POLSKA AKADEMIA NAUK, INSTYTUT MATEMATYCZNY

DISSERTATIONES MATHEMATICAE

(ROZPRAWY MATEMATYCZNE)

CCCLXXIV

GEGHAM GEVORKYAN and ANNA KAMONT

On general Franklin systems

WARSZAWA 1998

Gegham Gevorkyan
Erevan State University
Department of Mathematics
Alex Manoukian St. 1
375049 Erevan, Armenia
E-mail: ggg@instmath.sci.am

Anna Kamont
Instytut Matematyczny PAN
ul. Abrahama 18
81-825 Sopot, Poland
E-mail: ak@impan.gda.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by
Publishing House of the Warsaw University of Technology
ul. Polna 50, 00-644 Warszawa

P R I N T E D I N P O L A N D

ISSN 0012-3862

CONTENTS

Abstract

We study general Franklin systems, i.e. systems of orthonormal piecewise linear functions corresponding to quasi-dyadic sequences of partitions of $[0, 1]$. The following problems are treated: unconditionality of the general Franklin basis in L^p, $1 < p < \infty$, and H^p, $1/2 < p \le 1$; equivalent conditions for the unconditional convergence of the Franklin series in L^p for $0 < p \le 1$; relation between Haar and Franklin series with identical coefficients; characterization of the spaces BMO and Lip(α), $0 < \alpha < 1$, in terms of the Fourier–Franklin coefficients.

1991 *Mathematics Subject Classification*: Primary 42C10.
Key words and phrases: Franklin system, unconditional basis, unconditional convergence, L^p spaces, real Hardy spaces, Hölder classes, BMO space.
Part of this work was done while G. Gevorkyan was visiting the Mathematical Institute of the Polish Academy of Sciences in Sopot.
Received 6.11.1997.

1. Introduction

The classical Franklin system, introduced by Ph. Franklin in 1928 ([16]), is a complete orthonormal system of continuous, piecewise linear functions (with dyadic knots), obtained by means of Gram–Schmidt orthogonalization of Schauder functions. Since then, it has been studied by many authors from different points of view. The basic properties of this system, including exponential estimates for the Franklin functions and L^p-stability on dyadic blocks, have been obtained by Z. Ciesielski in [5] and [6]. These properties have turned out to be an important tool in further investigations of the Franklin system. It is known that this system is a basis in $C[0,1]$ and L^p for $1 \leq p < \infty$; moreover, the coefficients of a function in the Franklin basis give a linear isomorphism between the space of functions satysfying the Hölder condition in L^p norm with exponent α, $0 < \alpha < 1+1/p$, $1 \leq p \leq \infty$, and the appropriate sequence space ([5], [6]). The unconditionality of this basis in L^p, $1 < p < \infty$, has been proved by S. V. Bochkarev in [1]. P. Wojtaszczyk has obtained a characterization of the BMO space in terms of the coefficients of a function in the Franklin basis and has proved that this system is an unconditional basis in the real Hardy space H^1 ([29]; see [8] for a simplified proof). The unconditionality of the Franklin basis in real Hardy spaces H^p, $1/2 < p \leq 1$, has been obtained by P. Sjölin and J. Strömberg ([27]); they have also proved that for this range of p, the H^p quasi-norm of $f \in H^p$ is equivalent to the L^p quasi-norm of the square function of the Franklin series with coefficients $a_n = (f, f_n)$. Z. Ciesielski and Sun-Yung A. Chang have proved that $f \in H^1$ iff its Fourier–Franklin series is unconditionally convergent in L^1 (cf. [4]). The equivalence of the Franklin system with the Haar system and higher order orthonormal spline systems in L^p and H^p spaces has also beeen studied (see [7], [11], [26], [27]), and results concerning the boundedness of the translation operator are known as well (see for example [10], [17], [25]).

One of the authors of this paper has studied the unconditional convergence of Franklin series in L^p for $0 < p \leq 1$ ([18]–[21]). He has proved that the unconditional convergence of Franklin series in L^p is equivalent to each of the following conditions:

(i) the square function of the series is in L^p, and

(ii) the maximal function of the series is in L^p.

Moreover, the Franklin series converges unconditionally in L^p iff the Haar series with identical coefficients converges unconditionally in L^p. Analogous results concerning the convergence of Franklin series in Lorentz spaces can be found in the recent paper [22].

It should be mentioned that Franklin's construction has been later generalized to higher order spline functions, periodic splines, splines on $\mathbb{R}$ and for functions of several variables and defined on smooth compact manifolds; here one should mention names like Z. Ciesielski, J. Domsta, T. Figiel, P. Oswald, J. Strömberg.

In the present paper we study some of the above problems for general Franklin systems corresponding to quasi-dyadic sequences of partitions of the interval $[0,1]$. By a quasi-dyadic sequence of partitions we mean a sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ such that $\mathcal{P}_0 = \{0,1\}$, $\mathcal{P}_j \subset \mathcal{P}_{j+1}$ and $\mathcal{P}_{j+1}$ is obtained form $\mathcal{P}_j$ by adding 2^j new points (one new point between two consecutive points of $\mathcal{P}_j$), and the corresponding Franklin system is a sequence of orthonormal piecewise linear functions with knots from the sequence of partitions $\mathcal{P}_j$ (see Section 2.2 and Definition 2.1 in Section 2.3 for the precise formulation).

It follows from [5] that if $\lim_{j\to\infty} |\mathcal{P}_j| = 0$ ($|\mathcal{P}_j|$ denotes the diameter of the partition $\mathcal{P}_j$), then the corresponding Franklin system is a basis in $C[0,1]$ and L^p, $1 \leq p < \infty$. The main results of the present paper are the following. We prove that if the sequence of partitions is weakly regular (see Definition 2.2 for the weak, strong and strong periodic regularity of a quasi-dyadic sequence of partitions), then the corresponding Franklin system is an unconditional basis in L^p for $1 < p < \infty$ (Theorem 3.1). Next we show that if the sequence of partitions is strongly regular, then the Franklin system is an unconditional basis in H^p for $1/2 < p \leq 1$ (Theorem 4.2). Moreover, we prove that strong regularity of the sequence of partitions is a necessary condition for the corresponding Franklin system to be a basis in H^p for $1/2 < p \leq 1$ (Theorems 5.1 and 5.3). The question of the unconditional convergence of the Franklin series in L^p for $0 < p \leq 1$ is studied as well. We prove (see Theorem 4.1) that the unconditional convergence of the Franklin series in L^p is equivalent to each of the following conditions:

(i) the square function of the series is in L^p, and
(ii) the maximal function of the series is in L^p.

For $1/2 < p \leq 1$, all these conditions are equivalent to the fact that the series under consideration is a Fourier–Franklin series of some element of H^p (see Theorem 4.2).

Further, we compare the Franklin series and the Haar series with identical coefficients. We prove that, under suitable regularity of the sequence of partitions, the square functions of the Franklin and Haar series are equivalent in L^p, $0 < p < \infty$ (Propositions 6.1, 6.2; the Haar system under consideration corresponds to the same sequence of partitions as the Franklin system — for a detailed description see Section 2.4). As a consequence, under the assumption of strong periodic regularity of the sequence of partitions, we deduce that the Haar and Franklin systems are equivalent bases in L^p, $1 < p < \infty$, while for $0 < p \leq 1$ the Franklin series converges unconditionally in L^p iff the Haar series with the same coefficients converges unconditionally in L^p. Moreover, we get the boundedness of the associated translation operator in L^p, $1 < p < \infty$ (for both the Haar and Franklin systems), and in H^p, $1/2 < p \leq 1$ (for the Franklin system) — see Corollary 6.4. Finally, we obtain a characterization of the spaces BMO and $\mathrm{Lip}(\alpha)$, $0 < \alpha < 1$, in terms of the coefficients in the Franklin system corresponding to a strongly regular sequence of partitions — Theorems 7.1 and 7.2; recall that the spaces BMO and $\mathrm{Lip}(\alpha)$ are the dual spaces to H^1 and H^p, $1/2 < p < 1$, $\alpha = 1/p - 1$, respectively (cf. for example [13]).

It should be mentioned that analogous properties of the Haar system corresponding to a quasi-dyadic sequence of partitions (i.e. the unconditionality of the Haar system in

L^p for $1 < p < \infty$ and equivalence of the square and the maximal functions of a Haar series in L^p for $0 < p \le 1$) follow from general results on martingale transforms (see [2], [3]). For the reader's convenience, we recall these properties of the Haar system in Section 2.4.

The paper is organized as follows. In Section 2 we define the Franklin system corresponding to a quasi-dyadic sequence of partitions and summarize the properties of the Franklin and Haar systems needed for the purpose of this paper. In Section 3, we prove that the Franklin system corresponding to a weakly regular sequence of partitions is an unconditional basis in L^p for $1 < p < \infty$. In Section 4, for the Franklin system corresponding to a strongly regular sequence of partitions, we discuss conditions equivalent to the unconditional convergence of the Franlin series in L^p for $0 < p \le 1$, and the unconditionality of this system in H^p, $1/2 < p \le 1$, is proved. In Section 5 we prove that strong regularity of the quasi-dyadic sequence of partitions is a necessary condition for the corresponding Franklin system to be a basis in H^p, $1/2 < p \le 1$. In Section 6 the Franklin and Haar series with identical coefficients are discussed and results concerning the translation operators are formulated. Finally, in Section 7 we give a characterization of the spaces BMO and $\mathrm{Lip}(\alpha)$ with $0 < \alpha < 1$ in terms of the Fourier–Franklin coefficients of a function — again for the Franklin system corresponding to a strongly regular sequence of partitions.

1.1. Notation

Function spaces and H^p spaces. For the reader's convenience, we recall the definitions of the spaces we work with.

By L^p, $0 < p < \infty$, we denote the Lebesgue space of real-valued functions defined on $[0,1]$ for which $\|f\|_p = (\int_0^1 |f(u)|^p\,du)^{1/p} < \infty$. If $0 < p < 1$, then $\|f\|_p$ is not a norm, but then the space L^p is equipped with the metric $\varrho(f,g) = \|f-g\|_p^p$. Recall that L^p with this metric is a complete space.

By $C[0,1]$ we denote the space of continuous functions on $[0,1]$, and for $0 < \alpha < 1$, by $\mathrm{Lip}(\alpha) \subset C[0,1]$ we mean the subspace of functions satisfying the Hölder condition with exponent α. It is well known that $\mathrm{Lip}(\alpha)$, with the norm

$$\|f\|_{\mathrm{Lip}(\alpha)} = \|f\|_\infty + \sup_{0\le x,y\le 1} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \tag{1.1}$$

is a non-separable Banach space.

We need also the BMO space, i.e. the space of functions of bounded mean oscillation. If $f \in L^1$, then $f \in \mathrm{BMO}$ iff

$$\|f\|_{\mathrm{BMO}} = |(f,1)| + \sup_{\Gamma} \left(\frac{1}{|\Gamma|} \int_{\Gamma} |f(u) - f_\Gamma|^2\,du \right)^{1/2} < \infty. \tag{1.2}$$

where the supremum is taken over all subintervals $\Gamma \subset [0,1]$ and $f_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma f(v)\,dv$ ($|\Gamma|$ denotes the length of the inerval Γ); for equivalent definitions of the BMO space we refer to [13]. It is known that BMO is a non-separable Banach space.

Next, we recall the definition of real Hardy spaces on $[0,1]$, denoted by H^p, $1/2 < p \le 1$. We use the atomic definition, introduced in [12], and developed in [13]; for more details, we refer to [13].

First, recall the definition of p-atoms: a function $a : [0,1] \to \mathbb{R}$ is called a *p-atom* $(1/2 < p \le 1)$ iff either $a = 1$, or there is an interval $\Gamma \subset [0,1]$ such that $\operatorname{supp} a \subset \Gamma$, $\sup|a| \le |\Gamma|^{-1/p}$ and $\int_0^1 a(u)\,du = 0$; note that if a is a p-atom, then $\|a\|_p \le 1$.

For $p = 1$, a function $f \in L^1$ is said to belong to H^1 iff there are 1-atoms a_j and real coefficients c_j, $j \in \mathbb{N}$, with $\sum_{j=1}^\infty |c_j| < \infty$, such that $f = \sum_{j=1}^\infty c_j a_j$. The norm in H^1 is defined as $\|f\|_{H^1} = \inf(\sum_{j=1}^\infty |c_j|)$, where the infimum is taken over all atomic decompositions of f; H^1 with this norm is a Banach space.

The space H^p with $1/2 < p < 1$ is defined as a subspace of the dual of $\mathrm{Lip}(\alpha)$ with $\alpha = 1/p - 1$: $f \in (\mathrm{Lip}(\alpha))^*$ is said to belong to H^p if it admits an atomic decomposition $f = \sum_{j=1}^\infty c_j a_j$, where a_j are p-atoms and the real coefficients c_j satisfy $\sum_{j=1}^\infty |c_j|^p < \infty$; it should be noted that this condition implies the convergence of the series $\sum_{j=1}^\infty c_j a_j$ in the norm of $(\mathrm{Lip}(\alpha))^*$. For $f \in H^p$ we put $\|f\|_{H^p} = \inf(\sum_n |c_n|^p)^{1/p}$, with the infimum taken over all atomic decompositions of f. For $p < 1$, $\|\cdot\|_{H^p}$ is not a norm, but $\varrho(f,g) = \|f-g\|_{H^p}^p$ is a metric on H^p, and H^p with this metric is complete; thus, $(H^p, \|\cdot\|_{H^p}^p)$ is a Fréchet space. Moreover, a linear functional L on H^p is continuous iff there is a constant C_L such that $|Lf| \le C_L \|f\|_{H^p}$ for all $f \in H^p$; similarly, a linear operator $T : H^p \to H^p$ is continuous iff it is bounded, i.e. there is a constant C_T such that $\|Tf\|_{H^p} \le C_T \|f\|_{H^p}$ for all $f \in H^p$.

The spaces BMO and $\mathrm{Lip}(\alpha)$ are identified with the duals of H^1 and H^p, $\alpha = 1/p - 1$, respectively; cf. [13], Theorem B. More precisely: if $g \in \mathrm{BMO}$, $f \in H^1$ and $f = \sum_{j=1}^\infty c_j a_j$ is an atomic decomposition of f, then the formula

$$L_g(f) = \lim_{n\to\infty} \sum_{j=1}^n c_j \int_0^1 g(u) a_j(u)\,du$$

defines a continuous linear functional on H^1, and each continuous linear functional on H^1 is of this form; moreover, the norm of L_g in $(H^1)^*$ is equivalent to $\|g\|_{\mathrm{BMO}}$.

The dual of H^p, $1/2 < p < 1$, is identified with $\mathrm{Lip}(\alpha)$, where $\alpha = 1/p - 1$: if $g \in \mathrm{Lip}(\alpha)$, $f \in H^p$ and $f = \sum_{j=1}^\infty c_j a_j$ is an atomic decomposition of f, then the formula

$$L_g(f) = \sum_{j=1}^\infty c_j \int_0^1 g(u) a_j(u)\,du$$

defines a continuous linear functional on H^p, and each continuous linear functional on H^p is of this form; moreover, the "norm" of L_g in $(H^p)^*$ is equivalent to $\|g\|_{\mathrm{Lip}(\alpha)}$.

To shorten the notation, if $f \in H^1$ and $g \in \mathrm{BMO}$, or $f \in H^p$ and $g \in \mathrm{Lip}(\alpha)$ with $\alpha = 1/p - 1$, we denote by (f, g) the value of the functional L_g on f.

In Sections 3 and 4, the unconditional convergence of Franklin series in spaces L^p, $1 < p < \infty$ and H^p, $1/2 < p \le 1$, is studied. The *unconditional convergence* of a series $\sum_{n=1}^\infty x_n$ in a metric space (X, ϱ) means that for each permutation σ of $\mathbb{N}$, the series $\sum_{n=1}^\infty x_{\sigma(n)}$ is convergent in (X, ϱ). It is known that if (X, ϱ) is a complete linear metric space, then the series $\sum_{n=1}^\infty x_n$ is unconditionally convergent if and only if the series $\sum_{n=1}^\infty \varepsilon_n x_n$ converges in (X, ϱ) for each choice of the coefficients $\varepsilon_n \in \{-1, 1\}$ (cf. for example [24], Theorem 1 in Chapter 1).

Though $\|\cdot\|_{H^p}$ is not a norm for $1/2 < p < 1$, we use for the H^p spaces the same terminology as for Banach spaces; in particular, by a *basis* in H^p we mean a sequence of elements $y_n \in H^p$, $n \in \mathbb{N}$, such that for each $f \in H^p$ there is a unique sequence of coefficients $b_n(f)$ such that $f = \sum_{n=1}^{\infty} b_n(f) y_n$, with the series convergent in the metric $\|\cdot - \cdot\|_{H^p}^p$, and a basis is called *unconditional* if for each $f \in H^p$, the series $\sum_{n=1}^{\infty} b_n(f) y_n$ is unconditionally convergent in H^p.

Quasi-dyadic sequences of partitions. Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions of $[0,1]$. By this we mean that $\mathcal{P}_0 = \{0,1\}$, and

$$\mathcal{P}_j = \{t_{j,i} : 0 \leq i \leq 2^j\}, \quad \mathcal{P}_j \subset \mathcal{P}_{j+1} \quad \text{for } j \geq 0,$$
$$0 = t_{j,0} < \ldots < t_{j,2^j} = 1, \quad t_{j+1,2k} = t_{j,k} \quad \text{for all } j \geq 0 \text{ and } k = 0, \ldots, 2^j,$$

i.e. $\mathcal{P}_{j+1}$ is obtained from $\mathcal{P}_j$ by adding one point in each interval $(t_{j,k-1}, t_{j,k})$, $k = 1, \ldots, 2^j$. For $j \geq 0$ and $1 \leq k \leq 2^j$, we put $I_{j,k} = [t_{j,k-1}, t_{j,k}]$, and $I_{j,k}^{\circ} = (t_{j,k-1}, t_{j,k})$ is the interior of $I_{j,k}$. Moreover, we let

$$\mathcal{I}_j = \{I_{j,k} : 1 \leq k \leq 2^j\} \quad \text{and} \quad \mathcal{I} = \bigcup_{j \geq 0} \mathcal{I}_j.$$

The elements of $\mathcal{I}_j$ are called *intervals of rank* (or *order*) j.

Maximal and square functions. For $f \in L^1$, $\mathcal{M}(f,\cdot)$ denotes the Hardy–Littlewood maximal function of f over $[0,1]$, and $\mathcal{M}^*(f,\cdot)$ is the maximal function corresponding to the sequence of quasi-dyadic partitions $\{\mathcal{P}_j : j \geq 0\}$, i.e.

$$\mathcal{M}^*(f,x) = \sup_{\substack{I \ni x \\ I \in \mathcal{I}}} \frac{1}{|I|} \int_I |f(u)|\, du.$$

It is well known that the operator $\mathcal{M}$ is of type (p,p) for $p > 1$ and of weak type $(1,1)$ (cf. for example Theorem 1.3.1 in [28]). Clearly, $\mathcal{M}^*$ has the same properties.

For a given sequence of quasi-dyadic partitions $\{\mathcal{P}_j : j \geq 0\}$, the corresponding Franklin system, as introduced in Definition 2.1 below, is denoted by $\{f_n : n \geq 0\}$.

For a sequence of real numbers $(a_n)_{n \geq 0}$, the square function P and the maximal function S of the Franklin series with coefficients $(a_n)_{n \geq 0}$ are defined by the respective formulae

$$P(\cdot) = \Big(\sum_{n=0}^{\infty} a_n^2 f_n(\cdot)^2\Big)^{1/2} \quad \text{and} \quad S(\cdot) = \sup_{m \geq 0} \Big|\sum_{n=0}^{m} a_n f_n(\cdot)\Big|.$$

Moreover, for $f \in L^p$, $1 \leq p \leq \infty$, or $f \in H^p$, $1/2 < p \leq 1$, we denote by Pf and Sf the square function and the maximal function of the Franklin series with coefficients $a_n = (f, f_n)$, i.e. a_n are the Fourier coefficients of f with respect to the Franklin system corresponding to $\{\mathcal{P}_j : j \geq 0\}$ (note that the f_n's are Lipschitz functions, and therefore they define continuous linear functionals on H^p, $1/2 < p \leq 1$).

Abbreviations. To shorten the notation, we use the following abbreviations. For $a, b \in \mathbb{R}$, we put $a \vee b = \max(a,b)$, $a \wedge b = \min(a,b)$. We write $A \sim B$ if there are positive constants C_1, C_2 such that $C_1 A \leq B \leq C_2 B$. The letters C, C_p, $C_{\gamma,p}$ etc. denote various constants, the value of which may vary from line to line; the subscripts indicate the parameters on which the particular constant depends.

By χ_A we denote the indicator of a set $A \subset [0,1]$, A^c is the complement of A in $[0,1]$ and $|A|$ denotes the Lebesgue measure of A.

2. Definition and properties of general Franklin systems

2.1. Piecewise linear functions. We start with recalling some known facts concerning piecewise linear functions, which are needed for the purpose of this paper.

Let $\pi = \{t_i : 0 \le i \le n\}$ be a partition of $[0,1]$, $0 = t_0 < \ldots < t_n = 1$; for later convenience, we also put $t_{-1} = 0$ and $t_{n+1} = 1$,

$$\lambda_i = t_i - t_{i-1} \quad \text{for} \quad 0 \le i \le n+1, \qquad \nu_i = \frac{\lambda_i + \lambda_{i+1}}{2} \quad \text{for } 0 \le i \le n.$$

Let $\mathcal{S}_\pi$ be the space of piecewise linear, continuous functions on $[0,1]$ with knots π. Moreover, let N_i, $0 \le i \le n$, be the B-splines of order 2 corresponding to the partition π, i.e. N_i is the unique function from $\mathcal{S}_\pi$ satisfying $N_i(t_j) = \delta_{i,j}$. Note that $\operatorname{supp} N_i = [t_{i-1}, t_{i+1}]$, $\sum_{i=0}^n N_i(t) = 1$ for each $t \in [0,1]$, $\|N_i\|_1 = \nu_i$ and

$$(N_i, N_j) = \begin{cases} (\lambda_i + \lambda_{i+1})/3 & \text{for } i = j, \\ \lambda_{i+1}/6 & \text{for } i = j-1, \\ \lambda_i/6 & \text{for } i = j+1, \\ 0 & \text{for } |i-j| > 1. \end{cases} \tag{2.1}$$

Moreover, any function $f \in \mathcal{S}_\pi$ can be written in the form $f = \sum_{i=0}^n a_i N_i$, where $a_i = f(t_i)$, so the functions $\{N_i : 0 \le i \le n\}$ are a basis in $\mathcal{S}_\pi$.

Let $G_\pi = [(N_i, N_j) : 0 \le i, j \le n]$ be the Gram matrix of the system $\{N_i : 0 \le i \le n\}$, and define $G_\pi^{-1} = A_\pi = [a_{i,j} : 0 \le i, j \le n]$. In Proposition 2.1 we list some estimates for $a_{i,j}$, which are needed later on.

Proposition 2.1. *Let $\pi = \{t_i : 0 \le i \le n\}$ be a partition of $[0,1]$, and let $A_\pi = [a_{i,j} : 0 \le i, j \le n]$ be the inverse of the Gram matrix G_π defined above. Then the entries of the matrix A_π satisfy the following conditions*:

$$\tfrac{3}{2} \le a_{i,i} \cdot \nu_i \le 2 \quad \textit{for } 0 \le i \le n, \tag{2.2}$$

$$a_{i,j} = a_{j,i} \quad \textit{and} \quad a_{i,j} = (-1)^{i+j} |a_{i,j}| \quad \textit{for } 0 \le i, j \le n, \tag{2.3}$$

$$2|a_{i-1,j}| \le |a_{i,j}| \quad \textit{for } 0 < i \le j \le n, \tag{2.4}$$

$$2|a_{i+1,j}| \le |a_{i,j}| \quad \textit{for } 0 \le j \le i < n, \tag{2.5}$$

$$|a_{i,j}| \le \frac{2}{2^{|i-j|}} \cdot \frac{1}{\max_{i \le k \le j} \nu_k}, \quad 0 \le i \le j \le n. \tag{2.6}$$

Moreover,

$$|a_{i,j}| \big(\tfrac{3}{2}\lambda_i + 2\lambda_{i+1}\big) \le |a_{i+1,j}| \lambda_{i+1} \le 2|a_{i,j}| (\lambda_i + \lambda_{i+1}) \quad \textit{for } 0 \le i < j, \tag{2.7}$$

$$|a_{i,j}| \big(2\lambda_i + \tfrac{3}{2}\lambda_{i+1}\big) \le |a_{i-1,j}| \lambda_i \le 2|a_{i,j}| (\lambda_i + \lambda_{i+1}) \quad \textit{for } j < i \le n. \tag{2.8}$$

Proof. Properties (2.2)–(2.6) can be found for example in [9] and [24], or they are straightforward consequences of estimates given there, so their proof is omitted.

To check (2.7), note that for $i < j$,

$$\lambda_i a_{i-1,j} + 2(\lambda_i + \lambda_{i+1}) a_{i,j} + \lambda_{i+1} a_{i+1,j} = 0.$$

This together with (2.3) gives

$$2(\lambda_i + \lambda_{i+1})|a_{i,j}| = \lambda_{i+1}|a_{i+1,j}| + \lambda_i |a_{i-1,j}| \geq \lambda_{i+1}|a_{i+1,j}|.$$

On the other hand, applying (2.4) we get

$$2(\lambda_i + \lambda_{i+1})|a_{i,j}| = \lambda_{i+1}|a_{i+1,j}| + \lambda_i |a_{i-1,j}| \leq \lambda_{i+1}|a_{i+1,j}| + \tfrac{1}{2}\lambda_i |a_{i,j}|,$$

which gives the remaining inequality in (2.7).

