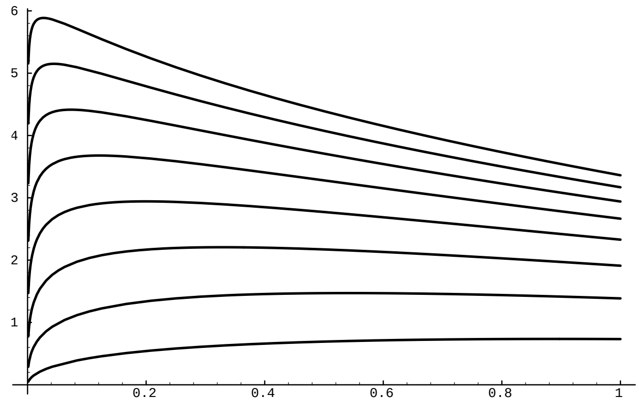

Fig. 15

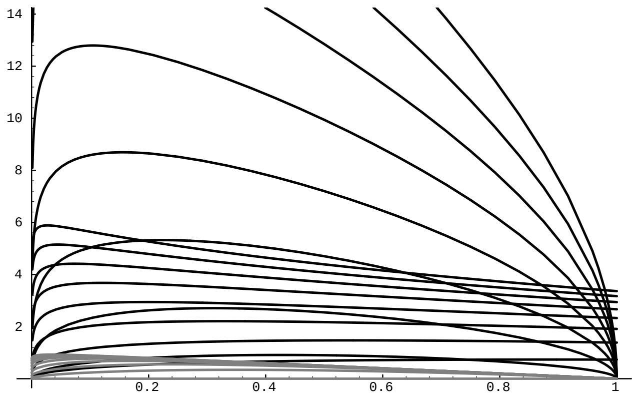

Fig. 16

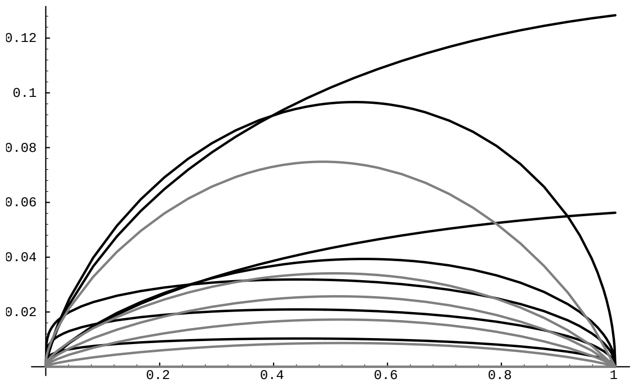

Fig. 17

In order to make use of (4.20) we shall first establish

Lemma 1.3. *For each $K > 0$ and $n = 2, 3, \ldots,$*

$$\lim_{L\to\infty} (\Phi^*_{L,n} \circ \varphi \circ \Phi^*_{L,n}(t)) = \Phi_{K,n}(t) \tag{4.25}$$

for $0 \le t \le 1$, *where* $\varphi : [0,1] \to [0,1]$ *is any function such that*

$$\lim_{t\to 0+} \frac{\log \varphi(t)}{\log t} = K_n. \tag{4.26}$$

P r o o f. Let $K > 0$ and $n = 2, 3, \ldots$ It follows from (4.26) and Theorem 3.1 in [P2] that, for $0 \le t \le 1$,

$$\lim_{L\to\infty} (\Phi_{L,n} \circ \varphi \circ \Phi_{1/L,n}(t)) = \Phi_{1/K,n}(t).$$

Hence, by Theorem 1.10, we get

$$\begin{aligned} \Phi^*_{L,n} \circ \varphi \circ \Phi^*_{L,n}(t) &= \Phi^*_{1,n} \circ (\Phi_{L,n} \circ \varphi \circ \Phi_{1/L,n}(t)) \circ \Phi^*_{1,n}(t) \\ &\to \Phi^*_{1,n} \circ \Phi_{1/K,n} \circ \Phi^*_{1,n}(t) = \Phi_{K,n}(t) \quad \text{ as } L \to \infty \end{aligned}$$

for $0 \le t \le 1$, which ends the proof.

Now we can prove

THEOREM 1.14. *For each* $K \ge 1$, $n = 2, 3, \ldots$ *and* $0 \le t \le 1$,

$$\begin{aligned} \lim_{L\to\infty} b_n[K,L](t) &= \lim_{L\to\infty} B_n[K,L](t) = \Phi_{K,n}(t), \\ \lim_{L\to\infty} \Phi^*_{1,n}(b_n[K,L](t)) &= \lim_{L\to\infty} \Phi^*_{1,n}(B_n[K,L](t)) = \Phi^*_{K,n}(t). \end{aligned} \tag{4.27}$$

P r o o f. Setting $\varphi_1(t) = t^{K_n}$ and $\varphi_2(t) = \lambda_n^{1-K_n} t^{K_n}$, we have

$$\lim_{t\to 0^+} \frac{\log \varphi_1(t)}{\log t} = \lim_{t\to 0^+} \frac{\log \varphi_2(t)}{\log t} = K_n, \qquad K_n = K^{1/(n-1)}.$$

This, in view of Lemma 1.3 and by (4.21) and (4.22), gives the desired result. The second statement can be derived from parallel properties of $\Phi^*_{K,n}$.

R e m a r k 1.9. Given $K \ge 1$, we may notice that the sequences $B_2[K, 2^i]$ and $b_2[K, 2^i]$ are defined by the use of *four elementary functions*. The passage from i to $i+1$ is obtained by the dynamics of $\Phi_2(t) = 2\sqrt{t}/(1+t)$, only.

Figures 18 and 19 show the graphs of $B_2[K, 2^i](t)$ and $b_2[K, 2^i](t)$ for $i = 1, 2, 3, 4$ near $t = 0$ with $K = 2$ and $K = 1.5$, respectively. The graphs of $B_2[K, 2^i](t)$ coincide for $i = 3$ and 4 in both cases.

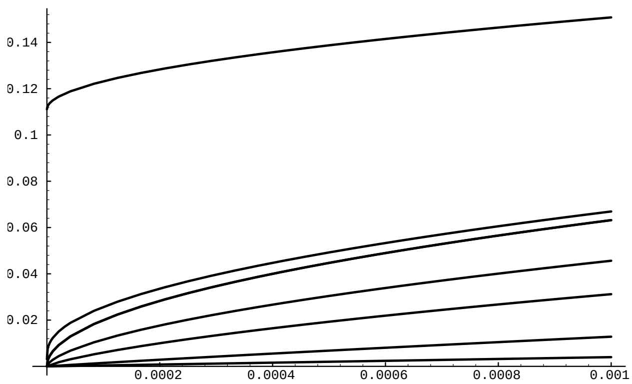

Fig. 18

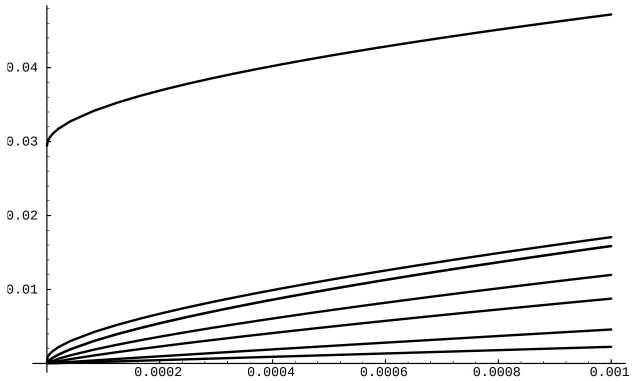

Fig. 19

In Figure 20 one may see the graphs of $B_2[K, 2^i](t)$ and $b_2[K, 2^i](t)$ for $i = 1, 2, 3$ and $K = 2$.

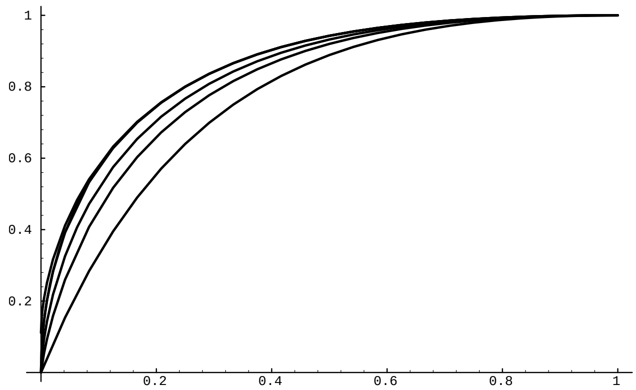

Fig. 20

5. Applications. A few very simple applications of the theory presented in the foregoing section can be obtained without any special constraints.

THEOREM 1.15. *For each* $\varrho \geq 1$ *and* $f \in Q^{\circ}_{\mathbb{R}}(\varrho)$, *there is a number* $K = K(\varrho)$ *such that*

$$16^{1-K} x^K \leq f(x) \leq 16^{1-1/K} x^{1/K} \tag{5.1}$$

for $0 \leq x \leq 1$ *and* $K(\varrho) \leq \nu(\varrho)$, *where* ν *is given by* (3.21).

P r o o f. The inequality (5.1) is an immediate consequence of Theorem 1.6 as well as the properties (ii) and (ii′).

THEOREM 1.16. *For each* $\varrho \geq 1$ *and* $f \in Q^{0}_{\mathbb{R}}(\varrho)$,

$$\max_{0 \leq t \leq 1} |f(t) - t| \leq M(\nu(\varrho)) \leq \Lambda(\nu(\varrho)). \tag{5.2}$$

Moreover, (5.2) *is sharp for* $\varrho = 1$.

P r o o f. By Theorem 1.6,

$$\max_{0\le t\le 1}|f(t)-t| \le \max_{0\le t\le 1}\max\{|\Phi_K(\sqrt{t})^2-t|, |\Phi_{1/K}(\sqrt{t})^2-t|\}$$
$$= \max_{0\le t\le 1}[\Phi_K(\sqrt{t})^2-t] = M(K) \le \Lambda(K)$$

with $K=\nu(\varrho)$, where Λ and ν are given by (2.37) and (3.21), respectively.

By Theorems 1.10 and 1.14 we have

COROLLARY 1.2. *Given $K\ge 1$, the sequences $b_2[K,2^i]$ and $B_2[K,2^i]$, $i=1,2,\ldots$, of elementary functions converge uniformly in $[0,1]$ to Φ_K.*

This gives a new, purely numerical, method to estimate Φ_K and any function of it with prescribed precision in terms of elementary functions.

From Theorem 1.3 in [P3], we have

COROLLARY 1.3. *A bound for the error in the approximation of Φ_K by $B_2[K,2^i]$ can be expressed in the form of the inequality*

$$\max_{0\le t\le 1}\{B_2[K,2^i](t)-\Phi_K(t)\} \le \max\{0.1^{2^i/K}, 3\cdot 0.1^{2^{i+1}}\},$$

which holds for every $i\ge 1$. The sequence of functions $b_2[K,2^i]$ converges to Φ_K more slowly; *cf. Figures* 18–20.

For every $0\le t<1$, $n=2,3,\ldots$, and $K,L>0$, define

$$\lambda_n[K,L](t)=\frac{\Phi_{K,n}(t)}{\Phi^*_{L,n}(t)}. \tag{5.3}$$

The function $\lambda_n[K,L]$ satisfies the functional identities

$$\lambda_n[K,L](\Phi_{M,n}(t))=\lambda_n[KM,LM](t) \tag{5.4}$$

and

$$\lambda_n[K,L](\Phi^*_{M,n}(t))=\frac{1}{\lambda_n[M/L,M/K](t)}. \tag{5.5}$$

By Theorem 1.10, Corollary 1.2 and (4.23), the inequalities

$$\frac{b_n[K,M](t)}{\Phi^*_{1,n}(b_n[L,M](t))}\le \lambda_n[K,L](t)\le \frac{B_n[K,M](t)}{\Phi^*_{1,n}(B_n[L,M](t))} \tag{5.6}$$

hold for every $0\le t<1$, $K,L\ge 1$ and $M>0$.

In connection with the study of quasisymmetric functions on the real line [LV] and the unit circle [Kr1], the distortion function $\lambda(K)$ introduced by Lehto, Virtanen and Väisälä (see [LV]) has found some applications. A generalization of $\lambda(K)$ introduced by S. Agard [Ag], namely $\lambda(K,t)$, has been studied by M. K. Vamanamurthy and M. Vuorinen [VV]. We have

$$\lambda_2[K,K](t)^2=\lambda(K,t) \tag{5.3$'$}$$

and

$$\lambda_2[K,K](1/\sqrt{2})^2=\lambda(K).$$

Setting $M = 4$ in (5.6) we get

$$\frac{(1/(1-t))^K[(1+\sqrt{t})^K - (1-\sqrt{t})^K]^4}{8[(1+\sqrt{t})^{2K} + (1-\sqrt{t})^{2K}]} \le \lambda(K,t) \le \frac{(2/(1-t))^K[(1+\sqrt{t})^K - 2^{1-K}(1-\sqrt{t})^K]^4}{16[(1+\sqrt{t})^{2K} + 4^{1-K}(1-\sqrt{t})^{2K}]}. \tag{5.7}$$

By (5.6) and Theorem 1.14 one can see that $\lambda_2[K,L](t)$ may be approximated by elementary functions.

R e m a r k 1.10. Setting $n = 2$, $M = 1$ and $t = 1/\sqrt{2}$ in (5.4) and in (5.5) one may see that the well-known properties of the classical function λ (cf. [LV]) follow immediately.

Figures 21–23 show the approximation of $\lambda_2[1.5,1.5](t)$, $\lambda(K)$ and $1/\lambda[1.5,1.5](t)$ obtained from (5.6). The graphs obtained by using $B_2[K,2^i]$, $i = 1,2,3$, coincide.

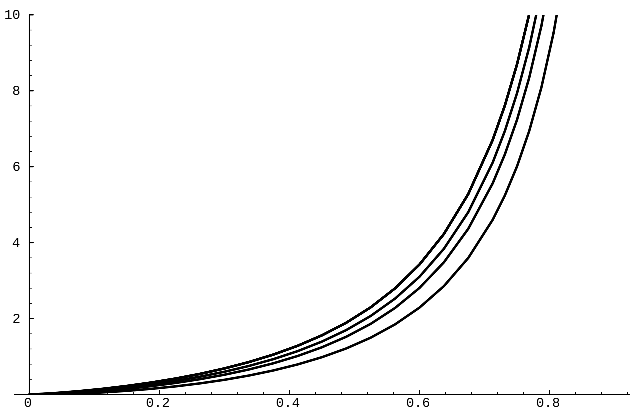

Fig. 21

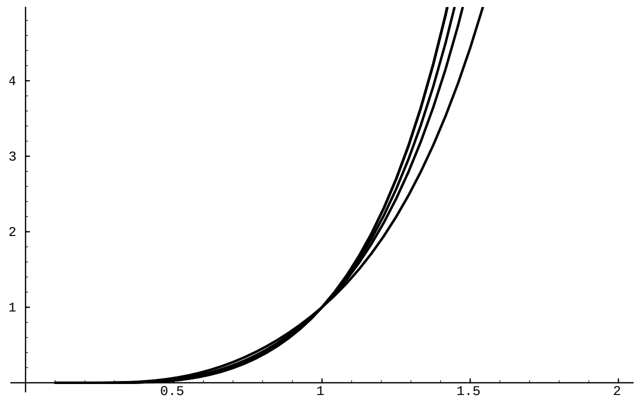

Fig. 22

Taking advantage of (4.20) we improve the inequality (ix) of Theorem 1.2 and obtain

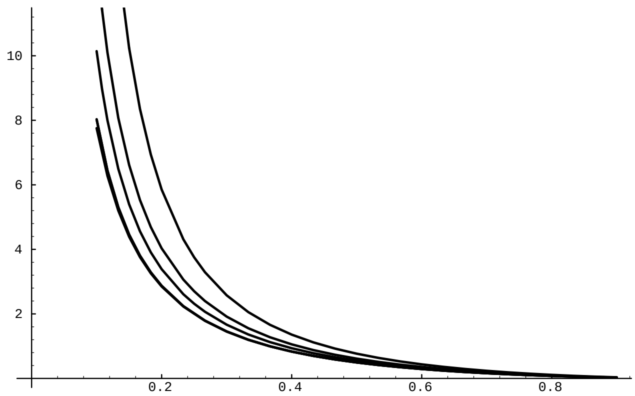

Fig. 23

THEOREM 1.17. *For each $K \geq 1$,*

$$\max_{0\leq t\leq 1}[\Phi_K(t) - t] \leq B_2[K,4](t_0) - t_0, \tag{5.8}$$

where t_0 is such that $B_2'[K,4](t_0) = 1$.

Proof. First we show that $B_2[K,4]$ is concave. To this end, let us differentiate $B_2[K,4](t)$ with respect to t, $0 < t < 1$. We obtain

$$B_2'[K,4](t) = 2^{3-K}K\frac{(1-t)^{K-1}}{\big[\big(1+\sqrt{t}\big)^K + 2^{1-K}\big(1-\sqrt{t}\big)^K\big]^2}\cdot\frac{1}{\sqrt{t}}\cdot\frac{1-2^{1-K}\big(\big(1-\sqrt{t}\big)/\big(1+\sqrt{t}\big)\big)^K}{1+2^{1-K}\big(\big(1-\sqrt{t}\big)/\big(1+\sqrt{t}\big)\big)^K}.$$

Introducing $x = \big(1-\sqrt{t}\big)/\big(1+\sqrt{t}\big)$ and considering

$$f(x) = \frac{1+x}{1-x}\cdot\frac{1-2^{1-K}x^K}{1+2^{1-K}x^K}, \qquad 0 < x < 1,$$

we can see that

$$\frac{d}{dx}\log f(x) = \frac{2}{1-x^2} - 2^{1-K}Kx^{K-1}\frac{2}{1-4^{1-K}x^{2K}} \geq 0 \tag{5.9}$$

for $0 < x < 1$ and $K \geq 1$.

We shall prove that

$$1 - 4^{1-K}x^{2K} \geq 2^{1-K}Kx^{K-1}(1-x^2) \quad \text{for } 0 \leq x \leq 1 \text{ and } K \geq 1. \tag{5.10}$$

Note that for $K = 1$ equality occurs in (5.10). Since

$$\partial_1 = \frac{\partial}{\partial K}(1 - 4^{1-K}x^{2K}) = -8\left(\frac{x}{2}\right)^{2K}\log\frac{x}{2} > 0$$

for $0 < x \leq 1$ and $K \geq 1$, and

$$\partial_2 = \frac{\partial}{\partial K}(2^{1-K}Kx^{K-1}(1-x^2)) = \left(\frac{x}{2}\right)^{K-1}(1-x^2)\left(1 + K\log\frac{x}{2}\right) \leq 0$$

for $0 < x < 2/e$ and $K \geq 1$, the inequality (5.10) remains true for $0 \leq x \leq 2/e$ and $K \geq 1$.

Let $2/e \le x \le 1$ and $1 \le K \le 3/2$. Then

$$8\left(\frac{x}{2}\right)^{2K} \ge \frac{8}{e^{K+1}}\left(\frac{x}{2}\right)^{K-1} \ge \frac{8}{e^{5/2}}\left(\frac{x}{2}\right)^{K-1} \ge \left(\frac{x}{2}\right)^{K-1}\left(1-\frac{4}{e^2}\right) \ge \left(\frac{x}{2}\right)^{K-1}(1-x^2)$$

and

$$-\log\frac{x}{2} > 1 + K\log\frac{x}{2} \Leftrightarrow (K+1)\log\frac{x}{2} < -1. \tag{5.11}$$

Thus $\partial_1 - \partial_2 \ge 0$ for $2/e \le x \le 1$ and $1 \le K \le 3/2$. Hence (5.10) holds for $2/e \le x \le 1$ and $1 \le K \le 3/2$.

Suppose now that $K > 3/2$ and $0 < x \le 1$. Then

$$1 + K\log\frac{x}{2} \le 1 + K\log\frac{1}{2} \le 1 + \log\frac{1}{2^{3/2}} < 0, \tag{5.12}$$

and thus $\partial_2 < 0$ for $K \ge 3/2$. In the proof of Theorem 1.2 it is shown that the left-hand ratio of $B_2'[K,4](t)$ is decreasing. This fact, together with our considerations on f, shows that $B_2[K,4]$ is concave, and our proof is complete.

II. Quasihomographies of a circle

1. Introduction. The main purpose of this chapter is to introduce a new and, as will be shown here, very flexible characterization of the boundary automorphisms of K-qc automorphisms of a disc on $\overline{\mathbb{C}}$. Before embarking on it, it is perhaps of interest to indicate that the boundary function of a qc automorphism of a Jordan domain D in $\overline{\mathbb{C}}$ can be singular. This is a remarkable feature of these boundary automorphisms. Using more subtle tools P. Tukia ([T1], [T3]) succeeded in showing that these boundary functions are even worse, i.e., they do not preserve sets of Hausdorff dimension 1.

On the contrary, several shortcomings of qs functions form a considerable difficulty, which is not so much natural for boundary functions of qc automorphisms of a given Jordan domain D in $\overline{\mathbb{C}}$.

We overcome these obstacles by using the original boundary condition (cf. [Le, p. 31]) without any simplification, assuming only that the domain D is bounded by a circle Γ in $\overline{\mathbb{C}}$.

The very special position of circles on $\overline{\mathbb{C}}$ with respect to the boundary value problem for K-qc automorphisms ensures simplicity and efficiency in our research at this stage.

Given a Jc Γ on $\overline{\mathbb{C}}$, let D and D^* be the complementary domains. Further, let $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$ be the classes of all K-qc automorphisms of D and D^*, respectively. These classes are equivalent by conformal reflection in Γ for every $K \ge 1$ if and only if Γ is a circle on $\overline{\mathbb{C}}$. Therefore, assuming that Γ is a circle on $\overline{\mathbb{C}}$, the boundary value problem for automorphisms in $\mathcal{F}_D(K)$ or $\mathcal{F}_{D^*}(K)$, $K \ge 1$, may be regarded simultaneously as the boundary value problem of automorphisms in

$$\mathcal{F}_\Gamma(K) := \{F \in \mathcal{F}_{\overline{\mathbb{C}}}(K) : F(\Gamma) = \Gamma\}. \tag{1.1}$$

2. Introduction to quasihomographies. Recall that in 1853 Möbius initiated the study of an equivalent class of geometrical transformations, which he called *Kreisver-*

wandtschaften. Other names are *homographies*, *homographic transformations*, *fractional linear transformations* and *Möbius transformations*; cf. [Hl] and [SZ]. It is worth explaining that we cannot use the names *quasi-Möbius* or *almost Möbius* for they have already been reserved for other classes of mappings introduced by J. Väisälä [Vä1] and D. H. Hamilton [Ha], respectively.

Suppose that Γ is a *circle* on the extended complex plane $\overline{\mathbb{C}}$, i.e., a stereographic projection of a circle on the sphere

$$B = \{(x, y, u) : x^2 + y^2 + u^2 - u = 0\}.$$

Then $\overline{\mathbb{C}} \setminus \Gamma = D \cup D^*$, where D and D^* are the *discs* on $\overline{\mathbb{C}}$ complementary to Γ.

Suppose that z_1, z_2, z_3, z_4 is an ordered quadruple of distinct points on Γ, and consider the expression

$$[z_1, z_2, z_3, z_4] = \left\{ \frac{z_3 - z_2}{z_3 - z_1} : \frac{z_4 - z_2}{z_4 - z_1} \right\}^{1/2}. \tag{2.1}$$

This expression is invariant under homographies and its values range over $(0, 1)$. Furthermore,

$$[z_1, z_2, z_3, z_4]^2 = 1 - [z_2, z_3, z_4, z_1]^2$$

by Ptolemy's theorem; cf. [Vo]. In view of the invariance of (2.1) under homographies, we may replace any circle Γ of the extended complex plane $\overline{\mathbb{C}}$ by any other. So, if necessary, without any loss of generality, we may confine ourselves to the circle $\Gamma = \overline{\mathbb{R}}$ or $\Gamma = T$.

Now, one can state

THEOREM 2.1. *Suppose that Γ is an arbitrary circle on $\overline{\mathbb{C}}$ and that F is an automorphism of $\mathcal{F}_\Gamma(K)$, $K \geq 1$. Then, for any distinct points $z_1, z_2, z_3, z_4 \in \Gamma$,*

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]) \leq [f(z_1), f(z_2), f(z_3), f(z_4)] \leq \Phi_K([z_1, z_2, z_3, z_4]), \tag{2.2}$$

where $f = F|_\Gamma$.

Proof. Let F be a K-quasiconformal mapping of the upper half-plane U and let x_1, x_2, x_3, x_4 be distinct points of $\Gamma = \overline{\mathbb{R}}$. Then the upper half-plane together with these points forms a quadrilateral $D = U(x_1, x_2, x_3, x_4)$ mapped by F onto $D' = U(f(x_1), f(x_2), f(x_3), f(x_4))$, where $f = F|_{\bar{\mathbb{R}}}$. By the definition of K-quasiconformal mappings,

$$\frac{1}{K} M(D) \leq M(D') \leq K M(D). \tag{2.3}$$

Let H_1 and H_2 be homographies of the upper half-plane onto itself such that $H_1(x_4) = H_2(f(x_4)) = \infty$. Then

$$\begin{aligned} M(D) = M(H_1(D)) &= \frac{2}{\pi}\mu\left(\sqrt{\frac{H_1(x_3) - H_1(x_2)}{H_1(x_3) - H_1(x_1)}}\right) \\ &= \frac{2}{\pi}\mu([H_1(x_1), H_1(x_2), H_1(x_3), \infty]) = \frac{2}{\pi}\mu([x_1, x_2, x_3, x_4]), \end{aligned} \tag{2.4}$$

and similarly,

$$M(D') = \frac{2}{\pi}\mu([f(x_1), f(x_2), f(x_3), f(x_4)]). \tag{2.4'}$$

Substituting these expressions into (2.3), we obtain

$$\frac{1}{K}\mu([x_1,x_2,x_3,x_4]) \le \mu([f(x_1),f(x_2),f(x_3),f(x_4)]) \le K\mu([x_1,x_2,x_3,x_4]). \tag{2.5}$$

Composing (2.5) with μ^{-1} we arrive at (2.2); see also [Z1]. The proof is complete.

Suppose now that Γ is a circle on $\overline{\mathbb{C}}$. Then there is a homography H mapping Γ onto $\overline{\mathbb{R}}$ and $H\circ F\circ H^{-1}\in\mathcal{F}_{\overline{\mathbb{R}}}(K)$ if and only if $F\in\mathcal{F}_\Gamma(K)$.

Let us give the following

Definition. Suppose that Γ is a circle on $\overline{\mathbb{C}}$. By $A_\Gamma(K)$ we denote the family of all sense-preserving automorphisms f of Γ such that (2.2) is satisfied for any distinct points $z_1,z_2,z_3,z_4\in\Gamma$ with a given constant $K\ge 1$. A function f from the class $A_\Gamma(K)$ is said to be a K-*quasihomography* (K-qh) of Γ. Further,

$$K(f)=\min\{K : f\in A_\Gamma(K)\}$$

is called the *maximal dilatation* of f.

Bolow we present some basic properties of K-qh. Theorems 2.2 and 2.3 follow immediately from (i) of Section I.2 and the definition of $A_\Gamma(K)$.

Theorem 2.2. *For every circle Γ on $\overline{\mathbb{C}}$, and $K_1,K_2\ge 1$, if $f_1\in A_\Gamma(K_1)$ and $f_2\in A_\Gamma(K_2)$, then $f_1\circ f_2\in A_\Gamma(K_1K_2)$.*

Theorem 2.3. *For every circle Γ on $\overline{\mathbb{C}}$, and $K\ge 1$, if $f\in A_\Gamma(K)$, then $f^{-1}\in A_\Gamma(K)$.*

Theorem 2.4. *For arbitrary circles Γ_1,Γ_2 in $\overline{\mathbb{C}}$, there exists a homography H such that*

(i) $H(\Gamma_1)=\Gamma_2$;

(ii) *for each $K\ge 1$ and $f\in A_{\Gamma_1}(K)$,*

$$S_H(f):=H\circ f\circ H^{-1}\in A_{\Gamma_2}(K), \tag{2.6}$$

$$S_H(A_{\Gamma_1}(K))=A_{\Gamma_2}(K). \tag{2.6$'$}$$

Proof. The condition (i) is obvious, while (ii) is a consequence of the fact that each homography preserves the cross-ratio.

Theorem 2.5. *For any circle Γ on $\overline{\mathbb{C}}$, a function f belongs to the class $A_\Gamma(1)$ if and only if f is a homography mapping Γ onto itself.*

Proof. Assume first that $f\in A_\Gamma(1)$. By Theorem 2.4 we can reduce our proof to the case $\Gamma=\overline{\mathbb{R}}$. Then there exists a homography $h_0:\overline{\mathbb{R}}\to\overline{\mathbb{R}}$ such that $h_0(f(0))=0$, $h_0(f(1))=1$ and $h_0(f(\infty))=\infty$. Since $h_0\circ f\in A_{\overline{\mathbb{R}}}(1)$, we have

$$\begin{aligned}[x,0,1,\infty]^2=\frac{1}{1-x}&=[(h_0\circ f)(x),(h_0\circ f)(0),(h_0\circ f)(1),(h_0\circ f)(\infty)]^2\\ &=\frac{1}{1-(h_0\circ f)(x)}\quad \text{for } x<0.\end{aligned}$$

By this argument, $x=(h_0\circ f)(x)$. For $0<x<1$,

$$[0,x,1,\infty]^2=1-x=1-(h_0\circ f)(x),$$

and hence $x = (h_0 \circ f)(x)$. For $x > 1$,

$$[0, 1, x, \infty]^2 = \frac{x-1}{x} = \frac{(h_0 \circ f)(x) - 1}{(h_0 \circ f)(x)},$$

and hence again $x = (h_0 \circ f)(x)$.

