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Lipschitz equisingularity

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CONTENTS

1. Introduction and statement of the results

The main purpose of this paper is to give a method of constructing Lipschitz homeomorphisms between germs of complex analytic sets. Neither Whitney's stratifications (Whitney, Thom, Mather) nor Zariski's equisingularity (Varchenko) offer such a possibility.

Lipschitz homeomorphisms are interesting because of (at least) two reasons. On the one hand, they have a lot of "good" properties which are of general interest (Lipschitz homeomorphisms preserve sets of measure zero, the group of all Lipschitz homeomorphisms has a natural topology, etc.).

On the other hand, analytic sets have interesting metric properties, such as for instance the Łojasiewicz property of regular separation: if X and Y are analytic, then for some $c > 0$, $C > 0$ we have, locally

$$\operatorname{dist}(x, Y) \geqslant C \operatorname{dist}(x, X \cap Y)^c$$

for $x \in X$. The best exponent $c = c(X, Y)$ (the Łojasiewicz exponent) is an interesting metric invariant of the pair (X, Y). Clearly, these properties are preserved under Lipschitz homeomorphisms, but not under arbitrary homeomorphisms.

It follows that there does not exist a Lipschitz homeomorphism $(\boldsymbol{C}^3, 0) \to (\boldsymbol{C}^3, 0)$ carrying $X_0 \cup Y_0$ onto $X_t \cup Y_t$ $(t \neq 0)$, where $X_t = \{y = z = 0\}$, $Y_t = \{y = x^3, z = tx\}$, although such a homeomorphism (even satisfying Hölder's condition) clearly exists.

To state our first results we need a definition.

Let $T \subset \boldsymbol{C}^m$ be an analytic set. A family of germs of analytic sets in $\boldsymbol{C}^n \times T$ over T is a germ X at $\{0\} \times T$ of a subset of $\boldsymbol{C}^n \times T$ such that the germ of X at every point $(0, t)$ is analytic. Let $p\colon \boldsymbol{C}^n \times T \to T$ be the standard projection; we put $X_t = p^{-1}(t) \cap X \subset \boldsymbol{C}^n \times \{t\}$.

We say that X is *Lipschitz-equisingular* at a point $t \in T$ if there exists a germ h of a Lipschitz homeomorphism (with a Lipschitz inverse)

$$h\colon (\boldsymbol{C}^n \times T, (0, t)) \to (\boldsymbol{C}^n \times T, (0, t))$$

such that $ph = p$ and $h(X) = X_t \times T$.

PROPOSITION 1.1 (Lipschitz equisingularity)(*). *Let* $X \subset C^n \times T$ *be a family of germs of analytic sets. Let* $t_0 \in T$. *Then there exists a neighbourhood* $U \subset T$ *of* t_0 *and an analytic set* $T' \subsetneq U$ *such that* X *is Lipschitz-equisingular at every point of* $U - T'$.

If we do not require h to be Lipschitz, then the above proposition is of course well-know (see e. g. [8]). Its proof (due to Thom) is based on some basic properties of Whitney's stratifications; h is constructed by integration of a discontinuous vector field (a different proof is given in [9]). Our Lipschitz homeomorphism will be constructed by integrating a Lipschitz vector field.

Proposition 1.1 is an easy consequence of the existence of a stratification of an analytic set with certain metric properties which we now describe.

To have a good model for these properties, consider for the moment the one-dimensional case, i.e. the germ at $0 \in C^n$ of an analytic curve X, singular at 0. It has a natural stratification into $X_{reg} = X - \{0\}$ and $X_{sing} = \{0\}$. Put $X^0 = \{0\}$, $X^1 = X$, $\mathring{X}^1 = X^1 - X^0$. For $q \in \mathring{X}^1$ let P_q be the orthogonal projection of $T_q C^n = C^n$ onto $T_q \mathring{X}^1$ (with respect to the standard hermitian metric on C^n).

It follows directly from Puiseux expansion that P_q satisfies the estimate

$$(1.1) \qquad |P_q - P_{q'}| \leqslant C|q-q'|/\mathrm{dist}(\{q, q'\}, X^0)$$

for $q, q' \in \mathring{X}^1$, for some constant C, depending only on X.

We are also interested in the derivative of the function P_q. Again it follows from Puiseux expansion that for some constants $C > 0$, $\alpha < 1$

$$(1.2) \qquad |D_v P(q)| \leqslant C|v|/\mathrm{dist}(q, X^0)^\alpha \quad \text{for all } q \in \mathring{X}^1 \text{ and } v \in T_q \mathring{X}^1.$$

To generalize these observations to higher dimensions we need some definitions.

Let $X \subset C^n$ be a germ at 0 of an analytic set. By a stratification of X we shall mean a decreasing sequence of germs of analytic sets

$$X = X^d \supset X^{d-1} \supset \ldots \supset X^l$$

such that $\mathring{X}^j = X^j - X^{j-1}$ is smooth and $\dim X^{j-1} < \dim X^j$.

For $q \in \mathring{X}^j$ let P_q: $T_q C^n = C^n \to T_q \mathring{X}^j$ be the orthogonal projection. Let $P_q^\perp = I - P_q$ be the orthogonal projection onto the normal spaces $T_q^\perp X^j$.

For technical reasons we shall replace the distance functions $\mathrm{dist}(q, X^j)$ by semianalytic functions $\varrho_j(q)$ (i. e. continuous functions with semianalytic graphs) such that

$$(1.3) \qquad (1/2n)\,\mathrm{dist}(q, X^j) \leqslant \varrho_j(q) \leqslant 2n\,\mathrm{dist}(q, X^j), \qquad \varrho_j \geqslant \varrho_{j+1}.$$

(*) This was recently proved by a different method by R. Hardt.

If one is prepared to use some basic properties of subanalytic sets [4] or if X^j's are algebraic, then the distance functions themselves can be used.

(To prove the existence of ϱ_j's we use the fact that for any semianalytic set A there is a semianalytic function $\tilde{\varrho}_A$ such that

$$(1/2)\operatorname{dist}(q, A) \leqslant \tilde{\varrho}_A(q) \leqslant 2\operatorname{dist}(q, A)$$

[7]; we put $\varrho_j = \sum_{k \geqslant j} \tilde{\varrho}_{X^k}$).

Let c_0 be a fixed constant, $c_0 \geqslant 2n$. A chain (more exactly, a c_0-chain) for a point $q \in \mathring{X}^j$ is a strictly decreasing sequence of indices j_s ($l \leqslant j_s \leqslant j$) and a sequence of points $q_{j_s} \in \mathring{X}^{j_s}$ such that $j_1 = j$, $q_{j_1} = q$ and

(1.4) j_s is the smallest integer for which $\varrho_k(q) \geqslant 2c_0^2 \varrho_{j_s}(q)$ for all $k < j_s$,

$$|q - q_{j_s}| \leqslant c_0 \varrho_{j_s}(q).$$

(The existence of a chain for a given point is clear. It is easy to verify the following inequalities:

(1.5) $(1/2c_0^4)\varrho_t(q) \leqslant (1/2c_0^3)\operatorname{dist}(q, X^t) \leqslant \varrho_t(q_{j_s}) \leqslant c_0^3 \operatorname{dist}(q, X^t) \leqslant c_0^4 \varrho_t(q)$

for all s and $t < j_s$.)

For the proofs of all the results except Proposition 1.4 only $2n$-chains are necessary.

The following proposition generalizes (1.1).