Inequalities (2.8) are obtained analogously.

It should be noted that, for fixed j, formulae (2.4), (2.5) and the fact that $a_{i,j}$ and $a_{i+1,j}$ have opposite signs follow just from the system of equations $\sum_{i=0}^n a_{i,j}(N_i, N_k) = \delta_{j,k}$, $k = 0, \ldots, n$, and these properties are sufficient to get (2.7) and (2.8). We refer to this fact in the proof of Lemma 5.2. ∎

In the sequel, we need the L^p-stability of the functions N_i, which can be checked by straightforward calculation:

Proposition 2.2. *Let π be a partition of* $[0,1]$, $f \in \mathcal{S}_\pi$, $f = \sum_{i=0}^n a_i N_i$. *Then for all* $1 \leq p \leq \infty$,

$$\left(\frac{1}{p+1}\right)^{1/p} \cdot \Big(\sum_{i=0}^n |a_i|^p \nu_i\Big)^{1/p} \leq \|f\|_p \leq \Big(\sum_{i=0}^n |a_i|^p \nu_i\Big)^{1/p}.$$

2.2. Franklin functions. Let $(\pi, \widetilde{\pi})$ be a pair of partitions of $[0,1]$ such that $\widetilde{\pi} \subset \pi$ and π is obtained from $\widetilde{\pi}$ by adding one knot τ, $\tau \neq 0, 1$. Then there is a unique, up to sign, function $\varphi \in \mathcal{S}_\pi$ such that $\varphi \perp \mathcal{S}_{\widetilde{\pi}}$ (in L^2) and $\|\varphi\|_2 = 1$; the sign of φ is chosen in such a way that $\varphi(\tau) > 0$.

The function φ is called *the Franklin function corresponding to the pair of partitions* $(\pi, \widetilde{\pi})$.

Let us formulate some properties of the Franklin function.

Let $\pi = \{t_i : 0 \leq i \leq n\}$; as $\tau \in \pi$, we have $\tau = t_k$ for some $0 < k < n$. Then $\widetilde{\pi} = \{t_i : 0 \leq i \leq n,\ i \neq k\}$, and for convenience we denote by $\widetilde{N}_i$, $i \neq k$, the B-splines corresponding to $\widetilde{\pi}$. Observe that

$$\widetilde{N}_i = N_i \quad \text{for } i < k-1 \text{ and } i > k+1,$$

$$\widetilde{N}_{k-1} = N_{k-1} + \frac{\lambda_{k+1}}{\lambda_k + \lambda_{k+1}} N_k \quad \text{and} \quad \widetilde{N}_{k+1} = N_{k+1} + \frac{\lambda_k}{\lambda_k + \lambda_{k+1}} N_k.$$

Define

$$w_i = -\frac{\lambda_{k+1}}{\lambda_k + \lambda_{k+1}} a_{i,k-1} + a_{i,k} - \frac{\lambda_k}{\lambda_k + \lambda_{k+1}} a_{i,k+1}, \tag{2.9}$$

where $A_\pi = [a_{i,j} : 0 \leq i, j \leq n]$ is the inverse of the Gram matrix G_π, and introduce the function

$$g = \sum_{i=0}^n w_i N_i. \tag{2.10}$$

Clearly, $g \in \mathcal{S}_\pi$, and it can be checked by straightforward calculation that $(g, \widetilde{N}_i) = 0$ for all $i \neq k$, whence $g \perp \mathcal{S}_{\widetilde{\pi}}$. Formula (2.9) and properties of $a_{i,j}$ from Proposition 2.1 imply the following:

$$|w_i| = \frac{\lambda_{k+1}}{\lambda_k + \lambda_{k+1}}|a_{i,k-1}| + |a_{i,k}| + \frac{\lambda_k}{\lambda_k + \lambda_{k+1}}|a_{i,k+1}|, \tag{2.11}$$

$$w_i = (-1)^{k-i}|w_i|, \quad \text{so in particular} \quad g(\tau) = w_k > 0, \tag{2.12}$$

$$a_{k,k} \le w_k \le \tfrac{3}{2}a_{k,k}, \quad \tfrac{1}{2}a_{k+l,k+l} \le |w_{k+l}| \le \tfrac{3}{2}a_{k+l,k+l} \quad \text{for } l = \pm 1 \tag{2.13}$$

(to check (2.13), note that

$$|a_{j,k}| = \frac{\lambda_k|a_{j,k-1}| + \lambda_{k+1}|a_{j,k+1}|}{2(\lambda_k + \lambda_{k+1})}$$

for $j \neq k$), which gives

$$\frac{3}{4}\,\frac{1}{\nu_i} \le |w_i| \le 3\frac{1}{\nu_i} \quad \text{for } i = k-1, k, k+1. \tag{2.14}$$

Thus, we have $\varphi = g/\|g\|_2$. Moreover, these formulae for $|w_i|$ (cf. (2.11), (2.14)), decay of $|a_{i,j}|$ from Proposition 2.1 (cf. (2.3)–(2.6)) and L^p stability of B-splines from Proposition 2.2 imply

$$\tfrac{1}{8}\mu^{1-1/p} \le \|g\|_p \le 15\mu^{1-1/p} \quad \text{for } 1 \le p \le \infty, \tag{2.15}$$

where $\mu = 1/\nu_{k-1} + 1/\nu_k + 1/\nu_{k+1}$.

As a consequence of formulae (2.9)–(2.15) and Proposition 2.1, we get the following pointwise estimates for the Franklin function φ:

Proposition 2.3 (Pointwise estimates for the Franklin function). *Let $(\pi, \widetilde{\pi})$ be a pair of partitions as above, and let φ be the Franklin function corresponding to $(\pi, \widetilde{\pi})$. Define $\xi_i = \varphi(t_i)$, i.e. $\varphi = \sum_{i=0}^n \xi_i N_i$, and $\mu = 1/\nu_{k-1} + 1/\nu_k + 1/\nu_{k+1}$. Then*

$$\tfrac{1}{120}\mu^{1/2-1/p} \le \|\varphi\|_p \le 120\mu^{1/2-1/p} \quad \text{for } 1 \le p \le \infty,$$
$$\xi_i = (-1)^{i+k}|\xi_i|, \quad i = 0, \dots, n,$$
$$\frac{1}{20}\,\frac{\mu^{-1/2}}{\nu_i} \le |\xi_i| \le 24\frac{\mu^{-1/2}}{\nu_i}, \quad i = k-1, k, k+1;$$

for $i \le k-1$,

$$|\xi_{i-1}| \le \frac{|\xi_i|}{2}, \quad |\xi_i| \le \frac{48}{2^{|i-k|}}\,\frac{\mu^{-1/2}}{\max_{i \le l \le k-1} \nu_l} \le \frac{96|i-k|}{2^{|i-k|}}\,\frac{\mu^{-1/2}}{t_k - t_{i-1}},$$
$$|\xi_{i-1}|\bigl(\tfrac{3}{2}\lambda_{i-1} + 2\lambda_i\bigr) \le |\xi_i|\lambda_i \le 2|\xi_{i-1}|(\lambda_{i-1} + \lambda_i);$$

and for $i \ge k+1$,

$$|\xi_{i+1}| \le \frac{|\xi_i|}{2}, \quad |\xi_i| \le \frac{48}{2^{|i-k|}}\,\frac{\mu^{-1/2}}{\max_{k+1 \le l \le i} \nu_l} \le \frac{96|i-k|}{2^{|i-k|}}\,\frac{\mu^{-1/2}}{t_{i+1} - t_k},$$
$$|\xi_{i+1}|\bigl(2\lambda_{i+1} + \tfrac{3}{2}\lambda_{i+2}\bigr) \le |\xi_i|\lambda_{i+1} \le 2|\xi_{i+1}|(\lambda_{i+1} + \lambda_{i+2}).$$

The pointwise estimates from Proposition 2.3 imply the following decay of norms of the Franklin function on the intervals from the partition π:

Proposition 2.4 (Decay of norms of the Franklin function on intervals). *Let $(\pi, \widetilde{\pi})$ be a pair of partitions as above, and let φ be the Franklin function corresponding to $(\pi, \widetilde{\pi})$. Then for $i < k-1$ we have*

$$3(\sqrt{2}-1) \int\limits_{t_{i-1}}^{t_i} |\varphi(u)|\,du \le \int\limits_{t_i}^{t_{i+1}} |\varphi(u)|\,du,$$

$$2 \max_{t_{i-1} \le t \le t_i} |\varphi(t)| \le \max_{t_i \le t \le t_{i+1}} |\varphi(t)|,$$

and for $i > k+1$,

$$3(\sqrt{2}-1) \int\limits_{t_i}^{t_{i+1}} |\varphi(u)|\,du \le \int\limits_{t_{i-1}}^{t_i} |\varphi(u)|\,du,$$

$$2 \max_{t_i \le t \le t_{i+1}} |\varphi(t)| \le \max_{t_{i-1} \le t \le t_i} |\varphi(t)|.$$

Proof. Consider the case $i < k-1$. By Proposition 2.3, $\xi_i = \varphi(t_i)$ and $\xi_{i+1} = \varphi(t_{i+1})$ have opposite signs and $|\xi_i| \le \frac{1}{2}|\xi_{i+1}|$, which implies

$$m_{i+1} = \int\limits_{t_i}^{t_{i+1}} |\varphi(u)|\,du = \frac{\lambda_{i+1}}{2} \frac{|\xi_i|^2 + |\xi_{i+1}|^2}{|\xi_i| + |\xi_{i+1}|} \ge (\sqrt{2}-1)\lambda_{i+1}|\xi_{i+1}|$$

and

$$m_i = \int\limits_{t_{i-1}}^{t_i} |\varphi(u)|\,du \le \frac{\lambda_i |\xi_i|}{2}.$$

Therefore, using the estimates from Proposition 2.3 we get

$$\frac{m_{i+1}}{m_i} \ge 2(\sqrt{2}-1)\frac{\lambda_{i+1}|\xi_{i+1}|}{\lambda_i|\xi_i|} \ge (\sqrt{2}-1)\frac{3\lambda_i + 4\lambda_{i+1}}{\lambda_i} \ge 3(\sqrt{2}-1).$$

The bound for $\max_{t_{i-1} \le t \le t_i} |\varphi(t)|$ is a straightforward consequence of Proposition 2.3. The case $i > k+1$ is treated analogously. ∎

Remark. The constants appearing in Propositions 2.3 and 2.4 are not sharp. Moreover, estimates analogous to those from Proposition 2.4 (i.e. with constants independent of the pair of partitions $(\pi, \widetilde{\pi})$) can be obtained for integrals of $|\varphi(\cdot)|^p$ for $1 < p < \infty$. However, for $0 < p < 1$, estimates of this type do not hold.

2.3. Sequences of partitions and Franklin functions. Let $\{\mathcal{P}_j : j \ge 0\}$ be a quasi-dyadic sequence of partitions, with

$$\mathcal{P}_j = \{t_{j,i} : 0 \le i \le 2^j\}.$$

Define $\pi_1 = \mathcal{P}_0$, and for $n \ge 2$, $n = 2^j + k$ with $1 \le k \le 2^j$,

$$\pi_n = \mathcal{P}_j \cup \{t_{j+1,2l-1} : 1 \le l \le k\}, \tag{2.16}$$

$$t_n = t_{j+1,2k-1}, \quad \{n\} = [t_{j+1,2k-2}, t_{j+1,2k}] = [t_{j,k-1}, t_{j,k}], \tag{2.17}$$

with $\{0\} = \{1\} = [0,1]$,

$$\{n_-\} = [t_{j+1,2k-3}, t_{j+1,2k-1}], \quad \{n_+\} = [t_{j+1,2k-1}, t_{j+1,2k+2}], \tag{2.18}$$

where for convenience we put $t_{j,-1} = 0$ and $t_{j,2^j+1} = 1$.

Note that π_n is obtained from π_{n-1} ($n \geq 2$) by adding exactly one point t_n. The Franklin system corresponding to the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ is defined as follows:

DEFINITION 2.1. Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions, and for $n \geq 1$, let π_n be as in (2.16). Then *the Franklin system* $\{f_n : n \geq 0\}$ *corresponding to* $\{\mathcal{P}_j : j \geq 0\}$ is the following family of functions:

$$f_0 = 1, \quad f_1(t) = 2\sqrt{3}(t - 1/2),$$

and for $n \geq 2$, f_n is the Franklin function corresponding to (π_n, π_{n-1}).

Note that this definition guarantees $\|f_n\|_2 = 1$ and $(f_n, f_m) = 0$ for $n \neq m$.

For a partition π, denote by Q_π the orthogonal (in L^2) projection onto $\mathcal{S}_\pi$. Note that Q_π is simultaneously a continuous linear operator on L^p, $2 \leq p \leq \infty$, and can be uniquely extended to a continous linear operator on L^p, $1 \leq p < 2$, and H^p, $1/2 < p \leq 1$; these extensions are denoted by Q_π as well. Clearly,

$$Q_{\pi_n} f = \sum_{i=0}^{n} (f, f_i) f_i.$$

Next, we list the properties of the projections Q_π which are needed for our purpose. For the proofs, we refer to [5] and [9].

THEOREM 2.5. (i) *For any partition* π *and* $f \in L^p$, $1 \leq p \leq \infty$,

$$\|Q_\pi f\|_p \leq 3\|f\|_p.$$

Moreover, *for each* $f \in L^1$ *we have*

$$|Q_\pi f(\cdot)| \leq 64\, \mathcal{M}(f, \cdot).$$

(ii) *Let* $\{\mathcal{P}_j : j \geq 0\}$ *be a quasi-dyadic sequence of partitions satisfying* $\lim_{n\to\infty} |\pi_n| = 0$. *Then for all* $f \in L^p$ *with* $1 \leq p < \infty$, *or* $f \in C[0,1]$ *for* $p = \infty$ *we have* $\lim_{n\to\infty} \|f - Q_{\pi_n} f\|_p = 0$. *Consequently*, *the Franklin system* $\{f_n : n \geq 0\}$ *corresponding to* $\{\mathcal{P}_j : j \geq 0\}$ *is a basis in* L^p, $1 \leq p < \infty$, *and* $C[0,1]$.

Moreover, *if* $f \in L^1$ *and* u *is a weak Lebesgue point of* f, *then* $f(u) = \lim_{n\to\infty} Q_{\pi_n} f(u)$.

2.3.1. *Regularity of sequences of partitions*. Recall that for a quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$,

$$I_{j,k} = [t_{j,k-1}, t_{j,k}], \quad \lambda_{j,k} = |I_{j,k}| = t_{j,k} - t_{j,k-1}, \quad k = 1, \ldots, 2^j.$$

When we pass from $\mathcal{P}_j$ to $\mathcal{P}_{j+1}$, then the interval $I_{j,k}$ is split into two intervals $I_{j+1,2k-1}$ and $I_{j+1,2k}$ with disjoint interiors, i.e. we have

$$I_{j,k} = I_{j+1,2k-1} \cup I_{j+1,2k}, \qquad \lambda_{j,k} = \lambda_{j+1,2k-1} + \lambda_{j+1,2k}.$$

Now, we introduce the weak, strong and strong periodic regularity of a quasi-dyadic sequence of partitions.

DEFINITION 2.2. Let $\gamma \geq 1$ and let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions of $[0,1]$.

(i) We say that the sequence $\{\mathcal{P}_j : j \ge 0\}$ satisfies the *weak regularity* condition with parameter γ if for all $j \ge 1$ and $k = 1, \dots, 2^{j-1}$,

$$\frac{1}{\gamma} \le \frac{\lambda_{j,2k-1}}{\lambda_{j,2k}} \le \gamma.$$

(ii) We say that the sequence $\{\mathcal{P}_j : j \ge 0\}$ satisfies the *strong regularity* condition with parameter γ if for all $j \ge 0$ and $k = 1, \dots, 2^j - 1$,

$$\frac{1}{\gamma} \le \frac{\lambda_{j,k+1}}{\lambda_{j,k}} \le \gamma.$$

(iii) We say that the sequence $\{\mathcal{P}_j : j \ge 0\}$ satisfies the *strong periodic regularity* condition with parameter γ if for all $j \ge 0$ and $k = 1, \dots, 2^j$,

$$\frac{1}{\gamma} \le \frac{\lambda_{j,k+1}}{\lambda_{j,k}} \le \gamma,$$

where by definition $\lambda_{j,2^j+1} = \lambda_{j,1}$.

Clearly, the sequence $\mathcal{P}_j = \{k/2^j : 0 \le k \le 2^j\}$ of dyadic partitions satisfies the strong periodic regularity condition with $\gamma = 1$. Another example of a strongly periodically regular quasi-dyadic sequence of partitions is the sequence of Chebyshev knots on $[0,1]$, i.e. with $t_{j,k} = (1 + \cos((2^j - k)\pi/2^j))/2 = \sin^2(k\pi/2^{j+1})$. The best approximation by spline functions with these knots appears to be closely related with the Ditzian–Totik modulus of smoothness with the step-weight function $w(x) = \sqrt{x(1-x)}$ and the best approximation by algebraic polynomials (see [23] for details and more examples).

In the sequel, the following estimates for the length of the intervals $I_{j,k}$ are used frequently:

Proposition 2.6. *Let $\gamma \ge 1$ and let $\{\mathcal{P}_j : j \ge 0\}$ be a quasi-dyadic sequence of partitions of* $[0,1]$.

(i) *Let the sequence $\{\mathcal{P}_j : j \ge 0\}$ satisfy the weak regularity condition with parameter γ. Then for all $j \ge 0$ and $k = 1, \dots, 2^j$,*

$$\frac{1}{\gamma+1}|I_{j,k}| \le |I_{j+1,2k-1}|,\ |I_{j+1,2k}| \le \frac{\gamma}{\gamma+1}|I_{j,k}|.$$

Consequently, if for some j,k,m,l we have $I_{m,l} \subset I_{j,k}$, then

$$\left(\frac{1}{\gamma+1}\right)^{m-j}|I_{j,k}| \le |I_{m,l}| \le \left(\frac{\gamma}{\gamma+1}\right)^{m-j}|I_{j,k}|.$$

(ii) *Let the sequence $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and $\alpha_\gamma = \log_2 \gamma$. Then for all $j \ge 0$ and $1 \le k,l \le 2^j$,*

$$\gamma^{-2}(|k-l|+1)^{-\alpha_\gamma}|I_{j,k}| \le |I_{j,l}| \le \gamma^2(|k-l|+1)^{\alpha_\gamma}|I_{j,k}|.$$

Proof. The inequalities from (i) are straightforward consequences of Definition 2.2(i).

To check (ii), let $1 \le k,l \le 2^j$, $k \ne l$, and choose μ, $0 \le \mu \le j-1$, such that $2^\mu \le |k-l| < 2^{\mu+1}$. Let a,b be such that

$$I_{j,k} \subset I_{j-\mu,a} \quad \text{and} \quad I_{j,l} \subset I_{j-\mu,b}.$$

Then $|a-b| \le 2$, and by strong regularity $\gamma^{-2}|I_{j-\mu,b}| \le |I_{j-\mu,a}| \le \gamma^2|I_{j-\mu,b}|$, so applying (i) we obtain

$$|I_{j,l}| \le \left(\frac{\gamma}{\gamma+1}\right)^{\mu}|I_{j-\mu,b}| \le \gamma^2\left(\frac{\gamma}{\gamma+1}\right)^{\mu}|I_{j-\mu,a}| \le \gamma^{2+\mu}|I_{j,k}|,$$

and we get (ii) by the choice of μ. ■

2.4. Sequences of partitions and general Haar systems. For a partition $\pi = \{t_i : 0 \le i \le m\}$ of $[0,1]$, let $\mathcal{H}_\pi$ be the space of functions constant on each interval $[t_{i-1}, t_i)$, $1 \le i \le m$, and continuous at 1. For a pair of partitions $(\pi, \widetilde{\pi})$ such that π is obtained from $\widetilde{\pi}$ by adding one point, there is a unique (up to sign) function $h \in \mathcal{H}_\pi$ with $h \perp \mathcal{H}_{\widetilde{\pi}}$ and $\|h\|_2 = 1$; it is called the *Haar function corresponding to* $(\pi, \widetilde{\pi})$.

Now, let $\{\mathcal{P}_j : j \ge 0\}$ be a quasi-dyadic sequence of partitions of $[0,1]$, and let the partitions π_n, $n \ge 1$, be as defined in (2.16). The Haar system $\{h_n : n \ge 1\}$ corresponding to $\{\mathcal{P}_j : j \ge 0\}$ is defined as follows: $h_1 = 1$, and for $n \ge 2$, h_n is the Haar function corresponding to (π_n, π_{n-1}). It can be calculated that for $n = 2^j + k$,

$$h_n(u) = \begin{cases} \sqrt{\dfrac{\lambda_{j+1,2k}}{\lambda_{j+1,2k-1}}}\dfrac{1}{\sqrt{\lambda_{j,k}}} & \text{for } u \in [t_{j+1,2k-2}, t_{j+1,2k-1}), \\ -\sqrt{\dfrac{\lambda_{j+1,2k-1}}{\lambda_{j+1,2k}}}\dfrac{1}{\sqrt{\lambda_{j,k}}} & \text{for } u \in [t_{j+1,2k-1}, t_{j+1,2k}), \\ 0 & \text{otherwise,} \end{cases} \tag{2.19}$$

and for $1 \le p \le \infty$,

$$\frac{1}{\sqrt{2}}(\lambda_{j+1,2k-1} \wedge \lambda_{j+1,2k})^{1/p-1/2} \le \|h_n\|_p \le 2(\lambda_{j+1,2k-1} \wedge \lambda_{j+1,2k})^{1/p-1/2}. \tag{2.20}$$

Therefore, if the sequence $\{\mathcal{P}_j : j \ge 0\}$ of partitions is weakly regular with parameter γ, then for $1 \le p \le \infty$,

$$\frac{1}{\sqrt{2(1+\gamma)}}|\{n\}|^{1/p-1/2} \le \|h_n\|_p \le 2\sqrt{1+\gamma}|\{n\}|^{1/p-1/2} \tag{2.21}$$

and moreover,

$$\frac{1}{\sqrt{\gamma}}|\{n\}|^{-1/2} \le |h_n(u)| \le \sqrt{\gamma}|\{n\}|^{-1/2} \quad \text{on } \{n\}. \tag{2.22}$$

Consider a quasi-dyadic sequence $\{\mathcal{P}_j : j \ge 0\}$ of partitions and the corresponding Haar system $\{h_n : n \ge 1\}$. Note that for any sequence $(a_n)_{n\ge1}$ of real coefficients, the sequences $\{S_m^H : m \ge 1\}$ and $\{S_{2^j}^H : j \ge 0\}$, where $S_m^H = \sum_{n=1}^m a_n h_n$, are martingales with respect to the σ-fields generated by the appropriate Haar functions. Clearly, if $a_n = (f, h_n)$ for all $n \in \mathbb{N}$ and some $f \in L^p$, $1 \le p < \infty$, then the S_m^H's are partial sums of the Fourier–Haar series of f, and for $f \in L^2$, S_m^H is the orthogonal projection of f onto the space spanned by $h_1, \dots, h_m$. Therefore, the results concerning the unconditional convergence of the Haar series follow from known results from martingale theory. The properties of Haar series which are needed later on (cf. Section 6) are summarized in Propositions 2.7 and 2.8. To formulate these propositions, we introduce the following notation: for a sequence $(a_n)_{n\geq 1}$ of real numbers, P^H and S^H denote the square and maximal functions of the corresponding Haar series, i.e.

$$P^H(\cdot) = \Big(\sum_{n=1}^{\infty} a_n^2 h_n(\cdot)^2\Big)^{1/2}, \quad S^H(\cdot) = \sup_{m\geq 1}\Big|\sum_{n=1}^{m} a_n h_n(\cdot)\Big|.$$

Moreover, for $f \in L^1$, we denote by $P^H f$, $S^H f$ the functions defined by the above formulae with the coefficients $a_n = (f, h_n)$.

If $\{\mathcal{P}_j : j \geq 0\}$ is a quasi-dyadic sequence of partitions of $[0,1]$ such that $|\mathcal{P}_j| \to 0$ as $j \to \infty$, then the corresponding Haar system is a basis in L^p for all $1 \leq p < \infty$. Combining this with D. L. Burkholder's result concerning martingales (cf. [2], Theorem 9), and Doob's inequality for submartingales (cf. for example [14], Theorem 3.4 in Chapter VII), we have

Proposition 2.7. *Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions of $[0,1]$ such that $|\mathcal{P}_j| \to 0$ as $j \to \infty$, and let $\{h_n : n \geq 1\}$ be the corresponding Haar system. Then $\{h_n : n \geq 1\}$ is a basis in L^p for all $1 \leq p < \infty$. This basis is unconditional in each L^p for $1 < p < \infty$, and for each p, $1 < p < \infty$, and $f \in L^p$,*

$$\|f\|_p \sim \|P^H f\|_p \sim \|S^H f\|_p,$$

with implied constants depending on p only.

Under the additional assumption of weak regularity of the sequence of partitions under consideration, applying Theorem 5.1 from [3], we get

Proposition 2.8. *Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions of $[0,1]$ satisfying the weak regularity condition with parameter γ, and let $\{h_n : n \geq 1\}$ be the corresponding Haar system. Then, for each sequence $(a_n)_{n\geq 1}$ of real coefficients and p, $0 < p \leq 1$, the following conditions are equivalent*:

(1) $P^H(\cdot) \in L^p$,
(2) $S^H(\cdot) \in L^p$,
(3) *the series $\sum_{n=1}^{\infty} a_n h_n$ converges unconditionally in L^p.*

2.5. Technical lemmas. For later reference, we present the formulation of Propositions 2.3 and 2.4 for sequences of Franklin functions.