By continuity, $(h_0 \circ f)(x) = x$ for each $x \in \overline{\mathbb{R}}$, and thus $f = h_0^{-1}$ is a homography.

The reverse implication is obvious since every homography preserves the cross-ratio of each ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \varGamma$. The proof is complete.

Problems of the kind presented in Theorem 2.4 were investigated by P. Tukia [T2] and A. Hinkkanen ([Hi1], [Hi2]) while working with K-qc mappings and ϱ-qs functions of $\mathbb{R}$, respectively.

3. Quasihomographies and quasisymmetric functions on the real line. Beurling and Ahlfors [BA] showed that the boundary values of the class of all K-qc normalized $(F(\infty) = \infty)$ automorphisms F of U can be represented uniformly by functions of the class $Q_{\mathbb{R}}(\varrho)$ with $\varrho = \lambda(K)$, where the function λ is defined in (5.3′) of Chapter I. Obviously, the study of the function space $Q_{\mathbb{R}}(\varrho)$ means the study of the boundary values of normalized K-qc automorphisms of U. By using the notion of ϱ-qs functions it has been shown (see [Ke]) that

- ∗ the theorem on removable singularities for K-qc mappings cannot be extended to boundary automorphisms. Even when a given f is ϱ-qs in the vicinity of a singular point x_0, it does not necessarily have to be a removable point for f;
- ∗ in contrast to the reflection principle for K-qc mappings, the qs constant of the reflected ϱ-qs function may increase;
- ∗ every ϱ-qs function f mapping $(0, 1)$ onto itself can be extended to a $\widetilde{\varrho}$-qs function on the real line and $\widetilde{\varrho} \le 28\varrho^4$, which is in contrast with the extension theorem for mappings of $\mathcal{F}_{D,\varDelta}(K)$, which assumes that ∂D must be a Q-quasicircle in order to guarantee an extension of functions of $\mathcal{F}_{D,\varDelta}(K)$ to $\widetilde{K}$-qc automorphisms of $\overline{\mathbb{C}}$, where $\widetilde{K} = \widetilde{K}(K, Q)$;
- ∗ the class of ϱ-qs functions that are differentiable infinitely many times on an interval (a, b) is dense in the class of ϱ-qs functions on (a, b), whereas the continuously differentiable K-qc mappings of a domain D are known to be dense in the class of K-qc mappings of D;
- ∗ for each $\varrho > 1$, each $p > 1$, and each compact set $E \subset \mathbb{R}$ of positive measure, there exists a function g, ϱ-qs on $\mathbb{R}$, such that $\int_E (g'(x))^p \, dx = \infty$, which contrasts with the local p-integrability of the Jacobian of a given K-qc mapping of a domain D on $\overline{\mathbb{C}}$;
- ∗ a strictly increasing continuous function f can be locally ϱ-qs but there does not necessarily exist any constant $\varrho^* \ge 1$ such that f is globally ϱ^*-qs, which contrasts with the theorem saying that locally K-qc mappings are also globally K-qc.

Notice that not always $\varrho(f^{-1}) = \varrho(f)$ nor $\varrho(f \circ g) \le \varrho(f)\varrho(g)$. For example $\varrho(x^2) = 3$, whereas $\varrho(x^{1/2}) = \sqrt{2}+1$ for $x > 0$. Similarly, $\varrho(x^4) = 15$, whereas $\varrho(x^2)\varrho(x^2) = 9$. These and other related consequences are results of the deformation (BA) only and their study does not seem to be very much motivated by topics of K-qc theory.

In the opposite direction, there is some evidence that boundary functions of K-qc automorphisms are more rigid in some respects when the mappings are themselves defined in a given Jordan domain D on $\overline{\mathbb{C}}$.

The following two theorems concern the relationship between the classes $A_{\overline{\mathbb{R}}}(K)$ and $Q_{\mathbb{R}}(\varrho)$. Let

$$A_{\mathbb{R}}(K) = \{f \in A_{\overline{\mathbb{R}}}(K) : f(\infty) = \infty\}. \tag{3.1}$$

THEOREM 2.6. *For each* $K \geq 1$, *there exists a constant* $\varrho = \varrho(K)$ *such that*

$$A_{\mathbb{R}}(K) \subset Q_{\mathbb{R}}(\varrho) \quad \text{with } \varrho = \lambda(K). \tag{3.2}$$

Moreover, *the constant* $\lambda(K)$ *defined by* (5.3′) *of Chapter* I *cannot be lowered.*

Proof. Suppose that $f \in A_{\mathbb{R}}(K)$, $K \geq 1$. Then f is a strictly increasing and continuous function in $\mathbb{R}$. Setting $z_1 = x - t$, $z_2 = x$, $z_3 = x + t$ and $z_4 = \infty$, $t > 0$, we see that (2.2) takes the form

$$\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right) \leq \left(\sqrt{1 + \frac{f(x) - f(x-t)}{f(x+t) - f(x)}}\right)^{-1} \leq \Phi_K\left(\frac{1}{\sqrt{2}}\right), \tag{3.3}$$

from which

$$\frac{\Phi_{1/K}\big(1/\sqrt{2}\big)^2}{1 - \Phi_{1/K}\big(1/\sqrt{2}\big)^2} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq \frac{\Phi_K\big(1/\sqrt{2}\big)^2}{1 - \Phi_K\big(1/\sqrt{2}\big)^2}.$$

Hence, by (iii) of Chapter I and by the functional identity

$$\Phi_K\big(1/\sqrt{2}\big)^2/\Phi_{1/K}\big(1/\sqrt{2}\big)^2 = \lambda(K)$$

we complete the proof; see (5.3′) in Chapter I and [LV].

THEOREM 2.7. *For each* $\varrho \geq 1$, *there exists a constant* $K = K(\varrho)$ *such that*

$$Q_{\mathbb{R}}(\varrho) \subset A_{\mathbb{R}}(K) \quad \text{with } K \leq \chi(\nu(\varrho)), \tag{3.4}$$

where χ *and* ν *are given by* (2.7) *and* (3.21) *in Chapter* I, *respectively. Moreover*, *this estimate is asymptotically sharp for* $\varrho = 1$.

Proof. Let $f \in Q_{\mathbb{R}}(\varrho)$, $1 \leq \varrho < \infty$, be an arbitrary ϱ-qs function of $\mathbb{R}$. Let $x_1, x_2, x_3, x_4 \in \overline{\mathbb{R}}$ be a positively ordered quadruple of distinct points with $y_i = f(x_i)$, $i = 1, 2, 3, 4$.

Suppose first that $x_4 = f(x_4) = \infty$. This means that we may confine ourselves to the case of three points $x_1 < x_2 < x_3$. Let L_1 and L_2 be linear functions such that $L_1(0) = x_1$, $L_1(1) = x_3$, $L_2(y_1) = 0$ and $L_2(y_3) = 1$. Since $g = L_2 \circ f \circ L_2 \in Q^0_{\mathbb{R}}(\varrho)$ there exists, by Theorem 2.4, a constant $K = K(\varrho)$ such that $g(t) \leq \Phi_K\big(\sqrt{t}\big)^2$, $0 \leq t \leq 1$. Substituting t for $(x_2 - x_1)/(x_3 - x_1)$, we get

$$\Phi_K\left(\sqrt{\frac{x_2 - x_1}{x_3 - x_1}}\right)^2 \geq g(t) = \frac{g(t) - g(0)}{g(1) - g(0)} = \frac{f(x_2) - f(x_1)}{f(x_3) - f(x_1)} = \frac{y_2 - y_1}{y_3 - y_1}. \tag{3.5}$$

In a similar way it can be shown that

$$\Phi_K\left(\sqrt{\frac{x_3-x_2}{x_3-x_1}}\right)^2 \ge \frac{f(x_3)-f(x_2)}{f(x_3)-f(x_1)} = \frac{y_3-y_2}{y_3-y_1}. \tag{3.6}$$

Since $[x_1,x_2,x_3,x_4]^2=(x_3-x_2)/(x_3-x_1)$ and $[y_1,y_2,y_3,y_4]^2=(y_3-y_2)/(y_3-y_1)$, we have

$$[y_1,y_2,y_3,y_4] \le \Phi_K([x_1,x_2,x_3,x_4]) \tag{3.7}$$

and

$$1-[y_1,y_2,y_3,y_4]^2 = \frac{y_2-y_1}{y_3-y_1} \le \Phi_K\left(\sqrt{\frac{x_2-x_1}{x_3-x_1}}\right)^2. \tag{3.8}$$

Thus, (3.7), (3.8), together with (iii) of Chapter I, give the inequality

$$\begin{aligned}[y_1,y_2,y_3,y_4] &\ge \left(1-\Phi_K\left(\sqrt{\frac{x_2-x_1}{x_3-x_1}}\right)^2\right)^{1/2} \\ &= \Phi_{1/K}\left(\sqrt{\frac{x_3-x_2}{x_3-x_1}}\right) = \Phi_{1/K}([x_1,x_2,x_3,x_4]).\end{aligned} \tag{3.9}$$

The inequalities (3.7) and (3.9), with the rule of cyclic permutation of the positively ordered quadruple of distinct points x_1, x_2, x_3, x_4, give the double inequality (2.2), which holds for a quadruple such that one of the points is at infinity.

Suppose now that $x_1<x_2<x_3<x_4$ are arbitrary points of $\mathbb{R}$. Let $a^2=(x_2-x_1)/(x_3-x_1)$ and $b^2=(x_4-x_3)/(x_4-x_2)$. Using (3.5), (3.6), (iii) of Chapter I and Theorem 1.1, one can see that there exists $K'\le\chi(K)$ such that

$$\begin{aligned}[y_1,y_2,y_3,y_4]^2 &= 1-\frac{y_2-y_1}{y_3-y_1}\cdot\frac{y_4-y_3}{y_4-y_2} \ge 1-\Phi_K(a)^2\Phi_K(b)^2 \\ &\ge 1-\Phi_{K'}(ab)^2 = \Phi_{1/K'}\big(\sqrt{1-a^2b^2}\big)^2 = \Phi_{1/K'}([x_1,x_2,x_3,x_4])^2\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}[y_1,y_2,y_3,y_4]^2 &\le 1-\Phi_{1/K}(a)^2\Phi_{1/K}(b)^2 \le 1-\Phi_{1/K'}(ab)^2 \\ &= \Phi_{K'}\big(\sqrt{1-a^2b^2}\,\big)^2 = \Phi_{K'}([x_1,x_2,x_3,x_4])^2,\end{aligned} \tag{3.11}$$

where χ is given by (2.7) of Chapter I. Using (3.10), (3.11) and cyclic permutation of x_1, x_2, x_3, x_4, we arrive at (3.4) with K' instead of K.

Both the cases considered, together with estimates (3.20) and (3.21), give the desired result. Since ν and χ are asymptotically sharp at $\varrho=1$, the estimate (3.4) is asymptotically sharp for $\varrho=1$. The proof is complete.

THEOREM 2.8. *Let Γ be an arbitrary circle on $\overline{\mathbb{C}}$, and let D, D^* be its complementary domains. For each $K\ge1$ and $f\in A_\Gamma(K)$, there exists a K^*-qc automorphism $F_f\in\mathcal{F}_\Gamma(K^*)$ such that $F_f|_\Gamma=f$ and K^* depends only on K.*

P r o o f. Suppose that $\Gamma=\overline{\mathbb{R}}$, and $f\in A_{\mathbb{R}}(K)$. Using the explicit Beurling–Ahlfors extension operator

$$\mathbb{F}_f(z)=\frac{1}{2}\int_0^1[f(x+ty)-f(x-ty)]\,dt+i\int_0^1[f(x+ty)-f(x-ty)]\,dt,$$

where $z = x + iy$, one may find that $\mathbb{F}_f \in \mathcal{F}_{\mathbb{R}}(K^*)$, where $K^* = K^*(K)$ is such that $K^*(1) = 1$; see [RZ2].

Suppose now that $f \in A_{\overline{\mathbb{R}}}(K)$. There exists a homography h such that $h \circ f \in A_{\mathbb{R}}(K)$. Hence, $F_{h\circ f} \in \mathcal{F}_{\mathbb{R}}(K^*)$ and $h^{-1} \circ F_{h\circ f}$ is the desired mapping of the same maximal dilatation.

To cover the case of an arbitrary circle Γ in $\overline{\mathbb{C}}$ one can use Theorem 2.4 and the fact that the composition of a K-qc mapping with a homography does not affect its maximal dilatation. The proof is complete.

It seems to be more natural when working with a given boundary function $f \in A_T(K)$ to use the harmonic Douady–Earle extension operator [DE], which is invariant under $A_T(1)$. The problem is that this extension has neither an explicit formula nor a nice qc order estimation; see [DE], [LP], [P1] and [SZ].

4. Quasihomographies and quasisymmetric functions on the unit circle. In the case $\Gamma = T$ we may easily notice that the classes $A_T(K)$ and $Q_T(\varrho)$ *differ much more* than in the case $\Gamma = \overline{\mathbb{R}}$, since we cannot obtain $Q_T(\varrho)$ from $A_T(K)$ by setting points z_1, z_2, z_3, z_4 at special positions on T, i.e. one cannot reduce $A_T(K)$ to $Q_T(\varrho)$ for any $K \geq 1$. Also, the notion of quasisymmetry of T is more rigid than quasisymmetry of $\mathbb{R}$; see [Z4].

To see the problem, look at the following

Example 2.1. Consider the sequence of homographies $\{h_n\}$ mapping the unit disc Δ onto itself such that

$$h_n(1) = 1, \quad h_n(-1) = -1 \quad \text{and} \quad h_n(i) = e^{i\pi n/(n+1)}, \qquad n = 1, 2, \ldots$$

For each $n = 1, 2, \ldots$ we have $h_n \in A_T(1)$, and

$$\frac{|\mathrm{arc}(h_n(1), h_n(i))|}{|\mathrm{arc}(h_n(i), h_n(-1))|} = n \to \infty.$$

Thus there does not exist any finite $\varrho \geq 1$ such that $h_n \in Q_T(\varrho)$ for $n = 1, 2, \ldots$ Hence

$$A_T(1) \setminus Q_T(\varrho) \neq \emptyset$$

for each $\varrho \geq 1$. This implies that

$$\mathcal{F}_\Delta(1)|_T \not\subset Q_T(\varrho)$$

for any $\varrho \geq 1$ and means that the family $\mathcal{F}_\Delta^\infty|_T$ *cannot be represented uniformly* by functions from $Q_T^\infty = \bigcup_{\varrho \geq 1} Q_T(\varrho)$.

We begin with proving

Theorem 2.9. *For each $K \geq 1$ and $f \in A_T(K)$, there exists a constant $\varrho = \varrho(f, K)$ such that $f \in Q_T(\varrho)$ and*

$$\varrho \leq \lambda(K) \cot(\varphi_f/4)^2, \tag{4.1}$$

where

$$\varphi_f = \min_{z\in T} \min\left\{ \arg\frac{f(-z)}{f(z)}, 2\pi - \arg\frac{f(-z)}{f(z)} \right\}. \tag{4.2}$$

P r o o f. Suppose that z_1, z_2, z_3, z_4 are distinct points of T. Then

$$\frac{[z_1,z_2,z_3,z_4]^2}{[z_2,z_3,z_4,z_1]^2}=\frac{|z_3-z_2||z_4-z_1|}{|z_4-z_3||z_2-z_1|}. \tag{4.3}$$

For $f\in A_T(K)$, let $w_i=f(z_i)$ and $\zeta_i=h(w_i)$, $i=1,2,3,4$, where h is a homography mapping T onto itself such that $\zeta_2=-\zeta_4$. Consider distinct points $z_1,z_2,z_3,z_4\in T$ such that $z_4=-z_2$ and $|z_2-z_1|=|z_3-z_2|$. Then $[z_1,z_2,z_3,z_4]=[z_2,z_3,z_4,z_1]$. By the definition of K-qh, $h\circ f\in A_T(K)$ and, because of (2.2),

$$\frac{1}{\lambda(K)}\le\frac{|\zeta_3-\zeta_2||\zeta_4-\zeta_1|}{|\zeta_4-\zeta_3||\zeta_2-\zeta_1|}\le\lambda(K), \tag{4.4}$$

where $\lambda(K)=\Phi_K\big(1/\sqrt{2}\big)^2/\Phi_{1/K}\big(1/\sqrt{2}\big)^2$.

Let

$$\alpha=\arg\frac{\zeta_2-\zeta_4}{\zeta_1-\zeta_4}\quad\text{and}\quad\beta=\arg\frac{\zeta_3-\zeta_4}{\zeta_2-\zeta_4},\qquad 0<\alpha,\beta<\frac{\pi}{2}. \tag{4.5}$$

Then

$$\frac{|\zeta_3-\zeta_2||\zeta_4-\zeta_1|}{|\zeta_4-\zeta_3||\zeta_2-\zeta_1|}=\frac{\tan\beta}{\tan\alpha} \tag{4.6}$$

and

$$\frac{1}{\lambda(K)}\le\frac{\tan\beta}{\tan\alpha}\le\lambda(K). \tag{4.7}$$

Now, by the concavity of $\arctan x$ for $x\ge 0$ and the Jensen inequality, we have

$$\beta\le\arctan(\lambda(K)\tan\alpha)\le\lambda(K)\arctan(\tan\alpha)=\lambda(K)\alpha \tag{4.8}$$

and similarly $\alpha\le\lambda(K)\beta$. Thus

$$1/\lambda(K)\le\beta/\alpha\le\lambda(K). \tag{4.9}$$

Let $\langle z_1,z_2\rangle=\{z\in T:\arg z_1<\arg z<\arg z_2\}$ and let $|\langle z_1,z_2\rangle|=|\arg z_2-\arg z_1|$ stand for its measure. Then, for every subarc η of T, we obtain

$$\frac{1-|a|}{1+|a|}|\eta|\le|h^{-1}(\eta)|=\int_\eta|(h^{-1})'(z)|\,|dz|\le\frac{1+|a|}{1-|a|}|\eta|, \tag{4.10}$$

where $a=|h(0)|$. Since $f=h^{-1}\circ(h\circ f)$, we have

$$\frac{|\langle w_2,w_3\rangle|}{|\langle w_1,w_2\rangle|}=\frac{|h^{-1}(\langle\zeta_2,\zeta_3\rangle)|}{|h^{-1}(\langle\zeta_1,\zeta_2\rangle)|}\le\left(\frac{1+|a|}{1-|a|}\right)^2\frac{|\langle\zeta_2,\zeta_3\rangle|}{|\langle\zeta_1,\zeta_2\rangle|}\le\lambda(K)\cot\left(\frac{\varphi_f}{4}\right)^2, \tag{4.11}$$

where φ_f is given by (4.2) (cf. [Vo], p. 13). This completes our proof.

The constant $\lambda(K)\cot(\varphi_f/4)^2$ may depend only on K when we confine ourselves to the normalized K-qh of T.

Let

$$A^\circ_T(K)=\{f\in A_T(K):f(z)=z,\ z^3=1\}. \tag{4.12}$$

Then we have

LEMMA 2.1. *For each* $K \ge 1$, $f \in A_T^\circ(K)$ *and* $z \in T$,

$$|f(z) - z| \le |\arg f(z) - \arg z| \le \frac{4}{\sqrt{3}} M(K), \tag{4.13}$$

where

$$M(K) := \max_{0 \le t \le 1} \left[\Phi_K\big(\sqrt{t}\,\big)^2 - t\right] = 2\Phi_{\sqrt{K}}\big(1/\sqrt{2}\,\big)^2 - 1 \tag{4.14}$$

is such that

$$M(K) = M(1/K) \le \Lambda(K); \tag{4.15}$$

see Chapter I, Theorem 1.3.

Proof. Without any loss of generality, suppose that $z \in T$ is such that $0 < \arg z < 2\pi/3$ and $\alpha = \arg z - \pi/3$, $|\alpha| \le \pi/3$. If $z_l = e^{2\pi i(l-1)/3}$, $l = 1, 2, 3$, then

$$[z_1, z, z_2, z_3]^2 = \frac{1 - \sqrt{3}\tan(\alpha/2)}{2}. \tag{4.16}$$

For $f \in A_T^\circ(K)$ and $\beta = \arg f(z) - \pi/3$, we have

$$\Phi_{1/K}\left(\frac{\big(1 - \sqrt{3}\tan(\alpha/2)\big)^{1/2}}{\sqrt{2}}\right)^2 \le \frac{1 - \sqrt{3}\tan(\beta/2)}{2} \le \Phi_K\left(\frac{\big(1 - \sqrt{3}\tan(\alpha/2)\big)^{1/2}}{\sqrt{2}}\right)^2. \tag{4.17}$$

On the other hand,

$$|f(z) - z| = 2\sin\frac{|\beta - \alpha|}{2} \le |\beta - \alpha| \le 2\tan\frac{|\beta - \alpha|}{2} \le 2\left|\tan\frac{\beta}{2} - \tan\frac{\alpha}{2}\right|. \tag{4.18}$$

Then, by (4.15) and (4.17), we have

$$|f(z) - z| \le \frac{4}{\sqrt{3}} \max_{0 \le t \le 1} \max\left\{\left|\Phi_K\big(\sqrt{t}\,\big)^2 - t\right|, \left|\Phi_{1/K}\big(\sqrt{t}\,\big)^2 - t\right|\right\} = \frac{4}{\sqrt{3}} M(K), \tag{4.19}$$

which completes the proof.

THEOREM 2.10. *For each* $K \ge 1$, *there exists a constant* $\varrho \ge 1$ *such that* $A_T^\circ(K) \subset Q_T^\circ(\varrho)$ *and*

$$\varrho \le \begin{cases} \lambda(K)\left(\dfrac{1 + \tan\big((2/\sqrt{3}\,)M(K)\big)}{1 - \tan\big((2/\sqrt{3}\,)M(K)\big)}\right)^2 & \text{for } 1 \le K \le K_0, \\ \frac{1}{3}\lambda(K)16^{K-1}\big(3 + 2\sqrt{2}\,\big)^{2K} & \text{for } K > K_0, \end{cases} \tag{4.20}$$

with Λ *given by Theorem* 1.3, *and* K_0, *satisfying* $1.326 < K_0 < 1.395$ *and*

$$1 + \tan\left(\frac{2}{\sqrt{3}}\Lambda(K_0)\right) = \frac{3 + 2\sqrt{2}}{\sqrt{3}}\left(1 - \tan\left(\frac{2}{\sqrt{3}}\Lambda(K_0)\right)\right). \tag{4.21}$$

Proof. With the notations of Lemma 2.1 we have for $z = e^{i\pi/3}$, by (4.17) and (iii) of Chapter I,

$$\left|\arg f(z) - \frac{\pi}{3}\right| = \beta \le 2\arctan\big(M(K^2)/\sqrt{3}\,\big). \tag{4.22}$$

The inequality (4.22) remains the same for $z = e^{\pi i}$ and $z = e^{5\pi i/3}$. Hence

$$\varphi_f > \min\left\{\frac{2\pi}{3}, \frac{2\pi}{3} - 4\arctan\big(M(K^2)/\sqrt{3}\,\big)\right\} = \frac{2\pi}{3} - 4\arctan\big(M(K^2)/\sqrt{3}\,\big). \tag{4.23}$$

If $r = \big(1 - \sqrt{3}\tan\frac{\alpha}{2}\big)/2$, then

$$\tan(\alpha/2) = (1-2r)/\sqrt{3}. \tag{4.24}$$

Setting $\beta = \pi/3 - \alpha$, we get

$$1 - \sqrt{3}\tan\frac{\beta}{2} = \frac{2-4r}{2-r}. \tag{4.25}$$

The properties (ii)–(iv) in Chapter I yield

$$\begin{aligned} \cot\left(\frac{\varphi_f}{4}\right)^2 &< \left(\frac{\sqrt{3} + M(K^2)/\sqrt{3}}{1 - M(K^2)}\right)^2 \\ &= \frac{1}{3}\left(\frac{1 + \Phi_K\big(1/\sqrt{2}\,\big)^2}{1 - \Phi_K^2\big(1/\sqrt{2}\,\big)}\right)^2 \\ &< \frac{1}{3}\Phi_{1/K}\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{-2} = \frac{1}{3}\Phi_{1/K}\left(\frac{1}{3+2\sqrt{2}}\right)^{-2} \\ &\le \frac{1}{3}\left(4^{1-K}\left(\frac{1}{3+2\sqrt{2}}\right)^K\right)^{-2} = \frac{1}{3}16^{K-1}\big(3+2\sqrt{2}\,\big)^{2K}. \end{aligned} \tag{4.26}$$

Applying the above estimate to (4.1) one obtains the case $K > K_0$ in (4.20). This result is not sharp since for $K = 1$ the upper bound is $\big(3 + 2\sqrt{2}\,\big)^2/3$.

To get a sharp estimate, we have to use Lemma 2.1. By (4.13) we have

$$\varphi_f > \min_{z\in T}(\pi - 2|\arg f(z) - \arg z|) \ge \pi - 2\frac{4}{\sqrt{3}}M(K) = \pi - \frac{8}{\sqrt{3}}M(K),$$

thus

$$\cot\left(\frac{\varphi_f}{4}\right)^2 \le \left(\frac{1 + \tan\big(2/\sqrt{3}M(K)\big)}{1 - \tan\big(2/\sqrt{3}M(K)\big)}\right)^2 < \left(\frac{1 + \tan\big(2/\sqrt{3}\Lambda(K)\big)}{1 - \tan\big(2/\sqrt{3}\Lambda(K)\big)}\right)^2 \tag{4.27}$$

Now, the case $1 \le K \le K_0$ in (4.20) follows by applying (4.27) to (4.1), where K_0 satisfies the equation

$$\frac{1 + \tan\big((2/\sqrt{3}\,)\Lambda(K)\big)}{1 - \tan\big((2/\sqrt{3}\,)\Lambda(K)\big)} = \min_{1\le K<\infty} 4^{K-1}\frac{\big(3+2\sqrt{2}\,\big)^K}{\sqrt{3}} = \frac{3+2\sqrt{2}}{\sqrt{3}}.$$

This makes our proof complete.

5. Quasisymmetric functions as quasihomographies. An opposite inclusion is presented in

THEOREM 2.11. *For each $\varrho \ge 1$, there exists a constant $K \ge 1$ such that $Q_T(\varrho) \subset A_T(K)$ and*

$$K \le \begin{cases} \chi(\nu(2C_\varrho^2 - 1)) & \text{for } 1 \le \varrho \le \varrho_0, \\ \chi(\nu(M_\varrho - 1)) & \text{for } \varrho > \varrho_0, \end{cases} \tag{5.1}$$

where $\varrho_0 = (50\pi + 1)/(50\pi - 1)$, χ *and* ν *are described by* (2.7) *of Chapter* I *and* (3.21) *of Chapter* I, *respectively*,

$$C_\varrho = \frac{64^{\nu(\varrho)-1}}{(1-(\pi/3)(\varrho-1)/(\varrho+1))^{\nu(\varrho)}} \cdot \frac{\sqrt{\varrho+1}+\sqrt{2\pi(\varrho-1)}}{\sqrt{\varrho+1}-4.1\sqrt{2\pi(\varrho-1)}}, \tag{5.2}$$

and

$$M_\varrho = \tfrac{1}{2}\pi^2 4^{7\nu(\varrho)-4}. \tag{5.3}$$

Proof. Let $f \in Q_T(\varrho)$, $1 \le \varrho < \infty$. Without any loss of generality, we may assume that $f(1) = 1$ and $f(-1) = -1$; cf. [K2]. Let $h(z) = i(1-z)/(1+z)$, $h(T) = \overline{\mathbb{R}}$. For every symmetric triple $a-t$, a, $a+t \in \mathbb{R}$ with $t > 0$, we have

$$[\infty, a-t, a, a+t]^2 = \tfrac{1}{2}. \tag{5.4}$$

For each quadruple of the form $\infty, a-t, a, a+t$, there exists a positively ordered quadruple of distinct points z_1, z_2, z_3, $z_4 \in T$ and positive numbers α, β, γ, δ such that $z_1 = h^{-1}(\infty) = -1$, $z_2 = h^{-1}(a-t) = e^{2i\alpha+i\pi}$, $z_3 = h^{-1}(a) = e^{2i(\alpha+\beta)+i\pi}$ and $z_4 = h^{-1}(a+t) = e^{2i(\alpha+\beta+\gamma)+i\pi}$. Moreover, $\alpha+\beta+\gamma = \pi-\delta$. Thus, by the invariance of the cross-ratio under homographies,

$$[z_1, z_2, z_3, z_4]^2 = \frac{\sin\beta}{\sin(\alpha+\beta)} \cdot \frac{\sin\delta}{\sin(\alpha+\delta)} = \frac{1}{2}. \tag{5.5}$$

Without loss of generality we assume that $\alpha \le \gamma$. Then $\alpha \le \pi/2$. Let $f(z_2) = e^{i\alpha'+i\pi}$, $f(z_3) = e^{2i(\alpha'+\beta')+i\pi}$ and $f(z_4) = e^{2i(\alpha'+\beta'+\gamma')+i\pi}$, where α', β', γ' are positive and such that there exists a positive δ' for which $\alpha'+\beta'+\gamma' = \pi-\delta'$. If $g : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $f(e^{ix}) = e^{ig(x)}$ and g is normalized ($g(0) = 0$, $g(\pi) = \pi$), then, by a result of J. Krzyż [Kr2], $g \in Q_{\mathbb{R}}(\varrho)$ and

$$|g(x) - x| \le \pi\frac{\varrho-1}{\varrho+1} \tag{5.6}$$

for $0 \le x \le \pi$.