PROPOSITION 1.2. *Let $X \subset C^n$ be a germ of an analytic set. There exists a stratification $\{X^j\}$ of X such that for some constant C and every k:*

(1.6, k) *for any j, any $q \in \mathring{X}^j$ and any chain $q = q_{j_1}, q_{j_2}, \ldots, q_{j_l}$ for q,*

$$|P_{q_{j_1}}^{\perp} P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant C|q_{j_1} - q_{j_2}|/\operatorname{dist}(q_{j_1}, X^{j_k - 1});$$

(1.7, k) *if, further, $q' \in \mathring{X}^j$ and $|q' - q| \leqslant (1/2c_0)\varrho_{j-1}(q)$, then*

$$|(P_{q'} - P_q) P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant C|q' - q|/\operatorname{dist}(q, X^{j_k - 1});$$

in particular,

(1.7,0) $$|P_{q'} - P_q| \leqslant C|q' - q|/\operatorname{dist}(q, X^{j-1}),$$

which is the most direct generalization of (1.1).

These estimates are sufficient for the prove of Proposition 1.1. To generalize (1.2), we define (for any N) the following neighbourhood of $\mathring{X}^r$ (for a given stratification $\{X^j\}$ of X):

$$U_{N,r} = \{x \in C^n\colon \operatorname{dist}(x, X^r) < \operatorname{dist}(x, X^{r-1})^N\}.$$

Remark that if $j > r$, $q \in \mathring{X}^j \cap U_{N,r}$ and $\{q_{j_s}\}$ is a chain for q, then $j_s = r$ for some s (provided that $N > 1$).

PROPOSITION 1.3. *There exists a stratification $\{X^j\}$ of X satisfying not* *only the conclusion of Proposition* 1.2, *but also such that for some* N, $\delta > 0$, $\alpha < 1$ *we have for every* k:

(1.8, k) *for any* $q \in \mathring{X}^j$, *any chain* $q_{j_1} = q$, $q_{j_2}, \ldots, q_{j_l}$ for q *and any* $v \in T_{q_{j_k}} \mathring{X}^{j_k}$, $|v| \leqslant 1$

$$|D_w P_q| \leqslant C/\operatorname{dist}(q, X^{j_k - 1}),$$

where $w = P_{q_{j_1}} P_{q_{j_2}} \ldots P_{q_{j_k}} v$;

(1.9, k) *if, further,* $q \in \mathring{X}^j \cap U_{N,j_k}$, *then there exists a* $w \in T_q X^j$, $|w| = 1$, *such that*

$$|\measuredangle (w, P_{q_{j_1}} P_{q_{j_2}} \ldots P_{q_{j_k}}(C^n))| \geqslant \delta$$

and

$$|D_w P_q| \leqslant C/\operatorname{dist}(q, X^{j_k})^\alpha.$$

The estimates in both of these propositions do not follow from Whitney's conditions. In fact, the stratification of $X = \{x^2 + y^2 = z^3\} \subset C^3$ into X and $\{0\}$ is a Whitney stratification, but (1.7,0) is not satisfied.

However, we shall prove that the estimates of Proposition 1.2 imply a very strong form of Whitney's condition A (implying also condition B); this should be compared with the complicated relations between Whitney's conditions and Zariski's equisingularity [3].

The construction of our stratifications is based on Łojasiewicz's „partitions normales". The difference is that, instead of using one projection $C^{i+1} \to C^i$ for every i, we use a finite number of them.

Finally, we mention an application of Proposition 1.3, not related to Lipschitz equisingularity. Let M be a complex manifold with a C^∞-hermitian metric. Let $X \subset M$ be a d-dimensional analytic set. Its regular part X_{reg} is a d-dimensional manifold; let Ω be the curvature form on X_{reg} of the induced hermitian metric. For every invariant homogeneous polynomial P of degree d we have the closed $2d$-form $P(\Omega)$ on X_{reg}.

PROPOSITION 1.4 (integrability of curvature forms). *If K is an open subset of X_{reg} such that the closure of K in M is compact, then $P(\Omega)$ is integrable over K.*

It is not clear to the author if Propositions 1.2 and 1.3 (and therefore also the other ones) hold in the real case.

Often we shall not distinguish between a set (or a function) and its germ at the origin.

The character C will stand for various constants.

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2. Lipschitz vector fields and stratifications

Here we show how Proposition 1.1 follows from Proposition 1.2.

The following proposition explains the relation between the estimates (1.6, k), (1.7, k) and Lipschitz vector fields.

PROPOSITION 2.1. *Let $\{X^j\}$ be a stratification of a set $X \subset C^n$ satisfying the assumptions of Proposition 1.2. Let v be a Lipschitz vector field on C^n such that $v(q) \in T_q \mathring{X}^k$ for $q \in \mathring{X}^k$, $k \leqslant j$. Then there exists a Lipschitz vector field w on C^n such that $w = v$ on X^j and for every k and $q \in \mathring{X}^k$ we have $w(q) \in T_q \mathring{X}^k$.*

Proof. It is enough to construct a w such that $w(q) \in T_q \mathring{X}^{j+1}$ for $q \in \mathring{X}^{j+1}$. First we define w on X^{j+1}: $w = v$ on X^j and $w(q) = P_q v(q)$ for $q \in \mathring{X}^{j+1}$. We prove that it is Lipschitz. Take a $q \in \mathring{X}^{j+1}$ and a chain $q = q_{j_1}, q_{j_2}, \ldots, q_{j_l}$. Writing $v(q_{j_s}) = v(q_{j_{s+1}}) + \tilde{v}_{j_s}$, $|\tilde{v}_{j_s}| \leqslant C|q_{j_s} - q_{j_{s+1}}|$, we get

$$v(q_{j_s}) = P_{q_{j_s}} v(q_{j_{s+1}}) + P_{q_{j_s}} \tilde{v}_{j_s}$$

and, by induction,

$$v(q_{j_2}) = P_{q_{j_2}} \cdots P_{q_{j_l}} v(q_{j_l}) + \sum_{s>l} P_{q_{j_2}} \cdots P_{q_{j_s}} \tilde{v}_{j_s}.$$

Therefore

$$\begin{aligned} |w(q) - w(q_{j_2})| &= |P_{q_{j_1}} v(q_{j_1}) - v(q_{j_2})| \leqslant |P_{q_{j_1}} (v(q_{j_1}) - v(q_{j_2}))| + |P^{\perp}_{q_{j_1}} v(q_{j_2})| \\ &\leqslant C|q_{j_1} - q_{j_2}| + |P^{\perp}_{q_{j_1}} P_{q_{j_2}} \cdots P_{q_{j_l}} v(q_{j_l})| + \sum_{s>l} |P^{\perp}_{q_{j_1}} P_{q_{j_2}} \cdots P_{q_{j_s}} \tilde{v}_{j_s}| \\ &\leqslant C|q_{j_1} - q_{j_2}| + \sum_{s>l} C|q_{j_1} - q_{j_2}|\, |\tilde{v}_{j_s}| / \operatorname{dist}(q, X^{j_s - 1}) \end{aligned}$$

and, since $|\tilde{v}_{j_s}|/\operatorname{dist}(q, X^{j_s-1}) \leqslant C$, $|w(q) - w(q_{j_2})| \leqslant C|q - q_{j_2}|$.