Proposition 2.9. *Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions satisfying the weak regularity condition with parameter γ and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. For $n \geq 2$, $n = 2^j + k$ with $1 \leq k \leq 2^j$, let t_n and $\{n\}$ be as in* (2.17). *Then there is a constant C_γ, depending only on γ, such that*

$$\frac{1}{C_\gamma}|\{n\}|^{1/p-1/2} \leq \|f_n\|_p \leq C_\gamma |\{n\}|^{1/p-1/2} \quad \text{for } 1 \leq p \leq \infty, \tag{2.23}$$

$$\frac{1}{C_\gamma}|\{n\}|^{-1/2} \leq f_n(t_n) \leq C_\gamma |\{n\}|^{-1/2}, \tag{2.24}$$

$$|f_n(t_{j+1,2k-2})| \leq C_\gamma |\{n\}|^{-1/2}, \quad |f_n(t_{j+1,2k})| \leq C_\gamma |\{n\}|^{-1/2}; \tag{2.25}$$

for $i \le 2k-2$,

$$f_n(t_{j+1,i}) = (-1)^{2k-1-i}|f_n(t_{j+1,i})|, \qquad |f_n(t_{j+1,i-1})| \le \tfrac{1}{2}|f_n(t_{j+1,i})|, \tag{2.26}$$

$$|f_n(t_{j+1,i})| \le C_\gamma \frac{|\{n\}|^{-1/2}}{2^{|2k-1-i|}} \le C_\gamma \frac{|\{n\}|^{-1/2}}{2^{|k-i/2|}}, \tag{2.27}$$

$$|f_n(t_{j+1,i})| \le C_\gamma \frac{|2k-1-i|}{2^{|2k-1-i|}} \frac{|\{n\}|^{1/2}}{t_{j+1,2k-1} - t_{j+1,i-1}}, \tag{2.28}$$

$$\begin{aligned} \int_0^{t_{j+1,i}} |f_n(u)|\,du &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} \int_{t_{j+1,i-1}}^{t_{j+1,i}} |f_n(u)|\,du \\ &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} (3\sqrt{2}-3)^{-|i-2k+2|} \|f_n\|_1 \end{aligned} \tag{2.29}$$

and for $i \ge 2k$, $i = 2l$,

$$f_n(t_{j+1,i}) = (-1)^{k-1-i/2}|f_n(t_{j+1,i})|, \qquad |f_n(t_{j+1,i+2})| \le \tfrac{1}{2}|f_n(t_{j+1,i})|, \tag{2.30}$$

$$|f_n(t_{j+1,i})| \le C_\gamma \frac{|\{n\}|^{-1/2}}{2^{|k-i/2|}}, \tag{2.31}$$

$$|f_n(t_{j+1,i})| \le C_\gamma \frac{|k-i/2-1|}{2^{|k-i/2|}} \frac{|\{n\}|^{1/2}}{t_{j+1,i+2} - t_{j+1,2k-1}}, \tag{2.32}$$

$$\begin{aligned} \int_{t_{j+1,i}}^{1} |f_n(u)|\,du &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} \int_{t_{j+1,i}}^{t_{j+1,i+2}} |f_n(u)|\,du \\ &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} (3\sqrt{2}-3)^{-|k-i/2|} \|f_n\|_1. \end{aligned} \tag{2.33}$$

Proof. By definition, f_n is the Franklin function corresponding to (π_n, π_{n-1}). Denote by μ_n the number μ from Proposition 2.3 chosen for $\pi = \pi_n$ and $\widetilde{\pi} = \pi_{n-1}$; then we have

$$\frac{\mu_n}{2} = \frac{1}{|\{n_-\}|} + \frac{1}{|\{n\}|} + \frac{1}{|\{n_+\}|},$$

with $\{n_-\}$ and $\{n_+\}$ given by (2.18). Note that formulae (2.18) and the definition of weak regularity imply $|\{n_-\}| \ge |\{n\}|/(\gamma+1)$ and $|\{n_+\}| \ge |\{n\}|/(\gamma+1)$. Thus

$$\frac{2}{|\{n\}|} \le \mu_n \le \frac{4\gamma+6}{|\{n\}|},$$

and now Proposition 2.9 is a consequence of Propositions 2.3 and 2.4. ∎

Lemma 2.10. *Let $\{\mathcal{P}_j : j \ge 0\}$ be a quasi-dyadic sequence of partitions. Let $n = 2^j + k$, $m = 2^j + l$ with $1 \le k < l \le 2^j$. Then there are two constants α, β, depending on n and m, such that*

$$f_n(u) = \alpha f_m(u) \quad \text{for } u \le t_{j+1,2k-2} \quad \text{and} \quad f_n(u) = \beta f_m(u) \quad \text{for } u \ge t_{j+1,2l}.$$

Proof. By definition, $f_n \in \mathcal{S}_{\pi_n}$, $f_n \perp \mathcal{S}_{\pi_{n-1}}$ and $f_m \in \mathcal{S}_{\pi_m}$, $f_m \perp \mathcal{S}_{\pi_{m-1}}$. Denote $\mathcal{N}_n = \{i : t_{j+1,i} \in \pi_n\}$, $\mathcal{N}_m = \{i : t_{j+1,i} \in \pi_m\}$, and let $N_{n,i}$, $i \in \mathcal{N}_n$, and $N_{m,i}$, $i \in \mathcal{N}_m$, be the corresponding B-splines. Thus, $f_n = \sum_{i\in\mathcal{N}_n} a_i N_{n,i}$ with $a_i = f_n(t_{j+1,i})$ and $f_m = \sum_{i\in\mathcal{N}_m} b_i N_{m,i}$ with $b_i = f_m(t_{j+1,i})$. Consider the functions f_n, f_m on $[0, t_{j+1,2k-2}]$.

Clearly, for $i \le 2k-3$ we have $N_{n,i} = N_{m,i}$ on $[0,1]$, and in addition $N_{n,2k-2} = N_{m,2k-2}$ on $[0, t_{j+1,2k-2}]$; moreover, the functions $N_{n,i} = N_{m,i}$ with $0 \le i \le 2k-3$ belong to both π_{n-1} and π_{m-1}. Therefore, formula (2.1) for inner products of B-splines and the orthogonality conditions imply that both $(a_i)_{0\le i\le 2k-2}$ and $(b_i)_{0\le i\le 2k-2}$ satisfy the following system of equations:

$$\begin{cases} 2x_0 + x_1 = 0, \\ \lambda_{j+1,i}x_{i-1} + 2(\lambda_{j+1,i} + \lambda_{j+1,i+1})x_i + \lambda_{j+1,i+1}x_{i+1} = 0 \quad \text{for } 1 \le i \le 2k-3. \end{cases}$$

Since this is a system of $2k-2$ equations with $2k-1$ variables, the dimension of the space of its solutions is 1. Since both $(a_i)_{0\le i\le 2k-2}$ and $(b_i)_{0\le i\le 2k-2}$ are non-zero, this implies that there is a constant α such that $a_i = \alpha b_i$ for all $0 \le i \le 2k-2$. This property and the representation of f_n and f_m imply that $f_n = \alpha f_m$ on $[0, t_{j+1,2k-2}]$.

The existence of a constant β such that $f_n = \beta f_m$ on $[t_{j+1,2l}, 1]$ follows by analogous arguments. ■

LEMMA 2.11. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and let $\{f_n : n \ge 0\}$ be the corresponding Franklin system. Let $0 < p \le 1$. Then there is a constant $C_{\gamma,p}$ such that for $I \in \mathcal{I}$,*

$$\int_{I^c} \sum_{\{n\}\subset I} |\{n\}|^{p/2} |f_n(u)|^p \, du \le C_{\gamma,p}|I|, \tag{2.34}$$

and moreover, for all $n \ge 0$,

$$\int_0^1 |f_n(u)|^p \, du \le C_{\gamma,p} |\{n\}|^{1-p/2}. \tag{2.35}$$

PROOF. To prove (2.34), let $I \in \mathcal{I}_{j_0}$. Note that if $\{n\} \subset I$ then $n = 2^j + m$ with $j \ge j_0$ and $\{n\} = [t_{j,m-1}, t_{j,m}]$ (cf. (2.17)). Let $I = I_{j_0,k} = [t_{j_0,k-1}, t_{j_0,k}] = [t_{j,2^{j-j_0}(k-1)}, t_{j,2^{j-j_0}k}]$. Thus, $\{n\} \subset I$ means that $2^{j-j_0}(k-1) < m \le 2^{j-j_0}k$, so we get for $u \in I_{j,l}$ with $l > 2^{j-j_0}k$ (cf. the decay of f_n — Proposition 2.9, formulae (2.30) and (2.31))

$$\begin{aligned}\sum_{\substack{2^j < n \le 2^{j+1} \\ \{n\}\subset I}} |\{n\}|^{p/2} |f_n(u)|^p &\le C_{\gamma,p} \sum_{2^{j-j_0}(k-1) < m \le 2^{j-j_0}k} 2^{-p|m-l|} \\ &\le C_{\gamma,p} 2^{-p|2^{j-j_0}k - l|}.\end{aligned}$$

As the sequence of partitions is strongly regular, applying the above inequality and Proposition 2.6(ii), we obtain

$$\begin{aligned}\int_{t_{j_0,k}}^1 \sum_{\substack{2^j < n \le 2^{j+1} \\ \{n\}\subset I}} |\{n\}|^{p/2} |f_n(u)|^p \, du &\le C_{\gamma,p} \sum_{l=2^{j-j_0}k+1}^{2^j} |I_{j,l}| 2^{-p|2^{j-j_0}k-l|} \\ &\le C_{\gamma,p} |I_{j,2^{j-j_0}k}| \sum_{l=2^{j-j_0}k+1}^{2^j} \frac{(l - 2^{j-j_0}k)^{\alpha_\gamma}}{2^{p|2^{j-j_0}k-l|}} \\ &\le C_{\gamma,p} \Big(\frac{\gamma}{\gamma+1}\Big)^{j-j_0} |I|,\end{aligned}$$

which gives

$$\int_{t_{j_0,k}}^{1} \sum_{n\subset I} |\{n\}|^{p/2}|f_n(u)|^p\,du \le \sum_{j=j_0}^{\infty} \int_{t_{j_0,k}}^{1} \sum_{\substack{2^j<n\le 2^{j+1}\\ \{n\}\subset I}} |\{n\}|^{p/2}|f_n(u)|^p\,du$$

$$\le C_{\gamma,p}|I| \sum_{j=j_0}^{\infty} \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0} \le C_{\gamma,p}|I|.$$

By analogous arguments we obtain

$$\int_{0}^{t_{j_0,k-1}} \sum_{\{n\}\subset I} |\{n\}|^{p/2}|f_n(u)|^p\,du \le C_{\gamma,p}|I|,$$

which implies inequality (2.34).

Inequality (2.35) follows from (2.34) (with $I=\{n\}$) and the fact that

$$\int_{\{n\}} |f_n(u)|^p\,du \le C_\gamma|\{n\}|^{1-p/2},$$

which in turn is an immediate consequence of the estimate for the supremum norm of the Franklin function (cf. Proposition 2.9, formula (2.23)). ■

Next, we formulate some technical estimates which are used frequently; their proofs are elementary and therefore the details are omitted.

PROPOSITION 2.12. *Let $\{\mathcal{P}_j : j\ge 0\}$ be a quasi-dyadic sequence of partitions satisfying the strong regularity condition with parameter γ and let $\{f_n : n\ge 0\}$ be the corresponding Franklin system. For $n\ge 2$, $n=2^j+k$ with $1\le k\le 2^j$, let $\{n\}$ and $t_n=t_{j+1,2k-1}$ be as in* (2.17)*. Then there is a constant C_γ, depending only on γ, such that*

$$\frac{1}{C_\gamma}|\{n\}|^{-1/2} \le |f_n(t_{j+1,2k-2})|,\ |f_n(t_{j+1,2k-1})|,\ |f_n(t_{j+1,2k})| \le C_\gamma|\{n\}|^{-1/2}. \tag{2.36}$$

Moreover, let $0<\alpha\le 1$. Then there is a positive constant $C_{\gamma,\alpha}$, depending only on γ and α, such that for all n and $A\subset\{n\}$ with $|A|\ge\alpha|\{n\}|$,

$$\int_A f_n^2(u)\,du \ge C_{\gamma,\alpha}.$$

PROOF. To get (2.36), apply the lower estimates for the values $|f_n(t_{j+1,2k-2})|$ and $|f_n(t_{j+1,2k})|$ from Proposition 2.3 and strong regularity of the sequence of partitions. The remaining part of Proposition 2.12 is a straightforward consequence of (2.36). ■

PROPOSITION 2.13. *Let $0<\alpha\le 1$. Then there is a positive constant C_α, depending only on α, such that for any interval $[a,b]$, $A\subset[a,b]$ with $|A|\ge\alpha(b-a)$ and a function f linear on $[a,b]$,*

$$\max_{u\in[a,b]}|f(u)| \le C_\alpha \max_{u\in A}|f(u)| \quad \textit{and} \quad \int_{[a,b]} |f(u)|\,du \le C_\alpha \int_A |f(u)|\,du.$$

Proposition 2.14. *Let the quasi-dyadic sequence $\{\mathcal{P}_j : j \ge 0\}$ of partitions satisfy the weak regularity condition with parameter γ, $E \subset [0,1]$ and*

$$B = \{u \in [0,1] : \mathcal{M}^*(\chi_E, u) > 1/(2\gamma+2)\}.$$

Let $I \in \mathcal{I}$, and let I^-, I^+ be the intervals in $\mathcal{I}$ obtained by splitting I. If $I \not\subset B$ then

$$|I^- \cap E^c| \ge \frac{1}{2\gamma}|I^-| \quad \textit{and} \quad |I^+ \cap E^c| \ge \frac{1}{2\gamma}|I^+|.$$

3. Franklin series in L^p, $< p < \infty$

The main result of this section is the following:

Theorem 3.1. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the weak regularity condition with parameter γ. Then the corresponding Franklin system is an unconditional basis in L^p for all $1 < p < \infty$.*

For the proof of Theorem 3.1 we need some auxiliary results. We start with a technical lemma.

Lemma 3.2. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the weak regularity condition with parameter γ. Let $I \in \mathcal{I}$ and let φ be a function such that*

$$\operatorname{supp}\varphi \subset I, \qquad \sup|\varphi| \le \frac{1}{|I|}, \qquad \int_I \varphi(u)\,du = 0.$$

Moreover, let $a_n = (\varphi, f_n)$. Then there is a constant C_γ, depending only on γ, such that

$$\int_{I^c} \sum_{n=0}^{\infty} |a_n f_n(u)|\,du \le C_\gamma.$$

Proof. First, observe that the conditions imposed on φ imply $a_0 = 0$ and $|a_1| \le \sqrt{3}$ (recall that f_0 and f_1 do not depend on the specific sequence of partitions — cf. Definition 2.1), so it is enough to consider the sum beginning with $n = 2$.

Since $I \in \mathcal{I}$, we have $I = I_{j_0,k}$ for some $j_0 \ge 0$ and $1 \le k \le 2^{j_0}$. To estimate $\int_{I^c} \sum_{n=0}^{\infty} |a_n f_n(u)|\,du$, we split it into several parts.

First, consider $\sum_{j=0}^{j_0} \sum_{n=2^j+1}^{2^{j+1}} |a_n| \|f_n\|_1$. To simplify the notation, let $I^- = I_{j_0+1,2k-1}$, $I^+ = I_{j_0+1,2k}$ and $\tau = t_{j_0+1,2k-1}$. Note that if $n \le 2^{j_0} + k - 1$, then f_n is linear on I, and for $2^{j_0} + k \le n \le 2^{j_0+1}$ the function f_n is linear on both subintervals I^- and I^+. Therefore, denoting by ξ_n, ζ_n the value of the derivative of f_n on $(I^-)^\circ$, $(I^+)^\circ$ respectively (clearly, $\xi_n = \zeta_n$ for $n \le 2^{j_0} + k - 1$) and using the properties of φ we get

$$\begin{aligned}
(3.1) \qquad |a_n| &= \Big|\int_0^1 \varphi(u) f_n(u)\,du\Big| = \Big|\int_I \varphi(u)\big(f_n(u) - f_n(\tau)\big)\,du\Big| \\
&\le \frac{1}{|I|}\Big(|\xi_n| \int_{I^-} |u-\tau|\,du + |\zeta_n| \int_{I^+} |u-\tau|\,du\Big) \\
&= \frac{|\xi_n| \cdot |I^-|^2 + |\zeta_n| \cdot |I^+|^2}{2|I|}.
\end{aligned}$$

To estimate $|\xi_n|$, let $n = 2^j + l$ with $1 \le l \le 2^j$, $j \le j_0$, and let Δ_j be the unique interval of order $j+1$ containing I^-; as $\Delta_j \in \mathcal{I}_{j+1}$, we have $\Delta_j = I_{j+1,k_j}$ for some k_j; moreover, f_n is linear on Δ_j. Applying the pointwise estimates for f_n from Proposition 2.9 (inequalities (2.27), (2.31)) we get

$$\sup_{u\in\Delta_j} |f_n(u)| \le C_\gamma 2^{-|l-k_j/2|} |\{n\}|^{-1/2},$$

which gives

$$|\xi_n| \le C_\gamma 2^{-|l-k_j/2|} |\{n\}|^{-1/2} \frac{1}{|\Delta_j|}. \tag{3.2}$$

Using the estimates for length of intervals from Proposition 2.6(i) we get

$$|\xi_n| \le C_\gamma 2^{-|l-k_j/2|} |\{n\}|^{-1/2} \left(\frac{\gamma}{\gamma+1}\right)^{j_0-j} \frac{1}{|I^-|},$$

and by similar arguments (note that $I^+ \subset \Delta_j$ for $j < j_0$),

$$|\zeta_n| \le C_\gamma 2^{-|l-k_j/2|} |\{n\}|^{-1/2} \left(\frac{\gamma}{\gamma+1}\right)^{j_0-j} \frac{1}{|I^+|}.$$

These estimates for $|\xi_n|$, $|\zeta_n|$ and (3.1) give

$$|a_n| \le C_\gamma 2^{-|l-k_j/2|} |\{n\}|^{-1/2} \left(\frac{\gamma}{\gamma+1}\right)^{j_0-j}.$$

As $\|f_n\|_1 \le C_\gamma |\{n\}|^{1/2}$ (cf. Proposition 2.9, (2.23)), the last inequality gives

$$\sum_{n=2^j+1}^{2^{j+1}} |a_n|\, \|f_n\|_1 \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j_0-j} \sum_{l=1}^{2^j} 2^{-|l-k_j/2|} \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j_0-j},$$

which implies

$$\sum_{j=0}^{j_0} \sum_{n=2^j+1}^{2^{j+1}} |a_n|\, \|f_n\|_1 \le C_\gamma \sum_{j=0}^{j_0} \left(\frac{\gamma}{\gamma+1}\right)^{j_0-j} \le C_\gamma. \tag{3.3}$$

Now, let $n = 2^j + l$ with $1 \le l \le 2^j$ and $j > j_0$. Then, by the properties of φ we have

$$|a_n| \le \int_I |\varphi(u)|\, |f_n(u)|\, du \le \frac{1}{|I|} \int_I |f_n(u)|\, du. \tag{3.4}$$

Recall that $I = [t_{j_0,k-1}, t_{j_0,k}] = [t_{j+1,2^{j+1-j_0}(k-1)}, t_{j+1,2^{j+1-j_0}k}]$. Consider n such that $t_n \in I$, i.e. $\{n\} \subset I$, or equivalently $2^{j+1-j_0}(k-1) < 2l-1 < 2^{j+1-j_0}k$. (It should be noted that we cannot use Lemma 2.11, which has been obtained for strongly regular partitions and $0 < p \le 1$; now we obtain an analogous estimate for weakly regular partitions, but for $p = 1$ only.) Then $\int_I |f_n(u)|\, du \le \|f_n\|_1$, and using the decay of the integrals of $|f_n|$ from Proposition 2.9 (cf. (2.29), (2.33)) we obtain

$$\begin{aligned}\int_{I^c} |f_n(u)|\, du &= \int_0^{t_{j+1,2^{j+1-j_0}(k-1)}} |f_n(u)|\, du + \int_{t_{j+1,2^{j+1-j_0}k}}^1 |f_n(u)|\, du \\ &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} \big(\theta^{|2^{j+1-j_0}(k-1)-2l+2|} + \theta^{|l-2^{j-j_0}k|}\big) \|f_n\|_1,\end{aligned}$$

where $\theta = 1/(3(\sqrt{2}-1)) < 1$. Moreover, for these n's we have (cf. formula (2.23) in Proposition 2.9 and Proposition 2.6(i))

$$\|f_n\|_1^2 \le C_\gamma |\{n\}| \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0} |I|.$$

By (3.4) we have $|a_n| \le |I|^{-1}\|f_n\|_1$, so applying the last two inequalities we get

$$\begin{aligned}\sum_{\substack{2^j<n\le 2^{j+1}\\ t_n\in I}} |a_n| \int_{I^c} |f_n(u)|\,du &\le \frac{C_\gamma}{|I|} \sum_{\substack{2^j<n\le 2^{j+1}\\ t_n\in I}} (\theta^{|2^{j+1-j_0}(k-1)-2l+2|} + \theta^{|l-2^{j-j_0}k|})\|f_n\|_1^2 \\ &\le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0} \sum_{l\in N} \theta^l,\end{aligned}$$

which means that

$$\sum_{\substack{2^j<n\le 2^{j+1}\\ t_n\in I}} |a_n| \int_{I^c} |f_n(u)|\,du \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0}. \tag{3.5}$$

Now, consider $n = 2^j + l$ with $2l-1 < 2^{j+1-j_0}(k-1)$, i.e. $t_n < t_{j+1,2^{j+1-j_0}(k-1)}$. Let J_j be the interval from $\mathcal{I}_j$ with left end coinciding with the left end of I. Since $J_j \subset I$, Proposition 2.6(i) implies that $|J_j| \le (\gamma/(\gamma+1))^{j-j_0}|I|$. Applying for these n's the estimates for the integral and pointwise decay of $|f_n|$ from Proposition 2.9 (cf. (2.33) and (2.31)), we get

$$\begin{aligned}\int_I |f_n(u)|\,du &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} \int_{J_j} |f_n(u)|\,du \\ &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} |J_j|\, |f_n(t_{j+1,2^{j+1-j_0}(k-1)})| \\ &\le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0} 2^{-|l-2^{j-j_0}(k-1)|} |\{n\}|^{-1/2} |I|.\end{aligned} \tag{3.6}$$

Clearly, $\int_{I^c} |f_n(u)|\,du \le \|f_n\|_1$, so the above inequlities, (3.4) and the estimate for $\|f_n\|_1$ (cf. (2.23)) give

$$\sum_{\substack{2^j<n\le 2^{j+1}\\ t_n<t_{j+1,2^{j+1-j_0}(k-1)}}} |a_n| \int_{I^c} |f_n(u)|\,du \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0} \sum_{l\in N} 2^{-l} \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0}.$$

By analogous arguments we check that

$$\sum_{\substack{2^j<n\le 2^{j+1}\\ t_n>t_{j+1,2^{j+1-j_0}k}}} |a_n| \int_{I^c} |f_n(u)|\,du \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0}.$$

The last two inequalities and (3.5) give

$$\sum_{n=2^j+1}^{2^{j+1}} \int_{I^c} |a_n|\, |f_n(u)|\,du \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0},$$

and summing over $j > j_0$ we get

$$\sum_{n=2^{j_0}+1}^{\infty} \int_{I^c} |a_n|\,|f_n(u)|\,du \le C_\gamma \sum_{j=j_0}^{\infty} \left(\frac{\gamma}{\gamma+1}\right)^{j-j_0} \le C_\gamma.$$

This and (3.3) complete the proof. ■

As a consequence of Lemma 3.2 we get

LEMMA 3.3. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the weak regularity condition with parameter γ. Let $\mathcal{T} \subset \mathcal{I}$ be a subset such that $I^\circ \cap \tilde{I}^\circ = \emptyset$ for all $I, \tilde{I} \in \mathcal{T}$, $I \ne \tilde{I}$, and let $B = \bigcup_{I\in\mathcal{T}} I$. Let ψ be a function such that*

$$\sup|\psi| \le b, \quad \operatorname{supp}\psi \subset B, \quad \forall_{I\in\mathcal{T}} \int_I \psi = 0,$$

where b is some nonnnegative number. Then there is a constant C_γ, depending only on γ, such that for any function ψ satisfying the above conditions,

$$\int_{B^c} \sum_{n=0}^{\infty} |a_n f_n(u)|\,du \le C_\gamma b|B|,$$

where $a_n = (\psi, f_n)$.