We intend to show that for every $1 \le \varrho \le \varrho_0$, where $2\pi(\varrho_0-1)/(\varrho_0+1) = 1/25$, there exists a constant C_ϱ, $1 \le C_\varrho < \infty$, such that

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{1}{C_\varrho} \cdot \frac{\sin\beta}{\sin(\alpha+\beta)} \tag{5.7}$$

for all admissible independent α, β and α', β' defined above. Assuming that $\varepsilon^2 = 2\pi \times (\varrho-1)/(\varrho+1)$, $0 \le \varepsilon \le 1/5 < \pi/4$, we will consider a few special cases:

(I) $\beta < \alpha+\beta \le \varepsilon$;
(II) $\beta \le \varepsilon < \alpha+\beta < \pi-\varepsilon$;
(III) $\varepsilon \le \beta < \alpha+\beta \le \pi-\varepsilon$;
(IV) $\varepsilon \le \beta \le \pi-\varepsilon \le \alpha+\beta$;
(V) $\pi-\varepsilon \le \beta < \alpha+\beta$.

(I) By (5.6), Theorem 1.6, and the inequality (ii) of Chapter I, there exists K', $1 \le K' \le \nu(\varrho)$, where ν is given by (2.7) of Chapter I, such that

$$\begin{aligned} \frac{\sin\beta'}{\sin(\alpha'+\beta')} &\ge \frac{\beta'}{\alpha'+\beta'}\cdot\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2} \ge \varPhi_{1/K'}\left(\sqrt{\frac{\beta}{\alpha+\beta}}\right)^2\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2} \\ &\ge 16^{1-K'}\left(\frac{\beta}{\alpha+\beta}\right)^{K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2} \\ &\ge 16^{1-K'}\left(\frac{\sin\beta}{\sin(\alpha+\beta)}\right)^{K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}. \end{aligned} \tag{5.8}$$

Since $\alpha \le \gamma$, we have $|z_2 - z_4| \ge \frac{1}{2}|z_1 - z_4|$, and by (5.5) we see that

$$\sin(\alpha+\beta) \le 4\sin\beta. \tag{5.9}$$

Therefore,

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge 64^{1-K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}\ . \tag{5.10}$$

(II) Using (5.6) and (5.9) we have

$$\sin\beta \ge \tfrac{1}{4}\sin(\alpha+\beta) \ge \tfrac{1}{4}\sin\varepsilon, \tag{5.11}$$

hence,

$$\frac{\sin\beta'}{\sin\beta} \ge \frac{\sin(\beta-\varepsilon^2)}{\sin\beta} = \cos\varepsilon^2 - \cot\beta\sin\varepsilon^2 \ge \cos\varepsilon^2 - 4\frac{\sin\varepsilon^2}{\sin\varepsilon} \tag{5.12}$$

and

$$\frac{\sin(\alpha'+\beta')}{\sin(\alpha+\beta)} \le \frac{\sin(\varepsilon+\varepsilon^2)}{\sin\varepsilon}. \tag{5.13}$$

Therefore,

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \left(\cos\varepsilon^2 - 4\frac{\sin\varepsilon^2}{\sin\varepsilon}\right)\frac{\sin\varepsilon}{\sin(\varepsilon+\varepsilon^2)}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}\ . \tag{5.14}$$

(III) It follows from (5.6) that

$$\frac{\sin\beta'}{\sin\beta} \ge \frac{\sin(\varepsilon-\varepsilon^2)}{\sin\varepsilon}, \tag{5.15}$$

and furthermore, by (5.13),

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{\sin(\varepsilon-\varepsilon^2)}{\sin(\varepsilon+\varepsilon^2)}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.16}$$

(IV) In this case we can see that $\gamma+\delta \le \varepsilon$, and since $\alpha \le \gamma$, it follows that $\alpha < \gamma+\delta \le \varepsilon$, hence $\alpha+\gamma+\delta < 2\varepsilon$. Using the same arguments as in (I), we conclude that

$$\begin{aligned} \frac{\sin(\gamma'+\delta')}{\sin(\gamma'+\delta'+\alpha')} &\le \frac{\gamma'+\delta'}{\gamma'+\delta'+\alpha'}\cdot\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)} \\ &\le \varPhi_{K'}\left(\sqrt{\frac{\gamma+\delta}{\gamma+\delta+\alpha}}\right)^2\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)} \end{aligned} \tag{5.17}$$

$$\le 16^{1-1/K'}\left(\frac{\gamma+\delta}{\gamma+\delta+\alpha}\right)^{1/K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}$$
$$\le 16^{1-1/K'}\left(\frac{\sin(\gamma+\delta)}{\sin(\gamma+\delta+\alpha)}\right)^{1/K'}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{1/K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}.$$

Since $\alpha\le\gamma$, we have $|z_1-z_3|\ge\frac12|z_2-z_3|$. Thus

$$\sin(\alpha+\beta)\ge\tfrac12\sin\beta. \tag{5.18}$$

Using (5.17) and (5.18), we get

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')}=\frac{\sin(\alpha'+\gamma'+\delta')}{\sin(\gamma'+\delta')} \ge 32^{-1+1/K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{1/K'}\frac{\sin(2\varepsilon+\varepsilon^2)}{2\varepsilon+\varepsilon^2}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.19}$$

(V) In this case we have $\gamma+\delta<\alpha+\gamma+\delta\le\varepsilon$, and following (IV), we arrive at

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')}\ge 32^{-1+1/K'}\left(\frac{\sin\varepsilon}{\varepsilon}\right)^{1/K'}\frac{\sin(\varepsilon+\varepsilon^2)}{\varepsilon+\varepsilon^2}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.20}$$

Hence, by (5.10), (5.14), (5.16), (5.19) and (5.20), we see that

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')}\ge\frac{1}{C_{K'}}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)} \tag{5.21}$$

with

$$\frac{1}{C_{K'}}=\min\left\{\frac{1}{C^1_{K'}},\frac{1}{C^2_{K'}},\frac{1}{C^3_{K'}},\frac{1}{C^4_{K'}},\frac{1}{C^5_{K'}}\right\}=\min\left\{\frac{1}{C^1_{K'}},\frac{1}{C^2_{K'}},\frac{1}{C^3_{K'}},\frac{1}{C^4_{K'}}\right\}, \tag{5.22}$$

where $C^l_{K'}$, $l=1,\dots,5$, are constants described by (5.10), (5.14), (5.16), (5.19) and (5.20), respectively.

In a similar way we get

$$\frac{\sin\delta'}{\sin(\alpha'+\delta')}\ge\frac{1}{C_{K'}}\cdot\frac{\sin\delta}{\sin(\alpha+\delta)}. \tag{5.23}$$

Therefore, by (5.21), (5.23) and (5.5), we have the inequality

$$[f(z_1),f(z_2),f(z_3),f(z_4)]^2=\frac{\sin\beta'}{\sin(\alpha'+\beta')}\cdot\frac{\sin\delta'}{\sin(\alpha'+\delta')} \ge\frac{1}{C^2_{K'}}[z_1,z_2,z_3,z_4]^2=\frac{1}{2C^2_{K'}} \tag{5.24}$$

for our quadruple $z_1,z_2,z_3,z_4\in T$. Similar arguments give

$$[f(z_2),f(z_3),f(z_4),f(z_1)]^2=\frac{\sin\gamma'}{\sin(\beta'+\gamma')}\cdot\frac{\sin\alpha'}{\sin(\beta'+\alpha')} \ge\frac{1}{C^2_{K'}}[z_2,z_3,z_4,z_1]^2=\frac{1}{C^2_{K'}}\left(1-\frac12\right)=\frac{1}{2C^2_{K'}} \tag{5.24'}$$

with the same points as in (5.24).

Let $F = h \circ f \circ h^{-1}$. Then F is a sense-preserving homeomorphism of $\mathbb{R}$ onto itself. Moreover, by (5.24) and (5.24′),

$$\text{(5.25)}\qquad \left(\frac{F(a+t)-F(a)}{F(a)-F(a-t)}+1\right)^{-1} = \frac{F(a)-F(a-t)}{F(a+t)-F(a-t)}$$
$$= [\infty, F(a-t), F(a), F(a+t)]^2$$
$$= [h\circ f(z_1), h\circ f(z_2), h\circ f(z_3), h\circ f(z_4)]^2$$
$$= [f(z_1), f(z_2), f(z_3), f(z_4)]^2 \ge \frac{1}{2C_{K'}^2}$$

and similarly

$$\text{(5.25′)}\qquad \left(\frac{F(a)-F(a-t)}{F(a+t)-F(a)}+1\right)^{-1} = \frac{F(a+t)-F(a)}{F(a+t)-F(a-t)}$$
$$= [F(a-t), F(a), F(a+t), \infty]^2$$
$$= [f(z_2), f(z_3), f(z_4), f(z_1)]^2 \ge \frac{1}{2C_{K'}^2}.$$

Using (5.25) and (5.25′), we see that

$$\text{(5.26)}\qquad \frac{1}{2C_{K'}^2-1} \le \frac{F(a+t)-F(a)}{F(a)-F(a-t)} \le 2C_{K'}^2-1,$$

hence $F \in Q_{\mathbb{R}}(2C_{K'}^2-1)$. Now, by Theorem 2.7, we see that there exists K, $1 \le K < \infty$, such that $F \in A_{\mathbb{R}}(K)$ and $K \le \chi(\nu(2C_{K'}^2-1))$. Thus, for any distinct $z_1, z_2, z_3, z_4 \in T$,

$$\text{(5.27)}\qquad [f(z_1), f(z_2), f(z_3), f(z_4)]^2 = [F\circ h(z_1), F\circ h(z_2), F\circ h(z_3), F\circ h(z_4)]^2$$
$$\le \Phi_K([h(z_1), h(z_2), h(z_3), h(z_4)])^2$$
$$= \Phi_K([z_1, z_2, z_3, z_4])^2,$$

hence $f \in A_T(K)$ for $1 \le \varrho \le \varrho_0$, where $\varrho_0 = (50\pi+1)/(50\pi-1)$.

Now let $\varrho > \varrho_0$. If $0 < \beta' < \pi/2$, then

$$\text{(5.28)}\qquad \frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{2}{\pi}\cdot\frac{\beta'}{\alpha'+\beta'}.$$

If $\pi/2 \le \beta' < \alpha'+\beta' < \pi$, then

$$\text{(5.29)}\qquad \frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge 1.$$

By applying (5.6), Theorem 13 in [Z3], Theorem 1.6 and the inequality (ii) of Chapter I, we see that there exists K', $1 \le K' < \nu(\varrho)$, such that

$$\text{(5.30)}\qquad \frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{2}{\pi}\cdot\frac{\beta'}{\alpha'+\beta'} \ge \frac{2}{\pi}\Phi_{1/K'}\left(\sqrt{\frac{\beta}{\alpha+\beta}}\right)^2 \ge \frac{2}{\pi}16^{1-K'}\left(\frac{\beta}{\alpha+\beta}\right)^{K'}.$$

We consider the following four possibilities:

(I′) $0 < \beta < \alpha+\beta \le \pi/2$;
(II′) $0 < \beta < \pi/4$ and $\pi/2 \le \alpha+\beta < \pi$;
(III′) $\pi/4 \le \beta \le \pi/2 < \alpha+\beta < \pi$;
(IV′) $\pi/2 \le \beta < \alpha+\beta < \pi$.

(I′) In this case,

$$\frac{\beta}{\alpha+\beta} \ge \frac{2}{\pi}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.31}$$

(II′) In view of $\alpha \le \pi/2$,

$$\frac{\beta}{\beta+\alpha} \ge \frac{4}{3\sqrt{2}\pi}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.32}$$

(III′) By (5.18),

$$\frac{\beta}{\alpha+\beta} \ge \frac{1}{8}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.33}$$

(IV′) Again by (5.18),

$$\frac{\beta}{\alpha+\beta} \ge \frac{1}{2} \ge \frac{1}{4}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}. \tag{5.34}$$

Since

$$\min\left\{\frac{2}{\pi},\frac{4}{3\sqrt{2}\pi} = \frac{1}{8},\frac{1}{8},\frac{1}{4}\right\} \tag{5.35}$$

we have, by (5.30),

$$\frac{\sin\beta'}{\sin(\alpha'+\beta')} \ge \frac{2}{\pi}16^{1-K'}\left(\frac{1}{8}\cdot\frac{\sin\beta}{\sin(\alpha+\beta)}\right)^{K'}. \tag{5.36}$$

In a similar way we can show that

$$\frac{\sin\delta'}{\sin(\delta'+\alpha')} \ge \frac{2}{\pi}16^{1-K'}\left(\frac{1}{8}\cdot\frac{\sin\delta}{\sin(\delta+\alpha)}\right)^{K'}. \tag{5.36′}$$

Following our considerations presented by (5.24), (5.24′), (5.25), (5.25′), and by (5.26) with (5.36) and (5.36′), and Theorem 2.7, we see that $F \in A_{\mathbb{R}}(K)$ and $1 \le K \le \chi(\nu(M_{K'}-1))$, where $M_{K'} = \frac{1}{2}\pi^2 4^{7K'-4}$.

Let us return to (5.22). For $0 \le \varepsilon \le \varepsilon_0 = 1/5$ we have

$$\max\left\{\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)},\frac{\sin(\varepsilon+\varepsilon^2)}{\sin\varepsilon\cos\varepsilon^2-4\sin\varepsilon^2}\right\} \le \frac{1+\varepsilon}{1-4\varepsilon/\cos\varepsilon} \le \frac{1+\varepsilon}{1-4.1\varepsilon}.$$

Hence,

$$\begin{aligned} C_{K'} &= \max\left\{64^{K'-1}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{K'}\frac{\varepsilon+\varepsilon^2}{\sin(\varepsilon+\varepsilon^2)},\frac{\sin(\varepsilon+\varepsilon^2)}{\sin\varepsilon}\cdot\frac{\sin\varepsilon}{\sin\varepsilon\cos\varepsilon^2-4\sin\varepsilon^2},\right.\\ &\qquad\left.\frac{\sin(\varepsilon+\varepsilon^2)}{\sin(\varepsilon-\varepsilon^2)},32^{1-1/K'}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{1/K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)}\right\}\\ &\le \max\left\{64^{K'-1}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{K'}\frac{2\varepsilon+\varepsilon^2}{\sin(2\varepsilon+\varepsilon^2)},\frac{\sin(\varepsilon+\varepsilon^2)}{\sin\varepsilon\cos\varepsilon^2-4\sin\varepsilon^2}\right\}\\ &\le 64^{K'-1}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{K'}\frac{1+\varepsilon}{1-4\varepsilon/\cos\varepsilon} \le 64^{K'-1}\left(\frac{\varepsilon}{\sin\varepsilon}\right)^{K'}\frac{1+\varepsilon}{1-4.1\varepsilon}\\ &\le 64^{K'-1}\frac{1}{\left(1-\frac{1}{6}\varepsilon^2\right)^{K'}}\frac{1+\varepsilon}{1-4.1\varepsilon} \end{aligned} \tag{5.37}$$

$$= \frac{64^{K'-1}}{(1-(\pi/3)(\varrho-1)/(\varrho+1))^{K'}} \cdot \frac{\sqrt{\varrho+1}+\sqrt{2\pi(\varrho-1)}}{\sqrt{\varrho+1}-4.1\sqrt{2\pi(\varrho-1)}}.$$

Since $\chi(K')$ and $\nu(K')$ are increasing functions, we obtain the estimate (5.1) in view of (5.2) and (5.3). This completes the proof of Theorem 2.11.

III. Distortion theorems for quasihomographies

1. Introduction. It is evident that the invariance of quasihomographies under composition with homographies is very challenging. Using this notion we prove a number of distortion theorems on K-qh automorphisms of an arbitrary circle Γ in $\overline{\mathbb{C}}$. Pursuing this idea and taking advantage of the fact that similarities linking the metric spaces $(A_{\Gamma_1}, d_{\Gamma_1})$ and $(A_{\Gamma_2}, d_{\Gamma_2})$ can be achieved by the use of homographies, we obtain a quantitative estimate of $d_\Gamma(f(z_1), f(z_2))$ depending on Γ and the qh constant $K(f)$. A remarkable feature of these distortion theorems for K-qh automorphisms is that they are all at least asymptotically sharp for $K = 1$. Moreover, some of the obtained estimates are the best or possess the best constants, which seems impossible to attain for ϱ-qs functions.

The property that a family $M \subset A_\Gamma$ is *closed* with respect to $A_\Gamma(1)$ also gives a *topological characterization* of K-qh automorphisms of a circle Γ in $\overline{\mathbb{C}}$.

The normal and compact sets of the metric space (A_Γ, d_Γ) are characterized by describing sufficient conditions.

2. Similarities. Let A_Γ be the family of all automorphisms of a circle Γ in $\overline{\mathbb{C}}$. It is evident that $(A_\Gamma, \circ)$ is a group under composition. For $f, g \in A_\Gamma$, we define

$$d_\Gamma(f, g) = \sup_{z\in\Gamma} d(f(z), g(z)), \tag{2.1}$$

where d is the *spherical chordal* metric defined by

$$\begin{aligned} &d(z, w) = |w-z|/\big(\sqrt{1+|z|^2}\sqrt{1+|w|^2}\,\big), \qquad z, w \in \mathbb{C}, \\ &d(z, \infty) = 1/\sqrt{1+|z|^2}, \qquad z, w \in \mathbb{C}. \end{aligned} \tag{2.1$'$}$$

Let $\widehat{A}_\Gamma$ denote the family of functions $f \in A_\Gamma$ such that there exist distinct points $z_1, z_2, z_3 \in \Gamma$ with

$$d(z_i, z_j) = d(z_j, z_k) = d(f(z_j), f(z_k)) \tag{2.2}$$

for $i,\ j,\ k = 1, 2, 3$.

Denote by $A_\Gamma^{z_1,z_2,z_3}$ the subclass of $\widehat{A}_\Gamma$ indicated by fixing three distinct points $z_1, z_2, z_3 \in \Gamma$ that remain fixed points of each automorphism $f \in A_\Gamma^{z_1,z_2,z_3}$.

Let

$$A_T^\circ = A_T^{z_1,z_2,z_3}, \tag{2.3}$$

where $z_i^3 = 1$, $i = 1, 2, 3$, and

$$A_{\bar{\mathbb{R}}}^\circ = A_{\bar{\mathbb{R}}}^{-1/\sqrt{3},\, 1/\sqrt{3},\, \infty}. \tag{2.3$'$}$$

The usual normalization for ϱ-qs functions of $\mathbb{R}$ by fixing 0 and 1 does not work here since $A_{\bar{\mathbb{R}}}^{0,1,\infty} \not\subset \widehat{A}_{\bar{\mathbb{R}}}$.

Finally, for a circle Γ in $\overline{\mathbb{C}}$, let

$$(2.4)\qquad \widehat{A}_\Gamma(K)=\widehat{A}_\Gamma\cap A_\Gamma(K),\qquad A^\circ_\Gamma(K)=A^\circ_\Gamma\cap A_\Gamma(K),\qquad A^\infty_\Gamma=\bigcup_{K\ge 1}A_\Gamma(K).$$

To have an explicit formula transforming results obtained for a subclass of $\widehat{A}_{\Gamma_1}(K)$ onto results for the corresponding subclass of $\widehat{A}_{\Gamma_2}(K)$, we begin with

LEMMA 3.1. *Let Γ be a circle on $\overline{\mathbb{C}}$, and let S be a similarity of (A_Γ,d_Γ) onto itself. Then S is an isometry.*

Proof. Suppose that Γ is a circle on $\overline{\mathbb{C}}$ and that there is a constant $c>1$ and a similarity S mapping (A_Γ,d_Γ) onto itself such that

$$d_\Gamma(S(f),S(g))=cd_\Gamma(f,g)$$

for all $f,g\in A_\Gamma$. If $f\ne g$, there exists $n\in\mathbb{N}$ such that

$$(2.5)\qquad d_\Gamma(S^n(f),S^n(g))=c^n d_\Gamma(f,g)>1.$$

This contradicts (2.1). If $0<c<1$, we may consider S^{-1}, whose constant is c^{-1}. By repeating the above argument we arrive at a contradiction again. The proof is finished.

Let

$$T_r:=\{z\in\mathbb{C}:|z|=r\},\qquad \overline{\mathbb{R}}_t:=\{z\in\mathbb{C}:\operatorname{Im}z=t\}.$$

Using this notation we prove

LEMMA 3.2. *For every $r>0$, there exists a similarity S mapping (A_{T_r},d_{T_r}) onto (A_T,d_T) with the constant*

$$(2.6)\qquad c=\frac{r^2+1}{2r}.$$

Proof. By (2.1) and the well-known properties of the stereographic projection of B onto $\overline{\mathbb{C}}$, there exists a homography H such that $H(T_r)=T_1=T$ and

$$(2.7)\qquad d(H(z_1),H(z_2))=cd(z_1,z_2)\quad\text{for } z_1,z_2\in T_r,$$

where c is described by (2.6). Let

$$(2.8)\qquad S_H(f)=H\circ f\circ H^{-1},\qquad f\in A_{T_r}.$$

Then, by (2.7), we get

$$(2.9)\qquad \begin{aligned} d_T(S_H(f),S_H(g))&=\sup_{w\in T}d((H\circ f\circ H^{-1})(w),(H\circ g\circ H^{-1})(w)),\\ &=c\sup_{z\in T_r}d(f(z),g(z))=cd_{T_r}(f,g).\end{aligned}$$

Thus, S_H is a similarity mapping (A_{T_r},d_{T_r}) onto (A_T,d_T) with the constant c described by (2.6).

Now we can prove

THEOREM 3.1. *For every circle Γ on $\overline{\mathbb{C}}$, there exists exactly one number r, $0<r\le 1$, such that the spaces (A_Γ,d_Γ) and (A_{T_r},d_{T_r}) are isometric.*

P r o o f. By the definition (2.1) of d_Γ, the basic properties of the stereographic projection of B onto $\overline{\mathbb{C}}$ and the euclidean isometry of B, there is a homography H such that $H(\Gamma) = T_r$, $0 < r \le 1$, and

$$d_\Gamma(H(z_1), H(z_2)) = d_\Gamma(z_1, z_2) \quad \text{for } z_1, z_2 \in \Gamma. \tag{2.10}$$

Hence

$$\begin{aligned} d_{T_r}(S_H(f), S_H(g)) &= \sup_{w\in T_r} d((H\circ f\circ H^{-1})(w), (H\circ g\circ H^{-1})(w)) \\ &= \sup_{w\in T_r} d((f\circ H^{-1})(w), (g\circ H^{-1})(w)) = \sup_{z\in\Gamma} d(f(z), g(z)) = d_\Gamma(f,g) \end{aligned}$$

for all $f, g \in A_\Gamma$, where $z = H^{-1}(w)$.

Suppose now that there is $r_1 \ne r$, $0 < r_1 \le 1$, and an isometry J of (A_Γ, d_Γ) onto $(A_{T_{r_1}}, d_{T_{r_1}})$. By Lemma 3.2, there exists a similarity $S : A_{T_{r_1}} \to A_{T_r}$ with $d_{T_r}(S(f), S(g)) = c d_{T_{r_1}}(f, g)$ for $f, g \in A_{T_{r_1}}$ with a constant $c \ne 1$, because $r_1 \ne r$. Then $S_H^{-1} \circ S \circ I$ is a similarity of (A_Γ, d_Γ) onto itself with a constant $c \ne 1$. By Lemma 3.1, this is impossible. Thus, the uniqueness of the constant r is proved, and the proof is complete.

By the above considerations, we may associate with every circle Γ in $\overline{\mathbb{C}}$ exactly one number $r = r(\Gamma)$ such that (A_Γ, d_Γ) and (A_{T_r}, d_{T_r}) are isometric. This number is called the *spherical radius of* Γ.

THEOREM 3.2. *For any circles* Γ_1, Γ_2 *on* $\overline{\mathbb{C}}$, *there exists a similarity* S *mapping* $(A_{\Gamma_1}, d_{\Gamma_1})$ *onto* $(A_{\Gamma_2}, d_{\Gamma_2})$ *with the constant*

$$c = \frac{r(\Gamma_1) + 1/r(\Gamma_1)}{r(\Gamma_2) + 1/r(\Gamma_2)}. \tag{2.11}$$

Moreover, *a similarity with a constant* $c_1 \ne c$ *does not exist.*

P r o o f. By Theorem 3.1, there exist isometries J_i of $(A_{\Gamma_i}, d_{\Gamma_i})$ onto $(A_{T_{r_i}}, d_{T_{r_i}})$ for $i = 1, 2$. By Lemma 3.2, there exist similarities S_i of $(A_{T_{r_i}}, d_{T_{r_i}})$ onto (A_T, d_T) with the constants $c_i = (r_i^2 + 1)/(2r_i)$, $i = 1, 2$. Hence

$$S = J_2^{-1} \circ S_2^{-1} \circ S_1 \circ J_1 \tag{2.12}$$

is a similarity mapping $(A_{\Gamma_1}, d_{\Gamma_1})$ onto $(A_{\Gamma_2}, d_{\Gamma_2})$ with the constant $c = c_1/c_2$. Thus we obtain (2.11).

If there exists a similarity S_1 of $(A_{\Gamma_1}, d_{\Gamma_1})$ onto $(A_{\Gamma_2}, d_{\Gamma_2})$ with a constant $c_1 \ne c$, then $S^{-1} \circ S_1$ is a similarity of $(A_{\Gamma_1}, d_{\Gamma_1})$ onto itself with the constant $c_1/c \ne 1$. This contradicts Lemma 3.1. Theorem 3.2 is proved.

COROLLARY 3.1. *For any circles* Γ_1, Γ_2 *in* $\overline{\mathbb{C}}$, *the spaces* $(A_{\Gamma_1}, d_{\Gamma_1})$ *and* $(A_{\Gamma_2}, d_{\Gamma_2})$ *are isometric if and only if* $r(\Gamma_1) = r(\Gamma_2)$.

EXAMPLE 3.1. By the definition of $r(\Gamma)$ it is evident that

$$r(T_r) = \begin{cases} r & \text{for } 0 < r \le 1, \\ 1/r & \text{for } r \ge 1 \end{cases} \tag{2.13}$$

and

$$r(\overline{\mathbb{R}}_t) = \sqrt{t^2+1} - |t| \quad \text{for } t \in \mathbb{R}. \tag{2.14}$$

Corollary 3.2. *The spaces* (A_{T_r}, d_{T_r}) *and* $(A_{\bar{\mathbb{R}}_t}, d_{\bar{\mathbb{R}}_t})$ *are isometric if and only if*

$$r + \frac{1}{r} = 2\sqrt{t^2 + 1}. \tag{2.15}$$

Example 3.2. The spaces (A_T, d_T) and $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$ are isometric.

The main result of this section can be formulated as follows:

Theorem 3.3. *For any circles* Γ_1, Γ_2 *on* $\overline{\mathbb{C}}$, *there exists a homography* H *such that*

(i) $H(\Gamma_1) = \Gamma_2$;

(ii) $d(H(z_1), H(z_2)) = cd(z_1, z_2)$ *for each* $z_1, z_2 \in \Gamma_1$, *with*

$$c = \frac{r(\Gamma_1) + 1/r(\Gamma_1)}{r(\Gamma_2) + 1/r(\Gamma_2)};$$

(iii) *for every* $f \in A_{\Gamma_1}$, *the mapping* $S_H = H \circ f \circ H^{-1}$ *is a similarity of* $(A_{\Gamma_1}, d_{\Gamma_1})$ *onto* $(A_{\Gamma_2}, d_{\Gamma_2})$ *with the constant* c *as in* (ii);

(iv) $S_H(\widehat{A}_{\Gamma_1}) = \widehat{A}_{\Gamma_2}$;

(v) $S_H(A_{\Gamma_1}(K)) = A_{\Gamma_2}(K)$, $K \geq 1$;

(vi) S_H *is an isomorphism of the groups* $(A_{\Gamma_1}, \circ)$ *and* $(A_{\Gamma_2}, \circ)$.

P r o o f. The statements (i)–(iii) follow from the proof of Lemma 3.2 and Theorems 3.1 and 3.2. To obtain (iv), suppose that $f \in \widehat{A}_{\Gamma_1}$. Then there exist distinct points $z_i \in \Gamma_1$ such that $d(z_i, z_j) = d(z_j, z_k) = d(f(z_j), f(z_k))$ for $i, j, k = 1, 2, 3$, $i \neq j \neq k$. Let $w_i = H(z_i)$, $i = 1, 2, 3$. Thus, by (ii), we have

$$d(w_i, w_j) = d(w_j, w_k) = d(S_H(f)(w_j), S_H(f)(w_k)), \quad i, j, k = 1, 2, 3.$$

The condition (v) follows immediately from (iii), the definition of $A_\Gamma(K)$ and the fact that each homography preserves the cross-ratio. Moreover,

$$S_H(f \circ g) = H \circ (f \circ g) \circ H^{-1} = (H \circ f \circ H^{-1}) \circ (H \circ g \circ H^{-1}) = S_H(f) \circ S_H(g)$$

for each $f, g \in A_{\Gamma_1}$. Thus (vi) is proved, which completes the proof.

3. Distortion theorems. First we prove the one-dimensional counterpart of the quasiconformal version of the Schwarz Lemma.