If $q' \in X^j$ is arbitrary, then $|q - q'| \geqslant C|q - q_{j_2}|$ and

$$|w(q) - w(q')| \leqslant |w(q) - w(q_{j_2})| + |w(q') - w(q_{j_2})| \leqslant C|q - q'|.$$

Let $q' \in \mathring{X}^{j+1}$. If $|q - q'| \leqslant (1/2)\operatorname{dist}(q, X^j)$, then

$$\begin{aligned} |w(q') - w(q)| &= |P_{q'} v(q') - P_q v(q)| \\ &\leqslant |v(q') - v(q)| + |(P_{q'} - P_q) v(q)| \\ &\leqslant C|q' - q| + |(P_{q'} - P_q) P_{q_{j_2}} \cdots P_{q_{j_l}} v(q_{j_l})| + \\ &\quad + \sum_s |(P_{q'} - P_q) P_{q_{j_2}} \cdots P_{q_{j_s}} \tilde{v}_{j_s}| \\ &\leqslant C|q' - q|. \end{aligned}$$

If $|q-q'| \geqslant (1/2)\operatorname{dist}(q, X^j)$, we select points $\tilde{q}, \tilde{q}' \in X^j$ closest to q, q'; we have $|w(q')-w(q)| \leqslant |w(q)-w(\tilde{q})|+|w(q')-w(\tilde{q}')|+|w(\tilde{q})-w(\tilde{q}')| \leqslant C|q'-q|$.

It is now enough to extend w to a Lipschitz vector field on C^n [2].

We now prove Proposition 1.1. Consider the germ of X at $(0, t_0)$ and let $\{X^j\}$ be a stratification of X satisfying (1.6, k) and (1.7, k). Let $J = \{j\colon \{0\} \times T \not\subset X^j\}$. Put

$$T' = \{t\colon (0, t) \in X^j \text{ for all } j \in J\} \cup T_{\text{sing}};$$

clearly, $T' \subsetneq T$.

Let $t \notin T'$. We identify a neighbourhood of t with an open subset of C^d (where $d = \dim T$). Let $e_1, \ldots, e_d$ be the standard "constant" vector fields on this neighbourhood. We shall show that e_l's can be lifted to Lipschitz vector fields v_l (i. e. $p_* v_l = e_l$) tangent to the strata $\mathring{X}^j$.

Let k be the smallest integer not belonging to J. In a neighbourhood of $(0, t)$, X^k is a smooth manifold containing $\{0\} \times T$ and $p\colon X^k \to T$ is a submersion. Thus e_l's can be lifted to smooth vector fields w_i tangent to X^k and (at points of $\{0\} \times T$) to $\{0\} \times T$. Using Proposition 2.1 we extend w_i's to Lipschitz vector fields, denoted again by w_i, defined in a neighbourhood $\{(x, t')\colon |x| < \varepsilon, |t'-t| < \varepsilon\}$ of $(0, t)$, tangent to the strata $\mathring{X}^j$. Since $p_* w_i$ are linearly independent at every point, we can replace them by suitable linear combinations and get the desired Lipschitz liftings v_l.

We can now construct the Lipschitz homeomorphism h; it will be defined in $\{(x, t')\colon |x| < \eta, |t'-t| < \varepsilon\}$ for some $\eta \leqslant \varepsilon$. For any t', $|t'-t| < \varepsilon$, let $v_{t'} = \sum (t'_l - t_l) v_l$. We integrate $v_{t'}$; let $H(s, t', x)$ be the solution of

$$\frac{\partial H}{\partial s}(s, t', x) = v_{t'}(s, H(s, t', x)), \qquad H(0, t', x) = x.$$

h is given by $(t', x) \mapsto (t', H(1, t', x))$.

3. Generalized normal partitions

We first introduce some notation. Let $T_1, \ldots, T_d$ be variables; we put

$$\tilde{\Delta}_k(T_1, \ldots, T_d) = \sum_{i_1, \ldots, i_k}^{*} \prod_{j_1, j_2}^{*} (T_{j_1} - T_{j_2}),$$

where $\sum^*$ denotes summation over all $i_1, \ldots, i_k$ such that $i_s \neq i_t$ for $s \neq t$ and $\prod^*$ multiplication over all j_1, j_2 such that $j_1 \neq j_2$, $j_s \neq i_t$ for $s = 1, 2$, $t = 1, \ldots, k$. If

$$P(T) = T^d + \sum_{i<d} a_i T^i$$

and $t_1, \ldots, t_d$ are the roots of P, then $\tilde{\Delta}_k(t_1, \ldots, t_d)$ is a polynomial in $a_0, \ldots, a_{d-1}$:

$$\tilde{\Delta}_k(t_1, \ldots, t_d) = \Delta_k(a_0, \ldots, a_{d-1}).$$

We shall call Δ_k's the generalized discriminants of P; Δ_0 is its discriminant. P has less than k different roots if and only if $\Delta_0 = 0, \ldots, \Delta_{d-k} = 0$.

Now we generalize Łojasiewicz's „partition normale" [7].

Let $Z \subset C^n$ be an analytic set; we decompose Z into equidimensional sets $Z = \bigcup Z^\alpha$, $\dim Z^\alpha = \alpha$. Put $Z_n = Z$.

Let $X_n^{n-1} \subset C^n$ be any hypersurface containing Z_n and let $F^n = 0$ be a reduced equation of X_n^{n-1}. We choose the x_n-axis so that the standard projection $C^n \to C^{n-1}$ gives a finite map $X_n^{n-1} \to C^{n-1}$. We identify C^{n-1} with $\{x_n = 0\}$. There exists a neighbourhood Ω_n of 0 in C^{n-1} such that for every $\xi \in \Omega_n$ the projection $\pi(\xi)$: $C^n \to C^{n-1}$ parallel to $(\xi, 1)$ induces a finite map $X_n^{n-1} \to C^{n-1}$.

Let $\xi_1^n, \xi_2^n, \ldots$ be a finite set (now arbitrary, but specified later on) of elements of Ω_n. Let

$$Z_{n-1} = \bigcup_i \pi(\xi_i^n)\Big(\bigcup_{\alpha < n-1} Z^\alpha\Big)$$

and let

$$X_i = \pi(\xi_i^n)\big(\{x \in X_n^{n-1}\colon \pi(\xi_i^n)\colon X_n^{n-1} \to C^{n-1} \text{ is not a local isomorphism at } x\}\big)$$

be the set of critical values of $\pi(\xi_i^n)|X_n^{n-1}$. We take for $X_{n-1}'^{n-2}$ any hypersurface in C^{n-1} containing Z_{n-1} and all X_i's. Let

$$X_n^{n-2} = \Big[\bigcup_i \pi(\xi_i^n)^{-1}(X_{n-1}'^{n-2})\Big] \cap X_n^{n-1},$$

$$X_{n-1}^{n-2} = \bigcup_i \pi(\xi_i^n)(X_n^{n-2});$$

let $F^{n-1} = 0$ be the reduced equation of X_{n-1}^{n-2}.

By decreasing induction on m we choose a suitable neighbourhood $\Omega_{m+1} \subset C^m$ of 0, a finite number of vectors $\xi_1^{m+1}, \xi_2^{m+1}, \ldots \in \Omega_{m+1}$, a hypersurface $X_{m+1}^m \subset C^{m+1}$ with a reduced equation $F^{m+1} = 0$ and decreasing sequences of analytic sets $X_{l+1}^j \subset X_{l+1}^l$, $l \geqslant m$, $m \leqslant j \leqslant l$.