PROOF. Note that $B^c \subset \bigcap_{I\in\mathcal{T}} I^c$; put $\psi_I = \psi\chi_I$ and $a_{I,n} = (\psi_I, f_n)$. Since the functions ψ_I satisfy the hypothesis of Lemma 3.2, we get

$$\int_{B^c} \sum_{n=0}^{\infty} |a_{I,n} f_n(u)|\,du \le \int_{I^c} \sum_{n=0}^{\infty} |a_{I,n} f_n(u)|\,du \le C_\gamma b|I|,$$

and summing over $I \in \mathcal{T}$ we obtain

$$\int_{B^c} \sum_{n=0}^{\infty} |a_n f_n(u)|\,du \le \sum_{I\in\mathcal{T}} \int_{B^c} \sum_{n=0}^{\infty} |a_{I,n} f_n(u)|\,du \le C_\gamma b \sum_{I\in\mathcal{T}} |I| = C_\gamma b|B|. \ ■$$

THEOREM 3.4. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the weak regularity condition with parameter γ. Let $1 < p < 2$ and $f \in L^p$, $f = \sum_{n=0}^{\infty} a_n f_n$. Let $\varepsilon = (\varepsilon_n)_{n\ge 0}$ with $\varepsilon_n \in \{-1, 1\}$ and*

$$T_\varepsilon f = \sum_{n=0}^{\infty} \varepsilon_n a_n f_n.$$

Then there is a constant $C_{\gamma,p}$, depending only on γ and p, such that for each $f \in L^p$ and each sequence ε,

$$\|T_\varepsilon f\|_p \le C_{\gamma,p} \|f\|_p.$$

PROOF. Let $f \in L^p$, $1 < p < 2$. For $x \in [0,1]$, define

$$\Phi(x) = \sup_{I: x\in I\in\mathcal{I}} \left| \frac{1}{|I|} \int_I f(u)\,du \right|.$$

First, observe that $\Phi(x) \le \mathcal{M}(f,x)$. Since for $p > 1$ the operator $\mathcal{M}(f,\cdot)$ is of type (p,p), this inequality implies

$$\|\Phi\|_p \le \|\mathcal{M}(f,\cdot)\|_p \le C_p \|f\|_p.$$

For $m \in \mathbb{Z}$ let

$$B_m = \{x \in [0,1] : \Phi(x) > 2^m\}.$$

Note that the set B_m is a sum of some intervals from $\mathcal{I}$, and let $\mathcal{T}_m$ be the set of maximal intervals from $\mathcal{I}$ contained in B_m; thus we have

$$B_m = \bigcup_{I \in \mathcal{T}_m} I, \quad \text{with} \quad I^\circ \cap \tilde{I}^\circ = \emptyset \text{ for } I, \tilde{I} \in \mathcal{T}_m,\ I \neq \tilde{I}.$$

Note that $B_{m+1} \subset B_m$; moreover, for each pair of different intervals from $\mathcal{I}$, either their interiors are disjoint or one is included in the other, which implies that for each interval $I \in \mathcal{T}_{m+1}$ there is a unique $J \in \mathcal{T}_m$ such that $I \subset J$. It is also clear that

$$\sum_{m \in Z} 2^{mp} |B_m| \le C_p \|\Phi\|_p^p \le C_p \|f\|_p^p. \tag{3.7}$$

Now, let

$$F_m(x) = \begin{cases} f(x) & \text{for } x \notin B_m, \\ \dfrac{1}{|I|} \displaystyle\int_I f(u)\, du & \text{for } x \in I,\ I \in \mathcal{T}_m. \end{cases}$$

We check that there is a constant C_γ such that $\|F_m\|_\infty \le C_\gamma 2^m$ for all m. Indeed, if $x \notin B_m$, then $\Phi(x) \le 2^m$, which means that

$$\left| \frac{1}{|J|} \int_J f(u)\, du \right| \le 2^m$$

for all $J \in \mathcal{I}$ with $x \in J$, and this implies $|f(x)| \le 2^m$ a.e. on B_m. On the other hand, if $I \in \mathcal{T}_m$ is an interval of rank j, I^* is the unique interval of rank $j-1$ containing I and I' is the other interval of rank j contained in I^*, then by maximality of I, neither I^* nor I' is included in B_m, which implies

$$\left| \frac{1}{|I'|} \int_{I'} f(u)\, du \right| \le 2^m \quad \text{and} \quad \left| \frac{1}{|I^*|} \int_{I^*} f(u)\, du \right| \le 2^m.$$

These inequalities and weak regularity of the sequence of partitions give

$$\left| \frac{1}{|I|} \int_I f(u)\, du \right| \le \frac{1}{|I|} \Big(\Big| \int_{I^*} f(u)\, du \Big| + \Big| \int_{I'} f(u)\, du \Big| \Big) \le 2^m \frac{|I^*| + |I'|}{|I|} \le (2\gamma + 1) 2^m.$$

Thus, for all $m \in \mathbb{Z}$ we have $\|F_m\|_\infty \le (2\gamma + 1) 2^m$.

Define $\psi_m = F_{m+1} - F_m$; then

$$\|\psi_m\|_\infty \le 3(2\gamma + 1) 2^m \quad \text{and} \quad \operatorname{supp} \psi_m \subset B_m. \tag{3.8}$$

Let us prove that the function ψ_m and set B_m satisfy the assumptions of Lemma 3.3 with constant $b_m = 3(2\gamma + 1)2^m$. It remains to check that $\int_I \psi_m(u)\, du = 0$ for all $I \in \mathcal{T}_m$, but this follows from the fact that for $I \in \mathcal{T}_m$ the set $I \cap B_{m+1}$ can be written as the union of some intervals from $\mathcal{T}_{m+1}$, and from the formulae for F_m and F_{m+1}; the technical details are omitted. Moreover, the function ψ_m is constant on the intervals $I \in \mathcal{T}_{m+1}$, which together with the previous property implies that $(\psi_m, \psi_{m'}) = 0$ for $m \neq m'$.

Note that by (3.7) and (3.8) we have $f = \sum_{m=-\infty}^{\infty} \psi_m$, with the series convergent in L^p. Thus, putting $a_{m,n} = (\psi_m, f_n)$, we obtain $a_n = \sum_{m=-\infty}^{\infty} a_{m,n}$ and $\varepsilon_n a_n =$

$\sum_{m=-\infty}^{\infty} \varepsilon_n a_{m,n}$. For $l \in \mathbb{Z}$ let

$$E_l = \Big\{u \in [0,1] : \Big|\sum_{n=0}^{\infty} \varepsilon_n a_n f_n(u)\Big| > 2^l\Big\},$$

$$X_l = \{u \in [0,1] : \Big|\sum_{n=0}^{\infty} \varepsilon_n \Big(\sum_{m\le l-1} a_{m,n}\Big) f_n(u)\Big| > 2^{l-1}\Big\},$$

$$Y_l = \Big\{u \in [0,1] : \Big|\sum_{n=0}^{\infty} \varepsilon_n \Big(\sum_{m\ge l} a_{m,n}\Big) f_n(u)\Big| > 2^{l-1}\Big\}.$$

Note that $E_l \subset X_l \cup Y_l$. Let us estimate $|X_l|$ and $|Y_l|$. First, using Chebyshev's inequality, estimates (3.8) for $\|\psi_m\|_\infty$ and the orthogonality of the functions ψ_m, we obtain

$$\begin{aligned}
|X_l| &= \Big|\Big\{u \in [0,1] : \Big|\sum_{n=0}^{\infty} \varepsilon_n \Big(\sum_{m\le l-1} a_{m,n}\Big) f_n(u)\Big|^2 > 2^{2l-2}\Big\}\Big| \\
&\le \frac{1}{2^{2l-2}} \sum_{n=0}^{\infty} \Big(\sum_{m\le l-1} a_{m,n}\Big)^2 \\
&= \frac{1}{2^{2l-2}} \Big\|\sum_{m\le l-1} \psi_m\Big\|_2^2 = \frac{1}{2^{2l-2}} \sum_{m\le l-1} \|\psi_m\|_2^2 \\
&\le \frac{C_\gamma}{2^{2l}} \sum_{m\le l-2} \|\psi_m\|_\infty^2 |B_m| \le \frac{C_\gamma}{2^{2l}} \sum_{m\le l-1} 2^{2m} |B_m|.
\end{aligned}$$

On the other hand, the functions ψ_m and the sets B_m satisfy the assumptions of Lemma 3.3, so using this lemma we get

$$\begin{aligned}
|Y_l| &\le |B_l| + \Big|\Big\{u \in B_l^c : \sum_{n=0}^{\infty} \sum_{m\ge l} |a_{m,n}|\,|f_n(u)| > 2^{l-1}\Big\}\Big| \\
&\le |B_l| + \frac{1}{2^{l-1}} \int_{B_l^c} \sum_{n=0}^{\infty} \sum_{m\ge l} |a_{m,n}|\,|f_n(u)|\,du \\
&\le |B_l| + \frac{1}{2^{l-1}} \sum_{m\ge l} \int_{B_m^c} \sum_{n=0}^{\infty} |a_{m,n}|\,|f_n(u)|\,du \\
&\le \frac{C_\gamma}{2^l} \sum_{m\ge l} 2^m |B_m|.
\end{aligned}$$

Thus, we have

$$|E_l| \le C_\gamma \Big(\frac{1}{2^{2l}} \sum_{m\le l-1} 2^{2m}|B_m| + \frac{1}{2^l} \sum_{m\ge l} 2^m |B_m|\Big).$$

Using this estimate and (3.7) we obtain (recall that $1 < p < 2$)

$$\begin{aligned}
\|T_\varepsilon f\|_p^p &\le C_p \sum_{l\in\mathbb{Z}} 2^{lp} |E_l| \\
&\le C_{\gamma,p} \Big(\sum_{l\in\mathbb{Z}} 2^{l(p-2)} \sum_{m\le l} 2^{2m}|B_m| + \sum_{l\in\mathbb{Z}} 2^{l(p-1)} \sum_{m\ge l} 2^m |B_m|\Big)
\end{aligned}$$

$$\leq C_{\gamma,p}\Big(\sum_{m\in\mathbb{Z}} 2^{2m}|B_m|\sum_{l\geq m} 2^{l(p-2)} + \sum_{m\in\mathbb{Z}} 2^{m}|B_m|\sum_{l\leq m} 2^{l(p-1)}\Big)$$
$$\leq C_{\gamma,p}\sum_{m\in\mathbb{Z}} 2^{mp}|B_m| \leq C_{\gamma,p}\|f\|_p^p. \ \blacksquare$$

Proof of Theorem 3.1. Recall that for each p, $1 \leq p < \infty$, the system $\{f_n : n \geq 0\}$ is a basis in L^p. As an orthonormal system, it is an unconditional basis in L^2. Its unconditionality in L^p for $1 < p < 2$ follows from Theorem 3.4, and then the unconditionality in L^p for $2 < p < \infty$ is obtained by a duality argument. ■

As a consequence of Theorem 3.4, using a well-known argument based on Khinchin's inequality and the maximal inequality from Theorem 2.5(i), we obtain

COROLLARY 3.5. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the weak regularity condition with parameter γ and $1 < p < \infty$. Then for $f \in L^p$ we have*

$$\|f\|_p \sim \|Pf\|_p \sim \|Sf\|_p,$$

with implied constants depending only on p and γ.

Moreover, for a real sequence $(a_n)_{n\geq 0}$, the following conditions are equivalent:

(1) *The series $\sum_{n=0}^{\infty} a_n f_n$ is unconditionally convergent in L^p.*
(2) *There is $f \in L^p$ such that $a_n = (f, f_n)$ for all $n \geq 0$.*
(3) $P(\cdot) = (\sum_{n=0}^{\infty} a_n^2 f_n^2(\cdot))^{1/2} \in L^p$.
(4) $S(\cdot) = \sup_{m\geq 0} |\sum_{n=0}^{m} a_n f_n(\cdot)| \in L^p$.

4. Franklin series in L^p, $0 < p \leq 1$, and H^p, $1/2 < p \leq 1$

Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. Let $0 < p < \infty$ and let $a = (a_n)_{n\geq 0}$ be a given sequence of real numbers. Consider the following conditions:

(A) $P(\cdot) = (\sum_{n=0}^{\infty} a_n^2 f_n^2(\cdot))^{1/2} \in L^p$.
(B) The series $\sum_{n=0}^{\infty} a_n f_n$ converges unconditionally in L^p.
(C) $S(\cdot) = \sup_{m\geq 0} |\sum_{n=0}^{m} a_n f_n(\cdot)| \in L^p$.

For $1 < p < \infty$, we have already proved the equivalence of (A)–(C) under the assumption of weak regularity of the sequence of partitions under consideration — cf. Corollary 3.5. In this section, we study the relations of the above conditions for $0 < p \leq 1$. Under the assumption of strong regularity of the sequence of partitions, we prove the following:

THEOREM 4.1. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the strong regularity condition with parameter γ and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. Then*, *for each p, $0 < p \leq 1$, conditions* (A), (B) *and* (C) *are equivalent.*

Moreover, we study the convergence of the Franklin series in H^p, $1/2 < p \leq 1$. We obtain the following result:

THEOREM 4.2. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the strong regularity condition with parameter γ and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. Let $1/2 < p \leq 1$ be given. Then the system $\{f_n : n \geq 0\}$ is an unconditional basis in H^p. Moreover, for $f \in H^p$,*

$$\|f\|_{H^p} \sim \|Pf\|_p \sim \|Sf\|_p \sim \sup_{\varepsilon} \Big\| \sum_{n=0}^{\infty} \varepsilon_n (f, f_n) f_n \Big\|_p, \tag{4.1}$$

with the supremum taken over $\varepsilon = (\varepsilon_n)_{n\geq 0}$ with $\varepsilon_n \in \{-1, 1\}$, and with implied constants depending only on p and γ. In addition, for the system $\{f_n : n \geq 0\}$, conditions (A)–(C) *are equivalent to*

(D) *There is $f \in H^p$ such that $a_n = (f, f_n)$ for all $n \geq 0$.*
(E) *The series $\sum_{n=0}^{\infty} a_n f_n$ converges unconditionally in H^p.*

The proofs of Theorems 4.1 and 4.2 are split into several lemmas. First, we state and prove the lemmas, and the proofs of the theorems are presented at the end of this section. The technique of the proofs is similar to that in [21]. For the convenience of the reader and for the sake of completeness, we present them in detail.

LEMMA 4.3. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the strong regularity condition with parameter γ and let $1/2 < p \leq 1$. Then there is a constant $C_{\gamma,p}$, depending only on p and γ, such that for every p-atom φ,*

$$\|S\varphi\|_p \leq C_{\gamma,p}, \quad \|P\varphi\|_p \leq C_{\gamma,p}.$$

PROOF. It is clear that for $\varphi = 1$, we have $\|S\varphi\|_p = \|P\varphi\|_p = 1$.

Now, let φ be a p-atom such that for some interval $\Gamma \subset [0, 1]$,

$$\operatorname{supp} \varphi \subset \Gamma, \quad \sup |\varphi| \leq |\Gamma|^{-1/p}, \quad \int_0^1 \varphi(x)\, dx = 0.$$

For the interval Γ, let

$$j_0 = \min\{j : \text{there is } I \in \mathcal{I}_j \text{ with } I \subset \Gamma\}.$$

Let I_{j_0,k_0} be an interval of rank j_0 included in Γ; it follows from the choice of j_0 that there can be at most two such intervals — it is possible that one of I_{j_0,k_0-1}, I_{j_0,k_0+1} is also included in Γ. Define

$$J = \bigcup_{|k-k_0|\leq 2} I_{j_0,k}.$$

By the choice of j_0 we have $\Gamma \subset J$, and strong regularity implies

$$|\Gamma| \sim |J| \sim |I_{j_0,k}| \quad \text{for } |k - k_0| \leq 2. \tag{4.2}$$

Note that $\|\varphi\|_2^2 \leq |\Gamma|^{1-2/p}$. Using (4.2) and Hölder's inequality with exponents $2/p$ and $2/(2-p)$ we get

$$\int_J (P\varphi(u))^p\, du \leq |J|^{1-p/2} \Big(\int_J (P\varphi(u))^2\, du \Big)^{p/2} \leq \|\varphi\|_2^p |J|^{1-p/2} \leq C_{\gamma,p}. \tag{4.3}$$

Since $S\varphi(u) \le 64\,\mathcal{M}(\varphi,u)$ (cf. Theorem 2.5(i)) and the operator $\mathcal{M}$ is of type $(2,2)$, we have $\|S\varphi\|_2 \le C\|\varphi\|_2$, so a similar argument gives

$$\int_J (S\varphi(u))^p\,du \le C_{\gamma,p}. \tag{4.4}$$

To estimate the corresponding integrals over J^{c}, note that

$$(P\varphi(u))^p \le \sum_{n=0}^{\infty} |a_n f_n(u)|^p \quad \text{and} \quad (S\varphi(u))^p \le \sum_{n=0}^{\infty} |a_n f_n(u)|^p, \tag{4.5}$$

where $a_n = (\varphi, f_n)$, and therefore it is enough to prove

$$\int_{J^{\mathrm{c}}} \sum_{n=0}^{\infty} |a_n f_n(u)|^p\,du \le C_{\gamma,p}. \tag{4.6}$$

The idea of the remaining part of the proof is similar to the proof of Lemma 3.2. First, consider $n \le 2^{j_0}$. Then f_n is linear on each interval $I_{j_0,k}$; denote by $\xi_{n,k}$ the value of the derivative of f_n on $I^{\circ}_{j_0,k}$ and let $\tau \in \Gamma$. Since $\Gamma \subset J$ and φ is an appropriate p-atom, by strong regularity we get

$$|a_n| = \Big|\int_J \varphi(u)(f_n(u) - f_n(\tau))\,du\Big| \le C_\gamma |\Gamma|^{2-1/p} \sum_{|k-k_0|\le 2} |\xi_{n,k}|. \tag{4.7}$$

For $|k-k_0| \le 2$ and $j \le j_0$, let $\Delta_{j,k} \in \mathcal{I}_j$ with $I_{j_0,k} \subset \Delta_{j,k}$, and let m_k be such that $\Delta_{j,k} = I_{j,m_k}$; let $n = 2^j + l$. Then we get (by arguments analogous to those used to obtain inequality (3.2) in the proof of Lemma 3.2)

$$|\xi_{n,k}| \le C_\gamma 2^{-|l-m_k|} |\{n\}|^{-1/2} \frac{1}{|\Delta_{j,k}|}.$$

Note that for $|k-k_0| \le 2$ we have $|m_k - m_{k_0}| \le 2$, so by strong regularity $|\Delta_{j,k}| \sim |\Delta_{j,k_0}|$; thus, the last inequality and (4.7) give

$$|a_n| \le C_\gamma 2^{-|l-m_{k_0}|} |\{n\}|^{-1/2} \frac{|\Gamma|^{2-1/p}}{|\Delta_{j,k_0}|}.$$

Since $\int_0^1 |f_n(u)|^p\,du \le C_{\gamma,p} |\{n\}|^{1-p/2}$ (cf. (2.35) in Lemma 2.11), the above inequality, Proposition 2.6(ii) and (4.2) imply

$$\begin{aligned}
\sum_{n=2^j+1}^{2^{j+1}} \int_0^1 |a_n f_n(u)|^p\,du &\le C_{\gamma,p} |\Gamma|^{2p-1} \sum_{l=1}^{2^j} 2^{-p|l-m_{k_0}|} \frac{|I_{j,l}|^{1-p}}{|I_{j,m_{k_0}}|^p} \\
&\le C_{\gamma,p} \frac{|\Gamma|^{2p-1}}{|I_{j,m_{k_0}}|^{2p-1}} \sum_{l=1}^{2^j} \frac{(|l-m_{k_0}|+1)^{\alpha_\gamma(1-p)}}{2^{p|l-m_{k_0}|}} \\
&\le C_{\gamma,p} \Big(\frac{\gamma}{\gamma+1}\Big)^{(j_0-j)(2p-1)} \frac{|\Gamma|^{2p-1}}{|I_{j_0,k_0}|^{2p-1}} \\
&\le C_{\gamma,p} \Big(\frac{\gamma}{\gamma+1}\Big)^{(j_0-j)(2p-1)}.
\end{aligned}$$

Thus, since $p > 1/2$, by summing over $j \le j_0 - 1$ we get

$$\sum_{n=0}^{2^{j_0}} \int_0^1 |a_n f_n(u)|^p \, du \le C_{\gamma,p}. \tag{4.8}$$

Consider now $n > 2^{j_0}$. Then we have

$$|a_n| \le |\Gamma|^{-1/p} \int_J |f_n(u)| \, du. \tag{4.9}$$

If $\{n\} \subset J$, then this inequality and the estimate for $\|f_n\|_1$ from Proposition 2.9 (cf. inequality (2.23)) give

$$|a_n| \le |\Gamma|^{-1/p} \|f_n\|_1 \le C_\gamma |\{n\}|^{1/2} |\Gamma|^{-1/p}.$$

Applying the last inequality, Lemma 2.11 (cf. inequality (2.34)) and (4.2) we get

$$\int_{J^c} \sum_{\{n\} \subset J} |a_n f_n(u)|^p \, du \le C_{\gamma,p}. \tag{4.10}$$

Finally, for $j \ge j_0$, the arguments including the decay of the L^1-norms of f_n over intervals from the partition π_n (i.e. inequalities (2.29) and (2.33) in Proposition 2.9; see also inequalities (3.6) in the proof of Lemma 3.2), the estimate for $\int_0^1 |f_n(u)|^p \, du$ from Lemma 2.11 (i.e. inequality (2.35)), the estimates for the length of the intervals from Proposition 2.6(i)–(ii) and (4.2) give

$$\sum_{\substack{2^j < n \le 2^{j+1} \\ \{n\} \not\subset J}} \int_0^1 |f_n(u)|^p \, du \Big(\int_J |f_n(u)| \, du \Big)^p \le C_{\gamma,p} \Big(\frac{\gamma}{\gamma+1} \Big)^{j-j_0} |\Gamma|,$$

which, together with (4.9), implies

$$\int_{J^c} \sum_{\substack{n > 2^{j_0} \\ \{n\} \not\subset J}} |a_n f_n(u)|^p \, du \le C_{\gamma,p}.$$

The last inequality, together with (4.8) and (4.10), gives (4.6). Lemma 4.3 now follows from inequalities (4.3)–(4.6). ∎

As a consequence of Lemma 4.3, we get

LEMMA 4.4. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and let $1/2 < p \le 1$. Then there is a constant $C_{\gamma,p}$, depending only on p and γ, such that for all $f \in H^p$,*

$$\|Sf\|_p \le C_{\gamma,p} \|f\|_{H^p}, \quad \|Pf\|_p \le C_{\gamma,p} \|f\|_{H^p}.$$

LEMMA 4.5. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ. Let $(a_n)_{n \ge 0}$ be a sequence of real coefficients such that $S(\cdot) = \sup_{m \ge 0} |\sum_{n=0}^m a_n f_n(\cdot)| \in L^1$. Then there is $f \in H^1$ such that $a_n = (f, f_n)$ for all $n \ge 0$. Moreover, there is a constant C_γ, depending only on γ, such that*

$$\|f\|_{H^1} \le C_\gamma \|S\|_1.$$

REMARK. Later on, we obtain a version of Lemma 4.5 for $1/2 < p < 1$ as well (cf. Lemma 4.9). However, the present version, for $p = 1$, is needed for the proof of Lemma 4.8 (i.e. the implication (C)⇒(A)), which in turn is used in the proof of Lemma 4.9.

Proof of Lemma 4.5. Since $S \in L^1$, there is a function $f \in L^1$ such that $a_n = (f, f_n)$, $n \geq 0$ — this follows from the Dunford–Pettis theorem (i.e. relative weak compactness in L^1 of a uniformly integrable subset, cf. for example [30]). Moreover, note that $\|f\|_1 \leq \|S\|_1$. It remains to check that $f \in H^1$.

For convenience, let $\|S\|_1 = 1$. Put $E_0 = B_0 = [0,1]$, and for $r \geq 1$,

$$E_r = \{u \in [0,1] : S(u) > 2^r\}, \quad B_r = \left\{u \in [0,1] : \mathcal{M}^*(\chi_{E_r}, u) > \frac{1}{2\gamma + 2}\right\}.$$

Since $\mathcal{M}^*$ is of weak type $(1,1)$, we have $|B_r| \leq C_\gamma |E_r|$.

Consider now the following decompositions of B_r:

$$B_r = \bigcup_{I \in \mathcal{T}_r} I = \bigcup_{\nu} \Gamma_{r,\nu}, \tag{4.11}$$

with the last union countable, where $\mathcal{T}_r$ is the family of maximal intervals from $\mathcal{I}$ included in B_r, and each $\Gamma_{r,\nu}$ is an interval which is a union of some intervals from $\mathcal{T}_r$, and no two $\Gamma_{r,\nu}$'s have a common endpoint. As for each $I \in \mathcal{T}_{r+1}$ there is $J \in \mathcal{T}_r$ such that $I \subset J$, it follows that for each $\Gamma_{r+1,\nu}$ there is $\Gamma_{r,\mu}$ with $\Gamma_{r+1,\nu} \subset \Gamma_{r,\mu}$.