Theorem 3.4. *For each* $K \geq 1$ *and* $f \in A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$, *we have*:

(i) $\Phi_{1/K}(\sqrt{x})^2 \leq f(x) \leq \Phi_K(\sqrt{x})^2$ *for* $0 \leq x \leq 1$;

(ii) $\Phi_K(1/\sqrt{x})^{-2} \leq f(x) \leq \Phi_{1/K}(1/\sqrt{x})^{-2}$ *for* $x \geq 1$;

(ii′) $\lambda(1/K, \sqrt{(x-1)/x}) \leq f(x) - 1 \leq \lambda(K, \sqrt{(x-1)/x})$ *for* $x \geq 1$;

(iii) $-\lambda(K, \sqrt{x/(x-1)}) \leq f(x) \leq -\lambda(1/K, \sqrt{x/(x-1)})$ *for* $x < 0$,

where $\lambda(K, t)$ *is defined by* (5.3′) *of Chapter I. Moreover, these estimates are the best possible.*

P r o o f. For $0 \leq x \leq 1$, we have $[\infty, 0, x, 1]^2 = x$. Then, by the definition of $A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$, we obtain (i). For $x \geq 1$, we have $[\infty, 0, 1, x]^2 = 1/x$. Thus $\Phi_{1/K}(1/\sqrt{x})^2 \leq 1/f(x) \leq$

$\Phi_K\big(1/\sqrt{x}\,\big)^2$, and hence (ii). Inequalities (ii′) follow from (ii) and from (5.3) in Chapter I. Suppose that $x<0$. The identity $[x,0,1,\infty]^2 = 1/(1-x)$ shows that

$$\Phi_{1/K}\left(\frac{1}{\sqrt{1-x}}\right)^2 \le \frac{1}{1-f(x)} \le \Phi_K\left(\frac{1}{\sqrt{1-x}}\right)^2.$$

Using again (5.3′) of Chapter I and (5.5) of Chapter I we arrive at (iii). Since equality may occur for each of the estimates and each value of arguments, they are the best possible. The proof is complete.

Using the estimate (ii) and (ii′) of Chapter I, one can easily prove

THEOREM 3.5. *For any* $K\ge 1$ *and* $f\in A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$, *we have*:

(i) $16^{1-K}x^K \le f(x) \le 16^{1-1/K}x^{1/K}$ *for* $0\le x\le 1$;

(ii) $16^{-1+1/K}x^{1/K} \le f(x) \le 16^{K-1}x^K$ *for* $x\ge 1$;

(ii′) $\dfrac{16^{1-K}(x-1)^K}{x^K-16^{1-K}(x-1)^K} \le f(x)-1 \le \dfrac{16^{1-1/K}(x-1)^{1/K}}{x^{1/K}-16^{1-1/K}(x-1)^{1/K}}$ *for* $x>1$;

(iii) $-\dfrac{16^{1-1/K}x^{1/K}}{(x-1)^{1/K}-16^{1-1/K}x^{1/K}} \le f(x) \le -\dfrac{16^{1-K}x^K}{(x-1)^K-16^{1-K}x^K}$ *for* $x<0$.

These inequalities are asymptotically sharp for $K=1$, *and the constants*: 16, K, $K-1$, $1/K$ *and* $1-1/K$ *in* (i) *and* (ii) *cannot be lowered in this type of estimates.*

Remark 3.1. Using the relationship between K-qh and ϱ-qs functions of $\mathbb{R}$ one can automatically obtain analogs Theorems 3.4 and 3.3 for ϱ-qs functions of $\mathbb{R}$. These estimates are asymptotically sharp for $\varrho=1$.

Remark 3.2. The inequalities of Theorem 3.5 can easily be improved by applying the approximations on Φ_K and $\lambda_2[K,L](t)$, presented in Section 4 of Chapter I.

COROLLARY 3.3. *For any* $K\ge 1$ *and* $f\in A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$,

$$(3.1)\qquad |f(x)-x| \le \begin{cases} \max\{16^{1-1/K}x^{1/K}-x,\, 16^{1-1/K}(1-x)^{1/K}-(1-x)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{for } 0\le x\le 1, \\ 16^{K-1}x^K - x \qquad\qquad\qquad \textit{for } x>1, \\ 16^{K-1}(1-x)^K-(1-x) \quad\ \textit{for } x<0. \end{cases}$$

Proof. (3.1) follows immediately from Theorem 3.4 if we note that for $0\le x\le 1$,

$$\begin{aligned} |f(x)-x| &\le \max\big\{\Phi_K\big(\sqrt{x}\,\big)^2 - x,\, x-\Phi_{1/K}\big(\sqrt{x}\,\big)^2\big\} \\ &= \max\big\{\Phi_K\big(\sqrt{x}\,\big)^2 - x,\, \Phi_K\big(\sqrt{1-x}\,\big)-(1-x)^2\big\} \\ &\le \{16^{1/K-1}x^{1/K}-x,\, 16^{1/K-1}(1-x)^{1/K}-(1-x)\}. \end{aligned}$$

The other two cases follow directly from Theorems 3.4 and 3.5. The proof is finished.

COROLLARY 3.4. *For any* $K\ge 1$ *and* $f\in A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$,

$$\max_x |f(x)-x| = \begin{cases} M(K) & \textit{for } 0\le x\le 1, \\ 16^{K-1}L^K - L & \textit{for } 1<x\le L \textit{ or } -L+1\le x<0. \end{cases}$$

The following lemma, with $M(K)$ given in Theorem 1.3, will be very useful in our considerations.

LEMMA 3.3. *For any* $K \ge 1$ *and* $f \in A_{\bar{\mathbb{R}}}(K)$ *with* $f(0) = 0$ *and* $f(\infty) = \infty$, *we have*

$$\Phi_{1/K}\left(\sqrt{\frac{x_1}{x_2}}\right)^2 \le \frac{f(x_1)}{f(x_2)} \le \Phi_K\left(\sqrt{\frac{x_1}{x_2}}\right)^2 \tag{3.2}$$

for $0 < x_1 < x_2$ *or* $x_2 < x_1 < 0$.

Proof. Use the identity $[\infty, 0, x_1, x_2]^2 = x_1/x_2 = [x_1, x_2, 0, \infty]$ and the definition of $A_{\bar{\mathbb{R}}}(K)$.

Remark 3.3. Under the assumptions of Lemma 3.3, by Theorem 3.3 it follows that

$$16^{1-K}\left(\frac{x_1}{x_2}\right)^K \le \frac{f(x_1)}{f(x_2)} \le 16^{1-1/k}\left(\frac{x_1}{x_2}\right) \quad \text{and} \quad M_{1/K}\left(\frac{x_1}{x_2}\right) \le \frac{f(x_1)}{f(x_2)} - \frac{x_1}{x_2} \le M_K\left(\frac{x_1}{x_2}\right).$$

Hence we have

$$\sup\left|\frac{f(x_1)}{f(x_2)} - \frac{x_1}{x_2}\right| \le M(K),$$

where the supremum is taken over all the possible x_1 and x_2 as in Lemma 3.3.

Now we prove

THEOREM 3.6. *For any* $K \ge 1$ *and* $f \in A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$, *we have*

$$|f(x_1) - f(x_2)| \le \begin{cases} 16^{2(1-1/K)}|x_2 - x_1|^{1/K} & \text{for } x_1, x_2 \in [0,1], \\ (16L)^{K-1/K}|x_2 - x_1|^{1/K} & \text{for } x_1, x_2 \in [-L+1, 1] \cup [0, L], \\ 16^{K-1}(16ML)^{K-1/K}|x_2 - x_1|^{1/K} & \text{for } x_1, x_2 \in [-L+1, M], \end{cases} \tag{3.3}$$

where $L, M > 1$. *Moreover*, *the exponent* $1/K$ *is the best possible.*

Proof. Let $f \in A^{0,1,\infty}_{\bar{\mathbb{R}}}(K)$ and $K \ge 1$. For $0 < x_1 < x_2 < 1$, it follows that

$$[0, x_1, x_2, \infty]^2 = \frac{x_2 - x_1}{x_2}. \tag{3.4}$$

By Lemma 3.3, the definition of $A_{\bar{\mathbb{R}}}(K)$ and (ii) of Chapter I, we have

$$|f(x_2) - f(x_1)| \le 16^{1-1/K}\frac{f(x_2)}{x_2^{1/K}}|x_2 - x_1|^{1/K} \le 16^{2(1-1/K)}|x_2 - x_1|^{1/K}.$$

Now let $0 < x_1 < x_2 \le L$ and $1 < x_2$. Then, by (3.2) and Theorem 3.5,

$$\begin{aligned} |f(x_2) - f(x_1)| &\le 16^{1-1/K}\frac{f(x_2)}{x_2^{1/K}}|x_2 - x_1|^{1/K} \\ &\le (16x_2)^{K-1/K}|x_2 - x_1|^{1/K} \le (16L)^{K-1/K}|x_2 - x_1|^{1/K}. \end{aligned}$$

Consider the case $-L + 1 \le x_1 \le x_2 < 1$. Then

$$[\infty, x_1, x_2, 1]^2 = \frac{x_2 - x_1}{1 - x_1}. \tag{3.5}$$

Using (2.4), we obtain

$$\begin{aligned} |f(x_2) - f(x_1)| &\le 16^{1-1/K}\frac{1 - f(x_1)}{(1 - x_1)^{1/K}}|x_2 - x_1|^{1/K} \\ &\le (16(1 - x_1))^{K-1/K}|x_2 - x_1|^{1/K} \le (16L)^{K-1/K}|x_2 - x_1|^{1/K}. \end{aligned}$$

In the case $-L+1\le x_1<0<1<x_2<M$, we have

$$[x_1,0,1,x_2]^2=\frac{x_2-x_1}{(1-x_1)x_2}. \tag{3.6}$$

This gives, by (2.3) and (ii) of Chapter I,

$$\begin{aligned}|f(x_2)-f(x_1)| &\le 16^{1-1/K}\frac{1-f(x_1)}{(1-x_1)^{1/K}}\frac{f(x_2)}{x_2^{1/K}}|x_2-x_1|^{1/K}\\ &\le 16^{1-1/K}16^{K-1}(1-x_1)^{K-1/K}16^{K-1}x_2^{K-1/K}|x_2-x_1|^{1/K}\\ &\le 16^{K-1}(16LM)^{K-1/K}|x_2-x_1|^{1/K}.\end{aligned}$$

Thus we have (3.3) and the theorem is proved.

By Theorem 2.3, for every $f\in A_\Gamma(K)$, $K\ge1$, there exists f^{-1} and it is also an element of $A_\Gamma(K)$, where Γ is a circle in $\overline{\mathbb{C}}$. Applying this fact to the previous theorem we immediately get

Corollary 3.5. *Under the assumption of Theorem* 3.6, *we have*

$$|f(x_2)-f(x_1)| \ge\begin{cases}16^{2(1-K)}|x_2-x_1|^K & \text{for } x_1,x_2\in[0,1],\\ (16L)^{1-K^2}|x_2-x_1|^K & \text{for } x_1,x_2\in[-L+1,1]\cup[0,L],\\ 16^{K-K^2}(16LM)^{1-K^2}|x_2-x_1|^K & \text{for } x_1,x_2\in[-L+1,M],\end{cases} \tag{3.7}$$

where $L,M>1$. Moreover, the exponent K is the best possible.

The main result of this section is

Theorem 3.7. *Let Γ be a circle in $\overline{\mathbb{C}}$. For any $f\in\widehat{A}_\Gamma(K)$ and $K\ge1$, the inequality*

$$d(f(z_1),f(z_2))\le C_K\left(\frac{2}{r(\Gamma)+1/r(\Gamma)}\right)^{1-1/K}d(z_1,z_2)^{1/K} \tag{3.8}$$

holds for all $z_1,z_2\in\Gamma$ with

$$C_K=\begin{cases}\big(2\sqrt3\big)^{-1+1/K}16^{3(1-1/K)}\left(\dfrac{4M(K)}{\big(\sqrt3\big)^{1+1/K}}+2^{1-1/K}\right)^2, & 1\le K<1.061,\\ \big(2/\sqrt3\big)^{1+1/K}16^{3(1-1/K)}, & 1.061\le K<1.899,\\ 2\big(\sqrt3\big)^{1+1/K}16^{2(1-1/K)}, & K\ge1.899,\end{cases} \tag{3.9}$$

$r(\Gamma)$ described by Theorem 3.1, *and Λ given by* (2.37) *of Chapter* I. *Asymptotically, $C_K\to1$ as $K\to1$ and $C_K\to512\sqrt3$ as $K\to\infty$.*

P r o o f. The proof is divided into a few steps. Confining ourselves to $A_{\overline{\mathbb{R}}}^{0,1,\infty}(K)$, we obtain (3.8). Changing this normalization we arrive at (3.8) with C_K given by the third line of (3.9) within $A_{\overline{\mathbb{R}}}^{\circ}(K)$. Using Theorem 3.3 we extend this result to $A_T^{\circ}(K)$. At the next step we begin with $A_T^{\circ}(K)$ and get the first two lines of C_K in (3.9). Then, using Theorem 3.3, we transform the obtained result to an arbitrary circle Γ in $\overline{\mathbb{C}}$.

Suppose that $f\in A_{\overline{\mathbb{R}}}^{0,1,\infty}(K)$, $K\ge1$, and $0<x_1<x_2$. Then, by (3.4), one can see that

$$d(f(x_2),f(x_1))\le C_K^1(x_2,x_1)d(x_2,x_1)^{1/K}, \tag{3.10}$$

where

$$C_K^1(x_2,x_1)=16^{1-1/K}\frac{f(x_2)}{x_2^{1/K}}\cdot\frac{(1+x_2^2)^{1/(2K)}(1+x_1^2)^{1/(2K)}}{(1+|f(x_2)|^2)^{1/2}(1+|f(x_1)|^2)^{1/2}}. \tag{3.11}$$

Using (ii) of Chapter I, we get

$$\begin{aligned}C_K^1(x_2,x_1)&\le\frac{(1+x_1^2)^{1/(2K)}}{(1+(16^{-1+1/K}x_1^{1/K})^2)^{1/2}}\left(1+\frac{1}{x_2^2}\right)^{1/(2K)}16^{1-1/K}\\&\le 16^{2(1-1/K)}\left(1+\frac{1}{x_1^2}\right)^{1/(2K)}2^{1/(2K)}\le 16^{2(1-1/K)}2^{1/K}\end{aligned} \tag{3.12}$$

for $1<x_1<x_2$.

In the case $0<x_1<1<x_2$, we have

$$C_K^1(x_2,x_1)\le 16^{2(1-1/K)}2^{1/(2K)}2^{1/(2K)}=16^{2(1-1/K)}2^{1/K}. \tag{3.13}$$

Let $x_1<x_2<1$. Using (3.5) we obtain

$$d(f(x_2),f(x_1))\le C_K^2(x_2,x_1)d(x_2,x_1)^{1/K}, \tag{3.14}$$

where

$$C_K^2(x_2,x_1)=16^{1-1/K}\frac{1+|f(x_1)|}{(1+|x_1|)^{1/K}}\cdot\frac{(1+|x_1|^2)^{1/(2K)}(1+|x_2|^2)^{1/(2K)}}{(1+|f(x_1)|^2)^{1/2}(1+|f(x_2)|^2)^{1/2}}. \tag{3.15}$$

By Theorem 3.4,

$$C_K^2(x_2,x_1)\le 16^{1-1/K}2^{1/2}\frac{(1+|x_2|)^{1/K}}{1+|f(x_2)|}2^{1/2}\le 2\cdot 16^{2(1-1/K)} \tag{3.16}$$

for $x_1<x_2<0$.

In the case $x_1<x_2<1$ and $x_2>0$,

$$C_K^2(x_2,x_1)\le 16^{1-1/K}2^{1/2}2^{1/(2K)}=16^{1-1/K}\big(\sqrt{2}\big)^{1+1/K}. \tag{3.17}$$

Let $x_1<0<1<x_2$. Using (3.6) we get

$$d(f(x_2),f(x_1))\le C_K^3(x_2,x_1)d(x_2,x_1)^{1/K}, \tag{3.18}$$

where

$$\begin{aligned}C_K^3(x_2,x_1)&=16^{1-1/K}\frac{(1+|f(x_1)|)f(x_2)}{(1+|x_1|)^{1/K}x_2^{1/K}}\cdot\frac{(1+|x_1|^2)^{1/(2K)}(1+|x_2|^2)^{1/(2K)}}{(1+|f(x_1)|^2)^{1/2}(1+|f(x_2)|^2)^{1/2}}\\&\le 16^{1-1/K}\left(1+\frac{1}{x_2^2}\right)^{1/(2K)}2^{1/2}\le 16^{1-1/K}\big(\sqrt{2}\big)^{1+1/K}.\end{aligned} \tag{3.19}$$

Setting $C_K=\max\{C_K^1,C_K^2,C_K^3\}$, one can see that

$$C_K\le 2\cdot 16^{2(1-1/K)}<512. \tag{3.20}$$

Let $f\in A_{\mathbb{R}}^{0,1,\infty}(K)$, $K\ge 1$, and let $l(x)=(2x-1)/\sqrt{3}$, $x\in\mathbb{R}$. Then

$$\frac{2}{3}\le\frac{1+x^2}{1+(l^{-1}(x))^2}\le 2\quad\text{for } x\in\overline{\mathbb{R}} \tag{3.21}$$

and

$$\frac{1}{\sqrt{3}}\, d(x_2, x_1) \le d(l^{-1}(x_2), l^{-1}(x_1)) \le \sqrt{3}\, d(x_2, x_1). \tag{3.22}$$

Let $g = L(f) = l \circ f \circ l^{-1}$. By (3.20) we get

$$\begin{aligned} d(g(x_2), g(x_1)) &= d((l \circ f \circ l^{-1})(x_2), (l \circ f \circ l^{-1})(x_1)) \\ &\le \sqrt{3} \cdot d((f \circ l^{-1})(x_2), (f \circ l^{-1})(x_1)) \\ &\le 2\sqrt{3} \cdot 16^{2(1-1/K)} d(l^{-1}(x_2), l^{-1}(x_1))^{1/K} \\ &\le 2\sqrt{3} \cdot 16^{2(1-1/K)} \big(\sqrt{3}\,\big)^{1/K} d(x_2, x_1)^{1/K} \\ &= 2\big(\sqrt{3}\,\big)^{1+1/K} 16^{2(1-1/K)} d(x_2, x_1)^{1/K}. \end{aligned} \tag{3.23}$$

Because $r(T) = r(\overline{\mathbb{R}})$, and by Theorem 3.3, there is a homography $h : \overline{\mathbb{R}} \to T$ such that

$$d(h(x_2), h(x_1)) = d(x_2, x_1) \quad \text{for } x_1, x_2 \in \overline{\mathbb{R}}, \tag{3.24}$$

and

$$h(\infty) = 1 = w_1, \quad h\big(-1/\sqrt{3}\,\big) = e^{2\pi i/3} = w_2, \quad h\big(1/\sqrt{3}\,\big) = e^{4\pi i/3} = w_3.$$

Hence $S_h(L(A_{\mathbb{R}}^{0,1,\infty}(K))) = A_T^{\circ}(K)$. Thus, for all $f \in A_T^{\circ}(K)$, we have $S_h^{-1}(f) = h^{-1} \circ f \circ h \in L(A_{\mathbb{R}}^{0,1,\infty}(K))$ and, in view of (3.23) and (3.24), we get

$$\begin{aligned} d(f(x_2), f(x_1)) &= d((h^{-1} \circ f \circ h)(h^{-1}(x_2)), (h^{-1} \circ f \circ h)(h^{-1}(x_1))) \\ &\le 2 \cdot 16^{2(1-1/K)} \big(\sqrt{3}\,\big)^{1+1/K} d(h^{-1}(x_2), h^{-1}(x_1))^{1/K} \\ &= 2\big(\sqrt{3}\,\big)^{1+1/K} 16^{2(1-1/K)} d(x_2, x_1)^{1/K}. \end{aligned} \tag{3.25}$$

By (3.25) we arrive at the third line of (3.9) for $\Gamma = T$.

Let now $f \in A_T^{\circ}(K)$. It is easy to see that we may confine our considerations only to the case where $z_1, z_2 \in \langle w_1, w_2 \rangle$. By the definition of $A_T^{\circ}(K)$ and by (ii) of Chapter I, we have

$$|f(z_2) - f(z_1)| \le 16^{1-1/K} \frac{|f(z_1) - w_1|}{|z_1 - w_1|^{1/K}} \cdot \frac{|f(z_2) - w_2|}{|z_2 - w_2|^{1/K}} |z_2 - z_1|^{1/K} \cdot \big(\sqrt{3}\,\big)^{1/K-1}. \tag{3.26}$$

Because

$$[w_1, z_1, z_2, w_2]^2 = \frac{z_2 - z_1}{z_2 - w_1} \cdot \frac{w_2 - w_1}{w_2 - z_1},$$

we have

$$\frac{|f(z_i) - w_i|}{|z_i - w_i|^{1/K}} = 16^{1-1/K} \frac{|f(z_i) - w_3|}{|z_i - w_3|^{1/K}} = 2\big(1/\sqrt{3}\,\big)^{1/K} 16^{1-1/K}, \quad i = 1, 2. \tag{3.27}$$

By (3.27) and (3.26), one can see that

$$|f(z_2) - f(z_1)| \le \frac{4}{\big(\sqrt{3}\,\big)^{1+1/K}} 16^{3(1-1/K)} |z_2 - z_1|^{1/K} \tag{3.28}$$

for all z_1, $z_2 \in T$.

Noting that $2d(z_1, z_2) = |z_2 - z_1|$ and using the above considerations we arrive at the second line of (3.9). To get the first row of (3.9), note that

$$\frac{|f(z_i) - w_3|}{|z_i - w_3|^{1/K}} \le \frac{|f(z_i) - z_i|}{|z_i - w_3|^{1/K}} + |z_i - w_3|^{1-1/K}, \quad i = 1, 2. \tag{3.29}$$

By Lemma 2.1,

$$|f(z) - z| \le \frac{4}{\sqrt{3}} M(K)$$

for any $f \in A_T^\circ(K)$, $K \ge 1$, and $z \in T$. Hence

$$\frac{|f(z_i) - w_3|}{|z_i - w_3|^{1/K}} \le \frac{4}{\sqrt{3}} M(K) \frac{1}{(\sqrt{3})^{1/K}} + 2^{1-1/K} = \frac{4}{(\sqrt{3})^{1+1/K}} M(K) + 2^{1-1/K} \tag{3.30}$$

for $i = 1, 2$. Using (3.26), (3.27) and (3.30), we get

$$|f(z_2) - f(z_1)| \le (\sqrt{3})^{-1+1/K} 16^{3(1-1/K)} \left(\frac{4}{(\sqrt{3})^{1+1/K}} M(K) + 2^{1-1/K} \right)^2 |z_2 - z_1|^{1/K} \tag{3.31}$$

and then

$$d(f(z_2), f(z_1)) \le (2\sqrt{3})^{-1+1/K} 16^{3(1-1/K)} \left(\frac{4}{(\sqrt{3})^{1+1/K}} M(K) + 2^{1-1/K} \right)^2 d(z_2, z_1)^{1/K}. \tag{3.32}$$

This gives the first line of (3.9) for $\Gamma = T$.

Let $f \in \widehat{A}_T(K)$. There exist $\alpha, \beta \in \mathbb{R}$ and $g \in A_T^\circ(K)$ such that $f(z) = e^{i\alpha} g(ze^{i\beta})$. We can see that (3.25), (3.28) and (3.32) hold for every $f \in \widehat{A}_T(K)$.

Suppose now that Γ is an arbitrary circle in $\overline{\mathbb{C}}$. By Theorem 3.3, there exists a homography $h : \Gamma \to T$ such that

$$\begin{aligned} d(f(z_2), f(z_1)) &= \frac{1}{C} d((h \circ f)(z_2), (h \circ f)(z_1)) \\ &= \frac{1}{C} d(S_h(f)(h(z_1)), S_h(f)(h(z_2))) \\ &\le \frac{C_K}{C} d(h(z_2), h(z_1))^{1/K} \\ &= C^{-1+1/K} C_K d(z_2, z_1)^{1/K} \\ &= C_K \left(\frac{2}{r(\Gamma) + 1/r(\Gamma)} \right)^{1-1/K} d(z_2, z_1)^{1/K}, \end{aligned} \tag{3.33}$$

where C_K, the minimum of the obtained estimates, is given by (3.9). This completes the proof of Theorem 3.7.

COROLLARY 3.6. *Under the assumptions of Theorem* 3.7, *we have*

$$d(f(z_2), f(z_1)) \ge C_K^{-K} \left(\frac{r(\Gamma) + 1/r(\Gamma)}{2} \right)^{K-1} d(z_2, z_1)^K. \tag{3.34}$$

Proof. This follows from the fact that $f^{-1} \in \widehat{A}_\Gamma(K)$ if and only if $f \in \widehat{A}_\Gamma(K)$ for every circle Γ in $\overline{\mathbb{C}}$ and $K \ge 1$.

Remark 3.4. Since $r(\Gamma) + 1/r(\Gamma) \ge 2$ for every circle Γ on $\overline{\mathbb{C}}$, it follovs that for every $f \in \widehat{A}_\Gamma(K)$, $K \ge 1$, the inequality

$$C_K^{-K} d(z_1, z_2)^{K-1} \le d(f(z_1), f(z_2)) \le C_K d(z_1, z_2)^{1/K} \tag{3.35}$$

holds for all $z_1, z_2 \in \Gamma$.

4. Normal and compact families of quasihomographies. Recall that $h \in A_\Gamma(1)$, where Γ is a circle on $\overline{\mathbb{C}}$, if and only if h is a homography mapping Γ onto itself (cf. Theorem 2.5). Thus one may begin this chapter with

Definition 3.1. A family $M \subset A_\Gamma$, where Γ is an arbitrary circle on $\overline{\mathbb{C}}$, is said to be *closed with respect to* $A_\Gamma(1)$ if $h_1 \circ f \circ h_2 \in M$ for all $f \in M$ and all $h_1, h_2 \in A_\Gamma(1)$.

It is worth noting that this definition generalizes the notion of closed families of qs functions introduced by A. Beurling and L. V. Ahlfors [BA].

Theorem 3.8. *Suppose that Γ is a circle on $\overline{\mathbb{C}}$ and that M is a non-empty subset of $A_\Gamma(K)$, $K \geq 1$. Then M is a normal set in (A_Γ, d_Γ) if and only if there exists a triple of distinct points $z_1, z_2, z_3 \in \Gamma$ and $\delta > 0$ such that $d(f(z_i), f(z_j)) \geq \delta$ for $i \neq j$, $i, j = 1, 2, 3$, and for each automorphism $f \in M$.*

Proof. Suppose that $\Gamma = \overline{\mathbb{R}}$ and that $M \subset A_{\overline{\mathbb{R}}}(K)$ is non-empty, $K \geq 1$.

We first prove the sufficiency. Let $h \in A_{\overline{\mathbb{R}}}(1)$ be such that $h(0) = x_1$, $h(1) = x_2$ and $h(\infty) = x_3$. Moreover, let $\{f_n\} \subset M$. Then there exists a sequence $\{h_n\} \subset A_{\overline{\mathbb{R}}}(1)$ of homographies such that $\{h_n \circ f_n \circ h\} \subset A_{\overline{\mathbb{R}}}^{0,1,\infty}(K)$. Making use of Theorem 3.7, with $\Gamma = \overline{\mathbb{R}}$, we can see that there exists a subsequence $\{h_{n_k} \circ f_{n_k} \circ h\}$ that converges in $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$ to an automorphism $f \in A_{\overline{\mathbb{R}}}^{0,1,\infty}(K)$. Because

$$h_{n_k}^{-1}(0) = f_{n_k}(x_1), \quad h_{n_k}^{-1}(1) = f_{n_k}(x_2), \quad h_{n_k}^{-1}(\infty) = f_{n_k}(x_3)$$

and

$$d(f_{n_k}(x_i), f_{n_k}(x_j)) \geq \delta \quad \text{for } i \neq j,\ i, j = 1, 2, 3,\ k \in \mathbb{N},$$

and by [LV, Thm. 5.1], there exists a subsequence $\{h_{n_{k_l}}^{-1}\}$ of $\{h_{n_k}^{-1}\}$ that converges in $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$ to a homography h_0. Note that $(A_{\overline{\mathbb{R}}}, \circ, d_{\overline{\mathbb{R}}})$ is a metric group and that $h_0(\overline{\mathbb{R}}) = \overline{\mathbb{R}}$. Thus the sequence

$$f_{n_{k_l}} \circ h = h_{n_{k_l}}^{-1} \circ (h_{n_{k_l}} \circ f_{n_{k_l}} \circ h), \quad l \in \mathbb{N},$$

converges in $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$ to the automorphism $h_0 \circ f$. Therefore $\{f_{n_{k_l}}\}$ converges in $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$ to $h_0 \circ f \circ h^{-1} \in A_{\overline{\mathbb{R}}}(K)$. The last conclusion follows from the fact that $A_{\overline{\mathbb{R}}}(K)$ is a closed subset of the metric space $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$.

For the necessity, suppose that $M \subset A_{\overline{\mathbb{R}}}(K)$ is a normal set in $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$ and that $\mathcal{F} : A_{\overline{\mathbb{R}}} \to \mathbb{R}$ is defined by

$$\mathcal{F}(f) = \min\{d(f(0), f(1)), d(f(1), f(\infty)), d(f(\infty), f(0))\}. \tag{4.1}$$

The function $\mathcal{F}$ is continuous on $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$, and $\mathcal{F}(f) > 0$ for each $f \in A_{\overline{\mathbb{R}}}$. Since $\overline{M}$, the closure of M in $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$, is compact, we have

$$\inf\{\mathcal{F}(f) : f \in M\} \geq \inf\{\mathcal{F}(f) : f \in \overline{M}\} = \delta > 0. \tag{4.2}$$

This completes the proof for $\Gamma = \overline{\mathbb{R}}$.

In the general case, i.e., when Γ is an arbitrary circle on $\overline{\mathbb{C}}$, the proof is a consequence of the already presented proof for $\Gamma = \overline{\mathbb{R}}$ and Theorem 3.3, which says that there exists a similarity S_H of $(A_{\overline{\mathbb{R}}}, d_{\overline{\mathbb{R}}})$ onto (A_Γ, d_Γ) satisfying (i), (ii) and (v) of Theorem 3.3. This finishes the proof of Theorem 3.8.