Suppose we have these objects for $m \geqslant k$. Take an x_{k+1}-axis so that the standard projection $C^{k+1} \to C^k$ induces a finite map $X_{k+1}^k \to C^k$. Let $\Omega_{k+1} \subset C^k$ be a neighbourhood of 0 such that for any $\xi \in \Omega_{k+1}$ the projection $\pi(\xi)$: $C^{k+1} \to C^k$ parallel to $(\xi, 1)$ induces a finite map $X_{k+1}^k \to C^k$. We take an arbitrary finite set of vectors $\xi_1^{k+1}, \ldots$ in Ω_{k+1}.

If $I(p, q)$ is a sequence of indices $(i_p, i_{p-1}, \ldots, i_{q+1})$, we put

$$\pi_{I(p,q)} = \pi(\xi_{i_{q+1}}^{q+1}) \circ \ldots \circ \pi(\xi_{i_p}^p)\colon C^p \to C^q,$$

$k \leqslant q < p \leqslant n$.

Let $Z_k = \bigcup_{I(n,k)} \pi_{I(n,k)}\big(\bigcup_{\alpha<k} Z^\alpha\big)$; let $X_l \subset C^k$ be the set of critical values of $\pi(\xi_i^{k+1})$: $X_{k+1}^k \to C^k$. We take for $X_k'^{k-1}$ any hypersurface in C^k containing Z_k and every X_l.

Now we define

$$X_n^{k-1} = X_n^k \cap \bigcup_{I(n,k-1)} \pi_{I(n,k-1)}^{-1}(X_k'^{k-1}), \tag{3.1}$$

$$X_m^{k-1} = \bigcup_{I(n,m)} \pi_{I(n,m)}(X_n^{k-1}), \quad m \geqslant k. \tag{3.2}$$

Let $F^k = 0$ be a reduced equation of X_k^{k-1}.

We put $\mathring{X}_m^j = X_m^j - X_m^{j-1}$.

$F^m(x+\lambda(\xi, 1))$ is equivalent to a distinguished polynomial with respect to λ; let $\Delta_s^m(x, \xi)$ be its generalized discriminants.

The following lemma lists all the simple properties of $\{X_m^j\}$ needed later. For the purpose of the proof we introduce

$$X_m^{*j} = X_m^j - \bigcup_{I(m,j+1)} \pi_{I(m,j+1)}^{-1}(X_{j+1,\mathrm{sing}}^j);$$

clearly $\mathring{X}_m^j$ is an open subset of X_m^{*j}.

LEMMA 3.1. 1) *$\pi(\xi_i^m)(X_m^j) \subset X_{m-1}^j$ and $\pi(\xi_i^m)(X_m^{*j}) \subset X_{m-1}^{*j}$ for all ξ_i^m;*

2_m) *every $\pi_{I(m,j)}$ induces a finite map $X_m^j \to C^j$ and a local isomorphism $X_m^{*j} \to C^j$; in particular $\mathring{X}_m^j$ is smooth and $\pi_{I(m,j)}$ induces a local isomorphism $\mathring{X}_m^j \to C^j$;*

3_m) *for every i and every topological component X' of X_m^{*j} there is an l such that for all $x \in X'$*

$$\Delta_s^m(x, \xi_i^m) = 0 \text{ for } s < l, \quad \Delta_l^m(x, \xi_i^m) \neq 0;$$

the same holds therefore for every topological component of $\mathring{X}_m^j$;

4) *for $x \in C^m \setminus X_m^{m-1}$ and for every i the equation $F^{m+1}(x+\lambda(\xi_i^{m+1}, 1)) = 0$ is nonsingular with respect to λ;*

5) *Z is the sum of (some) topological components of $\mathring{X}_n^j$'s.*

Proof. 1), 4), 5) are obvious. Let $x_0 \in X_m^{*j}$. Fix $\xi_{i_m}^m = \xi^m, \ldots, \xi_{i_{j+1}}^{j+1} = \xi^{j+1}$, let $\pi^s = \pi(\xi^s)$: $C^s \to C^{s-1}$ and let π^{pq}: $C^p \to C^q$ be their composition. For simplicity of notation assume that all $\xi^l = 0$, i.e. π^s are the standard projections. Let $y_1, \ldots, y_{j+1}$ be coordinates in a neighbourhood of $\pi^{m,j+1}(x_0)$ in C^{j+1} such that $X_{j+1}^j = \{y_{j+1} = 0\}$. Together with 2_m), 3_m) we shall prove (by increasing induction on m) that

6_m) *there exists a variety $V_m \subset C^m$ (a wing) such that $x_0 \in V_m$, $\pi^{m,j+1}$: $V_m \to C^{j+1}$ is finite and is a local isomorphism on $V_m \setminus (\pi^{m,j+1})^{-1}(X_{j+1}^j)$ and $V_m \setminus (\pi^{m,j+1})^{-1}(X_{j+1}^j) \subset C^m \setminus X_m^{m-1}$.*

We show that $6_m)$ implies $3_{m+1})$. Consider V_m as the graph of a multivalued function f, i.e. $V_m = \{(y, f(y))\}$, where $f\colon C^{j+1} \to C^{m-j-1}$ is analytic in $y_1, \ldots, y_j, y_{j+1}^{1/r}$. For any fixed $y' = (y_1, \ldots, y_j)$ the equation $F^{m+1}((y', y_{j+1}), f(y', y_{j+1}), \lambda) = 0$ is (by 4)) nonsingular with respect to λ for $y_{j+1} \neq 0$; thus every solution is of the form $\lambda = \lambda_\alpha(y', y_{j+1}^{1/s})$. Using for instance the argument of Varchenko [9], it is easy to see that the multiplicities of these roots for $y_{j+1} = 0$ do not depend on y'; this proves $3_{m+1})$.

$3_{m+1})$ implies that $\pi^{m+1}\colon X_{m+1}^j \to X_m^j$ is an isomorphism in a neighbourhood of x_0, which, together with $2_m)$, proves $2_{m+1})$.

To prove $6_{m+1})$, choose an α_0 so that x_0 is in the branch $\{(y, f(y), \lambda_{\alpha_0}(y))\}$. Let $\lambda_\alpha(y', y_{j+1}^{1/s}) = \mu_\alpha(y') + y_{j+1}^{1/s}\psi_\alpha(y', y_{j+1}^{1/s})$. Take a $t > s$ and define $V_{m+1} = \{(y, f(y), \mu_{\alpha_0}(y') + y_{j+1}^{1/t})\}$.

The collection of varieties $\{X_m^j\}$ and projections $\pi(\xi_i^m)$ (called admissible) having the properties as in the lemma will be called a generalized normal partition compatible with Z.

4. Regular projections

Let $X \subset C^n$ be a germ at 0 of a hypersurface with a reduced equation $F = 0$. Fix the x_n-axis so that F does not vanish on it. Let Ω' be a fixed neighbourhood of the origin in C^{n-1} such that for every $\xi \in \Omega'$ the projection $\pi(\xi)\colon C^n \to C^{n-1}$ induces a finite map $X \to C^{n-1}$. Then, by the preparation theorem, $F(x+\lambda(\xi, 1))$ is equivalent to a distinguished polynomial $W(x, \xi; \lambda)$ in λ:

$$F(x+\lambda(\xi, 1)) = Q(x, \xi, \lambda)\, W(x, \xi; \lambda), \qquad Q(0, 0, 0) \neq 0, \qquad W(0, 0; \lambda) = \lambda^d;$$

Let A be a germ of a subset of B at a point $q \in B$. We shall say that a $B \subset C^n$, $\Omega \subset C^{n-1}$, $\Lambda \subset C$ of the origins such that $F(x+\lambda(\xi, 1))$, $W(x, \xi; \lambda)$, $Q(x, \xi, \lambda)$ are defined for $x \in B$, $\xi \in \Omega$, $\lambda \in \Lambda$ and $Q \neq 0$ in $B \times \Omega \times \Lambda$.