Now, define the following sequence of functions: $g_0(u) = \int_0^1 f(t)\,dt$, and for $r \geq 1$,

$$g_r(u) = \begin{cases} f(u) & \text{for } u \notin B_r, \\ \dfrac{1}{|\Gamma_{r,\nu}|} \displaystyle\int_{\Gamma_{r,\nu}} f(t)\,dt & \text{for } u \in \Gamma_{r,\nu}. \end{cases} \tag{4.12}$$

Next, we prove that

$$|g_r(u)| \leq C_\gamma 2^r. \tag{4.13}$$

Once we have proved (4.13), we obtain the following representation of f:

$$f = g_0 + \sum_{r=0}^{\infty} (g_{r+1} - g_r) = g_0 + \sum_{r=0}^{\infty} \sum_{\nu} b_{r,\nu},$$

with $b_{r,\nu} = (g_{r+1} - g_r)\chi_{\Gamma_{r,\nu}}$. Then the functions

$$a_{r,\nu} = \frac{b_{r,\nu}}{C_\gamma 2^r |\Gamma_{r,\nu}|}$$

are 1-atoms; in fact,

$$\begin{aligned} \int_0^1 b_{r,\nu}(u)\,du &= \int_{\Gamma_{r,\nu} \cap B_{r+1}} g_{r+1}(u)\,du + \int_{\Gamma_{r,\nu} \setminus B_{r+1}} g_{r+1}(u)\,du - \int_{\Gamma_{r,\nu}} g_r(u)\,du \\ &= \int_{\Gamma_{r,\nu} \setminus B_{r+1}} f(u)\,du + \sum_{\mu : \Gamma_{r+1,\mu} \subset \Gamma_{r,\nu}} \int_{\Gamma_{r+1,\mu}} f(u)\,du - \int_{\Gamma_{r,\nu}} f(u)\,du = 0, \end{aligned}$$

and now it is easy to check that the $a_{r,\nu}$'s satisfy the conditions for 1-atoms. Since $a_{r,\nu}$'s are 1-atoms, we obtain

$$\begin{aligned}\|f\|_{H^1} &\le \Big|\int_0^1 f(u)\,du\Big| + C_\gamma \sum_{r=0}^{\infty}\sum_{\nu} 2^r|\varGamma_{r,\nu}| \\ &\le \|S\|_1 + C_\gamma \sum_{r=0}^{\infty} 2^r|B_r| \le \|S\|_1 + C_\gamma \sum_{r=0}^{\infty} 2^r|E_r| \le C_\gamma\|S\|_1.\end{aligned}$$

Therefore, it remains to prove (4.13). First, note that $|f(u)| \le S(u)$. For $u \in B_r^{\mathrm{c}}$ and given $m \ge 0$, let $J \in \mathcal{I}$ be such that $u \in J$ and $S_m = \sum_{n=0}^{m} a_n f_n$ is linear on J; then $J \not\subset B_r$, so by the definition of B_r we have

$$|J \cap E_r^{\mathrm{c}}| \ge \frac{2\gamma+1}{2\gamma+2}|J|.$$

Since on E_r^{c} we have $|S_m(\cdot)| \le 2^r$, by Proposition 2.13 we get $|S_m(\cdot)| \le C_\gamma 2^r$ on J, which implies $S(\cdot) \le C_\gamma 2^r$ and $|g_r(\cdot)| \le C_\gamma 2^r$ on B_r^{c}.

Consider now $|g_r(\cdot)|$ on B_r, and let $\varGamma_{r,\nu}$ be one of the intervals in the second representation of B_r in (4.11). Let

$$j_0 = \min\{j : \text{there is } I \in \mathcal{I}_j \text{ such that } I \subset \varGamma_{r,\nu}\}$$

and let I_{j_0} be an interval of rank j_0 included in $\varGamma_{r,\nu}$; note that there are at most two adjacent intervals with this property, and by strong regularity $|I_{j_0}| \sim |\varGamma_{r,\nu}|$. Moreover, denote by $t_{\mathrm{L}}, t_{\mathrm{R}}$ the left and right endpoints of $\varGamma_{r,\nu}$, and let I_j^{L}, I_j^{R} be intervals of rank $j \ge j_0$ containing t_{L} and t_{R} respectively; in case one of the points $t_{\mathrm{L}}, t_{\mathrm{R}}$ is a point from the partition $\mathcal{P}_j$, the corresponding interval is chosen in such a way that it is not contained in $\varGamma_{r,\nu}$. Note that $I_{j+1}^{\mathrm{L}} \subset I_j^{\mathrm{L}}$, $I_{j+1}^{\mathrm{R}} \subset I_j^{\mathrm{R}}$. By strong regularity we have $|I_{j_0}^{\mathrm{L}}| \sim |I_{j_0}^{\mathrm{R}}| \sim |\varGamma_{r,\nu}|$, and therefore, by Proposition 2.6(i),

$$|I_j^{\mathrm{L}}| \le C_\gamma\Big(\frac{\gamma}{\gamma+1}\Big)^{j-j_0}|\varGamma_{r,\nu}|, \qquad |I_j^{\mathrm{R}}| \le C_\gamma\Big(\frac{\gamma}{\gamma+1}\Big)^{j-j_0}|\varGamma_{r,\nu}|. \tag{4.14}$$

Now, we have

$$\int_{\varGamma_{r,\nu}} f(u)\,du = \int_{\varGamma_{r,\nu}} \sum_{n=0}^{2^{j_0}} a_n f_n(u)\,du + \sum_{j=j_0}^{\infty} \int_{\varGamma_{r,\nu}} \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u)\,du. \tag{4.15}$$

Let us estimate the first term in the sum (4.15). The interval $I_{j_0}^{\mathrm{L}}$ is not included in $\varGamma_{r,\nu}$, and as the intervals $\varGamma_{r,\cdot}$ do not have common endpoints, it follows that $I_{j_0}^{\mathrm{L}}$ is not included in B_r; therefore, by the definitions of the sets E_r and B_r, we get

$$|I_{j_0}^{\mathrm{L}} \cap E_r^{\mathrm{c}}| > \frac{2\gamma+1}{2\gamma+2}|I_{j_0}^{\mathrm{L}}|,$$

and for $u \in I_{j_0}^{\mathrm{L}} \cap E_r^{\mathrm{c}}$ we have $|\sum_{n=0}^{2^{j_0}} a_n f_n(u)| \le 2^r$; as the function $\sum_{n=0}^{2^{j_0}} a_n f_n(\cdot)$ is linear on $I_{j_0}^{\mathrm{L}}$, Proposition 2.13 implies that $|\sum_{n=0}^{2^{j_0}} a_n f_n(u)| \le C_\gamma 2^r$ on $I_{j_0}^{\mathrm{L}}$. The same argument gives $|\sum_{n=0}^{2^{j_0}} a_n f_n(u)| \le C_\gamma 2^r$ on $I_{j_0}^{\mathrm{R}}$. Moreover, note that, by the choice of

j_0, for $J \in \mathcal{I}_{j_0-1}$ with $I_{j_0} \subset J$ we have $J \not\subset B_r$; therefore, by Proposition 2.14 we have $|I_{j_0} \cap E_r^{\mathrm{c}}| \ge (1/(2\gamma))|I_{j_0}|$, and by the same argument as previously, $|\sum_{n=0}^{2^{j_0}} a_n f_n(u)| \le C_\gamma 2^r$ on I_{j_0}. The intervals $I_{j_0}, I_{j_0}^{\mathrm{L}}, I_{j_0}^{\mathrm{R}}$ cover $\Gamma_{r,\nu}$, so we get

$$\Big| \int_{\Gamma_{r,\nu}} \sum_{n=0}^{2^{j_0}} a_n f_n(u)\, du \Big| \le C_\gamma 2^r |\Gamma_{r,\nu}|. \tag{4.16}$$

Consider now $\sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u)$, $j \ge j_0$, on I_j^{L}; as $I_j^{\mathrm{L}} \not\subset B_r$, by Proposition 2.14 we have $|I_{j,-}^{\mathrm{L}} \cap E_r^{\mathrm{c}}| \ge (1/(2\gamma))|I_{j,-}^{\mathrm{L}}|$ and $|I_{j,+}^{\mathrm{L}} \cap E_r^{\mathrm{c}}| \ge (1/(2\gamma))|I_{j,+}^{\mathrm{L}}|$, where $I_{j,-}^{\mathrm{L}}, I_{j,+}^{\mathrm{L}}$ are intervals of rank $j+1$ included in I_j^{L}. Since $\sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u)$ is linear on both $I_{j,-}^{\mathrm{L}}$ and $I_{j,+}^{\mathrm{L}}$, by Proposition 2.13 we get

$$\Big| \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u) \Big| \le C_\gamma 2^r \quad \text{on } I_j^{\mathrm{L}},$$

and by the same argument we obtain an analogous estimate on I_j^{R}. Now, let Φ_j be the unique function from $\mathcal{S}_{\mathcal{P}_j}$ equal to 1 on $\Gamma_{r,\nu} \setminus (I_j^{\mathrm{L}} \cup I_j^{\mathrm{R}})$, and equal to 0 on $(\Gamma_{r,\nu} \cup I_j^{\mathrm{L}} \cup I_j^{\mathrm{R}})^{\mathrm{c}}$. Since all the functions f_n with $n > 2^j$ are orthogonal to Φ_j, we obtain (cf. (4.14))

$$\begin{aligned}
\Big| \int_{\Gamma_{r,\nu}} \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u)\, du \Big| &= \Big| \int_{\Gamma_{r,\nu}} \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u)\, du - \int_0^1 \Big(\sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u) \Big) \Phi_j(u)\, du \Big| \\
&\le \int_{I_j^{\mathrm{L}}} \Big| \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u) \Big|\, du + \int_{I_j^{\mathrm{R}}} \Big| \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u) \Big|\, du \\
&\le C_\gamma 2^r (|I_j^{\mathrm{L}}| + |I_j^{\mathrm{R}}|) \le C_\gamma 2^r \Big(\frac{\gamma}{\gamma+1} \Big)^{j-j_0} |\Gamma_{r,\nu}|,
\end{aligned}$$

which gives

$$\sum_{j=j_0}^{\infty} \Big| \int_{\Gamma_{r,\nu}} \sum_{n=2^j+1}^{2^{j+1}} a_n f_n(u)\, du \Big| \le \sum_{j=j_0}^{\infty} C_\gamma 2^r \Big(\frac{\gamma}{\gamma+1} \Big)^{j-j_0} |\Gamma_{r,\nu}| \le C_\gamma 2^r |\Gamma_{r,\nu}|.$$

The above inequality, (4.15), (4.16) and (4.12) give $|g_r(\cdot)| \le C_\gamma 2^r$ on $\Gamma_{r,\nu}$, and the proof of Lemma 4.5 is complete. ∎

LEMMA 4.6. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and $0 < p \le 1$. Let $(a_n)_{n\ge 0}$ be a sequence of real coefficients. Then* (A) *implies* (B) *and* (C). *Moreover, there is a constant $C_{\gamma,p}$, depending only on p and γ, such that*

$$\sup_{\varepsilon} \Big\| \sum_{n=0}^{\infty} \varepsilon_n a_n f_n \Big\|_p \le C_{\gamma,p} \|P\|_p, \qquad \|S\|_p \le C_{\gamma,p} \|P\|_p,$$

where the supremum is taken with respect to all sequences $\varepsilon = (\varepsilon_n)_{n\ge 0}$ with $\varepsilon_n \in \{1, -1\}$.

Proof. For convenience, let $\|P\|_p = 1$. Define $E_0 = [0,1]$,

$$E_r = \Big\{u \in [0,1] : \sum_{n=0}^{\infty} a_n^2 f_n^2(u) > 2^r\Big\}, \quad r = 1,2,\ldots,$$

$$B_r = \Big\{u \in [0,1] : \mathcal{M}^*(\chi_{E_r}, u) > \frac{1}{2\gamma+2}\Big\}, \quad r = 0,1,2,\ldots,$$

and let $\mathcal{T}_r$ be the set of maximal intervals from $\mathcal{I}$ included in B_r. First, we prove the following technical estimate:

$$\int_{I^c} \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} |a_n f_n(u)|^p\,du \le C_{\gamma,p} 2^{rp/2}|I| \quad \text{for } I \in \mathcal{T}_r. \tag{4.17}$$

Indeed, if $\{n\} \not\subset B_{r+1}$ then Proposition 2.14, the definition of E_{r+1}, Proposition 2.13 and the decay of $|f_n|$ (cf. Proposition 2.9, inequalities (2.26) and (2.30)) imply that $\|a_n f_n\|_\infty \le C_\gamma 2^{(r+1)/2}$. Using the bound for $\|f_n\|_\infty$ from Proposition 2.9 (inequality (2.23)) we get $|a_n| \le C_\gamma 2^{r/2}|\{n\}|^{1/2}$. Applying this estimate and Lemma 2.11 (inequality (2.34)) we obtain (4.17).

Now, for $I \in \mathcal{T}_r$ let

$$\psi_I = \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} a_n f_n.$$

We check that this formula defines an L^2 function. As for $\{n\} \not\subset B_{r+1}$ we have

$$|\{n\} \cap E_{r+1}^c| \ge \frac{2\gamma+1}{2\gamma+2}|\{n\}|,$$

by Proposition 2.12 it follows that $\int_{\{n\}\cap E_{r+1}^c} f_n^2(u)\,du \ge C_\gamma$. Therefore, we obtain (cf. the definition of E_{r+1})

$$\begin{aligned}\|\psi_I\|_2^2 &= \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} a_n^2 \le C_\gamma \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} a_n^2 \int_{\{n\}\cap E_{r+1}^c} f_n^2(u)\,du\\ &\le C_\gamma \int_{I\setminus E_{r+1}} \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\,du \le C_\gamma 2^r |I|.\end{aligned}$$

Thus, we have

$$\|\psi_I\|_2 \le C_\gamma 2^{r/2}|I|^{1/2} \quad \text{for } I \in \mathcal{T}_r. \tag{4.18}$$

We are ready to prove the unconditional convergence of $\sum_{n=0}^{\infty} a_n f_n$. Let $\varepsilon = (\varepsilon_n)_{n\ge 0}$ with $\varepsilon_n \in \{-1,1\}$ and

$$\psi_{I,\varepsilon} = \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} \varepsilon_n a_n f_n \quad \text{for } I \in \mathcal{T}_r.$$

The series defining $\psi_{I,\varepsilon}$ converges in L^2, so it converges in L^p as well (recall that $0 < p \le 1$). Therefore, it is sufficient to prove that there is a constant $C_{\gamma,p}$ such that

$$\sum_{r=0}^{\infty} \sum_{I\in\mathcal{T}_r} \|\psi_{I,\varepsilon}\|_p^p \le C_{\gamma,p}.$$

To estimate $\|\psi_{I,\varepsilon}\|_p^p$, note that by the Hölder inequality (with exponents $2/p$ and $2/(2-p)$) and (4.18) we get

$$\int_I |\psi_{I,\varepsilon}(u)|^p\,du \le \Big(\int_I |\psi_{I,\varepsilon}(u)|^2\,du\Big)^{p/2} |I|^{1-p/2} = \|\psi_I\|_2^p\,|I|^{1-p/2} \le C_{\gamma,p} 2^{rp/2}|I|.$$

Moreover, as for $p \le 1$,

$$|\psi_{I,\varepsilon}(u)|^p \le \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} |a_n f_n(u)|^p,$$

by (4.17) we have

$$\int_{I^c} |\psi_{I,\varepsilon}(u)|^p\,du \le C_{\gamma,p} 2^{rp/2}|I|.$$

Thus, $\|\psi_{I,\varepsilon}\|_p^p \le C_{\gamma,p}2^{rp/2}|I|$, which implies (cf. the definitions of the sets E_r and B_r)

$$\begin{aligned}\sum_{r=0}^{\infty}\sum_{I\in\mathcal{T}_r} \|\psi_{I,\varepsilon}\|_p^p &\le C_{\gamma,p}\sum_{r=0}^{\infty}\sum_{I\in\mathcal{T}_r} 2^{rp/2}|I| \le C_{\gamma,p}\sum_{r=0}^{\infty} 2^{rp/2}|B_r| \\ &\le C_{\gamma,p}\sum_{r=0}^{\infty} 2^{rp/2}|E_r| \le C_{\gamma,p}\|P\|_p^p \le C_{\gamma,p},\end{aligned}$$

which in turn implies the unconditional convergence of the series $\sum_{n=0}^{\infty} a_n f_n$ in L^p. Moreover, note that the last chain of inequalities also implies

$$\sup_{\varepsilon}\Big\|\sum_{n=0}^{\infty}\varepsilon_n a_n f_n\Big\|_p^p \le C_{\gamma,p}.$$

It remains to estimate $\|S\|_p$; clearly,

$$S(u) \le \sum_{r=0}^{\infty}\sum_{I\in\mathcal{T}_r} S\psi_I(u), \quad \text{which implies} \quad S(u)^p \le \sum_{r=0}^{\infty}\sum_{I\in\mathcal{T}_r} S\psi_I(u)^p.$$

The maximal inequality from Theorem 2.5(i) implies that $\|S\psi_I\|_2 \le C\|\psi_I\|_2$; moreover,

$$S\psi_I(u)^p \le \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} |a_n f_n(u)|^p.$$

Therefore, similar arguments to those used for estimating $\|\psi_{I,\varepsilon}\|_p^p$ give

$$\|S\psi_I\|_p^p \le C_{\gamma,p}2^{rp/2}|I|,$$

so summing over $r \ge 0$ and $I \in \mathcal{T}_r$ we get

$$\|S\|_p^p \le C_{\gamma,p},$$

which completes the proof of Lemma 4.6. ∎

Lemma 4.7. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and $0 < p \le 1$. Let $(a_n)_{n\ge 0}$ be a sequence of real coefficients. Then* (B) *implies* (A). *Moreover, there is a constant C_p, depending only on p,* *such that*

$$\|P\|_p \le C_{\gamma,p} \sup_{\varepsilon} \Big\| \sum_{n=0}^{\infty} \varepsilon_n a_n f_n \Big\|_p,$$

where the supremum is taken over all sequences $\varepsilon = (\varepsilon_n)_{n\ge 0}$ *with* $\varepsilon_n \in \{1,-1\}$.

Proof. Denote by $\{r_n : n \ge 1\}$ the sequence of Rademacher functions. Khinchin's inequalities state that for each p, $0 < p < \infty$, there are constants A_p, B_p (finite and positive) such that for any sequence $(c_n)_{n\ge1}$ of real coefficients with $\sum_{n=1}^{\infty} |c_n|^2 < \infty$,

$$A_p \Big(\sum_{n=1}^{\infty} |c_n|^2 \Big)^{1/2} \le \Big\| \sum_{n=1}^{\infty} c_n r_n \Big\|_p \le B_p \Big(\sum_{n=1}^{\infty} |c_n|^2 \Big)^{1/2}$$

(cf. for example [31], Chapter V, Theorem 8.4). The unconditional convergence of $\sum_{n=0}^{\infty} a_n f_n$ in L^p implies that for each $\varepsilon = (\varepsilon_n)_{n\ge0}$ with $\varepsilon_n \in \{-1,1\}$, the series $\sum_{n=0}^{\infty} \varepsilon_n a_n f_n$ converges in L^p (cf. [24]). This in turn implies that

$$\lim_{m\to\infty} \sup_{\varepsilon} \Big\| \sum_{n=m}^{\infty} \varepsilon_n a_n f_n \Big\|_p^p = 0 \quad \text{and} \quad \sup_{\varepsilon} \Big\| \sum_{n=0}^{\infty} \varepsilon_n a_n f_n \Big\|_p^p = M^p < \infty.$$

Now, a standard argument gives $P \in L^p$ and $\|P\|_p^p \le C_p M^p$. ∎

Lemma 4.8. *Let the quasi-dyadic sequence* $\{\mathcal{P}_j : j \ge 0\}$ *of partitions satisfy the strong regularity condition with parameter* γ *and* $0 < p \le 1$. *Let* $(a_n)_{n\ge0}$ *be a sequence of real coefficients. Then* (C) *implies* (A). *Moreover*, *there is a constant* $C_{\gamma,p}$, *depending only on* p *and* γ, *such that*

$$\|P\|_p \le C_{\gamma,p} \|S\|_p.$$

Proof. Suppose $\|S\|_p = 1$, and put $E_0 = [0,1]$,

$$E_r = \{u \in [0,1] : S(u) > 2^r\}, \qquad r = 1,2,\ldots,$$

$$B_r = \Big\{ u \in [0,1] : \mathcal{M}^*(\chi_{E_r}, u) > \frac{1}{2\gamma+2} \Big\}, \quad r = 0,1,2,\ldots$$

Consider the following decompositions of B_r:

$$B_r = \bigcup_{I\in\mathcal{T}_r} I = \bigcup_{\nu} \varGamma_{r,\nu}, \tag{4.19}$$

with the last union countable, where $\mathcal{T}_r$ is the family of maximal intervals from $\mathcal{I}$ included in B_r, and each $\varGamma_{r,\nu}$ is an interval which is a union of some intervals from $\mathcal{T}_r$, and no two $\varGamma_{r,\nu}$'s have a common endpoint. As for each $I \in \mathcal{T}_{r+1}$ there is $J \in \mathcal{T}_r$ such that $I \subset J$, it follows that for each $\varGamma_{r+1,\nu}$ there is $\varGamma_{r,\mu}$ with $\varGamma_{r+1,\nu} \subset \varGamma_{r,\mu}$.

Let us begin with some auxiliary calculations.

Auxiliary calculations — functions ζ_r, $\varphi_{r,\nu}$ *and* $\varPhi$. Denoting by j_I the rank of the interval $I \in \mathcal{I}$, we define

$$\zeta_r(u) = \begin{cases} 0 & \text{if } u \notin B_r, \\ \displaystyle\sum_{j\ge j_I} \sum_{\substack{n:2^j<n\le2^{j+1} \\ \{n\}\not\subset B_r}} |a_n f_n(u)| & \text{for } u \in I \text{ with } I \in \mathcal{T}_r. \end{cases}$$

In the sequel, the following estimate is needed: there is a constant C_γ such that

$$\int_0^1 \zeta_r(u)\,du \le C_\gamma 2^r |B_r|. \tag{4.20}$$

Indeed, we have

$$\int_0^1 \zeta_r(u)\,du = \sum_{I\in\mathcal{T}_r}\int_I \sum_{j\ge j_I} \sum_{\substack{n:2^j<n\le 2^{j+1}\\ \{n\}\not\subset B_r}} |a_n f_n(u)|\,du.$$

Observe that if $\{n\}\not\subset B_r$ then Proposition 2.14, the definition of E_r, Proposition 2.13 and the decay of f_n (cf. Proposition 2.9, inequalities (2.26) and (2.30)) imply that $\|a_n f_n\|_\infty \le C_\gamma 2^r$; this estimate and inequality (2.23) give $|a_n| \le C_\gamma 2^r |\{n\}|^{1/2}$. Moreover, if $\{n\}\not\subset B_r$ then $\{n\}\not\subset I$ for $I\in\mathcal{T}_r$. Denote by I_j^-, I_j^+, $j\ge j_I$, the intervals of rank j included in I and containing the left and right endpoints of I, respectively. Using the estimates from Proposition 2.9: the decay of integrals of Franklin functions (i.e. inequalities (2.29) and (2.33)) and the pointwise decay of Franklin functions (inequalities (2.27) and (2.31)), the above estimate for $|a_n|$, and the estimate for the length of I_j^-, I_j^+ from Proposition 2.6(i), we get

$$\begin{aligned}\sum_{j\ge j_I}\int_I \sum_{\substack{n:2^j<n\le 2^{j+1}\\ \{n\}\not\subset B_r}} |a_n f_n(u)|\,du &\le \frac{3\sqrt{2}-3}{3\sqrt{2}-4}\sum_{j\ge j_I}\Big(\int_{I_j^-} + \int_{I_{j+1}^+}\Big) \sum_{\substack{n:2^j<n\le 2^{j+1}\\ \{n\}\not\subset B_r}} |a_n f_n(u)|\,du\\ &\le C_\gamma 2^r \sum_{j\ge j_I}(|I_j^-| + |I_{j+1}^+|) \le C_\gamma 2^r |I|,\end{aligned}$$

so summing over $I\in\mathcal{T}_r$ we obtain (4.20).

Now, let $\Gamma_{r,\nu}$ be as in the second representation of B_r in (4.19); let $j_{r,\nu}$ be the minimal rank of an interval from $\mathcal{T}_r$ included in $\Gamma_{r,\nu}$; moreover, put

$$J_{r,\nu,j} = \bigcup_{\substack{I\in\mathcal{I}_j\\ I\subset\Gamma_{r,\nu}}} I \quad \text{for } j\ge j_{r,\nu}.$$

Note that $\{J_{r,\nu,j} : j\ge j_{r,\nu}\}$ is an increasing sequence of intervals and

$$\Gamma_{r,\nu} = \bigcup_{j\ge j_{r,\nu}} J_{r,\nu,j}.$$

Moreover, if $2^j < n \le 2^{j+1}$ and $\{n\}\subset\Gamma_{r,\nu}$, then $\{n\}\subset J_{r,\nu,j}$. Now, put

$$\sigma_{r,\nu,j,m}(u) = \sum_{\substack{n:2^j<n\le 2^{j+1}\wedge m\\ \{n\}\subset\Gamma_{r,\nu}}} a_n f_n(u), \quad m\ge 1,$$

$$\varphi_{r,\nu}(u) = \begin{cases} 0 & \text{if } u\in J_{r,\nu,j_{r,\nu}},\\ \displaystyle\sum_{i=j_{r,\nu}}^{j-1}\max_m |\sigma_{r,\nu,i,m}(u)| & \text{if } u\in J_{r,\nu,j}\setminus J_{r,\nu,j-1},\\ \displaystyle\sum_{i=j_{r,\nu}}^{\infty}\max_m |\sigma_{r,\nu,i,m}(u)| & \text{if } u\notin\Gamma_{r,\nu}.\end{cases}$$

In the sequel, the following estimate is needed: there is a constant C_γ such that

$$\int_0^1 \varphi_{r,\nu}(u)\,du \le C_\gamma 2^r |\Gamma_{r,\nu}|. \tag{4.21}$$

To check this inequality, note that

$$\int_0^1 \varphi_{r,\nu}(u)\,du = \sum_{j=j_{r,\nu}}^{\infty} \int_{J^{\mathrm{c}}_{r,\nu,j}} \max_m |\sigma_{r,\nu,j,m}(u)|\,du.$$

To estimate the terms appearing in the sum on the right-hand side of the above equality, let Δ_j^-, Δ_j^+ be the intervals of rank $j \ge j_{r,\nu}$ with the right endpoint of Δ_j^- being the left endpoint of $J_{r,\nu,j}$ and the left endpoint of Δ_j^+ being the right endpoint of $J_{r,\nu,j}$. As $\Delta_{j+1}^-, \Delta_j^+ \not\subset B_r$, and $\sigma_{r,\nu,j,m}(\cdot)$ is linear on these intervals, by Proposition 2.13 we get $|\sigma_{r,\nu,j,m}(\cdot)| \le C_\gamma 2^r$ on Δ_{j+1}^- and Δ_j^+. Denote by $J^{\mathrm{c,L}}_{r,\nu,j}, J^{\mathrm{c,R}}_{r,\nu,j}$ the left-hand part and right-hand part of $J^{\mathrm{c}}_{r,\nu,j}$, respectively. Consider the function $\sigma_{r,\nu,j,m}(\cdot)$ on $J^{\mathrm{c,R}}_{r,\nu,j}$. By Lemma 2.10, for all m this is a multiple of some Franklin function f_n with $2^j < n \le 2^{j+1}$ and $\{n\} \subset J_{r,\nu,j}$; this implies that $\max_m |\sigma_{r,\nu,j,m}(\cdot)|$ on this set is a multiple of $|f_n|$, and therefore the estimates for integrals of Franklin functions from Proposition 2.9 (i.e. inequality (2.33)) imply

$$\int_{J^{\mathrm{c,R}}_{r,\nu,j}} \max_m |\sigma_{r,\nu,j,m}(u)|\,du \le \frac{3\sqrt{2}-3}{3\sqrt{2}-4} \int_{\Delta_j^+} \max_m |\sigma_{r,\nu,j,m}(u)|\,du \le C_\gamma 2^r |\Delta_j^+|.$$

The integral over $J^{\mathrm{c,L}}_{r,\nu,j}$ is treated analogously, which gives

$$\int_{J^{\mathrm{c,L}}_{r,\nu,j}} \max_m |\sigma_{r,\nu,j,m}(u)|\,du \le C_\gamma 2^r |\Delta_{j+1}^-|.$$

Proposition 2.6(i) and strong regularity of the sequence of partitions imply that

$$|\Delta_j^-|, |\Delta_j^+| \le C_\gamma \left(\frac{\gamma}{\gamma+1}\right)^{j-j_{r,\nu}} |\Gamma_{r,\nu}|,$$

and we get

$$\sum_{j=j_{r,\nu}}^{\infty} \int_{J^{\mathrm{c}}_{r,\nu,j}} \max_m |\sigma_{r,\nu,j,m}(u)|\,du \le \sum_{j=j_{r,\nu}}^{\infty} C_\gamma 2^r \left(\frac{\gamma}{\gamma+1}\right)^{j-j_{r,\nu}} |\Gamma_{r,\nu}| \le C_\gamma 2^r |\Gamma_{r,\nu}|,$$

which proves (4.21).