As a consequence of Theorem 3.8 we have

COROLLARY 3.7. *Suppose that Γ is an arbitrary circle on $\overline{\mathbb{C}}$ and that $M \subset A_\Gamma(K)$ is non-empty for a given constant $K \geq 1$. If for each $f \in M$ we have $f(z_i) = w_i$, $i = 1,2,3$, where z_i, w_i are fixed points of Γ such that $z_i \neq z_j$ ($w_i \neq w_j$) for $i \neq j$, $i,j = 1,2,3$, then M is a normal family in (A_Γ, d_Γ).*

Let us note that $A_\Gamma^{z_1,z_2,z_3}(K)$ with $K \geq 1$ is closed in (A_Γ, d_Γ) for every circle Γ on $\overline{\mathbb{C}}$. Then, by Corollary 3.7, we have

COROLLARY 3.8. *For every circle Γ on $\overline{\mathbb{C}}$, $K \geq 1$ and any distinct points $z_1, z_2, z_3 \in \Gamma$, the set $A_\Gamma^{z_1,z_2,z_3}(K)$ is compact in (A_Γ, d_Γ).*

THEOREM 3.9. *For every circle Γ on $\overline{\mathbb{C}}$ and every $K \geq 1$, the family $\widehat{A}_\Gamma(K)$ is compact in (A_Γ, d_Γ).*

Proof. Let $z_1, z_2, z_3 \in \Gamma$ be as in Corollary 3.7, and let $\{f_n\} \subset \widehat{A}_\Gamma(K)$. For each $n \in \mathbb{N}$, there exist points $z_{1,n}, z_{2,n}, z_{3,n} \in \Gamma$ such that

$$d(z_{i,n}, z_{j,n}) = d(f_n(z_{i,n}), f_n(z_{j,n})) = \sqrt{3}\frac{r(\Gamma)}{1 + r(\Gamma)^2} = \delta \tag{4.3}$$

for $i \neq j$, $i,j = 1,2,3$. This is a consequence of Theorem 3.3, the definition of $\widehat{A}_\Gamma$ and the fact that $r(T) = 1$. Moreover, for each $n \in \mathbb{N}$, there are homographies $h_n \in A_\Gamma(1)$ described by $h_n(z_i) = z_{i,n}$, $i = 1,2,3$. Thus, by (4.3) and Theorem 3.8, there exist $g \in A_\Gamma(K)$, $h \in A_\Gamma(1)$, and a sequence $\{n_k\}$ such that

$$d_\Gamma(f_{n_k} \circ h_{n_k}, g) \to 0 \quad \text{and} \quad d_\Gamma(h_{n_k}, h) \to 0 \quad \text{as } k \to \infty. \tag{4.4}$$

From this and the fact that $(A_\Gamma, \circ, d_\Gamma)$ is a metric group, it follows that $f_{n_k} = (f_{n_k} \circ h_{n_k}) \circ h_{n_k}^{-1}$ converges in (A_Γ, d_Γ) to the automorphism $f = g \circ h^{-1} \in A_\Gamma(K)$. Moreover, by (4.3) and (4.4), one can see that

$$d(h(z_i), h(z_j)) = d(f(h(z_i)), f(h(z_j))) = d(g(z_i), g(z_j)) = \delta \tag{4.5}$$

for $i \neq j$, $i, j = 1,2,3$. This means that $f \in \widehat{A}_\Gamma(K)$. The theorem is proved.

LEMMA 3.4. *For every circle Γ on $\overline{\mathbb{C}}$, the set A_Γ^∞ is dense in (A_Γ, d_Γ).*

Proof. Suppose first that $\Gamma = T$ and let $f \in A_\Gamma$. Then

$$f(e^{2\pi i x}) = f(1)e^{2\pi i \sigma(x)}, \tag{4.6}$$

where σ is an increasing homeomorphism of $[0,1]$ onto itself. Let

$$\sigma_n(x) = \left(\sigma\left(\frac{k+1}{n}\right) - \sigma\left(\frac{k}{n}\right)\right)(nx - k) + \sigma\left(\frac{k}{n}\right) \quad \text{for } \frac{k}{n} \leq x \leq \frac{k+1}{n}, \tag{4.7}$$

$k = 0,1,\ldots,n-1$, for some $n \in \mathbb{N}$, and

$$f_n(e^{2\pi i x}) = f(1)e^{2\pi i \sigma_n(x)}. \tag{4.8}$$

Because $d_T(f_n, f) \to 0$ as $n \to \infty$, and by Theorem 2.11, the considerations in Section II.4, and the relation between qs functions on T and $\mathbb{R}$ given in (4.7), we can see that

$$f_n \in \bigcup_{\varrho \geq 1} Q_T(\varrho) = A_T^\infty, \tag{4.9}$$

where $Q_T(\varrho)$ denotes the class of ϱ-qs functions on T. This means that A_T^∞ is dense in (A_T, d_T). By Theorem 3.3, we can extend our considerations to the case of an arbitrary circle Γ on $\overline{\mathbb{C}}$. This ends the proof.

Using the above results one can prove

THEOREM 3.10. *For every circle Γ in $\overline{\mathbb{C}}$, $0 < r < r(\Gamma)/(1 + r(\Gamma)^2)$, $1 \le K < \infty$ and any $g \in A_\Gamma$, the set*

$$A_\Gamma(K) \cap \operatorname{cl} B(g, r) \tag{4.10}$$

is compact in (A_Γ, d_Γ), where $B(g, r) = \{f \in A_\Gamma : d_\Gamma(f, g) < r\}$.

Proof. Suppose first that $\Gamma = T$, $g = \mathrm{id}_T$, $z_1 = 1$, $z_2 = e^{2\pi i/3}$ and $z_3 = e^{4\pi i/3}$. Then, for $f \in \operatorname{cl} B(\mathrm{id}_T, r)$, we have

$$d(f(z_i), z_i) \le d_T(f, \mathrm{id}_T) \le r < 1/2 \quad \text{for } i = 1, 2, 3. \tag{4.11}$$

Hence,

$$d(f(z_i), f(z_j)) \ge \delta \tag{4.12}$$

for each $f \in A_T(K) \cap \operatorname{cl} B(\mathrm{id}_T, r)$, $i \ne j$, $i, j = 1, 2, 3$, where

$$\delta = \frac{\sqrt{3}}{2}(1 - 2r^2) - r\sqrt{1 - r^2}.$$

This means that $A_T(K) \cap \operatorname{cl} B(\mathrm{id}_T, r)$ is compact in (A_T, d_T).

Let now $g \in A_T$. Let r' satisfy $r < r' < 1/2$. By Lemma 3.4, there exist $K' \ge 1$ and $h \in A_T(K')$ such that

$$d_T(g, h) < r' - r.$$

Hence

$$d_T(f \circ h^{-1}, \mathrm{id}_T) = d_T(f, h) \le d_T(f, g) + d_T(g, h) \le r' < 1/2. \tag{4.13}$$

Making use of Theorem 2.2 we deduce that $f \circ h^{-1} \in A_T(KK')$ for all $f \in A_T(K) \cap \operatorname{cl} B(g, r)$. Let $J_h(f) = f \circ h^{-1}$, $f \in A_T$. It is evident that J_h is an isometry of (A_T, d_T) onto itself. By (4.13) and the fact that $f \circ h^{-1} \in A_T(KK')$, it follows that the family $J_h^{-1}(A_T(K) \cap \operatorname{cl} B(\mathrm{id}_T, r'))$ is compact and that

$$A_T(K) \cap \operatorname{cl} B(g, r) \subset J_h(A_T(KK') \cap B(\mathrm{id}_T, r')). \tag{4.14}$$

Thus $A_T(K) \cap \operatorname{cl} B(g, r)$ is compact, being a closed subset of $J_h(A_T(KK') \cap \operatorname{cl} B(\mathrm{id}_T, r'))$.

Using (i), (iii) and (v) of Theorem 3.3, one can see that our considerations remain true for an arbitrary circle Γ in $\overline{\mathbb{C}}$ and, consequently, we arrive at (4.10) with $0 \le r < r(\Gamma)/(1 + r(\Gamma)^2)$. The theorem is proved.

5. Topological characterization of quasihomographies

THEOREM 3.11. *Suppose that $M \subset A_\Gamma$ is non-empty and closed with respect to $A_\Gamma(1)$, where Γ is an arbitrary circle on $\overline{\mathbb{C}}$. Then $M \cap \widehat{A}_\Gamma$ is a normal set in (A_Γ, d_Γ) if and only if there is a constant $K \ge 1$ such that $M \subset A_\Gamma(K)$.*

P r o o f. Suppose first that $\Gamma = \overline{\mathbb{R}}$ and that $M \subset A_{\bar{\mathbb{R}}}$ is non-empty and closed with respect to $A_{\bar{\mathbb{R}}}(1)$ and such that $M \cap \widehat{A}_{\mathbb{R}}$ is a normal family in $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$. Then the set

$$M^0 = \big\{f \in M : f\big(\pm 1/\sqrt{3}\big) = \pm 1/\sqrt{3},\ f(\infty) = \infty\big\} \subset M \cap \widehat{A}_{\mathbb{R}} \tag{5.1}$$

is a normal family in $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$. Let $l(x) = (2x-1)/\sqrt{3}$, $x \in \mathbb{R}$. Thus the mapping $L(f) = l^{-1} \circ f \circ l$, defined for all $f \in A_{\bar{\mathbb{R}}}$, is an automorphism of the group $(A_{\bar{\mathbb{R}}}, \circ)$ and a homeomorphism of $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$ onto itself such that

$$\begin{aligned} d_{\bar{\mathbb{R}}}&(L(f), L(g)) \\ &= \sup d((l^{-1} \circ f \circ l)(x), (l^{-1} \circ g \circ l)(x)) \\ &= \sup d((l^{-1} \circ f)(y), (l^{-1} \circ g)(y)) \\ &= \frac{\sqrt{3}}{2} \sup\bigg(\bigg(\frac{1+(f(y))^2}{1+(l^{-1}(f(y)))^2} \cdot \frac{1+(g(y))^2}{1+(l^{-1}(g(y)))^2}\bigg)^{1/2} d(f(y), g(y))\bigg). \end{aligned} \tag{5.2}$$

From this and (3.21), we get

$$\frac{1}{\sqrt{3}}\, d_{\bar{\mathbb{R}}}(f,g) \le d_{\bar{\mathbb{R}}}(L(f), L(g)) \le \sqrt{3}\, d_{\bar{\mathbb{R}}}(f,g). \tag{5.3}$$

Thus the family $M^{0,1,\infty} = L(M^0) \subset A_{\bar{\mathbb{R}}}^{0,1,\infty} \cap M$ is normal in $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$. Moreover, since M is closed with respect to $A_{\bar{\mathbb{R}}}(1)$ and since L is an automorphism of $A_{\bar{\mathbb{R}}}$ such that $L(A_{\bar{\mathbb{R}}}(1)) = A_{\bar{\mathbb{R}}}(1)$, the family $M^{0,1,\infty}$ satisfies the following condition: *For all* $f \in M$ *and* $h_1, h_2 \in A_{\bar{\mathbb{R}}}(1)$,

$$\textit{if } h_1 \circ f \circ h_2 \in A_{\bar{\mathbb{R}}}^{0,1,\infty}, \quad \textit{then } h_1 \circ f \circ h_2 \in M^{0,1,\infty}. \tag{5.4}$$

For $x \in \overline{\mathbb{R}}$, let

$$\alpha(x) = \sup\{f(x) : f \in M^{0,1,\infty}\}, \qquad \beta(x) = \inf\{f(x) : f \in M^{0,1,\infty}\}. \tag{5.5}$$

Since $M^{0,1,\infty}$ is normal, it follows that

$$0 < \beta(1/2) = \beta \le 1/2 \le \alpha = \alpha(1/2) < 1. \tag{5.6}$$

In view of (5.4), define

$$f_n(x) = \frac{f(x/2^n)}{f(1/2^n)} \tag{5.7}$$

for $x \in \mathbb{R}$, $n = 1, 2, \ldots$ and $f \in M^{0,1,\infty}$. Then $f_n \in M^{0,1,\infty}$. Thus $\beta \le f_n(1/2) \le \alpha$ and

$$\beta^n \le f(1/2^n) \le \alpha^n, \qquad n \in \mathbb{N}. \tag{5.8}$$

Hence,

$$\alpha(x) \le \alpha(1/2^n) \le \alpha^n \le (2x)^{-\log_2 \alpha}, \tag{5.9}$$

for $1/2^{n+1} \le x \le 1/2^n$ and $n = 1, 2, \ldots$

Setting $K' = \big(-\frac{1}{2}\log_2 \alpha\big)^{-1}$, we obtain

$$\alpha(x) \le (2x)^{-\log_2 \alpha} \le x^{1/K'}, \qquad 0 \le x \le 1/4,$$

and

$$\alpha(x) \le \alpha(1/2) = \alpha = (1/4)^{1/K'} \le x^{1/K'}, \qquad 1/4 \le x \le 1/2.$$

Therefore

$$\alpha(x) \le x^{1/K'} \quad \text{for } 0 \le x \le 1/2. \tag{5.10}$$

By (5.8),

$$\beta(x) \ge \beta(1/2^{n+1}) \ge \beta^{n+1} \ge (x/2)^{-\log_2 \beta} \tag{5.11}$$

for $1/2^{n+1} \le x \le 1/2^n$ and $n = 1, 2, \ldots$ Setting $K'' = -2\log_2 \beta$, we have

$$\beta(x) \ge x^{K''} \quad \text{for } 0 \le x \le 1/2. \tag{5.12}$$

Now, by (5.10), (5.12) and (ii) and (ii′) of Chapter I it follows that

$$\Phi_{1/K}\big(\sqrt{x}\,\big)^2 \le f(x) \le \Phi_K\big(\sqrt{x}\,\big)^2 \tag{5.13}$$

for $f \in M^{0,1,\infty}$, $0 \le x \le 1/2$, with $K = \max\{K', K''\}$. By the property described in (5.4), it follows that for each $f \in M^{0,1,\infty}$, the automorphism $f^* = \omega \circ f \circ \omega$ belongs to $M^{0,1,\infty}$, where $\omega(x) = 1 - x$, $x \in \mathbb{R}$. Then, by (5.13), we have

$$\Phi_{1/K}\big(\sqrt{1-x}\,\big)^2 \le f^*(1-x) \le \Phi_K\big(\sqrt{1-x}\,\big)^2 \quad \text{for } 1/2 \le x \le 1. \tag{5.14}$$

Using (iii) of Chapter I with $x = t^2$, one can see that

$$\begin{aligned} \Phi_{1/K}\big(\sqrt{x}\,\big)^2 &= 1 - \Phi_K\big(\sqrt{1-x}\,\big)^2 \le f(x) = 1 - f^*(1-x) \\ &\le 1 - \Phi_{1/K}\big(\sqrt{1-x}\,\big)^2 = \Phi_K\big(\sqrt{x}\,\big)^2 \end{aligned} \tag{5.15}$$

for every $0 \le x \le 1$ and $K = \max\{K', K''\}$.

Let $f \in M$ and let $x_1, x_2, x_3, x_4 \in \overline{\mathbb{R}}$ be distinct points. There are $h_1, h_2 \in A_{\bar{\mathbb{R}}}(1)$ such that

$$h_2(0) = x_2, \quad h_2(1) = x_4, \quad h_2(\infty) = x_1 \quad \text{and} \quad h_1 \circ f \circ h_2 \in A_{\bar{\mathbb{R}}}^{0,1,\infty}.$$

By (5.4), the function $f^\circ = h_1 \circ f \circ h_2$ belongs to $M^{0,1,\infty}$. Setting $x = h_2^{-1}(x_3)$ and using (5.13) and (5.15), we have

$$\begin{aligned} [f(x_1), f(x_2), f(x_3), f(x_4)]^2 &= [f^\circ(\infty), f^\circ(0), f^\circ(x), f^\circ(1)]^2 \\ &= [\infty, 0, f^\circ(x), 1]^2 = f^\circ(x) \le \Phi_K(\sqrt{x}\,)^2 \\ &= \Phi_K([\infty, 0, x, 1])^2 = \Phi_K([x_1, x_2, x_3, x_4])^2. \end{aligned} \tag{5.16}$$

Making use of the other estimates in (5.13) and (5.15), we see that

$$\Phi_{1/K}([x_1, x_2, x_3, x_4]) \le [f(x_1), f(x_2), f(x_3), f(x_4)] \le \Phi_K([x_1, x_2, x_3, x_4]) \tag{5.17}$$

for any distinct points $x_1, x_2, x_3, x_4 \in \overline{\mathbb{R}}$, with $K = \max\{K', K''\}$. Thus $f \in A_{\bar{\mathbb{R}}}(K)$.

Suppose now that $M \subset A_{\bar{\mathbb{R}}}(K)$, where $K \ge 1$ is a given constant. Then

$$M \cap \widehat{A}_{\bar{\mathbb{R}}} \subset A_{\bar{\mathbb{R}}}(K) \cap \widehat{A}_{\bar{\mathbb{R}}} = \widehat{A}_{\mathbb{R}}(K) \tag{5.18}$$

and, by Theorem 3.9, the family $\widehat{A}_{\bar{\mathbb{R}}}(K)$ is compact in $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$. Thus $M \cap \widehat{A}_{\bar{\mathbb{R}}}$ is a normal family in $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$.

Let Γ be a circle on $\overline{\mathbb{C}}$. By Theorem 3.3 there is a similarity S_H mapping $(A_{\bar{\mathbb{R}}}, d_{\bar{\mathbb{R}}})$ onto (A_Γ, d_Γ) and satisfying the conditions (iv)–(vi) of that theorem. The mapping S_H, being a similarity, preserves normality and, being an isomorphism, preserves closedness with respect to $A_\Gamma(1)$. Then, by (iv) and (v) of Theorem 3.3, our proof remains true if $\overline{\mathbb{R}}$ is replaced by an arbitrary circle Γ on $\overline{\mathbb{C}}$. Theorem 3.11 is proved.

IV. Quasihomographies of a Jordan curve

1. Introduction. Originally defined for plane domains, the notion of K-qc mappings has been generalized to domains in $\mathbb{R}^n$; see [Ca] and [Vä1]. Recently Väisälä [Vä2] defined a counterpart of K-qc mappings for domains in a general Banach space. Also, K-qc mappings are well defined between topologically equivalent Riemann surfaces.

Unfortunately, the problem of describing an adequate counterpart of 1-dimensional K-qc mappings was open for a long time. The linearly invariant notion of ϱ-quasisymmetric (ϱ-qs) functions of line segments on $\mathbb{R}$, introduced by Beurling and Ahlfors [BA], *can be considered a particular example* of 1-dimensional K-qc mappings. Rotation-invariant ϱ-qs automorphisms of the unit circle T, introduced by Krzyż [Kr1], *cannot*, *in substance*, *be considered* 1-dimensional K-qc mappings. All the same, the family of quasisymmetric functions of T *can be identified* with the family of 1-dimensional qc mappings of T, whereas their inner structures remain incompatible.

The notion of K-quasihomographies of a circle Γ on $\overline{\mathbb{C}}$, defined in Chapter II, can be considered without constraints the 1-dimensional counterpart of K-qc mappings.

In this chapter we are going to extend the notion of K-quasihomographies to an oriented Jc Γ on $\overline{\mathbb{C}}$.

The study of how different properties of K-qc mappings behave with respect to the dimension seems to be one of the most interesting topics, particularly when the function space is formed of K-qc mappings on $\mathbb{R}^n$, $n = 1, 2, \ldots$, or on a Banach space. This topic becomes trivial in the case of conformal mappings for $n \neq 2$, which reduce to Möbius transformations.

2. Harmonic cross-ratio. Assume that Γ is an oriented Jordan curve on $\overline{\mathbb{C}}$, and D, D^* denote the left-hand and the right-hand complementary domains, i.e., $\overline{\mathbb{C}} \setminus \Gamma = D \cup D^*$. The correspondence $\Gamma \to (D, D^*)$ is unique in these circumstances. If Γ^* denotes the oriented Jc obtained from Γ by reversing the orientation, then the correspondence $\Gamma \to \Gamma^*$ is a conjugation and $\Gamma^* \to (D, D^*)^* = (D^*, D)$. Consider now a configuration $\Gamma(z_1, z_2, z_3, z_4)$ made up of a given oriented Jc Γ on $\overline{\mathbb{C}}$ and a quadruple of distinct points $z_1, z_2, z_3, z_4 \in \Gamma$ ordered according to the orientation of Γ. The quadrilaterals $D(z_1, z_2, z_3, z_4)$ and $D^*(z_1, z_2, z_3, z_4)$ are said to be *conjugate*. Let $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$ be the classes of all K-qc automorphisms of D and D^*, respectively. If Γ is a circle in $\overline{\mathbb{C}}$, then $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$ can be conformally identified for each $K \geq 1$. In the case when Γ is a Q-quasicircle, $Q \geq 1$, both the classes are related by a Q^2-qc reflection in Γ, and can be identified on the level of the *universal Teichmüller space*; see Theorem 4.12 below.

In the most general case when Γ is an arbitrary oriented Jc on $\overline{\mathbb{C}}$, we do not have any quasiconformal relation between $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$ preserving points of Γ. This is an obstacle to our research on the uniform boundary value problem for K-qc automorphisms. It means that we cannot simply start with a given and oriented Jc Γ on $\overline{\mathbb{C}}$ and a certain family of sense-preserving automorphisms of Γ representing boundary values of $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$, simultaneously.

The idea that the starting point should be an oriented Jc Γ on $\overline{\mathbb{C}}$, not a Jordan domain, when working with the boundary value problem for K-qc automorphisms, has its strong encouragement from universal Teichmüller space theory; see [Le, p. 97].

In spite of the simplicity of Definition 1 of K@-quasiconformal mappings it is not so easy to take advantage of this deformation when studying properties of these mappings.

To illustrate the difficulties we shall calculate the *modulus* of a given quadrilateral $D(z_1, z_2, z_3, z_4)$. To this end, note that by the Riemann mapping theorem there exists a conformal mapping H that maps D onto Δ. The mapping H is continuous on $\overline{D}$. Hence H maps $D(z_1, z_2, z_3, z_4)$ onto $\Delta(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, where $\zeta_i = H(z_i)$ for $i = 1, 2, 3, 4$. Furthermore, there exists a unique k with $0 < k < 1$ and a Möbius transformation M that maps $\Delta(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ onto $U(-1/k, -1, 1, 1/k)$, where U is the upper half-plane. Using the Schwarz–Christoffel transformation

$$S(z) := \int_0^z [(1-\zeta^2)(1-k^2\zeta^2)]^{-1/2}\, d\zeta$$

we map $U(-1/k, -1, 1, 1/k)$ conformally onto the rectangle $R\big(-\mathcal{K}(k) + i\mathcal{K}\big(\sqrt{1-k^2}\,\big), -\mathcal{K}(k), \mathcal{K}(k), \mathcal{K}(k) + i\mathcal{K}\big(\sqrt{1-k^2}\,\big)\big)$ with the given correspondence between vertices.

Therefore we have

$$m(D(z_1, z_2, z_3, z_4)) = \frac{\mathcal{K}\big(\sqrt{1-k^2}\,\big)}{2\mathcal{K}(k)}, \tag{2.1}$$

which is the ratio of $b = \mathcal{K}\big(\sqrt{1-k^2}\,\big)$ and $a = 2\mathcal{K}(k)$, the sides of the rectangle R.

This clearly shows difficulties that can be encountered when studying K-qc mappings. Obviously, the notion of modulus is determined on a given rectangle by the use of a more general conformal invariant, namely the *extremal length of the family of curves*, which, when specified, leads to the notion of the modulus without any loss of generality with respect to the K-qc theory. Intuitively this is fairly obvious, for if we want to find the *most nearly conformal* homeomorphism between pairs of rectangles on the plane whose vertices correspond to one another, the extremal one appears to be an affine mapping (i.e., linear quasiconformal), which reduces to a conformal mapping if and only if both the rectangles have the same ratio of the corresponding sides.

Before determining a direct generalization of the real-valued cross-ratio, it is perhaps necessary to recall another one-parameter configuration, namely a Jordan domain D with one interior point a and two ordered and distinguished boundary points, i.e., $D(a; z_1, z_2)$. This configuration carries a conformal invariant $\omega(a, \langle z_1, z_2\rangle; D)$ known as the *harmonic measure* of the oriented open arc $\langle z_1, z_2\rangle$, distinguished on the boundary ∂D by ordered points $z_1, z_2 \in \partial D$ as seen from the point $a \in D$. This is at the same time a harmonic function of $a \in D$ and a probability measure of the arc variable. Hence, if H maps $D(a; z_1, z_2)$ conformally onto $\Delta(0; \zeta_1, \zeta_2)$, we have

$$\omega(a, \langle z_1, z_2\rangle; D) = \omega(0, \langle \zeta_1, \zeta_2\rangle; \Delta) = \alpha/\pi, \tag{2.2}$$

where 2α is the Lebesgue measure of the angle $\measuredangle(\zeta_1, 0, \zeta_2)$. Hence we see that

$$|\zeta_2 - \zeta_1| = 2 \sin \pi\omega(0, \langle \zeta_1, \zeta_2\rangle; \Delta). \tag{2.3}$$

Consider a configuration $D(a; z_1, z_2, z_3, z_4)$ formed by the quadrilateral $D(z_1, z_2, z_3, z_4)$ and a distinguished point $a \in D$. This configuration is obviously characterized by three parameters. $D(a; z_1, z_2, z_3, z_4)$ will play an auxiliary role in our considerations of a new conformal invariant.

Let us associate with the configuration $D(a; z_1, z_2)$ the quantity

$$[z_1, z_2]_D^a := 2 \sin \pi\omega(a, \langle z_1, z_2\rangle; D), \tag{2.4}$$

which is an analogue of (2.3).

We associate with $D(a; z_1, z_2, z_3, z_4)$ the real number

$$\{z_1, z_2, z_3, z_4\}_D^a := \frac{[z_2, z_3]_D^a}{[z_1, z_3]_D^a} : \frac{[z_2, z_4]_D^a}{[z_1, z_4]_D^a}. \tag{2.5}$$

Then we have the following theorem.

THEOREM 4.1. *Let D be a Jordan domain on $\overline{\mathbb{C}}$. For every $a, b \in D$, the identity*

$$\{z_1, z_2, z_3, z_4\}_D^a = \{z_1, z_2, z_3, z_4\}_D^b \tag{2.6}$$

holds for any distinct points $z_1, z_2, z_3, z_4 \in \partial D$.

Proof. Let $a, b \in D$. By the Riemann mapping theorem, there are conformal mappings H_a and H_b that map Δ onto D with $H_a(0) = a$ and $H_b(0) = b$. Both can be regarded as homeomorphisms of $\overline{\Delta}$ onto $\overline{D}$. By the conformal invariance of the harmonic measure,

$$\begin{aligned} [H_a^{-1}(z'), H_a^{-1}(z'')]_\Delta^\circ &= 2 \sin \pi\omega(0, \langle H_a^{-1}(z'), H_a^{-1}(z'')\rangle; \Delta) \\ &= 2 \sin \pi\omega(H_a(0), \langle z', z''\rangle; H(\Delta)) \\ &= 2 \sin \pi\omega(a, \langle z', z''\rangle, D) = [z', z'']_D^a \end{aligned} \tag{2.7}$$

for all $z', z'' \in \Gamma$. The equality

$$[H_b^{-1}(z'), H_b^{-1}(z'')]_\Delta^\circ = [z', z'']_D^b \tag{2.7$'$}$$

holds by the same argument as for (2.4). Let $z_1, z_2, z_3, z_4 \in \Gamma$ be distinct. Setting $t_k = H_a^{-1}(z_k)$ and $r_k = H_b^{-1}(z_k)$, $k = 1, 2, 3, 4$, and using (2.4) and (2.4$'$), we obtain

$$\{z_1, z_2, z_3, z_4\}_D^a = \{t_1, t_2, t_3, t_4\}_\Delta^\circ = \{t_1, t_2, t_3, t_4\} \tag{2.8}$$

and

$$\{z_1, z_2, z_3, z_4\}_D^b = \{r_1, r_2, r_3, r_4\}_\Delta^\circ = \{r_1, r_2, r_3, r_4\}. \tag{2.8$'$}$$

Since $H_b^{-1} \circ H_a$ is a conformal automorphism of Δ, it is a homography mapping $\overline{\Delta}$ onto itself and thus it preserves cross-ratio. Therefore

$$\begin{aligned} &\{r_1, r_2, r_3, r_4\} \\ &= \{(H_b^{-1} \circ H_a)(t_1), (H_b^{-1} \circ H_a)(t_2), (H_b^{-1} \circ H_a)(t_3), (H_b^{-1} \circ H_a)(t_4)\} = \{t_1, t_2, t_3, t_4\}. \end{aligned} \tag{2.9}$$

This completes the proof.

The above theorem implies that the quantity $\{z_1, z_2, z_3, z_4\}_D^a$ is constant as a function of $a \in D$ for any fixed ordered quadruple of distinct points $z_1, z_2, z_3, z_4 \in \partial D$. Hence, we can define

(2.10) $$\{z_1, z_2, z_3, z_4\}_D := \{z_1, z_2, z_3, z_4\}^a_D$$

for any $a \in D$.

DEFINITION 4.1. Given a quadrilateral $D(z_1, z_2, z_3, z_4)$ we associate with it the number $\{z_1, z_2, z_3, z_4\}_D$, which is a conformal invariant called the *harmonic cross-ratio*.