Let A be a germ of a subset of B at a point $q \in B$. We shall say that a projection $\pi = \pi(\xi)$ is *ε-regular* at A with respect to X if there exists an integer l and a neighbourhood U of q such that for all η, $|\eta - \xi| < \varepsilon$,

$$\Delta_i(x, \eta) = 0 \quad \text{for } i < l, \qquad \Delta_l(x, \eta) \neq 0$$

for $x \in (U \setminus \{p\}) \cap A$. If $A = \{q\}$, we require that $\Delta_i(q, \eta) = 0$ for $i < l$, $\Delta_l(q, \eta) \neq 0$.

A projection is regular at A if it is ε-regular for some ε.

We shall be interested in the case of a germ of a curve A. By a (germ of a) curve in B we mean a germ at 0 of an analytic map $C \to B$, $t \mapsto q_t \in B$;

sometimes we shall identify such a curve with the image of the map q_t. The constant map will also be considered as a curve.

We shall give a geometric characterization of regular projections. Let $S_\varepsilon(q, \xi)$ denote the open cone in B:

$$S_\varepsilon(q, \xi) = \{q+\lambda(\eta, 1): |\eta-\xi| < \varepsilon, \lambda \in C\} \cap B.$$

PROPOSITION 4.1. *Let $\pi = \pi(\xi)$ be ε-regular with respect to X at a curve q_t. Then there exists a constant C such that for $|t| \neq 0$, sufficiently small, the intersection of $S_\varepsilon(q_t, \xi)$ with X consists of points of the form*

$$q_t + \lambda_{i,t}(\eta)(\eta, 1),$$

where $\lambda_{i,t}(\eta)$ are analytic for $|\eta-\xi| < \varepsilon$ and satisfy

$$\lambda_{i,t}(\eta) \neq \lambda_{j,t}(\eta) \quad \text{for all } \eta \text{ and } i \neq j,$$
$$|D\lambda_{i,t}| \leqslant C|\lambda_i|.$$

Proof. The equation $F(q_t+\lambda(\eta, 1)) = 0$ has (for $t \neq 0$) one root, $\lambda = 0$, with multiplicity, say l, while all the others are simple. Thus all these roots are of the form $\lambda = \lambda_i(t^{1/r}, \eta)$, λ_i analytic, and either $\lambda_i \equiv 0$ or $\lambda_i(t, \eta) \neq 0$ for all η and $t \neq 0$; also $\lambda_i(t, \eta) \neq \lambda_j(t, \eta)$, $i \neq j$. Thus $\lambda_{i,t}(\eta)$ are the branches of $\lambda_i(t^{1/r}, \eta)$. To prove the last statement observe that for those $\lambda_i(t, \eta)$ that are not $\equiv 0$ we have $\lambda_i(t, \eta) = t^{k(i)}\lambda_i^*(t, \eta)$, where $\lambda_i^*(0, \eta) \neq 0$. It follows that for some C $|D_\eta \lambda_i| \leqslant C|\lambda_i|$.

We are now interested in the existence of regular projections.

Let

$$X^{(i)} = \{x \in C^n: F \in \mathfrak{m}_x^i\},$$

where $\mathfrak{m}_x$ is the ideal of all germs at x of analytic functions vanishing at x. Thus $X^{(0)} = C^n \supset X^{(1)} \supset X^{(2)} \supset \ldots$ If $x \in X^{(i)}$ for some i, then $\lambda = 0$ is a root of $W(x, \xi; \lambda) = 0$ with multiplicity $\geqslant i$ for all ξ, so $\Delta_s(x, \xi) = 0$ for $s < i-1$. The following fact follows easily from Sard's theorem:

LEMMA 4.1. *If $x \in X^{(i)} \setminus X^{(i+1)}$, then $\Delta_{i-1}(x, \xi) \not\equiv 0$.*

Let q_t be a germ of a curve at $q \in B$, Γ its image (i.e. the germ of the set $\{q_t: t \in C\}$) and A an integer. We define a cusp-like neighbourhood of $\Gamma \setminus \{q\}$:

$$S_A(\Gamma) = \{x \in B: \operatorname{dist}(x, \Gamma) < |x-q|^A\}.$$

Recall that if X is a metric space, then a δ-net in X is a subset $A \subset X$ such that for every $x \in X$ we have $\operatorname{dist}(x, A) < \delta$.

PROPOSITION 4.2. *Let k be an integer and Ω_1 an open set in Ω. There exist ε, δ, such that if Ω_0 is a finite δ'-net in Ω_1 and $\Gamma_1, \ldots, \Gamma_k$ are germs of curves at $q_1, \ldots, q_k$, respectively, $(q_1 \in B)$ then there exists a $\xi \in \Omega_0$ and an A which determines an ε-regular projection with respect to X at every*

$S_A(\Gamma_i) \cap (X^{(j_i)} \setminus X^{(j_i+1)})$, where the indices j_i are such that $\Gamma_i \setminus \{q_i\} \subset X^{(j_i)} \setminus X^{(j_i+1)}$.

Proof. It suffices to prove the proposition for $k = 1$, for in the general case we can use induction on k: let ε', δ' satisfy the conclusion for $k-1$. We take a finite cover of Ω_1 by balls Ω^μ of radius $\varepsilon'/2$. Let ε_μ, δ_μ satisfy the conclusion for $\Omega_1 = \Omega^\mu$, $k = 1$. We take $\varepsilon = \min(\varepsilon'/2, \varepsilon_1, \varepsilon_2, \ldots)$, $\delta = \min(\delta', \delta_1, \delta_2, \ldots)$.

For $k = 1$ the proposition follows easily from the following lemma (we have to use it putting $X_0 = B$, $K = X^{(j_1)}$, $L = X^{(j_1+1)}$, $\Xi_0 = \Omega_1$, $\Delta = \Delta_{j_1-1}$):

LEMMA 4.2. *Let $X_0 \subset C^n$ be neighbourhood of the origin and $\Xi_0 \subset C^m$ a neighbourhood of a given point in C^m. Let $\Delta(x, \xi)$ be an analytic function defined in a neighbourhood of $\bar{X}_0 \times \bar{\Xi}_0$. Let K, L be analytic sets, $0 \in L \subset K \subset X_0$. Suppose that for every $x \in K \setminus L$ there exists a $\xi \in \Xi$ such that $\Delta(x, \xi) \neq 0$. Then there exist a finite number of points $\xi_1, \ldots, \xi_s \in \Xi_0$ and an $\varepsilon > 0$ such that for every curve q_t, $q_t \in K \setminus L$ for $t \neq 0$, there exists a j such that $\Delta(x, \xi) \neq 0$ for $|\xi - \xi_j| < \varepsilon$ and $x \in S_A(\Gamma)$ (where Γ is the image of q_t) for some A.*