Define

$$\Phi_r = \zeta_r + \sum_\nu \varphi_{r,\nu}.$$

It follows from (4.20) and (4.21) that

$$\int_0^1 \Phi_r(u)\,du \le C_\gamma 2^r |B_r|. \tag{4.22}$$

Auxiliary calculations — functions ψ_r. Now, introduce the auxiliary functions

$$\psi_r = \sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n f_n = \sum_{\{n\}\subset B_r} a_n f_n - \sum_{\{n\}\subset B_{r+1}} a_n f_n = \sum_{\{n\}\not\subset B_{r+1}} a_n f_n - \sum_{\{n\}\not\subset B_r} a_n f_n.$$

We want to estimate $S\psi_r$. First, note that for $u \notin B_r$,

$$\Big|\sum_{\substack{\{n\}\subset B_r\\ n\le m}} a_n f_n\Big| \le \sum_\nu \Big|\sum_{\substack{\{n\}\subset \Gamma_{r,\nu}\\ n\le m}} a_n f_n\Big| \le \sum_\nu \sum_{j=j_{r,\nu}}^\infty |\sigma_{r,\nu,j,m}(u)| \le \sum_\nu \varphi_{r,\nu}(u) \le \Phi_r(u).$$

As $u \notin B_r$ implies $u \notin B_{r+1}$, the above inequality and the second representation of ψ_r give

$$S\psi_r(u) \le \Phi_r(u) + \Phi_{r+1}(u) \quad \text{for } u \notin B_r. \tag{4.23}$$

Moreover, on B_r^{c} we have $S(\cdot) \le C_\gamma 2^r$ (cf. the analogous statement in the proof of Lemma 4.5), so for $u \notin B_r$ we get

$$\begin{aligned}\Big|\sum_{\substack{n\le m\\ \{n\}\not\subset B_r}} a_n f_n(u)\Big| &\le \Big|\sum_{n\le m} a_n f_n(u)\Big| + \Big|\sum_{\substack{n\le m\\ \{n\}\subset B_r}} a_n f_n(u)\Big| \\ &\le C_\gamma 2^r + \sum_\nu \Big|\sum_{\substack{n\le m\\ \{n\}\subset \Gamma_{r,\nu}}} a_n f_n(u)\Big| \\ &\le C_\gamma 2^r + \sum_\nu \varphi_{r,\nu}(u) \le C_\gamma 2^r + \Phi_r(u),\end{aligned}$$

which gives

$$S\Big(\sum_{\{n\}\not\subset B_r} a_n f_n, u\Big) \le C_\gamma 2^r + \Phi_r(u) \quad \text{for } u \notin B_r. \tag{4.24}$$

On the other hand, if $u \in B_r$ then $u \in I$ for some $I \in \mathcal{T}_r$; now, if $m \le 2^{j_I}$ then

$$\Big|\sum_{\substack{n\le m\\ \{n\}\not\subset B_r}} a_n f_n(u)\Big| \le \Big|\sum_{n=0}^m a_n f_n(u)\Big| + \Big|\sum_{\substack{n\le m\\ \{n\}\subset B_r}} a_n f_n(u)\Big|.$$

The first term on the right-hand side is bounded by $C_\gamma 2^r$, by the definition of E_r, B_r and maximality of I. Moreover, if $\{n\} \subset B_r$ and $n \le 2^{j_I}$ then $\{n\} \not\subset I$ and the second term can be bounded by $\sum_\nu \varphi_{r,\nu}(u)$, which gives

$$\Big|\sum_{\substack{n\le m\\ \{n\}\not\subset B_r}} a_n f_n(u)\Big| \le C_\gamma 2^r + \Phi_r(u).$$

For $m > 2^{j_I}$ we write

$$\Big|\sum_{\substack{n\le m\\ \{n\}\not\subset B_r}} a_n f_n(u)\Big| \le \Big|\sum_{n=0}^{2^{j_I}} a_n f_n(u)\Big| + \Big|\sum_{\substack{n\le 2^{j_I}\\ \{n\}\subset B_r}} a_n f_n(u)\Big| + \Big|\sum_{\substack{2^{j_I}<n\le m\\ \{n\}\not\subset B_r}} a_n f_n(u)\Big|.$$

As previously, the first two terms are bounded by $C_\gamma 2^r$ and $\sum_\nu \varphi_{r,\nu}(u)$, respectively, while the third term is bounded by $\zeta_r(u)$. Thus, we obtain

$$S\Big(\sum_{\{n\}\not\subset B_r} a_n f_n, u\Big) \le C_\gamma 2^r + \Phi_r(u) \quad \text{for } u \in B_r. \tag{4.25}$$

Applying (4.24) and (4.25) and using the third representation of ψ_r we get

$$S\psi_r(u) \le C_\gamma 2^r + \Phi_r(u) + \Phi_{r+1}(u) \quad \text{for } u \in B_r. \tag{4.26}$$

Therefore, (4.23) and (4.26) now give

$$S\psi_r(u) \le C_\gamma 2^r \chi_{B_r}(u) + \Phi_r(u) + \Phi_{r+1}(u).$$

Integrating the above inequality and using (4.22) we get

$$\int_0^1 S\psi_r(u)\,du \le C_\gamma 2^r |B_r|.$$

Applying Lemmas 4.5 and 4.4 we get the following bound for $\|P\psi_r\|_1$:

$$\int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\Big)^{1/2} du \le C_\gamma 2^r |B_r|. \tag{4.27}$$

Final part of the proof of Lemma 4.8. We need a bound of $\|P\psi_r\|_p^p$. First, using (4.27) and Hölder's inequality with exponents $1/p$ and $1/(1-p)$, we get

$$\int_{B_r} \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2} du \le |B_r|^{1-p} \Big(\int_{B_r} \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\Big)^{1/2} du\Big)^p \le C_{\gamma,p} 2^{rp} |B_r|.$$

It remains to estimate the integral over B_r^{c}. For $\{n\} \not\subset B_{r+1}$ we have $\|a_n f_n\|_\infty \le C_\gamma 2^r$ (by definition of B_{r+1}, E_{r+1} and Propositions 2.13 and 2.14). The estimate for $\|f_n\|_\infty$ (cf. inequality (2.23) in Proposition 2.9) now gives $|a_n| \le C_\gamma 2^r |\{n\}|^{1/2}$, so applying inequality (2.34) from Lemma 2.11 we get for $I \in \mathcal{T}_r$

$$\int_{I^{\mathrm{c}}} \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} |a_n f_n(u)|^p\,du \le C_{\gamma,p} 2^{rp} |I|.$$

Using this inequality we obtain

$$\begin{aligned}
\int_{B_r^{\mathrm{c}}} \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2} du &\le \sum_{I\in\mathcal{T}_r} \int_{B_r^{\mathrm{c}}} \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} |a_n f_n(u)|^p\,du \\
&\le \sum_{I\in\mathcal{T}_r} \int_{I^{\mathrm{c}}} \sum_{\substack{\{n\}\subset I\\ \{n\}\not\subset B_{r+1}}} |a_n f_n(u)|^p\,du \\
&\le C_{\gamma,p} \sum_{I\in\mathcal{T}_r} 2^{rp}|I| \le C_{\gamma,p} 2^{rp} |B_r|.
\end{aligned}$$

Thus, we have

$$\int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2}\,du \le C_{\gamma,p} 2^{rp}|B_r|,$$

and summing over $r \ge 0$ we obtain

$$\begin{aligned}\int_0^1 \Big(\sum_{n=0}^{\infty} a_n^2 f_n^2(u)\Big)^{p/2}\,du &\le \sum_{r=0}^{\infty}\int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2}\,du\\ &\le C_{\gamma,p}\sum_{r=0}^{\infty} 2^{rp}|B_r| \le C_{\gamma,p}. \ \blacksquare\end{aligned}$$

LEMMA 4.9. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and $1/2 < p \le 1$. Let $(a_n)_{n\ge 0}$ be a sequence of real coefficients such that $S(\cdot) = \sup_{m\ge 0} |\sum_{n=0}^m a_n f_n(\cdot)| \in L^p$. Then there is $f \in H^p$ such that $f = \sum_{n=0}^\infty a_n f_n$, with the series convergent in H^p. Moreover, there is a constant $C_{\gamma,p}$, depending only on p and γ, such that*

$$\|f\|_{H^p} \le C_{\gamma,p}\|S\|_p,$$

PROOF. First, consider a sequence $(a_n)_{n\ge 0}$ with a finite number of non-zero terms. Then, since $f_n \in H^p$, we have $h = \sum_{n=0}^\infty a_n f_n \in H^p \cap C[0,1]$, and moreover, by arguments analogous to those used in the proof of Lemma 4.5, we check that there is a constant $C_{\gamma,p}$, depending only on p and γ, such that

$$\|h\|_{H^p} \le C_{\gamma,p}\|S\|_p.$$

Now, let $(a_n)_{n\ge 0}$ be an arbitrary sequence with $S \in L^p$. Since $S \in L^p$, Lemma 4.8 gives $P \in L^p$. Thus, P is finite a.e.; define $P_m = (\sum_{n=m}^\infty a_n^2 f_n^2)^{1/2}$. Observe that $P_m \searrow 0$, which in turn gives $\|P_m\|_p \to 0$ as $m \to \infty$. Consider the sequence $h_k = \sum_{n=0}^k a_n f_n$. Combining Lemma 4.6, the decay of $\|P_m\|_p$ and the part of Lemma 4.9 just proved for finite sequences, we find that h_k is a Cauchy sequence in H^p, and by completeness of H^p, there is $f \in H^p$ with $f = \lim_{k\to\infty} h_k$. Moreover, combining the above calculations with Lemmas 4.6 and 4.8, we get

$$\begin{aligned}\|f\|_{H^p} &= \lim_{k\to\infty}\|h_k\|_{H^p} \le C_{\gamma,p}\sup_{k\ge 0}\|Sh_k\|_p\\ &\le C_{\gamma,p}\sup_{k\ge 0}\|Ph_k\|_p \le C_{\gamma,p}\|P\|_p \le C_{\gamma,p}\|S\|_p. \ \blacksquare\end{aligned}$$

Now, we are ready to prove Theorems 4.1 and 4.2.

Proof of Theorem 4.1. Theorem 4.1 is a consequence of Lemmas 4.6, 4.7 and 4.8. ∎

Proof of Theorem 4.2. Let $f \in H^p$ and $a_n = (f, f_n)$, $n \ge 0$. Put $s_m f = \sum_{n=0}^m a_n f_n$. By Lemmas 4.9, 4.6 and 4.4 we have

$$\|s_m f\|_{H^p} \le C_{\gamma,p}\|S(s_m f)\|_p \le C_{\gamma,p}\|P(s_m f)\|_p \le C_{\gamma,p}\|Pf\|_p \le C_{\gamma,p}\|f\|_{H^p}.$$

If f is a continuous function then $s_m f \to f$ in the uniform norm, which implies that $s_m f \to f$ in H^p. As the continuous functions are dense in H^p, this and the last inequalities imply that $s_m f \to f$ in H^p for all $f \in H^p$. Moreover, $(\cdot, f_n)$ is a continuous linear functional on H^p, which implies that if $f = \sum_{n=0}^{\infty} b_n f_n$, with the series convergent in H^p, then $b_n = (f, f_n)$. Thus, the system $\{f_n : n \geq 0\}$ is a basis in H^p. Its unconditionality follows from Lemmas 4.4, 4.6 and 4.9.

Equivalences (4.1) now follow from Lemmas 4.4, 4.6, 4.7 and 4.9.

Finally, the equivalence of conditions (A)–(E) follows from the unconditionality of the basis $\{f_n : n \geq 0\}$ in H^p, Theorem 4.1 and the lemmas just mentioned. ∎

5. The necessity of strong regularity in H^p, $1/2 < p \leq 1$

The main result of this section is the following:

THEOREM 5.1. *Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions and $1/2 < p \leq 1$. If the sequence $\{\mathcal{P}_j : j \geq 0\}$ does not satisfy the strong regularity condition, then the corresponding Franklin system is not a basis in H^p.*

The proof is based on the following lemma:

LEMMA 5.2. *Let $\varepsilon > 0$ and let $\pi = \{\tau_i : 0 \leq i \leq m\}$ be a partition of $[0,1]$ such that there exist three consecutive intervals $\Lambda_{k-1}, \Lambda_k, \Lambda_{k+1}$, where $\Lambda_l = [\tau_{l-1}, \tau_l]$, with the following property: either*

$$|\Lambda_{k+1}| \leq \varepsilon |\Lambda_{k-1}| \quad \text{and} \quad |\Lambda_k| \leq \varepsilon |\Lambda_{k-1}|,$$

or

$$|\Lambda_{k-1}| \leq \varepsilon |\Lambda_{k+1}| \quad \text{and} \quad |\Lambda_k| \leq \varepsilon |\Lambda_{k+1}|.$$

Let Q_π be the orthogonal (in L^2) projection onto $\mathcal{S}_\pi$ and for given p, $1/2 < p \leq 1$, define

$$\|Q_\pi\|_{H^p} = \sup_{f \in H^p} \frac{\|Q_\pi f\|_{H^p}}{\|f\|_{H^p}}.$$

Then for each p, $1/2 < p \leq 1$, there are ε_p and C_p, depending on p only, such that for all partitions π satisfying the above condition with $0 < \varepsilon \leq \varepsilon_p$,

$$\|Q_\pi\|_{H^1} \geq C_1 \log(1/\varepsilon) \quad \text{in case } p = 1,$$
$$\|Q_\pi\|_{H^p} \geq C_p \varepsilon^{1-1/p} \quad \text{in case } \tfrac{1}{2} < p < 1.$$

PROOF. For convenience, let $\lambda_l = |\Lambda_l|$. Suppose that the first set of inequalities is satisfied, i.e.

$$\lambda_{k+1} \leq \varepsilon \lambda_{k-1} \quad \text{and} \quad \lambda_k \leq \varepsilon \lambda_{k-1}. \tag{5.1}$$

We consider in detail the case when none of the intervals $\Lambda_{k-1}, \Lambda_k, \Lambda_{k+1}$ touches the boundary of $[0,1]$; the other cases, i.e. $k = 2$ or $k = m-1$, require only minor technical changes, and the detailed calculations are omitted.

Consider the function

$$\varphi(u) = \begin{cases} 1/\lambda_k & \text{for } u \in \Lambda_k = [\tau_{k-1}, \tau_k], \\ -1/\lambda_k & \text{for } u \in (\tau_{k-1} - \lambda_k, \tau_{k-1}), \\ 0 & \text{otherwise.} \end{cases}$$

Since $2^{-1/p}\lambda_k^{1-1/p}\varphi$ is a p-atom, we have

$$\|\varphi\|_{H^p} \le 2^{1/p}\lambda_k^{1/p-1}. \tag{5.2}$$

Let us estimate $\|Q_\pi\varphi\|_{H^p}$ from below. For this purpose, we use the following fact (an analog of a part of Theorem 11 of [15]):

Let $1/2 < p \le 1$ and $\psi(u) = \max(0, 1-|u|)$, $\psi_\zeta(u) = (1/\zeta)\psi(u/\zeta)$. For $f \in H^p$, define

$$f^*(u) = \sup_{\zeta>0} |(f, \psi_\zeta(u-\cdot))|.$$

Then there is a constant C_p, depending on p only, such that

$$\|f^*\|_p \le C_p\|f\|_{H^p}. \tag{5.3}$$

Since $Q_\pi\varphi \in \mathcal{S}_\pi$, we have $Q_\pi\varphi = \sum_{i=0}^m a_iN_i$, and as Q_π is the orthogonal projection onto $\mathcal{S}_\pi$, the coefficients a_i satisfy the equations $\sum_{i=0}^m a_i(N_i, N_j) = (\varphi, N_j)$, $j = 0, \ldots, m$. By straightforward calculation we get

$$(\varphi, N_j) = \begin{cases} 0 & \text{for } j \le k-3 \text{ and } j \ge k+1, \\ \dfrac{-\lambda_k}{2\lambda_{k-1}} & \text{for } j = k-2, \\ \dfrac{\lambda_k - \lambda_{k-1}}{2\lambda_{k-1}} & \text{for } j = k-1, \\ \dfrac{1}{2} & \text{for } j = k. \end{cases}$$

Thus, the equations for the a_i's take the following form (cf. formula (2.1) for (N_i, N_j)):

$$\begin{cases} 2a_0 + a_1 = 0, \\ \lambda_i a_{i-1} + 2(\lambda_i + \lambda_{i+1})a_i + \lambda_{i+1}a_{i+1} = 0 \quad \text{for } i \le k-3 \text{ or } i \ge k+1, \\ a_{m-1} + 2a_m = 0 \end{cases} \tag{5.4}$$

and

$$\begin{cases} \lambda_{k-2}a_{k-3} + 2(\lambda_{k-2} + \lambda_{k-1})a_{k-2} + \lambda_{k-1}a_{k-1} = -3\dfrac{\lambda_k}{\lambda_{k-1}}, \\ \lambda_{k-1}a_{k-2} + 2(\lambda_{k-1} + \lambda_k)a_{k-1} + \lambda_k a_k = 3\dfrac{\lambda_k - \lambda_{k-1}}{\lambda_{k-1}}, \\ \lambda_k a_{k-1} + 2(\lambda_k + \lambda_{k+1})a_k + \lambda_{k+1}a_{k+1} = 3. \end{cases} \tag{5.5}$$

Arguments analogous to those used in the proof of Proposition 2.1 now give:

$$\text{for } i \ge k: \quad \begin{cases} a_i a_{i+1} \le 0, \\ \left(2\lambda_{i+1} + \frac{3}{2}\lambda_{i+2}\right)|a_{i+1}| \le \lambda_{i+1}|a_i| \le 2(\lambda_{i+1} + \lambda_{i+2})|a_{i+1}|, \end{cases} \tag{5.6}$$

$$a_i a_{i+1} \le 0, \quad |a_i| \le \tfrac{1}{2}|a_{i+1}| \quad \text{for } i \le k-3. \tag{5.7}$$

Further, note that $\|\varphi\|_1 = 2$; as $\|Q_\pi\varphi\|_1 \le 3\|\varphi\|_1$ (see Theorem 2.5 (i)), we have

$$\frac{2}{5}\lambda_{k-1}\max(|a_{k-1}|, |a_{k-2}|) \le \int_{\tau_{k-2}}^{\tau_{k-1}} |Q_\pi\varphi(u)|\, du \le 6,$$

so

$$\max(|a_{k-1}|, |a_{k-2}|) \le 15/\lambda_{k-1} \tag{5.8}$$

and by (5.1)

$$|a_{k-1}|\lambda_k \le 15\varepsilon.$$

As a_{k+1} and a_k have opposite signs and $|a_{k+1}| \le \frac{1}{2}|a_k|$ (cf. (5.6)), the last equation in (5.5) can be written as

$$\eta + 2\lambda_k a_k + \xi\lambda_{k+1} a_k = 3,$$

with $|\eta| \le 15\varepsilon$ and $3/2 \le \xi \le 2$. Thus, if $\varepsilon \le 1/15$ then

$$\frac{1}{\lambda_k + \lambda_{k+1}} \le a_k. \tag{5.9}$$

Let y_i, $k \le i \le m$, be the point from Λ_i for which $Q_\pi\varphi(y_i) = 0$; in addition, let $y_{m+1} = \tau_m = 1$ and in case $a_{k-1} > 0$, put $y_k = \tau_{k-1}$. Denote

$$\Delta_i = \int\limits_{y_i}^{y_{i+1}} Q_\pi\varphi(u)\,du = \frac{1}{2}a_i(y_{i+1} - y_i) \quad \text{for } k \le i \le m.$$

Now, $Q_\pi\varphi$ is positive on (y_{k+2l}, y_{k+2l+1}), and negative on (y_{k+2l+1}, y_{k+2l+2}); this follows from the fact that $a_k > 0$, (5.6) and the choice of the y_i's, and we have

$$\Delta_{k+2l} \ge 0, \quad \Delta_{k+2l+1} \le 0 \quad \text{for } l \ge 0. \tag{5.10}$$

Further, for $l \ge 0$,

$$\Delta_{k+2l} \ge \int\limits_{\tau_{k+2l}}^{y_{k+2l+1}} Q_\pi\varphi(u)\,du = \frac{|a_{k+2l}|}{2}\,\frac{|a_{k+2l}|\lambda_{k+2l+1}}{|a_{k+2l}| + |a_{k+2l+1}|},$$

$$|\Delta_{k+2l+1}| \le \frac{|a_{k+2l+1}|}{2}\biggl(\frac{|a_{k+2l+1}|\lambda_{k+2l+1}}{|a_{k+2l}| + |a_{k+2l+1}|} + \lambda_{k+2l+2}\biggr).$$

These inequalities and (5.6) imply that for $l \ge 0$,

$$\Delta_{k+2l} \ge \tfrac{9}{8}|\Delta_{k+2l+1}|. \tag{5.11}$$

If in addition $\varepsilon < 1/60$, then $|a_{k-1}| \le \frac{1}{2}a_k$ (cf. (5.1), (5.8) and (5.9)), which implies $\tau_k - y_k \ge \frac{2}{3}\lambda_k$. Since $|a_{k+1}| \le \frac{1}{2}a_k$ by (5.6), which gives $y_{k+1} - \tau_k \ge \frac{2}{3}\lambda_{k+1}$, by the definition of Δ_k, (5.9) and (5.11) we have

$$\Delta_k \ge \tfrac{1}{3} \quad \text{and} \quad \Delta_k + \Delta_{k+1} \ge \tfrac{1}{27} \quad \text{for } \varepsilon \le \tfrac{1}{60}. \tag{5.12}$$

Now, we can estimate $\|(Q_\pi\varphi)^*\|_p$ from below. Choose ϱ with $\varrho \ge \lambda_k + \lambda_{k+1}$, and consider $u = \tau_{k-1} - \varrho$. Clearly, $(Q_\pi\varphi)^*(u) \ge |\int_0^1 Q_\pi\varphi(s)\psi_{3\varrho}(u-s)\,ds|$. Note that the choice of ϱ guarantees $\psi_{3\varrho}(u - y_{k+1}) \ge \psi_{3\varrho}(u - \tau_{k+1}) \ge 1/(9\varrho)$. Since ψ is increasing and nonnegative on $(-\infty, 0]$, from (5.11) and (5.12) we get

$$\begin{aligned}
\int\limits_{y_k}^{1} Q_\pi\varphi(s)\psi_{3\varrho}(u-s)\,ds &= \sum_{i=k}^{m} \int\limits_{y_i}^{y_{i+1}} Q_\pi\varphi(s)\psi_{3\varrho}(u-s)\,ds \\
&\ge \sum_{l\ge 0} \psi_{3\varrho}(u - y_{k+2l+1})(\Delta_{k+2l} + \Delta_{k+2l+1}) \\
&\ge \psi_{3\varrho}(u - y_{k+1})(\Delta_k + \Delta_{k+1}) \ge \frac{1}{243\varrho}.
\end{aligned}$$

On the other hand, by (5.8), (5.7) and the choice of y_k we get

$$\Big|\int\limits_0^{y_k} Q_\pi\varphi(s)\psi_{3\varrho}(u-s)\,ds\Big| \le \sup_{s\le y_k}|Q_\pi(s)|\int\limits_{-\infty}^{\infty}\psi_{3\varrho}(z)\,dz \le \frac{15}{\lambda_{k-1}}.$$

Thus, we obtain

$$(Q_\pi\varphi)^*(u) \ge \frac{1}{243\varrho} - \frac{15}{\lambda_{k-1}}.$$

Put $\beta = 1/(30\cdot 243)$ and take $\varepsilon < \beta/4$. Then the last inequality implies

$$(Q_\pi\varphi)^*(u) \ge \frac{1}{486\varrho} \quad \text{for } \lambda_k + \lambda_{k+1} \le \varrho \le \beta\lambda_{k-1}.$$

Using this inequality and (5.1), for $\varepsilon < \beta/4$ we get

$$\|(Q_\pi\varphi)^*\|_p^p \ge \int\limits_{\lambda_k+\lambda_{k+1}}^{\beta\lambda_{k-1}}\Big(\frac{1}{486\varrho}\Big)^p d\varrho \ge \int\limits_{2\varepsilon\lambda_{k-1}}^{\beta\lambda_{k-1}}\Big(\frac{1}{486\varrho}\Big)^p d\varrho.$$

Calculating this integral and applying (5.3) we get

$$\|Q_\pi\varphi\|_{H^p} \ge C_p\lambda_{k-1}^{1/p-1} \quad \text{for } 1/2 < p < 1,$$

or $\|Q_\pi\varphi\|_{H^1} \ge C_1\ln(\beta/2\varepsilon)$ in case $p = 1$, with the constant C_p depending on p but not on ε and π. Taking $\varepsilon < \beta/4$ and combining these inequalities with (5.1) and (5.2) we get $\|Q_\pi\|_{H^1} \ge C_1\ln(\beta/(2\varepsilon))$ and $\|Q_\pi\|_{H^p} \ge C_p\varepsilon^{1-1/p}$ in case $1/2 < p < 1$. ■

Proof of Theorem 5.1. Let $1/2 < p \le 1$ and suppose the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ does not satisfy the strong regularity condition.