As we may easily check, the harmonic cross-ratio reduces to the classical real-valued cross-ratio if D is a disc in $\overline{\mathbb{C}}$, and its values range over $(0, 1)$. We also introduce

(2.11) $$[z_1, z_2, z_3, z_4]_D := \{z_1, z_2, z_3, z_4\}^{1/2}_D.$$

We will call it the harmonic cross-ratio as well.

Now, several natural questions arise immediately. Before stating them, let us recall the already known situation when with a configuration made up of a Jordan D domain and a pair of distinct points $z_1, z_2 \in D$ there are associated two well-known conformal invariants, the *hyperbolic distance* $\mathbf{h}$ and the *Green's function* $\mathbf{g}$. In these circumstances they are related by the identity

(2.12) $$\tanh(\mathbf{h}(z_1, z_2)) = \exp(-\mathbf{g}(z_1, z_2)).$$

In the case of a quadrilateral we also have two conformal invariants, i.e., the *modulus* and the *harmonic cross-ratio*. Hence a natural question arises: Describe a function that expresses the relationship. An answer is

THEOREM 4.2. *Consider a quadrilateral* $D(z_1, z_2, z_3, z_4)$ *and let* $\mathbf{m}$ *and* $\mathbf{t}$ *denote its modulus and harmonic cross-ratio*, *respectively. Then*

(2.13) $$\mathbf{m} = \mu(\mathbf{t}) \quad \textit{and} \quad \mathbf{t} = \mu^{-1}(\mathbf{m}) = \Phi_{1/\mathbf{m}}\big(1/\sqrt{2}\,\big),$$

where μ *is given by* (2.2) *of Chapter* I.

Proof. By the Riemann mapping theorem there exists $0 < k < 1$ and a conformal mapping that maps $G(z_1, z_2, z_3, z_4)$ onto $U(-1/k, -1, 1, 1/k)$, where U is the upper half-plane. Then

$$\mathbf{t} = \left[\frac{-1}{k}, -1, 1, \frac{1}{k}\right]_U = \frac{2\sqrt{k}}{1+k} = \Phi_2(k) \quad \text{and} \quad \mathbf{m} = \frac{\mathcal{K}\big(\sqrt{1-k^2}\big)}{2\mathcal{K}(k)} = \frac{1}{2}\mu(k).$$

Since $k = \Phi_2^{-1}(\mathbf{t}) = \Phi_{1/2}(\mathbf{t})$, we have

$$\mathbf{m} = \tfrac{1}{2}\mu(\Phi_{1/2}(\mathbf{t})) = \tfrac{1}{2}\mu(\mu^{-1}(2\mu(\mathbf{t}))) = \mu(\mathbf{t}),$$

which proves the first equality in (2.13). The second identity is a consequence of the first one and certain identities satisfied by Φ_K; see Chapter I.

It is worth noting that the modulus of a given quadrilateral and its harmonic cross-ratio differ by a special function, whereas the hyperbolic distance and the Green's function differ by an elementary function.

Another natural question which arises after introducing harmonic cross-ratio is: How may this notion be used to define K-qc mappings? An answer is given by the following theorem.

THEOREM 4.3. *A mapping* $F \in \mathcal{F}_{D,D'}$ *is* K*-qc if and only if*

(2.14) $$\Phi_{1/K}(\mathbf{t}(G)) \le \mathbf{t}(F(G)) \le \Phi_K(\mathbf{t}(G))$$

for every quadrilateral $G := G(z_1, z_2, z_3, z_4)$ *such that* $\overline{G} \subset D$, *where* Φ_K *is given by* (2.1) *of Chapter* I, *with* $K \geq 1$, *and* $\mathbf{t}(G) := [z_1, z_2, z_3, z_4]_G$.

Proof. The proof is an immediate consequence of the first identity in (2.13), which transforms the condition of K-quasiconformality into (2.14). Other assumptions remain the same.

Obviously, one may regard Theorem 4.3 as a new definition of K-quasiconformality on the plane.

3. One-dimensional quasiconformal mappings. Suppose that Γ is an oriented Jc on $\overline{\mathbb{C}}$ and D, D^* are the domains complementary with respect to Γ.

Let A_Γ denote the family of all sense-preserving automorphisms of Γ. It is evident that $(A_\Gamma, \circ)$ is a group under composition. For $f \in A_\Gamma$, we call

$$\|f\|_\Gamma = d_\Gamma(f, \mathrm{id}_\Gamma) \tag{3.1}$$

the *chordal norm* of f. Then (A_Γ, d_Γ) is a metric space and $0 \leq d_\Gamma(f, g) \leq 1$ for any $f, g \in A_\Gamma$. Let $z_1, z_2, z_3 \in \Gamma$ be distinct points. By $A_\Gamma^{z_1,z_2,z_3}$ we denote the subspace of A_Γ consisting of all $f \in A_\Gamma$ that have z_1, z_2, z_3 as fixed points. Note also that

$$d_\Gamma(f, g) = d(f \circ g^{-1}, \mathrm{id}) = \|f \circ g^{-1}\|_\Gamma. \tag{3.2}$$

Thus, $\|f\|_\Gamma$ measures the maximum chordal deviation from the identity and $\|f\|_\Gamma = 1$ if and only if f maps one point of a pair of antipodal points of $\overline{\mathbb{C}}$ onto the other. Now we have

THEOREM 4.4. *For every Jc* Γ *on* $\overline{\mathbb{C}}$, $(A_\Gamma, \circ, d_\Gamma)$ *is a topological group.*

Proof. Let $f, g, u, v \in A_\Gamma$ and $\varepsilon > 0$. Since each $f \in A_\Gamma$ is uniformly continuous on Γ with respect to the d_Γ-metric, there is a constant $\eta > 0$ such that, for $z_1, z_2 \in \Gamma$,

$$d(f(z_1), f(z_2)) < \varepsilon/2 \quad \text{if} \quad d(z_1, z_2) < \eta.$$

Setting $\delta = \min(\varepsilon/2, \eta)$ we may then see that

$$\begin{aligned} d_\Gamma(f \circ g, u \circ v) &\leq d_\Gamma(f \circ g, f \circ v) + d_\Gamma(f \circ v, u \circ v) \\ &= \sup_{z \in \Gamma} d(f(g(z)), f(u(z))) + d_\Gamma(f, u) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

if $d_\Gamma(f, u) < \delta$ and $d_\Gamma(g, v) < \delta$. This means the continuity of the composition. Hence there exists $\delta > 0$ such that

$$d_\Gamma(u^{-1}, f^{-1}) = d_\Gamma(f^{-1}(f \circ u^{-1}), f^{-1}(\mathrm{id}_\Gamma)) < \varepsilon$$

if $d_\Gamma(f \circ u^{-1}, \mathrm{id}_\Gamma) < \delta$. The relation

$$d_\Gamma(f \circ u^{-1}, \mathrm{id}_\Gamma) = d_\Gamma(f \circ u^{-1}, u \circ u^{-1}) = d_\Gamma(f, u)$$

implies that $d_\Gamma(u^{-1}, f^{-1}) < \varepsilon$ if $d_\Gamma(f, u) < \delta$, which means the continuity of the inverse operation. This finishes the proof.

Given two oriented Jordan curves Γ and Γ' on the Riemann sphere $\overline{\mathbb{C}}$, let $A_{\Gamma,\Gamma'}$ be the family of all homeomorphisms mapping Γ onto Γ'. With any distinct points

$z_1, z_2, z_3, z_4 \in \Gamma$ ordered according to the orientation of Γ we associate the value

$$[z_1, z_2, z_3, z_4]_\Gamma := [z_1, z_2, z_3, z_4]_D, \tag{3.3}$$

where D is the domain that is complementary to Γ on the left, and $[z_1, z_2, z_3, z_4]_D$ is the harmonic cross-ratio of the quadrilateral $D(z_1, z_2, z_3, z_4)$. The correspondence is unique.

Given $K \geq 1$, we may state the following definition.

Definition 4.2. A sense-preserving homeomorphism $f \in A_{\Gamma,\Gamma'}$ is said to be a K-*quasihomography* (K-qh) if

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]_\Gamma) \leq [f(z_1), f(z_2), f(z_3), f(z_4)]_{\Gamma'} \leq \Phi_K([z_1, z_2, z_3, z_4]_\Gamma) \tag{3.4}$$

for any distinct points $z_1, z_2, z_3, z_4 \in \Gamma$ ordered according to the orientation of Γ.

We denote by $A_{\Gamma,\Gamma'}(K)$ the class of all the K-qh mappings $f \in A_{\Gamma,\Gamma'}$ with a given $K \geq 1$. These mappings share all the basic properties of K-qc mappings mentioned previously. Moreover, we let

$$A^\infty_{\Gamma,\Gamma'} = \bigcup_{K\geq 1} A_{\Gamma,\Gamma'}(K).$$

Given $f \in A^\infty_{\Gamma,\Gamma'}$, the number

$$K_{\Gamma,\Gamma'}(f) := \min\{K \geq 1 : f \in A_{\Gamma,\Gamma'}(K)\} \tag{3.5}$$

is said to be the *maximal dilatation* of f. Moreover,

$$K_\Gamma(f) = K_{\Gamma,\Gamma}(f).$$

In these circumstances, if $F \in \mathcal{F}^\infty_{D,D'}(K)$, where D and D' are Jordan domains on $\overline{\mathbb{C}}$, then $f := F|_\Gamma$ is obviously an element of $A_{\Gamma,\Gamma'}(K)$ for any $K \geq 1$, where $\Gamma := \partial D$ and $\Gamma' := \partial D'$, i.e.,

$$\mathcal{F}_{D,D'}(K)|_\Gamma \subset A_{\Gamma,\Gamma'}(K). \tag{3.6}$$

Conversely, given $f \in A_{\Gamma,\Gamma'}(K)$, where Γ and Γ' are oriented Jordan curves on $\overline{\mathbb{C}}$, and $K \geq 1$, we may find an element $F_f \in \mathcal{F}_{D,D'}(K^*)$, where $K^* = K^*(K)$ but D and D' are the left-hand domains complementary to Γ and Γ', respectively, such that $F_f|_\Gamma = f$. This can be achieved by the use of one of the well-known particular extensions (see [Le]); among them the conformally invariant Douady–Earle extension seems to be preferable (see [DE] and [SZ]).

Therefore, the family $A_{\Gamma,\Gamma'}(K)$, $K \geq 1$, represents uniformly the boundary values of $\mathcal{F}_{D,D'}(K)$ with the same K. Conversely, the number $K^*(K)$ can be explicitly estimated for any of the well-known extensions, and these estimates are asymptotically sharp, i.e., $K^*(K) \to 1$ as $K \to 1$; see [Le], [P1], [RZ2] and [SZ].

We may regard K-qh homeomorphisms as the 1-dimensional counterpart of K-qc mappings. Conformal invariance, transformation rules, topological and algebraic properties of K-qh mappings justify the notion of 1-dimensional K-qc mappings; see [Z1], [Z2], [Z3] and [Z6].

Some basic properties of K-quasihomographies of the class $A_{\Gamma,\Gamma'}(K)$ are presented in

Theorem 4.5. *For oriented Jc's Γ, Γ' and Γ'' on $\overline{\mathbb{C}}$ and for $K_1, K_2 \geq 1$, if $f_1 \in A_{\Gamma,\Gamma'}(K_1)$ and $f_2 \in A_{\Gamma',\Gamma''}(K_2)$, then $f_2 \circ f_1 \in A_{\Gamma,\Gamma''}(K_2 K_1)$.*

THEOREM 4.6. *For oriented Jc's Γ and Γ' on $\overline{\mathbb{C}}$, and $K \geq 1$, we have $f \in A_{\Gamma,\Gamma'}(K)$ if and only if $f^{-1} \in A_{\Gamma',\Gamma}(K)$.*

The proof of Theorem 4.5 follows immediately from the composition property of Φ_K and the definition of $A_\Gamma(K)$. Theorem 4.6 is a consequence of similar arguments.

By the Riemann mapping theorem a number of properties of the class $A_{\Gamma,\Gamma'}(K)$ can be reduced to

$$A_\Gamma(K) := A_{\Gamma,\Gamma}(K). \tag{3.7}$$

THEOREM 4.7. *Let Γ be an oriented Jc on $\overline{\mathbb{C}}$. A function f belongs to $A_\Gamma(1)$ if and only if f is the boundary value of a conformal automorphism of D, where $\partial D = \Gamma$.*

Proof. Let H map Δ conformally onto D, and let $f \in A_\Gamma(1)$, where $\partial D = \Gamma$. The mapping $h = S_H^{-1}(f)$ belongs to $A_T(1)$ if and only if it is a homography mapping T onto itself; cf. Theorem 2.5.

4. Complete boundary transformation. It is an obvious remark that a conformal mapping between two Jordan domains is determined by its boundary values. Therefore one may say that conformal mappings have the *boundary character*.

In contrast, quasiconformal mappings have the *domain character*. We show this in the examples below. Hence, the following considerations are strictly connected with the conformal theory and the boundary values of quasiconformal mappings.

EXAMPLE 4.1. The mapping $F_\varrho(z) = z - \varrho(1 - |z|^2)$ is obviously a quasiconformal automorphism of Δ, if ϱ is sufficiently close to the origin. But $F_\varrho|_T = \mathrm{id}_T$.

EXAMPLE 4.2. It is not easy to find a conformal mapping of the unit disc Δ onto an elliptic domain $E(0; a, b)$, whereas the simplest and extremal quasiconformal mapping is just the affine one, which obviously is not conformal if the ratio a/b of semiaxes of $E(0; a, b)$ is equal to 1.

Let Γ_i, $i = 1, 2, 3$, be oriented Jc's on $\overline{\mathbb{C}}$, and let A_{Γ_i} denote all sense-preserving automorphisms of Γ_i, $i = 1, 2, 3$. By D_i and D_i^* we denote the corresponding domains complementary with respect to Γ_i, $i = 1, 2, 3$. Moreover, let H, H_*, G and G_* be conformal mappings of D_1 onto D_2, D_1^* onto D_2^*, D_2 onto D_3 and D_2^* onto D_3^*, respectively. For every $f_{kl} \in A_{\Gamma_1}$, $k, l = 1, 2$, consider the transformation $\boldsymbol{S}_{H,H_*}$ described by

$$\boldsymbol{S}_{H,H_*}\left(\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\right) = (S^{kl}_{H,H_*}(f_{kl})) = \begin{pmatrix} H \circ f_{11} \circ H^{-1} & H \circ f_{12} \circ H_*^{-1} \\ H_* \circ f_{21} \circ H^{-1} & H_* \circ f_{22} \circ H_*^{-1} \end{pmatrix}. \tag{4.1}$$

Since conformal mappings between Jordan domains can be considered homeomorphisms of their closures, the transformation $\boldsymbol{S}_{H,H_*}$ is well defined, and will be called here the *complete boundary transformation* mapping $\prod_{l=1}^4 A_{\Gamma_1}$ onto $\prod_{l=1}^4 A_{\Gamma_2}$.

It is evident that

$$\boldsymbol{S}_{G\circ H, G_*\circ H_*} = \boldsymbol{S}_{G,G_*} \circ \boldsymbol{S}_{H,H_*}, \tag{4.2}$$

which yields

$$\boldsymbol{S}_{H^{-1},H_*^{-1}} = \boldsymbol{S}^{-1}_{H,H_*}. \tag{4.3}$$

Write

$$\widetilde{f} = \begin{pmatrix} f & f \\ f & f \end{pmatrix}, \qquad f \in A_{\Gamma_1},$$

and let G_{Γ_1} be the collection of all such elements. Given $\widetilde{f}$ and $\widetilde{g}$ from G_{Γ_1}, set

$$\widetilde{f} * \widetilde{g} = \begin{pmatrix} f \circ g & f \circ g \\ f \circ g & f \circ g \end{pmatrix}. \tag{4.4}$$

Hence, $(G_{\Gamma_1}, *)$ is a group and by defining

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \circ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} f_{11} \circ g_{11} & f_{12} \circ g_{12} \\ f_{21} \circ g_{21} & f_{22} \circ g_{22} \end{pmatrix}$$

we can see that

$$\boldsymbol{S}_{H,H_*}(\widetilde{f}) \circ \boldsymbol{S}_{H,H_*}(\widetilde{g}) = \boldsymbol{S}_{H,H_*}(\widetilde{f} * \widetilde{g}) \tag{4.5}$$

for every $\widetilde{f}$ and $\widetilde{g}$ from G_{Γ_1}. We have proved

THEOREM 4.8. *The complete boundary transformation $\boldsymbol{S}_{H,H_*}$ is an isomorphism between $(G_{\Gamma_1}, *)$ and $(\boldsymbol{S}_{H,H^*}(G_{\Gamma_1}), \circ)$.*

Let

$$S_H = S^{11}_{H,H_*}, \quad S_{H_*} = S^{22}_{H,H_*}, \quad D_{HH_*} = S^{12}_{H,H_*} \quad \text{and} \quad D_{H_*H} = S^{21}_{H,H_*}. \tag{4.6}$$

Define

$$R_{\Gamma_1,\Gamma_2} = H \circ H_*^{-1}; \tag{4.7}$$

we call R_{Γ_1,Γ_2} the *conformal representation of Γ_1 with respect to Γ_2*. Let $R_\Gamma := R_{T,\Gamma}$ for every Jc Γ on $\overline{\mathbb{C}}$. This R_Γ is known as the *sewing automorphism* or the *conformal representation* of Γ and denoted by CR[Γ]; cf. [P4]. Notice that

$$R_{\Gamma_1,\Gamma_2} = \{H \circ H_*^{-1}\}$$

if H and H_* are fixed.

THEOREM 4.9. *For any oriented Jc's Γ_1, Γ_2 on $\overline{\mathbb{C}}$, and any conformal H and H_* as described above, the solution set of*

$$S_H(f) = S_{H_*}(f) \tag{4.8}$$

contains infinitely many automorphisms of the form $(H_*^{-1} \circ H)^n$, where f^n means the n-fold composition of f, and where $f^{-n} = (f^{-1})^n$, $n = 0, \pm 1, \pm 2, \ldots$

Proof. This follows immediately, by checking that $H_*^{-1} \circ H$ satisfies (4.8). By using (4.2) and (4.3), we complete the proof.

The transformation

$$J_{HH_*} = S_{H_*} \circ S_H^{-1} \tag{4.9}$$

is a self-isomorphism of A_{Γ_2}.

It is obvious that all solutions of (4.8) form a group under composition. Denote by $R^\infty_{\Gamma_2,\Gamma_1}$ the family of all solutions obtained in Theorem 4.10 below, and set

$$R^\infty_{\Gamma_1,\Gamma_2} := S_H(R^\infty_{\Gamma_2,\Gamma_1}) = S_{H^*}(R^\infty_{\Gamma_2,\Gamma_1}). \tag{4.10}$$

Then we may easily see that $(R^{\infty}_{\Gamma_1,\Gamma_2},\circ)$ is a group generated by R_{Γ_1,Γ_2} and we call it the *fix-point group* of J_{HH_*}.

Not without justification we may call J_{HH_*} a *conjugation operator* in A_{Γ_2}, whose "real line" is $R^{\infty}_{\Gamma_1,\Gamma_2}$. It is very probable that the elements in $R^{\infty}_{\Gamma_2,\Gamma_1}$ are the only solutions of (4.8). It is obvious that (cf. Theorem 4.5)

$$K_{\Gamma_2}(H\circ H_*^{-1})=K_{\Gamma_2}(H_*\circ H^{-1}) \tag{4.11}$$

and

$$K_{\Gamma_2^*}(H\circ H_*^{-1})=K_{\Gamma_2^*}(H_*\circ H^{-1}), \tag{4.12}$$

where Γ_i^* is obtained from Γ_i by reversing the orientation. Moreover, each of the expressions in (4.11) and (4.12) is a constant considered as a function of H and H_*.

Let now

$$A^{\infty}_{\Gamma_i}=\bigcup_{K\ge 1}A_{\Gamma_i}(K)\quad\text{and}\quad A^{\infty}_{\Gamma_i^*}=\bigcup_{K\ge 1}A_{\Gamma_i^*}(K) \tag{4.13}$$

for $i=1,2,3$. Obviously, $(A^{\infty}_{\Gamma_i},\circ)$ and $(A^{\infty}_{\Gamma_i^*},\circ)$ are subgroups of $(A_{\Gamma_i},\circ)$ for $i=1,2,3$.

We have proved

THEOREM 4.10. *Let Γ_1 and Γ_2 be oriented Jc's on $\overline{\mathbb{C}}$, and let D_1, D_1^*, D_2, D_2^* denote the domains complementary to Γ_1 and Γ_2, respectively. Then*:

(i) *$H(\Gamma_1)=H_*(\Gamma_1)=\Gamma_2$ for every H and H_* mapping D_1 and D_1^* conformally onto D_2 and D_2^*, respectively*;

(ii) *the transformations S_H and S_{H_*} defined by* (4.6) *are homeomorphisms between $(A_{\Gamma_1},d_{\Gamma_2})$ and $(A_{\Gamma_2},d_{\Gamma_2})$, and isomorphisms between $(A_{\Gamma_1},\circ)$ and $(A_{\Gamma_2},\circ)$*;

(iii) *S_H is an isomorphism between $(A^{\infty}_{\Gamma_1},\circ)$ and $(A^{\infty}_{\Gamma_2},\circ)$, whereas S_{H_*} is an isomorphism between $(A^{\infty}_{\Gamma_1^*},\circ)$ and $(A^{\infty}_{\Gamma_2^*},\circ)$, such that*

$$S_H(A_{\Gamma_1}(K))=A_{\Gamma_2}(K)\quad\textit{and}\quad S_{H_*}(A_{\Gamma_1^*}(K))=A_{\Gamma_2^*}(K),\qquad K\ge 1;$$

(iv) *the transformations D_{HH_*} and D_{H_*H} are homeomorphisms between $(A_{\Gamma_1},d_{\Gamma_1})$ and $(A_{\Gamma_2},d_{\Gamma_2})$. Moreover*,

$$D_{HH_*}(f)=S_H(f)\circ r=r\circ S_{H_*}(f),\qquad D_{H_*H}(f)=r^{-1}\circ S_H(f)=S_{H_*}(f)\circ r^{-1}$$

and

$$(D_{HH_*}(f))^{-1}=D_{H_*H}(f^{-1})$$

for every $f\in A_{\Gamma_1}$, where $r=R_{\Gamma_1,\Gamma_2}$ for brevity;

(v) *the transformation J_{HH_*} defined by* (4.9) *is an automorphism of the metric group $(A_{\Gamma_2},\circ,\varrho_{\Gamma_2})$ and $R^{\infty}_{\Gamma_1,\Gamma_2}$ are fixed points of this transformation.*

Moreover,

$$S_H=S_r\circ S_{H_*}\quad\textit{and}\quad J_{HH_*}=S_r^{-1}.$$

5. Quasicircles. A Q-*quasicircle* on $\overline{\mathbb{C}}$ is the image of a circle (say, the unit circle) on $\overline{\mathbb{C}}$ under a Q-qc mapping of $\overline{\mathbb{C}}$. Quasiconformal mappings preserve sets of measure zero, so every quasicircle is of zero area. On the other hand, a quasicircle need not be rectifiable. Moreover, the Hausdorff dimension of a quasicircle may take any values from $[1, 2]$; see [GV]. A considerable number of properties of quasicircles can be found in [Ge3].

By a theorem of Ahlfors (see [Le]), a Jordan curve Γ on $\mathbb{C}$ is a quasicircle if and only if there exists a constant C such that

$$\min_{j=1,2} \operatorname{dia}(\Gamma_j) \le C|z_1 - z_2| \tag{5.1}$$

for any $z_1, z_2 \in \Gamma$, where Γ_1 and Γ_2 denote the components of $\Gamma \setminus \{z_1, z_2\}$. By (5.1) the property of a Jordan curve to be a K-quasicircle has an obvious geometrical meaning; see [Ge3] and [Le]. If Γ is a K-quasicircle, then

$$\dim_{\mathrm{H}}(\Gamma) \le 2 - \frac{2}{K+1}, \tag{5.2}$$

where $\dim_{\mathrm{H}}(\Gamma)$ denotes the Hausdorff dimension of Γ; see [As]. Therefore, K-quasicircles can be considered *fractals*.

Now, we shall obtain a few characterizations of quasicircles as applications of *harmonic cross-ratios* and the conjugate 1-dimensional K-qh automorphisms of a given oriented Jc Γ on $\overline{\mathbb{C}}$. In these characterizations, no special point in $\overline{\mathbb{C}}$ is distinguished, which is not always possible for several other characterizations (cf. [Ge3]), where the point at infinity plays an essential role.

THEOREM 4.11. *Let Γ be an oriented Jc on $\overline{\mathbb{C}}$. Then Γ is a quasicircle if and only if there exists a constant $K \ge 1$ such that*

$$\Phi_{1/K}([z_1, z_2, z_3, z_4]_\Gamma) \le [z_1, z_2, z_3, z_4]_{\Gamma^*} \le \Phi_K([z_1, z_2, z_3, z_4]_\Gamma) \tag{5.3}$$

for any distinct points $z_1, z_2, z_3, z_4 \in \Gamma$, where Γ^* is the conjugate Jc.

Proof. Suppose that Γ is a Q-quasicircle, $Q \ge 1$. Then there is a Q^2-qc reflection J_Γ in Γ. Let H and H_* be conformal mappings of Δ and Δ^* onto D and D^*, respectively, where $\Gamma = \partial D$ and $\Gamma^* = \partial D^*$. The mapping

$$F = J_T \circ H_*^{-1} \circ J_\Gamma \circ H \tag{5.4}$$

is a qc automorphism of Δ. Consider $f = F|_T$ and distinct points $w_1, w_2, w_3, w_4 \in T$. Then we have

$$\Phi_{1/Q^2}([w_1, w_2, w_3, w_4]) \le [f(w_1), f(w_2), f(w_3), f(w_4)] \le \Phi_{Q^2}([w_1, w_2, w_3, w_4]).$$

Due to the conformal invariance of harmonic cross-ratios, it follows that

$$[w_1, w_2, w_3, w_4] = [z_1, z_2, z_3, z_4]_\Gamma,$$

where $w_i = H^{-1}(z_i)$, $i = 1, 2, 3, 4$. The reflection J_Γ does not change the points of Γ, whereas

$$[f(w_1), f(w_2), f(w_3), f(w_4)] = [z_1, z_2, z_3, z_4]_{\Gamma^*}$$

holds by the conformal invariance of harmonic cross-ratios. Thus we have the necessity with $K = Q^2$.

To prove the sufficiency let Γ be an oriented Jc on $\overline{\mathbb{C}}$ such that the inequalities (5.3) hold for any distinct points $z_1, z_2, z_3, z_4 \in \Gamma$. Consider $h = H_*^{-1} \circ H$ on T. By (5.3) and the conformal invariance of the harmonic cross-ratios and the identity

$$[\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,]_T = [\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,]_{T^*},$$

the double inequality

$$\Phi_{1/K}([w_1,w_2,w_3,w_4]) \le [h(w_1),h(w_2),h(w_3),h(w_4)] \le \Phi_K([w_1,w_2,w_3,w_4]) \tag{5.5}$$

holds for $w_i = H^{-1}(z_i)$, $i=1,2,3,4$. Therefore, by Theorem 2.8, there exists a $K'(K)$-qc automorphism F_h of Δ with the boundary values given by h. Consider

$$G = H_* \circ J_T \circ F_h \circ H^{-1}. \tag{5.6}$$

One may see that G is a sense-reversing qc mapping of $\overline{D}$ onto $\overline{D^*}$ which is the identity on Γ. Defining $G(z) = G^{-1}(z)$ for $z \in D^*$, we see that G is a K'-qc reflection in Γ, where $K' \le \min\{\lambda^{3/2}(K),\ 2\lambda(K)-1\}$ with $\lambda(K) = \Phi_K\big(1/\sqrt{2}\big)^2/\Phi_{1/K}\big(1/\sqrt{2}\big)^2$. Consequently, Γ is a quasicircle. The proof is complete.

Using the notion of conformal representation we prove

THEOREM 4.12. *If an oriented Jc Γ on $\overline{\mathbb{C}}$ is a Q-quasicircle, $Q \ge 1$, then $R_\Gamma \in A_\Gamma(Q^2) \cap A_{\Gamma^*}(Q^2)$. Conversely, for each $K \ge 1$, there is a constant $Q = Q(K)$ such that if $R_\Gamma \in A_\Gamma(K) \cup A_{\Gamma^*}(K)$, then Γ is a $Q(K)$-quasicircle, where* $1 \le Q(K) \le \min\{\lambda^{3/2}(K), 2\lambda(K)-1\}$.

P r o o f. Suppose that an oriented Jc Γ on $\overline{\mathbb{C}}$ is a Q-quasicircle, $Q \ge 1$. Then there is a Q^2-qc reflection in Γ. The mapping F defined by (5.4) is a Q^2-qc automorphism of Δ. Thus $F|_T = H_*^{-1} \circ H \in A_T(Q^2)$. The automorphism

$$S_H(H_*^{-1} \circ H) = S_{H_*}(H_*^{-1} \circ H) = H \circ H_*^{-1} \tag{5.7}$$

is then an element of $A_\Gamma(Q^2) \cap A_{\Gamma^*}(Q^2)$.

Suppose now that

$$H \circ H_*^{-1} \in A_\Gamma(K) \cup A_{\Gamma^*}(K), \qquad K \ge 1.$$

The automorphism $H_*^{-1} \circ H$ belongs to $A_T(K) = A_{T^*}(K)$.

Hence, by Theorem 2.8, there exists a $Q(K)$-qc automorphism F_h of Δ with the boundary values given by $h = H_*^{-1} \circ H$. From this moment we follow the sufficiency proof of Theorem 4.11, starting from (5.5), to obtain the sufficiency in Theorem 4.12. This ends the present proof.