Proof. We can assume that Ξ_0 is a neighbourhood of the origin in C^m. Consider the Taylor expansion of Δ:

$$\Delta(x, \xi) = \sum \Delta_\alpha(x) \xi^\alpha.$$

Let I be the ideal in $\mathcal{O}(X_0)$ generated by all Δ_α's. There exists a blowing-up $h\colon M \to X_0$ such that $h^*(I)$ is locally principal. Thus there is a cover $\mathfrak{U}$ of M and for every $U \in \mathfrak{U}$ a function $\varphi_U \in \mathcal{O}(U)$ such that

$$\Delta(h(x), \xi) = \varphi_U(x) \Delta_U^*(x, \xi) \quad \text{for} \quad x \in U,$$

where $\Delta_U^*(x, \xi) = \sum \Delta_{U,\alpha}^*(x) \xi^\alpha$ and for every $x \in U$ there is an α such that $\Delta_{U,\alpha}^*(x) \neq 0$. Clearly $\{\varphi_U = 0\} \subset h^{-1}(L)$. Take an $x_0 \in U$ and the smallest ν for which there is an α, $|\alpha| = \nu$, such that $\Delta_{U,\alpha}^*(x_0) \neq 0$. Then, after a linear change in the ξ-variables: $\zeta = A_{x_0} \xi$, we can assume that $\Delta_U^*(x_0; 0, \ldots, 0, \zeta_m) = \zeta_m^\nu$ (unit), so $\Delta_U^*(x, \zeta)$ is equivalent in a neighbourhood of x_0 to a distinguished polynomial

$$P_{x_0}(x, \zeta', \zeta_m) = \zeta_m^\nu + \sum a_i(x, \zeta') \zeta_m^i$$

and $a_i(x_0, \zeta') = 0$ for all $\zeta' = (\zeta_1, \ldots, \zeta_{m-1})$. Thus every solution of $P_{x_0}(x, \zeta', \zeta_m) = 0$ satisfies $|\zeta_m| \leqslant C|x - x_0|^{1/\nu}$. Let $\xi_0 = A_{x_0}^{-1}(0, \ldots, 0, \mu)$, where $|\mu|$ is so small that $\xi_0 \in \Xi_0$ and $\mu \neq 0$. Then there are neighbourhoods of x_0 and of ξ_0 such that $\Delta_U^*(x, \xi) \neq 0$ in these neighbourhoods. Therefore there exists a finite number of points $\xi_1, \ldots, \xi_s \in \Xi_0$ and a number ε such that for every $U \in \mathfrak{U}$ and every $x_0 \in U$ there is a j such that $\Delta_U^*(x, \xi) \neq 0$ for x in a neighbourhood of x_0 and for $|\xi - \xi_j| < \varepsilon$. If q_t is a curve in X_0, $q_t \in K \setminus L$

for $t \neq 0$, we lift it to a curve $\tilde{q}_t$ in M, take a $U \in \mathfrak{U}$ containing $\tilde{q}_0$ and find a ξ_j for $\tilde{q}_0$. If A is big enough, the lifting of $S_A(\Gamma)$ lies in U.

Suppose now that the standard projection $\pi = \pi(0)$: $C^n \to C^{n-1}$ is ε-regular with respect to X at a point $p \in C^n$. We want to describe $S_{\varepsilon'}(p, 0) \cap X$, where $\varepsilon' \leqslant \varepsilon$ will be specified later on. It is the disjoint sum of manifolds $M_j = \{p + \lambda_j(\xi)(\xi, 1): |\xi| < \varepsilon'\}$, where $\lambda_j(\xi) \neq \lambda_k(\xi)$ for all ξ and $j \neq k$, and $|D\lambda_j| \leqslant C|\lambda_j|$. We want to represent M_j's as the graphs of functions $x_n = \varphi_j(x')$, $x' = (x_1, \ldots, x_{n-1})$ and to define precisely the domains of φ_j's. We shall use the implicit function theorem in the following form:

IFT: Let $B(q, \delta)$ be the ball $\{x \in C^r: |x - q| < \delta\}$ and let G: $B(q, \delta) \to C^r$ be an analytic function such that $DG(q) = I$, $|D^2 G| \leqslant L = \text{const}$. Then there exist constants $\delta' = \delta'(\delta, L, r)$ and $K = K(\delta, L, r)$ such that G^{-1} is defined on $B(G(q), \delta')$ and $|D(G^{-1})| \leqslant K$.

For those λ_j's which are not identically 0, we have $|D(\lg \lambda_j)| \leqslant C$, so

$$e^{-C\varepsilon}|\lambda_j(0)| \leqslant |\lambda_j(\xi)| \leqslant e^{C\varepsilon}|\lambda_j(0)|, \quad |\xi| < \varepsilon.$$

Let $p' = \pi(p)$. Consider the map

$$G(\xi) = p' + [\lambda_j(\xi)/\lambda_j(0)]\,\xi, \quad |\xi| < \varepsilon.$$

Then $G(0) = 0$, $DG(0) = I$. For $|\xi| < \varepsilon/2$ we have

$$|D^2 \lambda_j(\xi)| \leqslant (2^n/\varepsilon) \sup_{|\zeta| = \varepsilon} |D\lambda_j(\zeta)| \leqslant C(2^n/\varepsilon) \sup_{|\zeta| = \varepsilon} |\lambda_j(\zeta)| \leqslant Ce^{C\varepsilon}(2^n/\varepsilon)|\lambda_j(0)|.$$

Therefore for $|\xi| < \varepsilon/2$

$$|D^2 G(\xi)| \leqslant (2|D\lambda_j(\xi)| + |D^2 \lambda_j(\xi)|\,|\xi|)/|\lambda_j(0)| \leqslant 2C + Ce^{C\varepsilon} 2^{n-1} = L.$$

Thus the equation $H(\xi) \equiv p' + \lambda_j(\xi)\xi = p' + q'$ can be solved (with respect to ξ) for $|q'| \leqslant \delta'|\lambda_j(0)|$. This shows that M_j can be described as $\{x_n = \varphi_j(x')\}$, where φ_j is defined on $B(p', |\lambda_j(0)|\delta')$ and $|D\varphi_j| = |D(\lambda_j \circ H^{-1})| \leqslant C|\lambda_j \circ H^{-1}|\,|D(H^{-1})| \leqslant CK|\lambda_j \circ H^{-1}|/|\lambda_j(0)| \leqslant CKe^{C\varepsilon}$.

Choose now ε' so small that $e^{C\varepsilon}\varepsilon' < \delta'$. Then for $|\xi| < \varepsilon'$ we have $|\lambda_j(\xi)\xi| < \delta'|\lambda_j(0)|$, i.e. $p' + \lambda_j(\xi)\xi \in B(p', |\lambda_j(0)|\delta')$. Summarizing, we have proved the following proposition:

PROPOSITION 4.2. *Let $\pi = \pi(0)$ be ε-regular with respect to X at p. Then there exist constants ε', δ', M, depending only on C, ε, n (where C is as in Proposition 4.1) such that $S_{\varepsilon'}(p, 0) \cap X$ can be described as follows. Let $p' = \pi(p)$, $p = (p', p_n)$, $\pi^{-1}(p') \cap X = \{p + \lambda_j^0 e_n,\ j = 1, \ldots, k\}$. There exist functions φ_j: $B(p', |\lambda_j^0|\delta') \to C$ such that $|D\varphi_j| \leqslant M$ and $S_{\varepsilon'}(p, 0) \cap X$ is the (disjoint) sum of $\{x_n = p_n + \varphi_j(x')\}$.*

5. Quasi-wings

These will be a modification of Whitney's wings.

We fix a splitting $C^m = C^k \oplus C^{m-k}$. Let $\pi\colon C^m \to C^k$ be the standard projection and let $x_1, \ldots, x_k$ (resp. $y_1, \ldots, y_{m-k}$) be coordinates in C^k (resp. in C^{m-k}).