If the corresponding Franklin system $\{f_n : n \ge 0\}$ were a basis in H^p, and for $f \in H^p$ and a sequence of coefficients $(b_n)_{n\ge 0}$ we had $f = \sum_{n=0}^{\infty} b_n f_n$, then the continuity of the liner functional $(\cdot, f_n)$ would imply $b_n = (f, f_n)$, and $Q_{\mathcal{P}_j} f = \sum_{n=0}^{2^j} b_n f_n$, i.e. for each $f \in H^p$, the sequence $Q_{\mathcal{P}_j} f$ would converge to f in H^p.

Now, let ε_p be as in Lemma 5.2. As the sequence of partitions is not strongly regular, it follows that for each ε, $0 < \varepsilon \le \varepsilon_p$, we can find j_ε and a pair of adjacent intervals $I_{j_\varepsilon-1,l}, I_{j_\varepsilon-1,l+1} \in \mathcal{I}_{j_\varepsilon-1}$ such that

$$\text{either} \quad |I_{j_\varepsilon-1,l}| \le \varepsilon^2|I_{j_\varepsilon-1,l+1}| \quad \text{or} \quad |I_{j_\varepsilon-1,l+1}| \le \varepsilon^2|I_{j_\varepsilon-1,l}|.$$

Then, passing to a splitting of $I_{j_\varepsilon-1,l}$ and $I_{j_\varepsilon-1,l+1}$, we can find three consecutive intervals $I_{j_\varepsilon,k-1}, I_{j_\varepsilon,k}, I_{j_\varepsilon,k+1}$ such that either

$$|I_{j_\varepsilon,k+1}| \le \varepsilon|I_{j_\varepsilon,k-1}| \quad \text{and} \quad |I_{j_\varepsilon,k}| \le \varepsilon|I_{j_\varepsilon,k-1}|,$$

or

$$|I_{j_\varepsilon,k-1}| \le \varepsilon|I_{j_\varepsilon,k+1}| \quad \text{and} \quad |I_{j_\varepsilon,k}| \le \varepsilon|I_{j_\varepsilon,k+1}|.$$

Thus, $\mathcal{P}_{j_\varepsilon}$ satisfies the assumptions of Lemma 5.2, which gives $\|Q_{\mathcal{P}_{j_\varepsilon}}\|_{H^p} \ge C_p\varepsilon^{1-1/p}$ for $1/2 < p < 1$, or $\|Q_{\mathcal{P}_{j_\varepsilon}}\|_{H^1} \ge C_1\ln(1/\varepsilon)$ in case $p = 1$. As $\varepsilon \le \varepsilon_p$ is arbitrary, this implies the existence of a sequence $g_m \in H^p$ and a sequence of indices i_m such that $g_m \to 0$ in H^p and $\|Q_{\mathcal{P}_{i_m}} g_m\|_{H^p} \to \infty$, so the operators $Q_{\mathcal{P}_j}$ are not equicontinuous. Therefore, the sequence $Q_{\mathcal{P}_j} f$ cannot converge in H^p for each $f \in H^p$, and consequently the system $\{f_n : n \ge 0\}$ cannot be a basis in H^p. ■

The results of Theorems 4.2 and 5.1 can be summarized in the following form:

THEOREM 5.3. *Let $\{\mathcal{P}_j : j \geq 0\}$ be a quasi-dyadic sequence of partitions of* $[0,1]$, *and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. Then the following conditions are equivalent*:

(1) $\{\mathcal{P}_j : j \geq 0\}$ *satisfies the strong regularity condition for some parameter γ.*
(2) $\{f_n : n \geq 0\}$ *is a basis in H^p for each p, $1/2 < p \leq 1$.*
(3) $\{f_n : n \geq 0\}$ *is a basis in H^p for some p, $1/2 < p \leq 1$.*
(4) $\{f_n : n \geq 0\}$ *is an unconditional basis in H^p for each p, $1/2 < p \leq 1$.*
(5) $\{f_n : n \geq 0\}$ *is an unconditional basis in H^p for some p, $1/2 < p \leq 1$.*

6. Haar and Franklin series with identical coefficients

Now, we compare the behaviour of the Haar and Franklin series with identical coefficients. We start with the following:

PROPOSITION 6.1. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the weak regularity condition with parameter γ, and let $\{f_n : n \geq 0\}$ and $\{h_n : n \geq 1\}$ be the corresponding Franklin and Haar systems*, *respectively. Then for each p, $1 < p < \infty$, and a sequence $(a_n)_{n\geq 1}$ of real coefficients*,

$$\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 f_n^2\Big)^{1/2}\Big\|_p \sim \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_p,$$

with implied constants depending only on p and γ.

PROOF. It is enough to note that if the sequence of partitions is weakly regular with parameter γ, then there is a constant C_γ such that for all $n \geq 1$ and $u \in [0,1]$,

$$|h_n(u)| \leq C_\gamma \mathcal{M}(f_n, u) \quad \text{and} \quad |f_n(u)| \leq C_\gamma \mathcal{M}(h_n, u).$$

To check these inequalities, recall the pointwise estimates for the Franklin function in Proposition 2.9 (inequalities (2.28) and (2.32)), and the estimates for the Haar functions (cf. (2.19), (2.22)). To complete the proof, apply the following maximal inequality of Fefferman and Stein (cf. for example [28], Theorem 2.1.1): for each $1 < p < \infty$, there is a constant C_p such that for every sequence of functions $\{g_n : n \geq 0\}$ with $(\sum_{n=0}^{\infty} g_n^2(\cdot))^{1/2} \in L^p$,

$$\Big\|\Big(\sum_{n=0}^{\infty} (\mathcal{M}(g_n, \cdot))^2\Big)^{1/2}\Big\|_p \leq C_p \Big\|\Big(\sum_{n=0}^{\infty} g_n^2(\cdot)\Big)^{1/2}\Big\|_p. \ \blacksquare$$

The inequality of Fefferman–Stein does not hold for $0 < p \leq 1$. To obtain an analogous result in this case, the technique similar to that from [19] is used.

PROPOSITION 6.2. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the strong regularity condition with parameter γ, and let $\{f_n : n \geq 0\}$ and $\{h_n : n \geq 1\}$ be the corresponding Franklin and Haar systems*, *respectively. Then for each p, $0 < p \leq 1$,* *and a sequence* $(a_n)_{n\ge 1}$ *of real coefficients*,

$$\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 f_n^2\Big)^{1/2}\Big\|_p \sim \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_p,$$

with implied constants depending only on p *and* γ.

Proof. Let us start with the proof of the bound of the Haar square function by the Franklin square function. Define

$$\begin{aligned} E_r &= \Big\{u \in [0,1] : \sum_{n=1}^{\infty} a_n^2 f_n^2(u) > 2^r\Big\},\\ B_r &= \{u \in [0,1] : \mathcal{M}^*(\chi_{E_r}, u) > 1/2\},\\ \psi_r &= \sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n h_n. \end{aligned}$$

For $\{n\} \not\subset B_{r+1}$ we have $|\{n\} \cap E_{r+1}^{\mathrm{c}}| > \frac{1}{2}|\{n\}|$, whence by Proposition 2.12 there is a constant C_γ such that $\int_{\{n\}\cap E_{r+1}^{\mathrm{c}}} f_n^2(u)\,du \ge C_\gamma$. Using this we get

$$\begin{aligned} \|\psi_r\|_2^2 &= \sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 \le C_\gamma \sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 \int\limits_{\{n\}\cap E_{r+1}^{\mathrm{c}}} f_n^2(u)\,du\\ &\le C_\gamma \int\limits_{B_r\cap E_{r+1}^{\mathrm{c}}} \sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 f_n^2(u)\,du \le C_\gamma 2^r |B_r|. \end{aligned}$$

Since $\operatorname{supp}\psi_r \subset B_r$, using the above estimate and Hölder's inequality with exponents $2/p$ and $2/(2-p)$ we get

$$\int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 h_n^2(u)\Big)^{p/2} du \le |B_r|^{1-p/2}\|\psi_r\|_2^p \le C_{\gamma,p} 2^{rp/2}|B_r|.$$

Summing over r we obtain

$$\begin{aligned} \int_0^1 \Big(\sum_{n=0}^{\infty} a_n^2 h_n^2(u)\Big)^{p/2} du &\le \sum_r \int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r\\ \{n\}\not\subset B_{r+1}}} a_n^2 h_n^2(u)\Big)^{p/2} du\\ &\le C_{\gamma,p} \sum_r 2^{rp/2}|B_r| \le C_{\gamma,p}\sum_r 2^{rp/2}|E_r|, \end{aligned}$$

and therefore

$$\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_p \le C_{\gamma,p}\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 f_n^2\Big)^{1/2}\Big\|_p.$$

It remains to prove the converse inequality. To this end, let

$$\widetilde{E}_r = \Big\{u \in [0,1] : \sum_{n=1}^{\infty} a_n^2 h_n^2(u) > 2^r\Big\},$$
$$\widetilde{B}_r = \{u \in [0,1] : \mathcal{M}^*(\chi_{\widetilde{E}_r}, u) > 1/2\},$$
$$\widetilde{\psi}_r = \sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n f_n.$$

As previously, if $\{n\} \not\subset \widetilde{B}_{r+1}$, then $|\{n\}\cap \widetilde{E}^{c}_{r+1}| \geq \frac{1}{2}|\{n\}|$, and therefore $\int_{\{n\}\cap \widetilde{E}^{c}_{r+1}} h_n^2(u)\,du \geq C_\gamma$ (cf. (2.22)). Using this we get

$$\begin{aligned}\|\widetilde{\psi}_r\|_2^2 &= \sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 \leq C_\gamma \sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 \int\limits_{\{n\}\cap E^{c}_{r+1}} h_n^2(u)\,du \\ &\leq C_\gamma \int\limits_{\widetilde{B}_r\cap \widetilde{E}^{c}_{r+1}} \sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 h_n^2(u)\,du \leq C_\gamma 2^r |\widetilde{B}_r|.\end{aligned}$$

Using the last estimate and applying Hölder's inequality with exponents $2/p$ and $2/(2-p)$ we get

$$\int\limits_{\widetilde{B}_r} \Big(\sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2} du \leq |\widetilde{B}_r|^{1-p/2} \|\widetilde{\psi}_r\|_2^p \leq C_{\gamma,p} 2^{rp/2} |\widetilde{B}_r|. \tag{6.1}$$

To prove the analogous bound for the integral over $\widetilde{B}^{c}_r$ note that if $\{n\} \not\subset \widetilde{B}_{r+1}$ then $|\{n\} \cap \widetilde{E}^{c}_{r+1}| \geq \frac{1}{2}|\{n\}|$, and as h_n is constant on both subintervals of $\{n\}$ from the next partition, we have $\|a_n^2 h_n^2\|_\infty \leq C_\gamma 2^{r+1}$. Since $\|h_n\|_\infty \sim |\{n\}|^{-1/2}$ (cf. (2.21)), we get $|a_n| \leq C_\gamma 2^{r/2}|\{n\}|^{1/2}$. Let $\widetilde{\mathcal{T}}_r$ be the set of maximal intervals from $\mathcal{I}$ included in $\widetilde{B}_r$. Applying the last estimate for $|a_n|$ and inequality (2.34) from Lemma 2.11, for $I \in \widetilde{\mathcal{T}}_r$ we get

$$\int\limits_{I^{c}} \sum_{\substack{\{n\}\subset I \\ \{n\}\not\subset \widetilde{B}_{r+1}}} |a_n f_n(u)|^p\,du \leq C_{\gamma,p} 2^{rp/2} |I|,$$

which gives

$$\int\limits_{\widetilde{B}^{c}_r} \Big(\sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2} du \leq \sum_{I\in \widetilde{\mathcal{T}}_r} \int\limits_{I^{c}} \sum_{\substack{\{n\}\subset I \\ \{n\}\not\subset \widetilde{B}_{r+1}}} |a_n f_n(u)|^p\,du \leq C_{\gamma,p} 2^{rp/2} |\widetilde{B}_r|.$$

This inequality and (6.1) imply

$$\int_0^1 \Big(\sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2} du \leq C_{\gamma,p} 2^{rp/2} |\widetilde{B}_r|,$$

and summing over r we get

$$\int_0^1 \Big(\sum_{n=0}^{\infty} a_n^2 f_n^2(u)\Big)^{p/2} du \le \sum_r \int_0^1 \Big(\sum_{\substack{\{n\}\subset \widetilde{B}_r \\ \{n\}\not\subset \widetilde{B}_{r+1}}} a_n^2 f_n^2(u)\Big)^{p/2} du$$
$$\le C_{\gamma,p} \sum_r 2^{rp/2} |\widetilde{B}_r| \le C_{\gamma,p} \sum_r 2^{rp/2} |\widetilde{E}_r|,$$

which gives

$$\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 f_n^2\Big)^{1/2}\Big\|_p \le C_{\gamma,p} \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_p. \ \blacksquare$$

PROPOSITION 6.3. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong periodic regularity condition with parameter γ, and let $\{h_n : n \ge 1\}$ be the corresponding Haar system. Then for each p, $0 < p < \infty$, and a sequence $(a_n)_{n\ge 1}$ of real coefficients*,

$$\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_n^2\Big)^{1/2}\Big\|_p \sim \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 h_{n+1}^2\Big)^{1/2}\Big\|_p,$$

with implied constants depending only on p and γ.

PROOF. It is enough to check the equivalence for $0 < p < 2$; for $p = 2$ the equivalence is clear, and for $2 < p < \infty$ it follows from the equivalence for $1 < p < 2$ by the unconditionality of the Haar system (cf. Proposition 2.7) and the duality argument.

For $r \in \mathbb{Z}$ let

$$E_r = \Big\{u \in [0,1] : \sum_{n=1}^{\infty} a_n^2 h_n^2(u) > 2^r\Big\},$$
$$B_r = \{u \in [0,1] : \mathcal{M}^*(\chi_{E_r}, u) > 1/2\},$$
$$\psi_r = \sum_{\substack{\{n\}\subset B_r \\ \{n\}\not\subset B_{r+1}}} a_n h_{n+1}.$$

For $\{n\} \not\subset B_{r+1}$ we have $|E_{r+1}^{\mathrm{c}} \cap \{n\}| \ge \frac{1}{2}|\{n\}|$, so $\int_{\{n\}\cap E_{r+1}^{\mathrm{c}}} h_n^2(u)\,du \ge C_\gamma$ (cf. (2.22)). Using this inequality and the definition of E_{r+1} we get

$$\|\psi_r\|_2^2 = \sum_{\substack{\{n\}\subset B_r \\ \{n\}\not\subset B_{r+1}}} a_n^2 \le C_\gamma \int_{B_r\setminus E_{r+1}} \sum_{\substack{\{n\}\subset B_r \\ \{n\}\not\subset B_{r+1}}} a_n^2 h_n^2(u)\,du \le C_\gamma 2^r |B_r|. \tag{6.2}$$

On the other hand, B_r is a union of some intervals from $\mathcal{I}$; let $\mathcal{T}_r$ be the family of maximal intervals from $\mathcal{I}$ included in B_r. For an interval $I \in \mathcal{I}$, $I = I_{j,k}$ for some $1 \le k \le 2^j$, define

$$I^+ = \begin{cases} I_{j,k+1} & \text{if } k < 2^j, \\ I_{j+1,1} & \text{if } k = 2^j, \end{cases}$$

and put

$$B_r^* = \sum_{I\in\mathcal{T}_r} (I \cup I^+).$$

Then the periodic strong regularity of the sequence of partitions implies that $|B_r^*| \le (\gamma+1)|B_r|$; moreover, note that $\operatorname{supp}\psi_r \subset B_r^*$. Therefore, using estimate (6.2) for $\|\psi_r\|_2^2$

and Hölder's inequality with exponents $2/(2-p)$ and $2/p$, we get

$$\int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r \\ \{n\}\not\subset B_{r+1}}} a_n^2 h_{n+1}^2(u)\Big)^{p/2} du = \int_{B_r^*} \Big(\sum_{\substack{\{n\}\subset B_r \\ \{n\}\not\subset B_{r+1}}} a_n^2 h_{n+1}^2(u)\Big)^{p/2} du$$
$$\le C_{\gamma,p} |B_r|^{1-p/2} \|\psi_r\|_2^p \le C_{\gamma,p} 2^{rp/2} |B_r|.$$

Therefore, as $p<2$, we obtain

$$\int_0^1 \Big(\sum_{n=1}^{\infty} a_n^2 h_{n+1}^2(u)\Big)^{p/2} du \le \sum_{r\in\mathbb{Z}} \int_0^1 \Big(\sum_{\substack{\{n\}\subset B_r \\ \{n\}\not\subset B_{r+1}}} a_n^2 h_{n+1}^2(u)\Big)^{p/2} du$$
$$\le C_{\gamma,p} \sum_{r\in\mathbb{Z}} 2^{rp/2} |B_r| \le C_{\gamma,p} \sum_{r\in\mathbb{Z}} 2^{rp/2} |E_r|$$
$$\le \int_0^1 \Big(\sum_{n=1}^{\infty} a_n^2 h_n^2(u)\Big)^{p/2} du,$$

and one of the inequalities in the equivalence is proved. The opposite inequality is proved in the analogous way. ■

Combining Propositions 6.1, 6.2 and 6.3 with Propositions 2.7, 2.8, Corollary 3.5 and Theorems 4.1, 4.2, one can obtain various results on the simultaneous unconditional convergence in L^p of Haar and Franklin series with identical, or shifted, coefficients, equivalent conditions for such convergence, or boundedness of shift operators. Now, we formulate one of the possible versions of such a result.

Corollary 6.4. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong periodic regularity condition with parameter γ. Let $\{f_n : n \ge 0\}$ and $\{h_n : n \ge 1\}$ be the corresponding Franklin and Haar systems, respectively. Then*

(i) *For each p, $1<p<\infty$, the systems $\{f_n : n\ge 0\}$ and $\{h_n : n \ge 1\}$ are equivalent bases in L^p, i.e. for each sequence $(a_n)_{n\ge 1}$ of real coefficients, the series $\sum_{n=0}^{\infty} a_{n+1} f_n$ converges in L^p iff $\sum_{n=1}^{\infty} a_n h_n$ converges in L^p, and moreover $\|\sum_{n=0}^{\infty} a_{n+1} f_n\|_p \sim \|\sum_{n=1}^{\infty} a_n h_n\|_p$.*

(ii) *Let $(a_n)_{n\ge 1}$ be a sequence of real coefficients and $0<p\le 1$. Then the series $\sum_{n=1}^{\infty} a_n h_n$ converges unconditionally in L^p if and only if $\sum_{n=0}^{\infty} a_{n+1} f_n$ converges unconditionally in L^p.*

(iii) *The shift operator U, defined by $Uh_n = h_{n+1}$, is a bounded linear operator on L^p for each p, $1<p<\infty$.*

(iv) *The shift operator T, defined by $Tf_n = f_{n+1}$, is a bounded linear operator on L^p for each p, $1<p<\infty$, and on H^p for each p, $1/2<p\le 1$.*

To comment on point (i) of Corollary 6.4 recall that even the classical Franklin and Haar systems (i.e. corresponding to the sequence of dyadic partitions) are not equivalent bases in L^1 (cf. [26]).

Recall that if the Haar system corresponding to a quasi-dyadic sequence of partitions is a basis in L^p, $1<p<\infty$, then it is an unconditional basis in this space (cf. Proposition 2.7). By Theorem 5.3 we know that the Franklin system corresponding to a quasi-dyadic sequence of partitions is a basis in H^p, $1/2 < p \le 1$, iff it is an unconditional basis in this space. For the Franklin system in L^p, $1 < p < \infty$, the weak regularity of the sequence of partitions is sufficient for the unconditionality of the corresponding Franklin system, but we do not know whether it is necessary as well. Clearly, the weak regularity condition is stronger than the condition $|\mathcal{P}_j| \to 0$, which in turn is a necessary and sufficient condition for the Franklin system to be a basis in L^p. However, some of the above statements need not hold for an arbitrary sequence of quasi-dyadic partitions:

PROPOSITION 6.5. (i) *If the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ is weakly regular but not strongly periodically regular, then the equivalence from Proposition* 6.3 *does not hold.*

(ii) *If $\{\mathcal{P}_j : j \ge 0\}$ is not weakly regular, then the equivalence from Proposition* 6.1 *need not be true.*

PROOF. To check (i), note that if the sequence of partitions is not strongly periodically regular, then for each $M > 0$ there is an interval $I \in \mathcal{I}$ such that

$$\text{either} \quad |I| > M|I^+| \quad \text{or} \quad |I^+| > M|I|, \tag{6.3}$$

where I^+ is defined as in the proof of Proposition 6.3. Moreover, denoting by h_I the Haar function with support equal to I, we know that for a weakly regular system, $\|h_I\| \sim |I|^{1/p-1/2}$ (see (2.21)). This and (6.3) imply that in this case the equivalence from Proposition 6.3 cannot hold. An example of such a sequence of partitions is $\mathcal{P}_j = \{t_{j,k} : 0 \le k \le 2^j\}$ with $t_{j,k} = 2\sin^2(k\pi/2^{j+2})$. This sequence is even strongly regular, but not periodically strongly regular.

To show (ii), we give an example of a sequence which is not weakly regular and for which the L^p norms of the Haar and Franklin functions corresponding to the same interval from $\mathcal{I}$ are not equivalent, and therefore the equivalence from Proposition 6.1 does not hold. Let $(M_j : j \ge 0)$ be any sequence of positive numbers such that $\lim_{j\to\infty} M_j = \infty$. Then we put $\mathcal{P}_0 = \{[0,1]\}$, $\mathcal{P}_{2j+1}$ is obtained from $\mathcal{P}_{2j}$ by splitting each interval from $\mathcal{I}_{2j}$ into equal parts, and $\mathcal{P}_{2j+2}$ is obtained from $\mathcal{P}_{2j+1}$ by splitting each interval $I \in \mathcal{I}_{2j+1}$ into left and right parts $I_{\mathrm{L}}, I_{\mathrm{R}}$ in such a way that $|I_{\mathrm{L}}| = M_j|I_{\mathrm{R}}|$. Now, comparing the formulae for the L^p norms of the corresponding Haar and Franklin functions (cf. (2.20) and Proposition 2.3), we see that for this sequence of partitions and $2 < p < \infty$,

$$\limsup_{n\to\infty} \frac{\|h_n\|_p}{\|f_n\|_p} = \infty,$$

while for $1 < p < 2$,

$$\liminf_{n\to\infty} \frac{\|h_n\|_p}{\|f_n\|_p} = 0. \ \blacksquare$$

7. Characterization of the spaces BMO and Lip(α), $0 < \alpha < 1$

In this section we characterize the spaces dual to H^p with $1/2 < p \le 1$ in terms of the Fourier–Franklin coefficients. Recall that the dual space to H^1 is BMO, and the dual to H^p with $1/2 < p < 1$ is $\mathrm{Lip}(\alpha)$ with $\alpha = 1/p - 1$ (cf. Section 1.1 and the references given there).

Let us start with the characterization of $\mathrm{Lip}(\alpha)$.

THEOREM 7.1. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the strong regularity condition with parameter γ and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. Moreover, let $0 < \alpha < 1$ and $f \in C[0,1]$, $f = \sum_{n=0}^{\infty} a_n f_n$. Then $f \in \mathrm{Lip}(\alpha)$ iff*

$$\sup_{n\geq 0} \frac{|a_n|}{|\{n\}|^{1/2+\alpha}} < \infty,$$

where $\{n\}$ is defined by formula (2.17). Moreover,

$$\|f\|_{\mathrm{Lip}(\alpha)} \sim \sup_{n\geq 0} \frac{|a_n|}{|\{n\}|^{1/2+\alpha}},$$

with implied constants depending only on γ and α.