Using the notions of Section 4 of Chapter IV, we assume that $\Gamma_1 = T$ and that Γ_2 is denoted by Γ. So let $D_1 = \Delta$, $D_1^* = \Delta^*$, $D_2 = D$ and $D_2^* = D^*$. Recall that $A_T(K) = A_{T^*}(K)$ for every $K \ge 1$. It is an easy observation that now we also have

$$K_\Gamma(H_* \circ H^{-1}) = K_{\Gamma^*}(H_* \circ H^{-1}). \tag{5.8}$$

This equality, together with (4.11) and (4.12), enables us to state the following

DEFINITION 4.3. Let Γ be an oriented Jc on $\overline{\mathbb{C}}$. The common value described by (5.8), (4.11) and (4.12) is denoted by K_Γ.

Then, as an immediate consequence of the previous considerations and Theorem 4.12, one obtains

THEOREM 4.13. *An oriented Jc Γ on $\overline{\mathbb{C}}$ is a quasicircle if and only if* $K_\Gamma < \infty$.

It is worth noting that an oriented Jc Γ on $\overline{\mathbb{C}}$ is a circle on $\overline{\mathbb{C}}$ if and only if

$$A_\Gamma(K) = A_{\Gamma^*}(K) \tag{5.9}$$

for each $K \geq 1$.

Two Jc's Γ_1 and Γ_2 on $\overline{\mathbb{C}}$ are said to be *equivalent* ($\Gamma_1 \sim \Gamma_2$) if there is a homography H such that $\Gamma_2 = H(\Gamma_1)$. If $\Gamma_1 \sim \Gamma_2$, then $K_{\Gamma_1} = K_{\Gamma_2}$. Let $\boldsymbol{\Gamma}$ be the family of all Jc's in $\overline{\mathbb{C}}$, and let

$$\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}/\sim . \tag{5.10}$$

DEFINITION 4.5. For each equivalence $[\Gamma] \in \boldsymbol{\Gamma}_1$, the value

$$\|[\Gamma]\| = \tfrac{1}{2}\log K_\Gamma \tag{5.11}$$

is called the *norm* in $\boldsymbol{\Gamma}_1$.

Let $\boldsymbol{\Gamma}^\infty$ denote the family of all Jc's Γ in $\overline{\mathbb{C}}$ with K_Γ finite, and let $\Gamma_1, \Gamma_2 \in \boldsymbol{\Gamma}^\infty$. Then

$$q(\Gamma_1, \Gamma_2) = \frac{1}{2}\left|\log\frac{K_{\Gamma_1}}{K_{\Gamma_2}}\right| \tag{5.12}$$

is a pseudometric in $\boldsymbol{\Gamma}^\infty$.

To make q a metric we shall introduce a much weaker equivalence relation on $\boldsymbol{\Gamma}^\infty$. First we introduce a new equivalence relation on $\boldsymbol{\Gamma}$ by saying that two Jc's Γ_1 and Γ_2 are $\boldsymbol{w}$-equivalent ($\Gamma_1 \approx \Gamma_2$) if $K_{\Gamma_1} = K_{\Gamma_2}$. Let

$$\boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}/\approx .$$

We denote the equivalence classes in this space by $[[\Gamma]]$ and let $\boldsymbol{\Gamma}_2^\infty = \boldsymbol{\Gamma}^\infty/\approx$. Putting

$$q^*([[\Gamma_1]], [[\Gamma_2]]) = q(\Gamma_1, \Gamma_2) \tag{5.13}$$

one obtains

THEOREM 4.14. *The space* $(\boldsymbol{\Gamma}_2^\infty, q^*)$ *is a metric space.*

Additionally, one has

THEOREM 4.15. *If a Jc Γ on $\overline{\mathbb{C}}$ is a quasicircle, then*

$$A_\Gamma^\infty = A_{\Gamma^*}^\infty. \tag{5.14}$$

Proof. Suppose that Γ on $\overline{\mathbb{C}}$ is a Jc, and H and H_* are conformal mappings of Δ and Δ^* onto D and D^*, respectively. Assume that Γ is a Q-quasicircle, $Q \geq 1$, and that $f \in A_D^\infty$. Then there is $K \geq 1$ such that $f \in A_D(K)$.

Let

$$f_* = J_{HH_*}(f). \tag{5.15}$$

By the previous considerations and Theorems 4.9 and 4.11, it follows that

$$K_{\Gamma^*}(f) \leq Q^4 K_\Gamma(f). \tag{5.16}$$

Hence, there is a constant L, $1 \le L \le Q^4 K$, such that $f \in A_{\Gamma^*}(L)$. Starting with any $f \in A^\infty_{\Gamma^*}$, and using the fact that

$$J^{-1}_{HH_*} = J_{H_* H}, \tag{5.17}$$

we may obtain a similar inclusion from the identity (5.14). This ends the proof.

Conjecture. *Suppose that Γ is a Jc on $\overline{\mathbb{C}}$ for which* (5.14) *holds*, *where D and D^* denote the complementary domains. Then Γ is a quasicircle.*

V. The universal Teichmüller space

1. Introduction. The universal Teichmüller space was introduced by L. Bers in the study of the group of qc automorphisms of a fixed domain D in $\overline{\mathbb{C}}$. We assume here that D is a Jordan domain in $\overline{\mathbb{C}}$, and denote by $\mathcal{F}^\infty_{\overline{D}}$ the family of all qc automorphisms of D, which are regarded as automorphisms of the closure $\overline{D} = D \cup \Gamma$, where $\Gamma = \partial D$.

Two mappings F and G in $\mathcal{F}^\infty_{\overline{D}}$ are said to be *equivalent* ($F \backsim G$) if they differ by a conformal automorphism of D. The equivalence classes can be represented by normalized qc automorphisms of $\overline{D}$; cf. [Le].

By the existence theorem for the Beltrami equation (cf. [Bo] and [LV]), there is a unique correspondence between the equivalence classes of the above relation and the open unit ball of the Banach space consisting of all L^∞-functions on D, whose elements are called *complex dilatations*.

A much more interesting space is obtained by saying that two mappings F and G in $\mathcal{F}^\infty_{\overline{D}}$ are *equivalent* ($F \sim G$) if they differ by the boundary values of a conformal automorphism of $\overline{D}$. This boundary relation is apparently weaker than the previous one. The space of the corresponding equivalence classes is the *universal Teichmüller space* (UTS). We emphasize the fact that this equivalence relation is defined by the boundary value of the previous relation, i.e. $\backsim|_\Gamma = \sim$. Moreover, the corresponding equivalence relation is induced in the unit ball of the Banach space L^∞_D of complex dilatations. However, it seems plausible that there does not exist any intrinsic characterization of functions in L^∞_D.

Let A^∞_Γ denote the class of *qh automorphisms* of an oriented Jc Γ on $\overline{\mathbb{C}}$. It is obvious that equivalence classes of the latter relation can be represented as equivalence classes of A^∞_Γ. These in turn can be represented by those normalized automorphisms in A^∞_Γ that keep three given distinct points of Γ fixed. Hence, the boundary type normalization for the qc automorphisms from $\mathcal{F}^\infty_{\overline{D}}$ seems to be justified, where $\Gamma = \partial D$.

Similarly to the theory of quasicircles, we define the weakest equivalence relation by setting $F \approx G$ for two mappings F and G in $\mathcal{F}^\infty_{\overline{D}}$ if $K(F) = K(G)$. This induces a corresponding relation in the family of complex dilatations, which divides the unit ball of L^∞ in D into spheres centered at the origin. This relation may be considered also in A^∞_Γ. It is worth noting that, in the case of $D = U$ or of $D = \Delta$, two equivalent automorphisms may have different qs constants; cf. Example 2.1, p. 48.

A remarkable fact about the second relation is that it defines the UTS on the boundary curve Γ. Therefore, the conformally invariant characterization of the class A^∞_Γ, representing boundary values of qc automorphisms of D, is important in the UTS theory. Another important fact about this characterization is that we may define the *Teichmüller distance* for automorphisms in A_Γ^∞ without any reference to qc extensions. This makes the space of equivalence classes, obtained from A_Γ^∞ by dividing by the second equivalence relation, a metric space. This illustrates vividly the boundary character of the UTS or, more precisely, of the *universal Teichmüller metric space* (UTMS).

Unfortunately, each oriented Jc Γ on $\overline{\mathbb{C}}$ has two complementary domains. Pursuing the idea of the pure boundary character of the UTMS, we shall complete it on the other side of Γ.

We have already considered simultaneously the boundary value problem for qc automorphisms of D and D^*, provided that Γ is a circle in $\overline{\mathbb{C}}$ (see Chapter II). This simultaneous treatment is based on the fact that $\mathcal{F}_D(K)$ and $\mathcal{F}_{D^*}(K)$, $K \geq 1$, may be identified by a conformal reflection if and only if Γ is a circle in $\overline{\mathbb{C}}$. Hence, $A_\Gamma(K) = A_{\Gamma^*}(K)$ for every $K \geq 1$ if and only if Γ is a circle in $\overline{\mathbb{C}}$.

Assuming that Γ is a quasicircle in $\overline{\mathbb{C}}$, one may identify $\mathcal{F}_D^\infty$ and $\mathcal{F}_{D^*}^\infty$ by a qc reflection in Γ. Moreover, we have proved that $A_\Gamma^\infty = A_{\Gamma^*}^\infty$ provided Γ is a quasicircle in $\overline{\mathbb{C}}$ and Γ^* denotes Γ with the opposite orientation.

Unfortunately, if Γ is an arbitrary oriented Jc on $\overline{\mathbb{C}}$, there does not exist any suitable relation between $\mathcal{F}_D^\infty$ and $\mathcal{F}_{D^*}^\infty$ that preserves all points of Γ; cf. [ST]. Hence, we shall consider A_Γ^∞ and $A_{\Gamma^*}^\infty$ separately.

On the other hand, let an oriented Jc Γ on $\overline{\mathbb{C}}$ be given, with D and D^* the complementary domains. We shall consider the problem of distinguishing between D and D^*. If Γ is an oriented Jc in the open complex plane $\mathbb{C}$, the correspondence $\Gamma \leftrightarrow (D, D^*)$ is fairly obvious and can be achieved by assuming that the point at infinity is in D^*. This defines the "positive" orientation of Γ.

This idea does not work if Γ is on $\overline{\mathbb{C}}$, which case is of our special interest. We overcome this difficulty by assuming that Γ is on the Riemann sphere, i.e., $\overline{\mathbb{C}}$ equipped with a fixed conformal structure; cf. the Introduction. Given an oriented Jc Γ on the Riemann sphere one may uniquely associate with Γ the complementary domains D and D^* defining the *left-hand domain* by D and the *right-hand domain* by D^*. Hence, the correspondence $\Gamma \leftrightarrow (D, D^*)$ is unique.

For an oriented Jc Γ on the Riemann sphere $\overline{\mathbb{C}}$, it is justified to introduce the notions $\mathcal{F}_\Gamma^\infty$, $\mathcal{F}_{\Gamma^*}^\infty$ instead of $\mathcal{F}_{D|\Gamma}^\infty$ and $\mathcal{F}_{D^*|\Gamma^*}^\infty$, respectively. The already solved *general boundary value problem* for qc mappings means a characterization by which $(\mathcal{F}_\Gamma^\infty, \mathcal{F}_{\Gamma^*}^\infty)$ is identified with $(A_\Gamma^\infty, A_{\Gamma^*}^\infty)$ on each oriented Jc Γ on the Riemann sphere. It becomes *uniform* if $(\mathcal{F}_\Gamma(K), \mathcal{F}_{\Gamma^*}^\infty(K))$ is represented by $(A_\Gamma(K), A_{\Gamma^*}(K))$ for every $K \geq 1$.

Another motivation for this notion arises from the fact that the conformal mappings H and H^* of a given pair of complementary Jordan domains D and D^* onto Δ and Δ^*, respectively, are described by the oriented unit circle T and the oriented Jc Γ as the common boundaries of the corresponding domains.

2. The universal Teichmüller space of a circle. Suppose now that Γ is an arbitrary circle in $\overline{\mathbb{C}}$. Therefore, as mentioned in the Introduction to Chapter IV, $A_\Gamma(K) =$

$A_{\Gamma^*}(K)$ for every $K \geq 1$, and then

$$A_\Gamma^\infty = A_{\Gamma^*}^\infty.$$

Two automorphisms $f, g \in A_\Gamma^\infty$ are said to be *equivalent* ($f \sim g$) if $f \circ g^{-1} \in A_\Gamma(1)$. The set

$$T_\Gamma = A_\Gamma^\infty / \sim \tag{2.1}$$

of equivalence classes is the *universal Teichmüller space of* Γ. For $f, g \in A_\Gamma^\infty$, one defines

$$\tau(f, g) = \tfrac{1}{2} \log K(f \circ g^{-1}). \tag{2.2}$$

Hence, $0 \leq \tau(f, g) \leq \log K$ for each $f, g \in A_\Gamma(K)$, $1 \leq K < \infty$. Suppose that $f_i \sim g_i$ for $f_i, g_i \in A_\Gamma^\infty$, $i = 1, 2$. There exist $h_1, h_2 \in A_\Gamma(1)$ such that $g_i = h_i \circ f_i$, $i = 1, 2$. Hence

$$\tau(g_1, g_2) = \tfrac{1}{2} \log K(h_1 \circ f_1 \circ f_2^{-1} \circ h_2^{-1}) = \tfrac{1}{2} \log K(f_1 \circ f_2^{-1}) = \tau(f_1, f_2).$$

Thus we can define

$$\tau^*([f], [g]) := \tau(f, g) \tag{2.3}$$

for any $f \in [f]$ and $g \in [g]$. This is the *Teichmüller distance* between two points of T_Γ.

We have proved

THEOREM 5.1. *The space* (T_Γ, τ^*) *is a metric space.*

The importance of the fact that we may define τ^* without any help of quasiconformal extensions cannot be overestimated. This mainly enables us to call T_Γ a *boundary model of the universal Teichmüller metric space.*

Let Γ_1 and Γ_2 be circles in $\overline{\mathbb{C}}$. By Theorem 3.3, there exists a homography H such that $H(\Gamma_1) = \Gamma_2$ and that the transformation

$$S_H(f) = H \circ f \circ H^{-1}$$

maps $A_{\Gamma_1}^\infty$ onto $A_{\Gamma_2}^\infty$ (cf. Theorem 4.11). Suppose that $f, g \in A_{\Gamma_1}^\infty$ are equivalent ($f \sim g$). Then there exists a constant $\eta \in A_{\Gamma_1}(1)$ such that $f = \eta \circ g$ and

$$S_H(f) \circ (S_H(g))^{-1} = H \circ f \circ H^{-1} \circ H \circ g^{-1} \circ H^{-1} = H \circ \eta \circ H^{-1} = S_H(\eta) \tag{2.4}$$

is an element of $A_{\Gamma_2}(1)$. Hence $S_H(f) \sim S_H(g)$, and one can define $S_H^* : T_{\Gamma_1} \to T_{\Gamma_2}$ by setting

$$S_H^*([f]) := [S_H(f)]. \tag{2.5}$$

This yields

THEOREM 5.2. *For any circles* Γ_1, Γ_2 *on* $\overline{\mathbb{C}}$, *the transformation* S_H^* *of* (T_{Γ_1}, τ_1^*) *onto* (T_{Γ_2}, τ_2^*) *is an isometry.*

Proof. Suppose that $[f], [g] \in T_{\Gamma_1}$. Then

$$\begin{aligned}\tau_2^*(S_H^*([f]), S_H^*([g])) &= \tau_2(S_H(f), S_H(g)) \\ &= \tfrac{1}{2} \log K(H \circ f \circ H^{-1} \circ H \circ g^{-1} \circ H^{-1}) \\ &= \tfrac{1}{2} \log K(H \circ f \circ g^{-1} \circ H^{-1}) = \tfrac{1}{2} \log K(S_H(f \circ g^{-1})) \\ &= \tfrac{1}{2} \log K(f \circ g^{-1}) = \tau_1(f, g) = \tau_1^*([f], [g]).\end{aligned}$$

THEOREM 5.3. *For every circle Γ on $\overline{\mathbb{C}}$, the space (T_Γ, τ^*) is a complete metric space.*

Proof. In view of Theorems 3.3 and 5.2, we may confine our considerations to the case $\Gamma = T$. Let $w_k = e^{i(2\pi k/3)}$, $k = 0, 1, 2$. Consider a sequence $f_n \in A_T^\infty$ such that

$$\tau^*([f_n], [f_m]) \to 0 \quad \text{as } n, m \to \infty.$$

For each $n \in \mathbb{N}$, there exists a function $g_n \in [f_n]$ such that $g_n(w_k) = w_k$ for $k = 0, 1, 2$. By the definition of τ^*, we see that

$$\tau(g_n, g_m) = \tau^*([f_n], [f_m]) \to 0 \quad \text{as } n, m \to \infty. \tag{2.6}$$

Thus, there exists $n_0 \in \mathbb{N}$ such that $\tau(g_n, g_{n_0}) < 1/2$ for every $n \geq n_0$. Hence,

$$K(g_n) \leq e \max\{K(g_1), \ldots, K(g_{n_0})\} < \infty \tag{2.7}$$

for each $n \in \mathbb{N}$.

By Corollary 3.8, there exists an automorphism $g \in A_T^\infty$ and a subsequence g_{n_l}, $l = 1, 2, \ldots$, such that

$$d_T(g_{n_l}, g) \to 0 \quad \text{as } l \to \infty. \tag{2.8}$$

By (2.6), for $\varepsilon > 0$, there exists l_0 such that

$$\tfrac{1}{2} \log K(g_{n_k} \circ g_{n_l}^{-1}) = \tau(g_{n_k}, g_{n_l}) < \varepsilon \tag{2.9}$$

for $k, l > l_0$. Fixing $l > l_0$, we get

$$K(g_{n_k} \circ g_{n_l}^{-1}) < e^{2\varepsilon} \quad \text{for } k > l_0. \tag{2.10}$$

Hence

$$d_T(g_{n_k} \circ g_{n_l}^{-1}, g \circ g_{n_l}^{-1}) \to 0 \quad \text{as } k \to \infty. \tag{2.11}$$

By (2.11) and (2.10), one can see that

$$\tau(g, g_{n_l}) = \tfrac{1}{2} \log K(g \circ g_{n_l}^{-1}) \leq \varepsilon \quad \text{for } l > l_0. \tag{2.12}$$

This means that $\tau(g, g_{n_l}) \to 0$ as $l \to \infty$. Then, using (2.6), we arrive at

$$\tau^*([f_n], [g]) = \tau(g_n, g) \to 0 \quad \text{as } n \to \infty, \tag{2.13}$$

which is the desired result.

3. The universal Teichmüller space of an oriented Jordan curve. Suppose that Γ is an oriented Jc on $\overline{\mathbb{C}}$ and D, D^* are its left- and right-hand domains, respectively. Set $\mathbb{A}_\Gamma = A_\Gamma \times A_\Gamma$. For all pairs $f = (f_1, f_2)$ and $g = (g_1, g_2)$ from $\mathbb{A}_\Gamma$, put

$$f \circ g = (f_1, f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2). \tag{3.1}$$

Then $(\mathbb{A}_\Gamma, \circ)$ is a group. Introducing

$$\boldsymbol{d}_\Gamma(f, g) = \max\{d_\Gamma(f_1, g_1), d_\Gamma(f_2, g_2)\} \tag{3.2}$$

one makes $\mathbb{A}_\Gamma$ a metric space, so that $(\mathbb{A}_\Gamma, \circ, \boldsymbol{d}_\Gamma)$ is a metric group. Moreover, let

$$A_\Gamma^\infty = \bigcup_{K \geq 1} A_\Gamma(K), \qquad A_{\Gamma^*}^\infty = \bigcup_{K \geq 1} A_{\Gamma^*}(K) \tag{3.3}$$

and then

$$\mathbb{A}_\Gamma(K) = (A_\Gamma(K), A_{\Gamma^*}(K)), \qquad \mathbb{A}_\Gamma^\infty = (A_\Gamma^\infty, A_{\Gamma^*}^\infty). \tag{3.4}$$

Hence, $(\mathbb{A}_\Gamma^\infty, \circ)$ is a group as well. Note that each element of $\mathbb{A}_\Gamma^\infty$ is an automorphism of $\Gamma \times \Gamma$.

We say that two automorphisms $f_1, g_1 \in A_\Gamma^\infty$ are *equivalent* ($f_1 \sim g_1$) if $f_1 \circ g_1^{-1} \in A_\Gamma(1)$. Also, we say that two automorphisms f_2, $g_2 \in A_{\Gamma^*}^\infty$ are *equivalent* ($f_2 \backsim^* g_2$) if $f_2 \circ g_2^{-1} \in A_{\Gamma^*}(1)$. Let

$$T_\Gamma = A_\Gamma^\infty / \sim \quad \text{and} \quad T_{\Gamma^*} = A_{\Gamma^*}^\infty / \backsim^* . \tag{3.5}$$

We say that

$$\mathbf{T}_\Gamma = (T_\Gamma, T_{\Gamma^*}) \tag{3.6}$$

is the *universal Teichmüller space of* Γ.

For all pairs $f = (f_1, f_2)$ and $g = (g_1, g_2)$ from $\mathbb{A}_\Gamma^\infty$, we define

$$\tau_\Gamma(f_1, g_1) = \tfrac{1}{2} \log K_\Gamma(f_1 \circ g_1^{-1}), \qquad \tau_{\Gamma^*}(f_2, g_2) = \tfrac{1}{2} \log K_{\Gamma^*}(f_2 \circ g_2^{-1}). \tag{3.7}$$

and

$$\boldsymbol{\tau}_\Gamma(f, g) = \tfrac{1}{4} \log K_\Gamma(f_1 \circ g_1^{-1}) K_{\Gamma^*}(f_2 \circ g_2^{-1}), \tag{3.8}$$

which is a pseudometric in $\mathbb{A}_\Gamma^\infty$; moreover, $0 \le \boldsymbol{\tau}_\Gamma(f, g) \le \log K$ for all $f, g \in \mathbb{A}_\Gamma(K)$.

THEOREM 5.4. *For an oriented Jc* Γ *on* $\overline{\mathbb{C}}$ *and every* $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathbb{A}_\Gamma^\infty$, *we have*, *with* Γ^* *the Jc conjugate to* Γ:

(i) $\boldsymbol{\tau}_\Gamma(f, g) = 0$ *if and only if* $f \circ g^{-1} \in \mathbb{A}_\Gamma(1)$,
(ii) $\boldsymbol{d}_{\Gamma^*}(f, g) = \boldsymbol{d}_\Gamma(f, g)$,
(iii) $\boldsymbol{\tau}_{\Gamma^*}(f, g) \le \boldsymbol{\tau}_\Gamma(f, g) + 2 \log Q$ *if* Γ *is a* Q*-quasicircle.*

Proof. The identity

$$\boldsymbol{\tau}_\Gamma(f, g) = 0$$

is equivalent to

$$\tau_\Gamma(f_1, g_1) = 0 \quad \text{and} \quad \tau_{\Gamma^*}(f_2, g_2) = 0$$

and hence to

$$f_1 \circ g_1^{-1} \in A_\Gamma(1) \quad \text{and} \quad f_2 \circ g_2^{-1} \in A_{\Gamma^*}(1)$$

and to

$$f \circ g^{-1} = (f_1 \circ g_1^{-1}, f_2 \circ g_2^{-1}) \in \mathbb{A}_\Gamma(1).$$

Obviously, $\mathbb{A}_\Gamma = \mathbb{A}_{\Gamma^*}$ and, because of (3.2), the assertion (ii) follows. If Γ is a Q-quasicircle, then, by Theorem 4.15, we have

$$A_\Gamma(K) \subset A_{\Gamma^*}(L_1) \quad \text{and} \quad A_{\Gamma^*}(K) \subset A_\Gamma(L_2),$$

where $1 \le L_1 \le Q^4 K$ and $1 \le L_2 \le Q^4 K$, $K \ge 1$. Now, (iii) follows by elementary calculations.

If Γ is a circle, then $Q = 1$, and, because of (iii),

$$\boldsymbol{\tau}_{\Gamma^*}(f, g) = \boldsymbol{\tau}_\Gamma(f, g).$$

The theorem is proved.

As a completion of Theorem 5.4, one may state

COROLLARY 5.1. *We have*:

(i) $\mathbb{A}_{\Gamma^*} = \mathbb{A}_\Gamma$ *always*;
(ii) $\mathbb{A}^\infty_{\Gamma^*} = \mathbb{A}^\infty_\Gamma$ *if* Γ *is a quasicircle in* $\overline{\mathbb{C}}$;
(iii) $T_{\Gamma^*} = T_\Gamma$ *if and only if* Γ *is a circle in* $\overline{\mathbb{C}}$.

Proof. The assertion (i) is obvious; (ii) follows by Theorem 4.15. The assertion (iii) is a result of the observation that $[f] \in T_\Gamma$ inherits the group structure if and only if $K_\Gamma(f) = K_{\Gamma^*}(f) = 1$.

Assume now that $[f^1]$, $[f^2]$, $[g^1]$ and $[g^2]$ are elements of T_Γ. Then

$$[f^1] = [f^2] \Leftrightarrow f^1 \sim f^2 \Leftrightarrow f^1 = h^1 \circ f^2$$

and

$$[g^1] = [g^2] \Leftrightarrow g^1 \sim g^2 \Leftrightarrow g^1 = h^2 \circ g^2,$$

where h^1 and h^2 are elements of $A_\Gamma(1)$ and $A_{\Gamma^*}(1)$, respectively. Thus

$$f^1 = (f^1_1, f^1_2) = (h^1_1 \circ f^2_1, h^1_2 \circ f^2_2) \quad \text{and} \quad g^1 = (g^1_1, g^1_2) = (h^2_1 \circ g^2_1, h^2_2 \circ g^2_2).$$

Hence

$$\begin{aligned}(3.9)\qquad \boldsymbol{\tau}_\Gamma(f^1, g^1) &= \tfrac{1}{4}\log K_\Gamma(h^1_1 \circ f^2_1 \circ (g^2_1)^{-1} \circ (h^2_1)^{-1}) K_{\Gamma^*}(h^1_2 \circ f^2_2 \circ (g^2_2)^{-1} \circ (h^2_2)^{-1}) \\ &= \tfrac{1}{4}\log K_\Gamma(f^2_1 \circ (g^2_1)^{-1}) K_{\Gamma^*}(f^2_2 \circ (g^2_2)^{-1}) = \boldsymbol{\tau}_\Gamma(f^2, g^2).\end{aligned}$$

So we can define

$$\boldsymbol{\tau}^*_\Gamma([f],[g]) = \boldsymbol{\tau}_\Gamma(f, g). \tag{3.10}$$

Evidently, this expression is well defined and independent of the representation. Thus, we have proved

THEOREM 5.5. *For an oriented Jc* Γ *on* $\overline{\mathbb{C}}$, *the space* $(\mathbf{T}_\Gamma, \boldsymbol{\tau}^*_\Gamma)$ *is a metric space.*

Suppose now that $[f] = [g]$. Then $f_1 \sim g_1$ and $f_2 \sim g_2$. Let H and H_* map, as usual, Δ and Δ^* onto D and D^*, respectively. Define

$$\widetilde{f}_2 = J_{HH_*}(f_1) \quad \text{and} \quad \widetilde{g}_2 = J_{HH_*}(g_1).$$

Then

$$\widetilde{f}_2 \circ (\widetilde{g}_2)^{-1} = J_{HH_*}(f_1 \circ (g_1)^{-1}) \tag{3.11}$$

and, because of Theorem 4.11, $\widetilde{f}_2 \circ (\widetilde{g}_2)^{-1} \in A_{\Gamma^*}(1)$. Moreover, let

$$\widetilde{f}_1 = J_{H_*H}(f_2) \quad \text{and} \quad \widetilde{g}_1 = J_{H_*H}(g_2);$$

then

$$\widetilde{f}_1 \circ (\widetilde{g}_1)^{-1} = J_{H_*H}(f_2 \circ (g_2)^{-1}). \tag{3.11$'$}$$

Hence one can see that $\widetilde{f}_1 \circ (\widetilde{g}_1)^{-1} \in A_\Gamma(1)$.

Let us introduce

$$\boldsymbol{J}_{HH_*}(f) = (J_{H_*H}(f_2), J_{HH_*}(f_1)). \tag{3.12}$$

As justified by the above considerations, we can define

$$\boldsymbol{J}^*_{HH_*}([f]) = [\boldsymbol{J}_{HH_*}(f)]. \tag{3.13}$$

This is a well-defined transformation of $\mathbf{T}_\Gamma$ onto $\mathbf{T}_{\Gamma^*}$.

THEOREM 5.6. *For an oriented Jc Γ on $\overline{\mathbb{C}}$, we have*:

(i) *the transformation $\boldsymbol{J}_{HH_*}$ defined by* (3.12) *is an automorphism of the metric group $(\mathbb{A}_\Gamma, \circ, d_\Gamma)$, and $\mathbb{R}^\infty_\Gamma$ are fixed points of this transformation*;

(ii) $(\boldsymbol{J}_{HH_*})^{-1} = \boldsymbol{J}_{H_*H}$, *and $\boldsymbol{J}_{HH_*}$ is an involution of $\mathbb{A}^\infty_\Gamma$, provided Γ is a quasicircle*;

(iii) $\boldsymbol{J}^*_{HH_*}$ *is an isometry between $(\mathbf{T}_\Gamma, \boldsymbol{\tau}^*_\Gamma)$ and $(\mathbf{T}_{\Gamma^*}, \boldsymbol{\tau}^*_\Gamma)$.*

Proof. The condition (i) follows from (ii) of Theorem 4.10 and by Corollary 5.1. The identity of (ii) is a simple consequence of the definition of $\boldsymbol{J}_{HH_*}$ and Theorem 4.10. The assertion (iii) follows by the definition of $\mathbf{l}\tau^*_\Gamma$ and (iii) of Theorem 4.10. The identity (ii) of Corollary 5.1 and the identity in (ii) of that theorem guarantee the second statement of (ii). This finishes the proof.