Suppose we are given a germ Γ at 0 of a curve in C^k with equations $x_i = x_i(t)$, $x_i(0) = 0$ $(t \in C)$. Assume that $p = \operatorname{ord} x_1(t) \leqslant \operatorname{ord} x_i(t)$ for all i. Let s be a fixed positive integer, $s \geqslant p$. Put $u = (u_2, \ldots, u_k)$ and let

$$x(t, u) = \big(x_1(t), x_2(t) + u_2 t^s, \ldots, x_k(t) + u_k t^s\big). \tag{5.1}$$

We shall work with neighbourhoods of $\Gamma \setminus \{0\}$ of the form

$$U_\varepsilon = \{x \in C^k\colon\ x = x(t, u),\ t \neq 0,\ |u| < \varepsilon\}. \tag{5.2}$$

A quasi-wing V (of dimension k) in C^m over U_ε is the image of a map of the form

$$(C \setminus \{0\}) \times C^{k-1} \ni (t, u) \mapsto \big(x(t^a, u), f(t, u)\big),$$

where f is a germ of an analytic function $C^k \to C^{m-k}$ satisfying (for some C) the following estimates:

$$|D_t f(t, u)| \leqslant C|t|^{pa-1}, \qquad |D_u f(t, u)| \leqslant C|t|^{sa}. \tag{5.3}$$

We shall write

$$V = \big\{\big(x(t^a, u), f(t, u)\big)\colon\ t \neq 0,\ |u| < \varepsilon\big\}.$$

f will be called a *representation* of V (it is not uniquely determined by V). We shall identify V with its representation.

V can also be represented in the form $\big\{\big(x(t^{qa}, u), f(t^q, u)\big)\big\}$ for any integer $q > 0$; such representations will be called equivalent.

V can be broken into a sum of smooth manifolds, not necessarily disjoint. In fact, let $x_0 = (x_{01}, \ldots, x_{0k}) \in U_\varepsilon$; put

$$\begin{aligned} D(x_0) &= \{x \in U_\varepsilon\colon\ |x_1 - x_{01}| < (1/2)|x_{01}|\}, \\ D'(x_0) &= \{x \in U_{\varepsilon/2}\colon\ |x_1 - x_{01}| < (1/3)|x_{01}|\}. \end{aligned} \tag{5.4}$$

In $D(x_0)$ we select a branch of $x_1^{1/pa}$ and solve $x(t^a, u) = x$ with respect to (t, u): $t = \psi(\varrho x_1^{1/pa})$, where ϱ is a root of unity of order pa and ψ is analytic, $\psi(0) = 0$,

$$u_2 = \big[x_2 - x_2\big(\psi(\varrho x_1^{1/pa})^a\big)\big]/\psi(\varrho x_1^{1/pa})^s,$$

etc. Inserting this into f we get functions $f_\varrho\colon D(x_0) \to C^{m-k}$ such that

$$V \cap [D(x_0) \times C^{m-k}] = \bigcup_\varrho \operatorname{graph} f_\varrho.$$

It is trivial to show that estimates (4.3) are equivalent to the condition

(5.5) every f has first derivatives bounded by a constant independent of x_0.

The manifolds $V_\varrho = \text{graph } f_\varrho$ will be called *branches* of V.

A. Selection of quasi-wings

LEMMA 5.1. *Let $V^{(\alpha)} = \{(x(t^a, u), f^{(\alpha)}(t, u))\}$ be quasi-wings in C^m over U_ε such that $f_i^{(\alpha)}(t, u) \neq f_i^{(\beta)}(t, u)$ for $\alpha \neq \beta$, $t \neq 0$ and all u, i ($f_i^{(\alpha)}$'s are components of $f^{(\alpha)}$). Let $\tilde{\Gamma}$ be a germ at 0 of a curve in C^m given by $(x(t^a, u(t)), v(t))$, $x(t^a, u(t)) \in U_\varepsilon$. Let $\vec{t}(t)$ be the unit tangent vector to $\tilde{\Gamma}$ at the point corresponding to t and π_1 the projection on to the x_1-axis. Assume that $|\pi_1(\vec{t})| \geqslant C > 0$ for all t. Suppose that for $t \neq 0$, for every i and every α we have $v_i(t) \neq f_i^{(\alpha)}(t, u(t))$. Then there exists a quasi-wing $W = \{(x(t^a, u), g(t, u))\}$ over U_ε such that $v(t) = g(t, u(t))$ and $f_i^{(\alpha)}(t, u) \neq g_i(t, u)$ for all α, i, u, $t \neq 0$.*

In such a situation we say that W contains $\tilde{\Gamma}$.

An easy proof is left to the reader; in § 8 we shall prove a stronger selection lemma.

B. Lifting of quasi-wings

LEMMA 5.2. *Let $X \subset C^{m+1}$ be a hypersurface, $\pi_0: C^{m+1} \to C^m$ the standard projection, $\pi_0: X \to C^m$ a finite map. Let $V = \{(x(t^a, u), f(t, u))\}$ be a quasi-wing in C^m over U_ε. Assume that, for every $x_0 \in U_\varepsilon$ and every branch V_ϱ of V, π_0 induces a local isomorphism $\pi_0^{-1}(V_\varrho) \cap X \to V_\varrho$. Let $\tilde{\Gamma}$ be a germ of a curve in C^{m+1} given by $(x(t^a, u(t)), v(t), w(t))$, $u(0) = 0$. If π_0 is regular with respect to X at $\tilde{\Gamma}$, then there are quasi-wings in C^{m+1} over $U_{\varepsilon/2}$ of the form*

$$W_i = \{(x(t^{aq}, u), f(t^q, u), g_i(t, u))\}$$

such that

$$(x(t^{aq}, u), f(t^q, u), z) \in X, \ |u| < \varepsilon/2, \ t \neq 0 \Leftrightarrow z = g_i(t, u) \text{ for some } i.$$

Proof. Let $F = 0$ be a reduced equation of X. Then the equation $F(x(t^a, u), f(t, u), z) = 0$ can be solved as $z = g_i(t^{1/q}, u)$ for some q. We have only to show that W_i are quasi-wings over $U_{\varepsilon/2}$. Put $g = g_i$ for simplicity, take a point

$$p = (x(t_0^a, u(t_0)), v(t_0), w(t_0)) \in \tilde{\Gamma},$$

$t_0 \neq 0$; let $p'' = x(t_0^a, u(t_0))$, $\tilde{D}(p'') = \{x \in U_\varepsilon: |x_1 - p_1| < (2/3)|p_1|\}$. Take an aq-th root of unity ϱ and construct f_ϱ and g_ϱ in $\tilde{D}(p'')$; we shall show that

$$|g_\varrho(x) - g_\varrho(p'')| \leqslant K|x - p''| \quad \text{for} \quad x \in \tilde{D}(p''), \tag{5.6}$$

where the constant K does not depend on p. This of course will imply estimates (5.5).

We have $|f_\varrho(x)-f_\varrho(p'')| \leqslant A|x-p''|$ for $x \in \tilde{D}(p'')$, where A does not depend on p.

Let us return to the situation of Proposition 4.2. Let $p' = \pi_0(p)$, $\pi_0^{-1}(p') \cap X = \{p+\lambda_i^0 e_{m+1}\}$ and let $\varphi_i\colon B(p', |\lambda_i^0|\delta') \to C$ have the same meaning. Suppose that $\lambda_1^0 = g_\varrho(p'')-p_{m+1}$; then in a neighbourhood of p'' we have $g_\varrho(x) = \varphi_1(x, f_\varrho(x))+p_{m+1}$. Let $x \in \tilde{D}(p'')$. We distinguish two cases.