PROOF. First, let $f \in \mathrm{Lip}(\alpha)$, so

$$|f(u) - f(s)| \leq L|u-s|^{\alpha}.$$

Clearly, for $n = 0,1$ we have $|a_n| \leq \sqrt{3}\,\|f\|_\infty$ (cf. the formulae for f_0, f_1 in Definition 2.1). Now, let $n \geq 2$, $n = 2^j + k$ with $1 \leq k \leq 2^j$. Using the pointwise estimates for Franklin functions from Proposition 2.9 (inequalities (2.26), (2.27), (2.30) and (2.31)) and the estimates for the length of intervals from Proposition 2.6(ii) we obtain

$$\begin{aligned}
|a_n| &= \Big|\int_0^1 f(u) f_n(u)\,du\Big| = \Big|\int_0^1 (f(u) - f(t_n)) f_n(u)\,du\Big| \\
&\leq L \sum_{l=1}^{2^{j+1}} \int_{I_{j+1,l}} |u - t_n|^{\alpha} |f_n(u)|\,du \\
&\leq C_\gamma L |\{n\}|^{-1/2} \sum_{l=1}^{2^{j+1}} |I_{j+1,l}| (|I_{j+1,l\wedge 2k}| + \ldots + |I_{j+1,l\vee 2k-1}|)^{\alpha} 2^{-|2k-l|/2} \\
&\leq L C_{\gamma,\alpha} |\{n\}|^{1/2+\alpha} \sum_{l=1}^{2^{j+1}} (|2k-l|+1)^{\alpha_\gamma + \alpha(\alpha_\gamma+1)} 2^{-|2k-l|/2} \\
&\leq L C_{\gamma,\alpha} |\{n\}|^{1/2+\alpha}.
\end{aligned}$$

To prove the converse inequality, let $|a_n| \leq L|\{n\}|^{1/2+\alpha}$. For $u \in [0,1]$, let $I_j(u)$ be the interval from $\mathcal{I}_j$ containing u. Consider a pair of points $u, s \in [0,1]$; let

$$j_0 = \max\{j : \text{there is at most one point from } \mathcal{P}_j \text{ between } s \text{ and } u\}.$$

Note that for $j \leq j_0$, either $I_j(u) = I_j(s)$, or $I_j(u)$ and $I_j(s)$ are adjacent intervals; therefore, by strong regularity $|I_j(u)| \sim |I_j(s)|$ and moreover $|I_{j_0}(u)| \sim |I_{j_0}(s)| \sim |s-u|$. Let τ_j^*, $j \leq j_0$, be the common endpoint of $I_j(u)$ and $I_j(s)$, $\tau_j^* = \tau_{j,l_j}$. Since for $n \leq 2^{j_0}$ the function f_n is linear on both $I_j(u)$ and $I_j(s)$, the pointwise estimates for f_n from

Proposition 2.9 (inequalities (2.27) and (2.31)) imply

$$|f_n(u) - f_n(s)| \le C_\gamma |s-t| \frac{|\{n\}|^{-1/2}}{2^{|k_n - l_j|}} \left(\frac{1}{|I_j(u)|} + \frac{1}{|I_j(s)|} \right), \quad \text{where } n = 2^j + k_n$$

(for more details, cf. the calculations for inequality (3.2) in the proof of Lemma 3.2). This inequality and the estimates for the length of intervals from Proposition 2.6(ii), (i) give

$$\begin{aligned}
(7.1) \quad & \sum_{j=0}^{j_0-1} \sum_{n=2^j+1}^{2^{j+1}} |a_n|\,|f_n(u) - f_n(s)| \\
&\le C_\gamma\, L|s-u| \sum_{j=0}^{j_0-1} \sum_{n=2^j+1}^{2^{j+1}} |\{n\}|^{1/2+\alpha} \left(\frac{1}{|I_j(u)|} + \frac{1}{|I_j(s)|} \right) \frac{|\{n\}|^{-1/2}}{2^{|k_n - l_j|}} \\
&\le C_{\gamma,\alpha}\, L|s-u| \sum_{j=0}^{j_0-1} |I_j(u)|^{\alpha-1} \sum_{k=1}^{2^j} (|k-l_j|+1)^{\alpha_\gamma \alpha} 2^{-|k-l_j|} \\
&\le C_{\gamma,\alpha}\, L|s-u|\,|I_{j_0}(u)|^{\alpha-1} \sum_{j=0}^{j_0-1} \left(\frac{\gamma}{\gamma+1} \right)^{(1-\alpha)(j_0-j)} \le C_{\gamma,\alpha}\, L|s-u|^\alpha.
\end{aligned}$$

Now, let $n > 2^{j_0}$, $n = 2^j + k_n$. Choosing m_j in such a way that $I_j(u) = I_{j,m_j}$ for $j > j_0$ and using again the estimates from Propositions 2.9 and 2.6 mentioned above, we obtain

$$\begin{aligned}
\sum_{j=j_0}^{\infty} \sum_{n=2^j+1}^{2^{j+1}} |a_n|\,|f_n(u)| &\le C_\gamma\, L \sum_{j=j_0}^{\infty} \sum_{n=2^j+1}^{2^{j+1}} |\{n\}|^\alpha 2^{-|k_n - m_j|} \\
&\le C_{\gamma,\alpha}\, L \sum_{j=j_0}^{\infty} |I_j(u)|^\alpha \sum_{k=1}^{2^j} (|k-m_j|+1)^{\alpha_\gamma \alpha} 2^{-|k-m_j|} \\
&\le C_{\gamma,\alpha}\, L|I_{j_0}(u)|^\alpha \sum_{j=j_0}^{\infty} \left(\frac{\gamma}{\gamma+1} \right)^{\alpha(j-j_0)} \le C_{\gamma,\alpha}\, L|I_{j_0}(u)|^\alpha.
\end{aligned}$$

As an analogous inequality holds for s as well, and by the choice of j_0 we have $|I_{j_0}(u)| \sim |I_{j_0}(s)| \sim |s-u|$, we obtain

$$\sum_{j=j_0}^{\infty} \sum_{n=2^j}^{2^{j+1}} |a_n|\,|f_n(u) - f_n(s)| \le C_{\gamma,\alpha}\, L\,|s-u|^\alpha.$$

Thus, the last inequality and (7.1) give

$$\sum_{j=0}^{\infty} \sum_{n=2^j+1}^{2^{j+1}} |a_n|\,|f_n(u) - f_n(s)| \le C_{\gamma,\alpha}\, L|s-u|^\alpha.$$

It should be clear that analogous arguments imply the uniform convergence of the series $\sum_{n=0}^{\infty} a_n f_n$, so we get $f = \sum_{n=0}^{\infty} a_n f_n \in \mathrm{Lip}(\alpha)$.

The equivalence of the norms follows from the above estimates as well. ∎

Now, we present a characterization of BMO. The proof is an adaptation of the proof from [29], but it is presented for the sake of completeness.

THEOREM 7.2. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \geq 0\}$ satisfy the strong regularity condition with parameter γ and let $\{f_n : n \geq 0\}$ be the corresponding Franklin system. Let $f \in L^1$, $f = \sum_{n=0}^{\infty} a_n f_n$. Then $f \in \mathrm{BMO}$ iff*

$$\sup_{n\geq 0}\Big(\frac{1}{|\{n\}|}\sum_{\{m\}\subset\{n\}} a_m^2\Big)^{1/2} < \infty,$$

where $\{n\}$ is defined by (2.17). *Moreover*,

$$\|f\|_{\mathrm{BMO}} \sim \sup_{n\geq 0}\Big(\frac{1}{|\{n\}|}\sum_{\{m\}\subset\{n\}} a_m^2\Big)^{1/2},$$

with implied constants depending only on γ.

PROOF. First, let the sequence of coefficients be such that for some constant M,

$$\sum_{\{m\}\subset\{n\}} a_m^2 \leq M^2|\{n\}|, \quad n \geq 0. \tag{7.2}$$

We are going to show that there is a constant C_γ such that for any interval $\Gamma \subset [0,1]$ and $f = \sum_{n=0}^{\infty} a_n f_n$,

$$\exists_{c_\Gamma} \int_\Gamma |f(u) - c_\Gamma|^2\,du \leq C_\gamma M^2|\Gamma|, \tag{7.3}$$

which implies $f \in \mathrm{BMO}$ and one of the inequalities for the norms.

Consider an interval $\Gamma \subset [0,1]$, and let

$$j_0 = \min\{j : \text{there is } I \in \mathcal{I}_j \text{ such that } I \subset \Gamma\}.$$

Let $I_{j_0,k} \in \mathcal{I}_{j_0}$, $I_{j_0,k} \subset \Gamma$. The choice of j_0 implies that there are at most two adjacent intervals from $\mathcal{I}_{j_0}$ included in Γ. Define

$$J = \bigcup_{|l-k|\leq 2} I_{j_0,l}, \qquad \widetilde{J} = \bigcup_{|l-k|\leq 3} I_{j_0,l}.$$

Then $\Gamma \subset J \subset \widetilde{J}$, and by strong regularity

$$|\Gamma| \leq |J| \leq |\widetilde{J}| \leq C_\gamma|\Gamma|.$$

Define

$$\psi_1 = \sum_{j\geq j_0}\sum_{\substack{2^j<n\leq 2^{j+1}\\ \{n\}\subset\widetilde{J}}} a_n f_n, \qquad \psi_2 = \sum_{j\geq j_0}\sum_{\substack{2^j<n\leq 2^{j+1}\\ \{n\}\not\subset\widetilde{J}}} a_n f_n, \qquad \psi_3 = \sum_{n=0}^{2^{j_0}} a_n f_n.$$

Clearly, $f = \psi_1 + \psi_2 + \psi_3$. First, by (7.2) and the definition of $\widetilde{J}$ we get

$$\int_\Gamma \psi_1^2(u)\,du \leq \sum_{j\geq j_0}\sum_{\substack{2^j<n\leq 2^{j+1}\\ \{n\}\subset\widetilde{J}}} a_n^2 \leq \sum_{|l-k|\leq 3}\sum_{\{n\}\subset I_{j_0,l}} a_n^2 \leq M^2|\widetilde{J}| \leq C_\gamma M^2|\Gamma|. \tag{7.4}$$

Moreover, (7.2) implies that $|a_n| \leq M|\{n\}|^{1/2}$. Let $u \in \Gamma$; for $j \geq j_0$, choose l_j such that $u \in I_{j,l_j}$. Note that if $n = 2^j + k$, $1 \leq k \leq 2^j$, and $\{n\} \not\subset \widetilde{J}$, then $|k - l_j| \geq 2^{j-j_0}$.

Therefore, the bound for $|a_n|$ and the decay of $|f_n|$ from Proposition 2.9 (i.e. inequalities (2.27) and (2.31)) give

$$|\psi_2(u)| \le C_\gamma M \sum_{j\ge j_0} \sum_{|k-l_j|\ge 2^{j-j_0}} 2^{-|k-l_j|} \le C_\gamma M \sum_{j\ge j_0} 2^{-2^{j-j_0}} \le C_\gamma M,$$

which implies

$$\int_\Gamma \psi_2^2(u)\,du \le C_\gamma M^2|\Gamma|. \tag{7.5}$$

To estimate ψ_3, let τ be the point from $\mathcal{P}_{j_0-1}\cap\Gamma$, if it exists, or any point from Γ otherwise. Since any function f_n with $n \le 2^{j_0}$ is linear on $\Gamma\cap\{u\le\tau\}$ and $\Gamma\cap\{u\ge\tau\}$, using the above estimate for $|a_n|$ and the decay of $|f_n|$ from Proposition 2.9 (cf. the calculations for inequality (7.1) in the proof of Theorem 7.1) we get

$$|\psi_3(u)-\psi_3(\tau)| \le C_\gamma\, M \quad \text{for } u\in\Gamma,$$

and clearly

$$\int_\Gamma |\psi_3(u)-\psi_3(\tau)|^2\,du \le C_\gamma\, M^2|\Gamma|. \tag{7.6}$$

Inequalities (7.4)–(7.6) imply that (7.3) holds with $c_\Gamma = \psi_3(\tau)$, which completes the first part of the proof.

To prove the converse inequality, let $f\in \mathrm{BMO}$, $\|f\|_{\mathrm{BMO}} = K$. Denote by L_f the functional on H^1 corresponding to f (cf. Section 1.1). Since $\|f_n\|_{H^1} \sim \|f_n\|_1 \sim |\{n\}|^{1/2}$ (cf. Theorem 4.2 and inequality (2.23) in Proposition 2.9), we have

$$|a_n| = |(f,f_n)| = |L_f f_n| \le C\|f\|_{\mathrm{BMO}}\|f_n\|_{H^1} \le C_\gamma K|\{n\}|^{1/2}. \tag{7.7}$$

Let $n\ge 2$, $2^i < n \le 2^{i+1}$. Consider the following decomposition of f: $f=\varphi_1+\varphi_2+\varphi_3$, with

$$\varphi_1 = \sum_{\{m\}\subset\{n\}} a_m f_m, \qquad \varphi_2 = \sum_{j=i}^{\infty} \sum_{\substack{2^j<m\le 2^{j+1}\\ \{m\}\not\subset\{n\}}} a_m f_m, \qquad \varphi_3 = \sum_{m=0}^{2^i} a_m f_m.$$

Note that

$$\sum_{\{m\}\subset\{n\}} a_m^2 = \int_0^1 \varphi_1^2(u)\,du. \tag{7.8}$$

The integrals of φ_1^2 over $\{n\}$ and $\{n\}^{\mathrm{c}}$ are treated separately.

We start with some technical estimate. Let $I_{i,k}, I_{i,l}\in\mathcal{I}_i$ with $l\ne k$. Then there is a constant C_γ such that

$$\sum_{\{m\}\subset I_{i,k}} |\{m\}|^{1/2}\Big(\int_{I_{i,l}} f_m^2(u)\,du\Big)^{1/2} \le C_\gamma 2^{-|k-l|}|I_{i,l}|^{1/2}. \tag{7.9}$$

The proof is similar to that of inequality (2.34) in Lemma 2.11, so we give just a sketch. For simplicity, suppose $k<l$ (the other case is considered analogously). Let $m=2^j+s$, $\{m\}\subset I_{i,k}$. Then, using the rate of decay of f_m^2 from Proposition 2.9 and the estimates

for the lengths of intervals from Proposition 2.6(ii) we get

$$\int_{I_{i,l}} f_m^2(u)\,du \le C_\gamma |\{m\}|^{-1} |I_{j,2^{j-i}(l-1)+1}| \cdot 2^{-2|2^{j-i}(l-1)+1-s|}.$$

Using this inequality and the estimates for the lengths of intervals from Proposition 2.6(i) we obtain

$$\begin{aligned}\sum_{\substack{\{m\}\subset I_{i,k}\\ 2^j<m\le 2^{j+1}}} |\{m\}|^{1/2}\Big(\int_{I_{i,l}} f_m^2(u)\,du\Big)^{1/2} &\le C_\gamma \sum_{s\le 2^{j-i}k} |I_{j,2^{j-i}(l-1)+1}|^{1/2} 2^{-|2^{j-i}(l-1)+1-s|}\\ &\le C_\gamma \Big(\frac{\gamma}{\gamma+1}\Big)^{(j-i)/2} |I_{i,l}|^{1/2} 2^{-2^{j-i}(l-1-k)}\\ &\le C_\gamma \Big(\frac{\gamma}{\gamma+1}\Big)^{(j-i)/2} |I_{i,l}|^{1/2} 2^{-|l-k|},\end{aligned}$$

and summing over $j \ge i$ we get (7.9).

Let $n = 2^i + k$, so $\{n\} = I_{i,k}$. Using (7.7), (7.9) and Proposition 2.6(ii) we get

$$\begin{aligned}\Big(\int_{\{n\}^c} \varphi_1^2(u)\,du\Big)^{1/2} &\le C_\gamma K \sum_{\{m\}\subset I_{i,k}} |\{m\}|^{1/2} \Big(\int_{I_{i,k}^c} f_m^2(u)\,du\Big)^{1/2}\\ &\le C_\gamma K \sum_{l\ne k} \sum_{\{m\}\subset I_{i,k}} |\{m\}|^{1/2} \Big(\int_{I_{i,l}} f_m^2(u)\,du\Big)^{1/2}\\ &\le C_\gamma K \sum_{l\ne k} |I_{i,l}|^{1/2} 2^{-|k-l|}\\ &\le C_\gamma K |I_{i,k}|^{1/2} \sum_{l\ne k} (|k-l|+1)^{\alpha_\gamma/2} 2^{-|k-l|} \le C_\gamma K |I_{i,k}|^{1/2},\end{aligned}$$

which gives

$$\int_{\{n\}^c} \varphi_1^2(u)\,du \le C_\gamma K^2 |\{n\}|. \tag{7.10}$$

Similarly,

$$\begin{aligned}\Big(\int_{\{n\}} \varphi_2^2(u)\,du\Big)^{1/2} &\le C_\gamma K \sum_{l\ne k} \sum_{\{m\}\subset I_{i,l}} |\{m\}|^{1/2} \Big(\int_{I_{i,k}} f_m^2(u)\,du\Big)^{1/2}\\ &\le C_\gamma K \sum_{l\ne k} |I_{i,k}|^{1/2} 2^{-|k-l|} \le C_\gamma K |I_{i,k}|^{1/2},\end{aligned}$$

so we get

$$\int_{\{n\}} \varphi_2^2(u)\,du \le C_\gamma K^2 |\{n\}|. \tag{7.11}$$

Moreover, since $(f_m, 1) = 0$ for $m \ge 1$, we have

$$\Big|\int_{\{n\}} \varphi_1(u)\,du\Big| \le \sum_{\{m\}\subset\{n\}} |a_m| \Big|\int_{\{n\}} f_m(u)\,du\Big| \le C_\gamma K \sum_{\{m\}\subset\{n\}} |\{m\}|^{1/2} \int_{\{n\}^c} |f_m(u)|\,du,$$

so by inequality (2.34) in Lemma 2.11,

$$\Big| \int_{\{n\}} \varphi_1(u)\,du \Big| \le C_\gamma K |\{n\}|. \tag{7.12}$$

Since each function f_m with $m \le 2^i$ is linear on $\{n\}$, we get (cf. the analogous calculations for ψ_3 in the first part of the proof)

$$|\varphi_3(u) - \varphi_3(s)| \le C_\gamma K \quad \text{for } u, s \in \{n\}. \tag{7.13}$$

Let $\xi \in \{n\}$. Using (7.13), (7.12) and (7.11), we get

$$\begin{aligned} \Big| \varphi_3(\xi) - \frac{1}{|\{n\}|} \int_{\{n\}} f(u)\,du \Big| &\le \Big| \varphi_3(\xi) - \frac{1}{|\{n\}|} \int_{\{n\}} \varphi_3(u)\,du \Big| \\ &\quad + \Big| \frac{1}{|\{n\}|} \int_{\{n\}} \varphi_1(u)\,du \Big| + \Big| \frac{1}{|\{n\}|} \int_{\{n\}} \varphi_2(u)\,du \Big| \\ &\le C_\gamma K + \Big(\frac{1}{|\{n\}|} \int_{\{n\}} \varphi_2^2(u)\,du \Big)^{1/2} \le C_\gamma K. \end{aligned}$$

By the definition of BMO we have

$$\Big(\frac{1}{|\{n\}|} \int_{\{n\}} \Big| f(t) - \frac{1}{|\{n\}|} \int_{\{n\}} f(u)\,du \Big|^2 dt \Big)^{1/2} \le K,$$

which together with the preceding inequality implies

$$\Big(\frac{1}{|\{n\}|} \int_{\{n\}} |f(t) - \varphi_3(\xi)|^2\,dt \Big)^{1/2} \le K.$$

This, together with (7.11) and (7.13), gives

$$\begin{aligned} \Big(\int_{\{n\}} \varphi_1^2(u)\,du \Big)^{1/2} &\le \Big(\int_{\{n\}} |f(u) - \varphi_3(\xi)|^2\,du \Big)^{1/2} \\ &\quad + \Big(\int_{\{n\}} \varphi_2^2(u)\,du \Big)^{1/2} + \Big(\int_{\{n\}} |\varphi_3(u) - \varphi_3(\xi)|^2\,du \Big)^{1/2} \\ &\le C_\gamma K |\{n\}|^{1/2}. \end{aligned}$$

The last inequality together with (7.8) and (7.10) gives

$$\sum_{\{m\} \subset \{n\}} a_m^2 \le C_\gamma K^2 |\{n\}|. \ \blacksquare$$

Finally, consider the spaces VMO and $\mathrm{lip}(\alpha)$, $0 < \alpha < 1$, which are separable subspaces of BMO and $\mathrm{Lip}(\alpha)$, respectively. They can be considered as the closure in the norms $\|\cdot\|_{\mathrm{BMO}}$ and $\|\cdot\|_{\mathrm{Lip}(\alpha)}$ of the space of functions satisfying the Lipschitz condition. We have the following characterization of these spaces:

COROLLARY 7.3. *Let the quasi-dyadic sequence of partitions $\{\mathcal{P}_j : j \ge 0\}$ satisfy the strong regularity condition with parameter γ and let $\{f_n : n \ge 0\}$ be the corresponding Franklin system.*

(i) *Let* $0 < \alpha < 1$ *and* $f \in \mathrm{Lip}(\alpha)$. *Then* $f \in \mathrm{lip}(\alpha)$ *iff*

$$\lim_{n\to\infty} \frac{|a_n|}{|\{n\}|^{1/2+\alpha}} = 0.$$

(ii) *Let* $f \in \mathrm{BMO}$. *Then* $f \in \mathrm{VMO}$ *iff*

$$\lim_{n\to\infty} \left(\frac{1}{|\{n\}|} \sum_{\{m\}\subset\{n\}} a_m^2 \right)^{1/2} = 0.$$

References

[1] S. V. Bochkarev, *Some inequalities for the Franklin series*, Anal. Math. 1 (1975), 249–257.

[2] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. 37 (1966), 1494–1504.

[3] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124 (1970), 249–304.

[4] S.-Y. A. Chang and Z. Ciesielski, *Spline characterizations of* H^1, Studia Math. 75 (1983), 183–192.

[5] Z. Ciesielski, *Properties of the orthonormal Franklin system*, ibid. 23 (1963), 141–157.

[6] —, *Properties of the orthonormal Franklin system II*, ibid. 27 (1966), 289–323.

[7] —, *Equivalence, unconditionality and convergence a.e. of the spline bases in* L^p *spaces*, in: Approximation Theory, Banach Center Publ. 4, PWN, Warszawa 1979, 55–68.

[8] —, *The Franklin orthogonal system as unconditional basis in* $\mathrm{Re}\, H^1$ *and VMO*, in: Functional Analysis and Approximation, Internat. Ser. Numer. Math. 60, Birkhäuser, Basel, 1981, 117–125.

[9] Z. Ciesielski and A. Kamont, *Projections onto piecewise linear functions*, Funct. Approx. Comment. Math. 25 (1997), 129–143.

[10] Z. Ciesielski and S. Kwapień, *Some properties of the Haar*, *Walsh–Paley*, *Franklin and the bounded polygonal orthonormal bases in* L_p *spaces*, Comment. Math. Prace Mat., Tomus Specialis in honorem Ladislai Orlicz II (1979), 37–42.

[11] Z. Ciesielski, P. Simon and P. Sjölin, *Equivalence of Haar and Franklin bases in* L^p *spaces*, Studia Math. 60 (1977), 195–210.

[12] R. R. Coifman, *A real variable characterization of* H^p, ibid. 51 (1974), 268–274.

[13] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[14] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

[15] C. Fefferman and E. M. Stein, H^p *spaces of several variables*, Acta Math. 129 (1972), 137–193.

[16] Ph. Franklin, *A set of continuous orthogonal functions*, Math. Ann. 100 (1928), 522–529.

[17] G. G. Gevorkyan, *Unboundedness of the shift operator in the Franklin system in the space* L^1, Mat. Zametki 38 (1985), 523–533 (in Russian).

[18] —, *Theorems on the modified Franklin–Strömberg system*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. 24 (1) (1989), 69–92 (in Russian).

[19] —, *On Haar and Franklin series with the same coefficients*, Erevan. Gos. Univ. Uchen. Zap. Estestv. Nauki 3 (1989), 3–9 (in Russian).

[20] —, *Theorems on the modified Franklin–Strömberg system II*, Izv. Akad. Nauk Armenii Mat. 26 (1) (1991), 31–51 (in Russian).

[21] —, *Some theorems on unconditional convergence and the majorant of Franklin series and their application to the spaces* $\mathrm{Re}(H^p)$, Proc. Steklov Inst. Math. (1992), 49–76.

[22] G. G. Gevorkyan, *Franklin and Haar series in Lorentz spaces* $L_{p,q}[0,1]$, $0 < q \le p < 1$, East J. Approx. 3 (1997), 419–444.

[23] A. Kamont, *Spline spaces and weighted moduli of smoothness*, J. Approx. Theory, submitted.

[24] B. Kashin and A. Saakyan, *Orthogonal Series*, Nauka, Moscow, 1984 (in Russian).

[25] P. Simon, *Remarks on the shift operators with respect to the Haar and Franklin systems*, Acta Math. Acad. Sci. Hungar. 39 (1982), 251–254.

[26] P. Sjölin, *The Haar and Franklin systems are not equivalent bases in* L^1, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25 (1977), 1099–1100.

[27] P. Sjölin and J. O. Strömberg, *Basis properties of Hardy spaces*, Ark. Mat. 21 (1983), 111–125.

[28] E. Stein, *Harmonic Analysis*: *Real-Variable Methods*, *Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, 1993.

[29] P. Wojtaszczyk, *The Franklin system is an unconditional basis in* H^1, Ark. Mat. 20 (1982), 293–300.

[30] —, *Banach Spaces for Analysts*, Cambridge Univ. Press, Cambridge, 1991.

[31] A. Zygmund, *Trigonometric Series*, Vol. 1 (2nd ed.), Cambridge Univ. Press, Cambridge, 1959.