By (iv) and (v) of Theorem 4.10 and by Theorem 4.14, it follows that D_{HH_*} and D_{H_*H} map $A_T(K)$ onto $A_\Gamma(K \cdot K_\Gamma)$ and into $A_{\Gamma^*}(K \cdot K_\Gamma)$, $K \geq 1$, respectively. If, in addition, Γ is a quasicircle in $\overline{\mathbb{C}}$, then both the transformations map A^∞_T onto $A^\infty_\Gamma = A^\infty_{\Gamma^*}$; cf. Theorem 4.16.

In order to lift D_{HH_*} and D_{H_*H} to the respective UTS, assume that $f, g \in A^\infty_T$ are such that $f \circ g^{-1} \in A_T(1)$. Then

$$D_{HH_*}(f) \circ (D_{HH_*}(g))^{-1} = S_H(f \circ g^{-1}) \in A_\Gamma(1) \tag{3.14}$$

and

$$D_{H_*H}(f) \circ (D_{H_*H}(g))^{-1} = S_{H_*}(f \circ g^{-1}) \in A_{\Gamma^*}(1). \tag{3.14$'$}$$

In accordance with the previous cases, one may define $D^*_{HH_*}$ and $D^*_{H_*H}$ that map T_T onto T_Γ and T_{Γ^*}, respectively. Hence, we can set

$$\boldsymbol{S}^*_{H,H_*} := \begin{pmatrix} S^*_H & D^*_{HH_*} \\ D^*_{H_*H} & S^*_{H_*} \end{pmatrix}. \tag{3.15}$$

Then we have a function

$$\boldsymbol{S}^*_{H,H_*} : \begin{pmatrix} T_T & T_T \\ T_T & T_T \end{pmatrix} \to \begin{pmatrix} T_D & T_D \\ T_{D^*} & T_{D^*}. \end{pmatrix}. \tag{3.16}$$

Let, as before, Γ_1 and Γ_2 be oriented Jc's on the Riemann sphere, and let H, H_* and G, G_* be conformal mappings of Δ and Δ_* onto D_1, D_1^* and D_2, D_2^*, respectively. Consider the transformation

$$\boldsymbol{J}_{\Gamma_1\Gamma_2}(f) = (J_{H_*G}(f_2), J_{HG_*}(f_1)), \tag{3.17}$$

where

$$J_{H_*G} = S_G \circ S^{-1}_{H_*} \quad \text{and} \quad J_{HG_*} = S_{G_*} \circ S^{-1}_H \tag{3.18}$$

map $\mathbb{A}^\infty_{\Gamma_1}$ onto $\mathbb{A}^\infty_{\Gamma_2}$. By arguments related to those that we used defining $\boldsymbol{\tau}^*$ and $\boldsymbol{J}^*_{HH_*}$, we may also define

$$\boldsymbol{J}^*_{\Gamma_1\Gamma_2}([f]) = [\boldsymbol{J}_{\Gamma_1\Gamma_2}(f)]. \tag{3.19}$$

Then we have

R e m a r k 5.1. If $\Gamma_1 = \Gamma_2 = \Gamma$, one may identify G and G_* with H and H_*, respectively, and set

$$\boldsymbol{J}_\Gamma := \boldsymbol{J}_{\Gamma\Gamma} := \boldsymbol{J}_{HH_*}. \tag{3.20}$$

Henceforth, the symbol $\boldsymbol{J}_\Gamma$ will be used instead of $\boldsymbol{J}_{HH_*}$, as more adequate in these circumstances.

R e m a r k 5.2. By the previous arguments, one can define

$$\boldsymbol{J}^*_\Gamma = \boldsymbol{J}^*_{\Gamma\Gamma}. \tag{3.21}$$

THEOREM 5.7. *For oriented Jc's Γ_1 and Γ_2 on $\overline{\mathbb{C}}$, we have*:

(i) $\boldsymbol{J}_{\Gamma_1\Gamma_2}$ *is an isomorphism between* $(\mathbb{A}^\infty_{\Gamma_1}, \circ)$ *and* $(\mathbb{A}^\infty_{\Gamma_2^*}, \circ)$;
(ii) $\boldsymbol{J}^*_{\Gamma_1\Gamma_2}$ *is an isometry between* $(\boldsymbol{T}_{\Gamma_1}, \boldsymbol{\tau}^*_{\Gamma_1})$ *and* $(\mathbf{T}_{\Gamma_2^*}, \boldsymbol{\tau}^*_{\Gamma_2^*})$;
(iii) $(\boldsymbol{J}_{\Gamma_1\Gamma_2})^{-1} = \boldsymbol{J}_{\Gamma_2\Gamma_1}$ *and* $(\boldsymbol{J}^*_{\Gamma_1\Gamma_2})^{-1} = \boldsymbol{J}^*_{\Gamma_2\Gamma_1}$.

P r o o f. The condition (i) is a simple consequence of the respective condition in Theorem 4.9. Since $\boldsymbol{S}_{\Gamma_1\Gamma_2}$ preserves the qh constant, (ii) follows by an easy calculation similar to those we used before. The assertion (iii) can be checked immediately.

R e m a r k 5.3. If Γ is an oriented circle on $\overline{\mathbb{C}}$, then $A_\Gamma(K) = A_{\Gamma^*}(K)$ for every $K \geq 1$. Hence, $\mathbb{A}_\Gamma(K) = A_\Gamma(K) \times A_{\Gamma^*}(K)$ can be identified with $A_\Gamma(K)$ on Γ. Since $K_\Gamma(f) = K_{\Gamma^*}(f)$ if and only if Γ is a circle on $\overline{\mathbb{C}}$, the metric $\boldsymbol{\tau}^*_\Gamma$ is isometric to τ^*_Γ. Hence we may identify $(\mathbf{T}_\Gamma, \boldsymbol{\tau}^*_\Gamma)$ with $(T_\Gamma, \tau^*_\Gamma)$ defined in Section V.2.

Suppose that Γ_1 and Γ_2 are oriented Jc's on $\overline{\mathbb{C}}$ with D_1, D_1^* and D_2, D_2^* the respective left-hand and right-hand domains. Let H and H_* map D_1 and D_1^* conformally onto D_2 and D_2^*, respectively. One may then consider the *parallel transformations*

$$\boldsymbol{S}_{\Gamma_1\Gamma_2} = (S_H, S_{H_*}) \tag{3.22}$$

and

$$\boldsymbol{S}^*_{\Gamma_1\Gamma_2} = (S^*_H, S^*_{H_*}) \tag{3.23}$$

that map $\mathbb{A}^\infty_{\Gamma_1}$ and T_{Γ_1} onto $\mathbb{A}^\infty_{\Gamma_2}$ and T_{Γ_2}, respectively.

Further development of the ideas presented in this chapter, including the case when Γ is a Jordan curve on a closed Riemann surface, will be presented in [Z15].

4. The space of normalized quasihomographies. In some parts of this presentation we have used certain convenient normalization conditions (see Chapter III). To simplify our examination of the universal Teichmüller space of an oriented Jc Γ on the Riemann sphere $\overline{\mathbb{C}}$, and to clarify certain situations, different aspects of normalization will be considered.

The first point to be considered is the unique correspondence between a given Jc Γ on $\overline{\mathbb{C}}$ and its complementary domains D and D^*. As mentioned in the introduction to this chapter, the relation

$$\Gamma \leftrightarrow (D, D^*) \tag{4.1}$$

is obvious, provided that Γ lies in the open complex plane $\mathbb{C}$, and assuming that the point at infinity is in D^*. But this excludes cases such as $\Gamma = \overline{\mathbb{R}}$ that are of special interest.

To make the relation (4.1) uniquely defined on $\overline{\mathbb{C}}$, we introduce two conditions:

* Γ is an oriented Jc on $\overline{\mathbb{C}}$;
* $\overline{\mathbb{C}}$ is equipped with a fixed conformal structure making it the Riemann sphere.

Under these circumstances we define D and D^* as the left-hand and right-hand complementary domains, respectively.

With this convention we adjust all notations related to the complementary domains in terms of the given oriented Jc Γ. Hence, for example, instead of

$$\mathcal{F}^{\infty}_{D|\Gamma}, \quad \mathcal{F}^{\infty}_{D^*|\Gamma}, \quad K_D \quad \text{and} \quad K_{D^*}$$

one may write, respectively,

$$\mathcal{F}^{\infty}_{\Gamma}, \quad \mathcal{F}^{\infty}_{\Gamma^*}, \quad K_\Gamma \quad \text{and} \quad K_{\Gamma^*}.$$

The universal Teichmüller space $\mathbf{T}_\Gamma$ of an oriented Jc Γ on the Riemann sphere is purely of boundary character and may be considered as the *boundary model* of the universal Teichmüller space.

The second aspect is normalization in the most usual sense, i.e. of representing every element of the universal Teichmüller space of an oriented Jc Γ on the Riemann sphere by a unique automorphism from $\mathbb{A}^{\infty}_{\Gamma}$. This can be achieved by considering

* all automorphisms from $\mathbb{A}^{\infty}_{\Gamma}$ that leave three given points of Γ fixed.

The correspondence between $\mathbb{A}^{\infty}_{\Gamma}$ and the above-mentiond normalized subfamily is provided by the transformations S_H and S_{H_*}, where H and H_* map D and D^* conformally onto themselves. It follows that

* $\mathbf{T}_\Gamma$ is the set of all pairs (f_1, f_2) of normalized quasihomographies such that $f_1 \in A^{\infty}_{\Gamma}$ and $f_2 \in A^{\infty}_{\Gamma^*}$.

In this way one may simplify several problems and notions used in the previous section.

The third normalization issue arises in a very natural way when one considers real functionals on a space of normalized K-qc automorphisms of a given domain D in $\overline{\mathbb{C}}$, or normalized K-qh automorphisms of a given Jc Γ in $\overline{\mathbb{C}}$; see [LV] and our Chapter III. It is easy to see that values of those functionals are related to certain normalization conditions in the previous sense.

To make our consideration clearer, consider the following problem: Suppose that G is a set and $F : G \to \mathbb{R}$ is a function. Let $\mathbb{H}$ be a family of subsets H of G, and let

$$F^+(H) = \sup_{x\in H} F(x) \quad \text{and} \quad F^-(H) = \inf_{x\in H} F(x).$$

Then one may consider the following extremal problems:

(i) $\sup_{H\in\mathbb{H}} F^+(H) = F^{++}_{\mathbb{H}}$;
(ii) $\inf_{H\in\mathbb{H}} F^+(H) = F^{+-}_{\mathbb{H}}$;
(iii) $\sup_{H\in\mathbb{H}} F^-(H) = F^{-+}_{\mathbb{H}}$;
(iv) $\inf_{H\in\mathbb{H}} F^-(H) = F^{--}_{\mathbb{H}}$.

Given G, $\mathbb{H}$ and F, we state

DEFINITION 5.2. A set $H_0 \in \mathbb{H}$ such that $F^+(H_0) = F^{++}_{\mathbb{H}}$ is called the *extremal set* of F in $\mathbb{H}$ with respect to (i). The other extremal sets of F in $\mathbb{H}$ can be defined analogously. A set that is extremal with respect to one of these extremal problems is called an *extremal set of F in the family* $\mathbb{H}$.

Given an oriented Jc Γ on $\overline{\mathbb{C}}$ and $K \geq 1$, let $G_K = A_\Gamma(K)$ be the family of all K-qh automorphisms of Γ. Let $F : A_\Gamma(K) \to \mathbb{R}$ be a functional and let $\mathbb{H}_K$ be the family of all $A_\Gamma^{z_1,z_2,z_3}(K)$ normalized by fixing three given distinct points $z_1, z_2, z_3 \in \Gamma$.

If there is an extremal set of F in $\mathbb{H}$, it is distinguished by the triple $z_1^0, z_2^0, z_3^0 \in \Gamma$, which is called the *extremal normalization* of $A_\Gamma(K)$ with respect to F.

Given distinct points $\check{z}_1, \check{z}_2, \check{z}_3 \in \Gamma$, it is easy to see that for any distinct points $z_1, z_2, z_3 \in \Gamma$ there exists $h \in A_\Gamma(1)$ such that $\check{z}_i = h(z_i)$, $i = 1, 2, 3$. Therefore, instead of extremalizing F in the previous sense, one may consider $F \circ S_{h^{-1}}$ on $A_\Gamma^{\check{z}_1,\check{z}_2,\check{z}_3}(K)$ and extremalize this with respect to the family of all $h \in A_\Gamma(1)$. Hence,

- $*$ in these circumstances, a functional F that is constant as a function of normalization points is said to be *essential*.

Clearly, all essentially quasihomographical functionals must have the form

$$F([z_1, z_2, z_3, z_4]_\Gamma, [f(z_1), f(z_2), f(z_3), f(z_4)]_\Gamma), \tag{4.2}$$

where z_1, z_2, z_3, z_4 are distinct points on Γ and $f \in A_\Gamma(K)$, $K \geq 1$.

Assume now that F is also defined for a family of oriented Jc's Γ in $\overline{\mathbb{C}}$. An additional extremalization may be carried out, for instance, with respect to the family Q-quasicircles. Problems of this kind were already considered by O. Martio and J. Sarvas [MS] and later [PZ3].

An example is the following

THEOREM 5.8. *For every circle* (1-*quasicircle*) Γ *on* $\overline{\mathbb{C}}$, $f \in \widehat{A}_\Gamma(K)$ *and* $K \geq 1$, *the inequality*

$$d(f(z_1), f(z_2)) \leq C_K d(z_1, z_2)^{1/K} \tag{4.3}$$

holds for any $z_1, z_2 \in \Gamma$, *where* d *is the chordal spherical distance and* C_K *is a constant described in Theorem* 3.7.

Proof. This follows from Theorem 3.7 and the fact that $r(\Gamma) + 1/r(\Gamma) \geq 2$.

In the family of all 1-quasicircles on $\overline{\mathbb{C}}$ there are many extremals with respect to (4.3), including the unit circle T and any line containing the origin. Fortunately, all these extremals are isometrically equivalent; see Corollary 3.2.

In this way we arrive at the normalization of the family of all Jc's Γ in $\overline{\mathbb{C}}$ by

- $*$ fixing three points on $\overline{\mathbb{C}}$ and considering all Jc's Γ on $\overline{\mathbb{C}}$ including these points.

The family of all normalized Q-quasicircles will be of interest subsequently; see [Z13].

Let Γ be an oriented Jc on $\overline{\mathbb{C}}$. For simplicity suppose it is the unit circle T. Between the family of all normalized quasihomographies of T and the family of all normalized quasicircles there is a unique and natural correspondence by the use of the conformal representation. Together with the previous considerations, this gives interesting possibilities when working with functionals depending also on the family of all normalized quasicircles.

The principle of the second and the third normalizations can be applied to the family of quasiconformal mappings of a given Jordan domain D in $\overline{\mathbb{C}}$. Here one of the normalizing points can be taken in D and one on the boundary.

5. A linearization formula. A *linearization formula* for the universal Teichmüller space of an oriented Jc $\varGamma$ on the Riemann sphere will now be considered. This construction, based on the idea of the *free abelian group*, gives a possibility to extend our considerations from the group of quasihomographies of $\varGamma$ to a *Banach space* related to $\varGamma$. Introducing a scalar product we make it a *Hilbert space*.

The aim of this consideration is to present certain constructions and very basic results showing new possibilities when working with this boundary model of the universal Teichmüller space. For further results we refer to [Z12].

Given an oriented Jc $\varGamma$ on $\overline{\mathbb{C}}$, let $\mathcal{M}_\varGamma$ be the family of all functions mapping $A_\varGamma$ onto the real line, and equal to zero except for a countable set of arguments. Thus $\mathcal{M}_\varGamma$ is a real linear space. Identifying each $f \in A_\varGamma$ with δ_f, where

$$\delta_f(g) = \begin{cases} 1 & \text{for } g = f, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

we uniquely associate with $A_\varGamma$ the real linear space $\mathcal{M}_\varGamma$.

Let

$$\widehat{\mathcal{M}}_\varGamma = \{\varphi : \mathcal{M}_\varGamma \to \mathbb{R} : \varphi \text{ is linear}\}. \tag{5.2}$$

Denote by $\widehat{\delta}_f$ the element of $\widehat{\mathcal{M}}_\varGamma$ which satisfies

$$\widehat{\delta}_f(\delta_g) = \begin{cases} 1 & \text{if } f = g, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

For every $\widehat{f} \in \mathcal{M}_\varGamma$, one has

$$\widehat{f} = \sum_{h \in A_\varGamma} \widehat{\delta}_h(\widehat{f})\delta_h \tag{5.4}$$

or, equivalently,

$$\widehat{f} = \sum_{n=1}^{\infty} \widehat{\delta}_{h_n}(\widehat{f})\delta_{h_n}. \tag{5.4′}$$

Two linear subspaces $\mathcal{M}_\varGamma^\infty$ and $\mathcal{M}_{\varGamma^*}^\infty$ of the space $\mathcal{M}_\varGamma$ that correspond to $A_\varGamma^\infty$ and $A_{\varGamma^*}^\infty$, where $\varGamma^*$ is the Jc conjugate to $\varGamma$, will be of interest. Given $\widehat{f} \in \mathcal{M}_\varGamma^\infty$ and ${}^*\widehat{f} \in \mathcal{M}_{\varGamma^*}^\infty$, define

$$\|\widehat{f}\|_\varGamma = \sum_{h \in A_\varGamma^\infty} |\widehat{\delta}_h(\widehat{f})| \, \|\delta_h\|_\varGamma \tag{5.5}$$

and

$$\|{}^*\widehat{f}\|_{\varGamma^*} = \sum_{h \in A_{\varGamma^*}^\infty} |\widehat{\delta}_h({}^*\widehat{f})| \, \|\delta_h\|_{\varGamma^*},$$

where $\|\delta_h\|_\Gamma = K_\Gamma(h)$ and $\|\delta_h\|_{\Gamma^*} = K_{\Gamma^*}(h)$. Both these expressions define norms. In the product space

$$\underline{\mathcal{M}}_\Gamma^\infty = \mathcal{M}_\Gamma^\infty \times \mathcal{M}_{\Gamma^*}^\infty \tag{5.6}$$

one can define a norm of $\widehat{\boldsymbol{f}} = (\widehat{f}, {}^*\widehat{f}) \in \mathcal{M}_\Gamma^\infty$ by

$$\|\widehat{\boldsymbol{f}}\|_\Gamma = \|\widehat{f}\|_\Gamma + \|{}^*\widehat{f}\|_{\Gamma^*}. \tag{5.7}$$

Let

$$\widetilde{\mathcal{M}}_\Gamma^\infty = \{\widehat{f} \in \mathcal{M}_\Gamma^\infty : \|\widehat{f}\|_\Gamma < \infty\}, \tag{5.8}$$

$$\widetilde{\mathcal{M}}_{\Gamma^*}^\infty = \{{}^*\widehat{f} \in \mathcal{M}_{\Gamma^*}^\infty : \|{}^*\widehat{f}\|_{\Gamma^*} < \infty\} \tag{5.8$'$}$$

and

$$\underline{\widetilde{\mathcal{M}}}_\Gamma^\infty = \widetilde{\mathcal{M}}_\Gamma^\infty \times \widetilde{\mathcal{M}}_{\Gamma^*}^\infty.$$

THEOREM 5.9. *Let Γ be a Jc in $\overline{\mathbb{C}}$. Then $(\widetilde{\mathcal{M}}_\Gamma^\infty, \|\cdot\|_\Gamma)$ is a Banach space.*

Proof. Assume that $\widehat{f}_n \in \widetilde{\mathcal{M}}_\Gamma^\infty$, $n = 1, 2, \ldots$, is a sequence such that $\|\widehat{f}_n - \widehat{f}_m\|_\Gamma \to 0$ as $m, n \to \infty$. Let

$$\mathcal{A}_n = \{h \in A_\Gamma^\infty : \widehat{\delta}_h(\widehat{f}_n) \neq 0\}.$$

Obviously, $\mathcal{A}_n$ is countable and hence so is $\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{A}_n$. Let

$$\widetilde{\mathcal{M}}_\Gamma^\infty(\mathcal{A}) = \{\widehat{f} \in \widetilde{\mathcal{M}}_\Gamma^\infty : \widehat{\delta}_h(\widehat{f}) = 0 \text{ for } h \in A_\Gamma^\infty \setminus \mathcal{A}\}.$$

There is a bijection $\varphi : \mathbb{N} \to \mathcal{A}$, which includes an isometry J between $\widetilde{\mathcal{M}}_\Gamma^\infty(\mathcal{A})$ and the space l^1, given by

$$J(\widehat{f}) = (a_n)_{n\in\mathbb{N}},$$

where $a_n = \widehat{\delta}_{J(n)}(\widehat{f})\|\delta_{J(n)}\|_\Gamma$. Hence

$$\begin{aligned}\|J(\widehat{f})\|_{l^1} = \|(a_n)\|_{l^1} &= \sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty |\widehat{\delta}_{J(n)}(\widehat{f})|\, \|\delta_{J(n)}\|_\Gamma \\ &= \sum_{h\in\mathcal{A}} |\delta_h(\widehat{f})|\, \|\delta_h\|_\Gamma = \sum_{h\in A_\Gamma^\infty} |\widehat{\delta}_h(\widehat{f})|\, \|\widehat{\delta}_h\|_\Gamma = \|\widehat{f}\|_\Gamma.\end{aligned}$$

Therefore,

$$\|J(\widehat{f}_n) - J(\widehat{f}_m)\|_{l^1} = \|\widehat{f}_n - \widehat{f}_m\|_\Gamma \to 0 \quad \text{as } m, n \to \infty.$$

By the completeness of $(l^1, \|\cdot\|_{l^1})$, there exists $c = \{c_n\}_{n\in\mathbb{N}} \in l^1$ such that $\|J(\widehat{f}_n) - c\|_{l^1} \to 0$ as $n \to \infty$. Hence $\|\widehat{f}_n - J^{-1}(c)\|_\Gamma \to 0$ as $n \to \infty$, and $J^{-1}(c) \in \widetilde{\mathcal{M}}_\Gamma^\infty(\mathcal{A}) \subset \widetilde{\mathcal{M}}_\Gamma^\infty$.

Since $\widetilde{\mathcal{M}}_\Gamma^\infty$ is a subspace of the real linear space $\mathcal{M}_\Gamma^\infty$, the proof is complete.

Remark 5.5. Under the assumption of Theorem 5.9 one may show that $(\widetilde{\mathcal{M}}_{\Gamma^*}^\infty, \|\cdot\|_{\Gamma^*})$ is a Banach space. Hence so is $(\underline{\widetilde{\mathcal{M}}}_\Gamma^\infty, \|\cdot\|_\Gamma)$.

Let L_Γ be the linear quotient space obtained from $\underline{\widetilde{\mathcal{M}}}_\Gamma^\infty = \widetilde{\mathcal{M}}_\Gamma^\infty \times \widetilde{\mathcal{M}}_{\Gamma^*}^\infty$ when dividing by the equivalence relation induced by the Teichmüller equivalence relation from A_Γ^∞ to $\widetilde{\mathcal{M}}_\Gamma^\infty$. Obviously, this corresponds to the universal Teichmüller space $\mathbf{T}_\Gamma$. The transformation

$$\mathbf{T}_\Gamma \to \mathbf{L}_\Gamma \tag{5.9}$$

is called the *linearization* of $\mathbf{T}_\Gamma$.

Moreover, one may prove

THEOREM 5.10. *The space* $(\mathbf{L}_\Gamma, \|\cdot\|_\Gamma)$ *is a Banach space.*

Given $\widehat{f}, \widehat{g} \in \mathcal{M}_\Gamma^\infty$ and ${}^*\widehat{f}, {}^*\widehat{g} \in \mathcal{M}_{\Gamma^*}^\infty$, we define a scalar product as

$$\widehat{f} \circ \widehat{g} = \sum_{h \in A_\Gamma^\infty} \widehat{\delta}_h(\widehat{f})\widehat{\delta}_h(\widehat{g}) K_\Gamma(h)^2 \tag{5.10}$$

and

$$ {}^*\widehat{f} \circ {}^*\widehat{g} = \sum_{h \in A_{\Gamma^*}^\infty} \widehat{\delta}_h({}^*\widehat{f})\widehat{\delta}_h({}^*\widehat{g}) K_{\Gamma^*}(h)^2. \tag{5.10$'$}$$

Hence, the spaces $(\mathcal{M}_\Gamma^\infty, \circ)$ and $(\mathcal{M}_{\Gamma^*}^\infty, \circ)$ become *unitary spaces*. Observe now that

$$(\delta_h \circ \delta_h)^{1/2} = \|\delta_h\| = \begin{cases} K_\Gamma(h) & \text{if } h \in A_\Gamma^\infty, \\ K_{\Gamma^*}(h) & \text{if } h \in A_{\Gamma^*}^\infty. \end{cases}$$

The norms arising from the scalar multiplication are

$$|\!|\!| \widehat{f} |\!|\!|_\Gamma = \Big\{ \sum_{h \in A_\Gamma^\infty} |\widehat{\delta}_h(\widehat{f})|^2 K_\Gamma(h)^2 \Big\}^{1/2} \tag{5.11}$$

and

$$|\!|\!| {}^*\widehat{f} |\!|\!|_{\Gamma^*} = \Big\{ \sum_{h \in A_{\Gamma^*}^\infty} |\widehat{\delta}_h({}^*\widehat{f})|^2 K_{\Gamma^*}(h)^2 \Big\}^{1/2}. \tag{5.11$'$}$$

A simple observation shows that

$$|\!|\!| \widehat{f} |\!|\!|_\Gamma \le \|\widehat{f}\|_\Gamma \quad \text{and} \quad |\!|\!| {}^*\widehat{f} |\!|\!|_{\Gamma^*} \le \|{}^*\widehat{f}\|_{\Gamma^*} \tag{5.12}$$

for every $\widehat{f} \in \mathcal{M}_\Gamma^\infty$ and ${}^*\widehat{f} \in \mathcal{M}_{\Gamma^*}$.

A scalar multiplication in $\mathcal{M}_\Gamma^\infty$ can be defined as

$$\widehat{\boldsymbol{f}} \circ \widehat{\boldsymbol{g}} = (\widehat{f}, {}^*\widehat{f}) \circ (\widehat{g}, {}^*\widehat{g}) = \widehat{f} \circ \widehat{g} + {}^*\widehat{f} \circ {}^*\widehat{g}, \tag{5.13}$$

which leads to the corresponding norm in $\underline{\mathcal{M}}_\Gamma^\infty$:

$$|\!|\!| \widehat{\boldsymbol{f}} |\!|\!|_\Gamma = |\!|\!| \widehat{f} |\!|\!|_\Gamma + |\!|\!| {}^*\widehat{f} |\!|\!|_\Gamma^*. \tag{5.14}$$

Confining ourselves to L_Γ, we have

THEOREM 5.11. *The space* $(\widetilde{\underline{\mathcal{M}}}_\Gamma^\infty, \circ)$ *is a Hilbert space.*

Suppose now that Γ_1 and Γ_2 are arbitrary Jc's on $\overline{\mathbb{C}}$ with D_1, D_1^*, D_2, D_2^* the corresponding complementary domains. Moreover, let H and H^* be conformal mappings of D_1 onto D_2 and D_1^* onto D_2^*, respectively. One may lift the transformations S_{H,H_*}^{ij} to a linear mapping $\widehat{S}_{H,H_*}^{ij}$ of $\mathcal{M}_{\Gamma_1}$ onto $\mathcal{M}_{\Gamma_2}$ by setting

$$\widehat{S}_{H,H_*}^{ij}(\widehat{f}) = \sum_{h \in A_{\Gamma_1}} \widehat{\delta}_h(\widehat{f}) \delta_{S_{H,H_*}^{ij}(h)}, \qquad i,j = 1,2. \tag{5.15}$$

Following (4.1) in Section IV.4 , one may also define $\widehat{\mathbf{S}}_{H,H_*}$.

Suppose now that Γ_1 is the unit circle T, and let Γ_2 be denoted by Γ. Then the norms of $\widehat{S}_{H,H_*}^{ij}$, $i,j = 1,2$, can be defined as *sup-norms*. These norms are finite if and only if

K_Γ is. Hence, we may define a norm of $\widehat{\mathbf{S}}_{H,H_*}$. Under the assumptions of the previous section, we may define

$$\widehat{J}_{HH_*} = \widehat{S}^{22}_{H,H_*} \circ (\widehat{S}^{11}_{H,H_*})^{-1}, \tag{5.16}$$

$$\widehat{\boldsymbol{J}}_{HH_*} = (\widehat{S}_{H_*H}, \widehat{S}_{HH_*}) \tag{5.17}$$

and

$$\widehat{\boldsymbol{J}}_{\Gamma_1\Gamma_2} = (\widehat{S}_{H_*G}, \widehat{S}_{HG_*}). \tag{5.18}$$

In this way we arrived at the most reasonable arrangement of linear transformations between certain Banach spaces. It is worthwhile to note that the Banach–Mazur distance between L_{Γ_1} and L_{Γ_2} can be estimated. A Teichmüller-type norm may also be considered instead of the norms introduced in this section; see [Z12]. Unfortunately, this is not convenient when considering the sup-norms of these transformations.

Applying methods of functional analysis to these objects one may derive further results on the universal Teichmüller space of an oriented Jordan curve Γ on the Riemann sphere.

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