1° $|x-p''| \leqslant |\lambda_1^0|\delta'/2(1+A)$. Then $|x-p''|+|f_\varrho(x)-f_\varrho(p'')| \leqslant |\lambda_1^0|\delta'$, so $g_\varrho(x) = \varphi_1(x, f_\varrho(x))+p_{m+1}$ and

$$|g_\varrho(x)-g_\varrho(p'')| \leqslant M(|x-p''|+|f_\varrho(x)-f_\varrho(p'')|) \leqslant M(1+A)|x-p''|.$$

2° $|x-p''| > |\lambda_1^0|\delta'/2(1+A)$. We shall show that for any $\mu \leqslant \varepsilon$ satisfying (5.7) below (which can therefore be made independent of p and x) $(x, f_\varrho(x), g_\varrho(x))$ is not in the cone $S_\mu(p, 0) = \{p+\lambda(\xi, 1)\colon \xi \in C^m, |\xi| < \mu, \lambda \in C\}$. For suppose that $(x, f_\varrho(x), g_\varrho(x)) \in S_\mu(p, 0)$. Then $g_\varrho(x) = \varphi_j(x, f_\varrho(x))+p_{m+1}$ for some j. We estimate $|\lambda_j^0|$. Since

$$|\varphi_j(x, f_\varrho(x))| \geqslant (|x-p''|+|f_\varrho(x)-f_\varrho(p'')|)/\mu,$$

we have

$$\begin{aligned}|\lambda_j^0| = |\varphi_j(p'', f_\varrho(p''))| &\geqslant |\varphi_j(x, f_\varrho(x))|-|\varphi_j(x, f_\varrho(x))-\varphi_j(p'', f_\varrho(p''))| \\ &\geqslant (|x-p''|/\mu)-M(1+A)|x-p''| = |x-p''|[(1/\mu)-M(1+A)].\end{aligned}$$

In particular, $|\lambda_j^0| > |\lambda_1^0|$ if μ satisfies

$$\delta'[(1/\mu)-M(1+A)]/2(1+A) > 1. \tag{5.7}$$

Further, φ_j is defined in the ball around p' of radius $= |\lambda_j^0|\delta' \geqslant |x-p''|\delta'[(1/\mu)-M(1+A)]$; this ball contains $(x, f_\varrho(x))$ provided that

$$|x-p''|\delta'[(1/\mu)-M(1+A)] > |x-p''|+|f_\varrho(x)-f_\varrho(p'')|,$$

which is satisfied if μ satisfies (5.7). Thus we get a contradiction since $j \neq 1$. So we have

$$|g_\varrho(x)-g_\varrho(p'')| \leqslant (|x-p''|+|f_\varrho(x)-f_\varrho(p'')|)/\mu \leqslant |x-p''|(1+A)/\mu.$$

C. Nicely situated quasi-wings. Two quasi-wings $V = \{(x(t^a, u), f(t, u))\}$, $W = \{(x(t^a, u), g(t, u))\}$ over U_ε will be called *nicely situated* if for some $i_1, \ldots, i_l$

$$f_{i_\beta}(t, u) \neq g_{i_\beta}(t, u) \quad \text{for } t \neq 0, \beta = 1, \ldots, l,$$

while for all other i

$$|f_i(t, u)-g_i(t, u)| \leqslant C\sum_\beta |f_{i_\beta}(t, u)-g_{i_\beta}(t, u)|.$$

LEMMA 5.3. *Let* $X \subset C^{m+1}$ *be a hypersurface,*

$$\tilde{V} = \{(x(t^a, u), f(t, u), \tilde{f}(t, u))\},$$
$$\tilde{W} = \{(x(t^a, u), g(t, u), \tilde{g}(t, u))\}$$

two quasi-wings over U_ε *in* C^{m+1} *such that*

1° $\tilde{V}, \tilde{W} \subset X$,

2° *the quasi-wings* $V = \{(x(t^a, u), f(t, u))\}$, $W = \{(x(t^a, u), g(t, u))\}$ *in* C^m *are nicely situated,*

3° *the standard projection* $\pi_0: C^{m+1} \to C^m$ *is regular with respect to* X *at every point of* $\tilde{V}$,

4° $\pi_0: \pi_0^{-1}(V) \cap X \to X$ *is a local isomorphism.*

Then $\tilde{V}, \tilde{W}$ *are nicely situated.*

Proof. Let k be the number of points in $\pi_0^{-1}(p) \cap X$ for $p \in V$; it is independent of p. Take a $\tilde{p} \in \tilde{V}$; let $p = \pi_0(\tilde{p})$. There exist k distinct numbers $\lambda_1(p) = 0$, $\lambda_2(p), \ldots, \lambda_k(p)$ such that $\tilde{p} + \lambda_i(p) e_{m+1}$ are all the points of $\pi_0^{-1}(p) \cap X$ (where $e_{m+1} = (0, \ldots, 0, 1)$). Since π_0 is regular at every such $\tilde{p}$, every $\lambda_i(p)$ (for $i \geq 2$) has an extension to an analytic function defined in a neighbourhood of p whose graph is a subset of X. Thus (after replacing t by its power, if necessary) there exist analytic functions $\lambda_2(t, u), \ldots, \lambda_k(t, u)$ such that

$$\lambda_i(t, u) \neq 0 \text{ for } t \neq 0, \quad \lambda_i(t, u) \neq \lambda_j(t, u) \text{ for } t \neq 0 \text{ and } i \neq j,$$

and

$$\big(x(t^a, u), f(t, u), \tilde{f}(t, u) + \lambda_i(t, u)\big)$$

are all the points of X in π_0^{-1} $\big(x(t^a, u), f(t, u)\big)$ distinct from $\big(x(t^a, u), f(t, u), \tilde{f}(t, u)\big)$. We have

$$\lambda_i(t, u) = t^{r_i} \bar{\lambda}_i(t, u), \quad |\bar{\lambda}_i(t, u)| \geq \varepsilon_0 > 0.$$

Again by regularity of π_0 at points of $\tilde{V}$ there exist analytic functions $\varphi_i(t, u, z)$ (for $i \geq 2$), defined for $z \in C^m$, $|z| < \varepsilon_1 |t|^{r_i}$ (for some constant ε_1) such that

$$\varphi_i(t, u, 0) = \lambda_i(t, u),$$
$$\big((x(t^a, u), f(t, u)) + z, \tilde{f}(t, u) + \varphi_i(t, u, z)\big) \in X,$$
$$\varphi_i(t, u, z)| \geq (1/2)|\lambda_i(t, u)|.$$

Now let

$$f_{i_\beta}(t, u) - g_{i_\beta}(t, u) = t^{\alpha(\beta)} h_\beta(t, u),$$

where h_β are invertible; let $a_0 = \min_\beta a(\beta)$. We have to prove that either $\tilde{f}(t, u) \neq \tilde{g}(t, u)$ for all u and $t \neq 0$, or that

$$|\tilde{f}(t, u) - \tilde{g}(t, u)| \leqslant A\,|t|^{a_0}, \qquad A = \text{const}.$$

Suppose that the latter condition is not satisfied. Then there is a function $u(t)$, which can be assumed to be analytic (after, maybe, replacing t by its power) such that

$$\big|\tilde{f}\big(t, u(t)\big) - \tilde{g}\big(t, u(t)\big)\big| \geqslant |t|^b, \qquad \text{for some } b < a_0.$$

Let

$$\tilde{p}_t = \big(x\big(t^a, u(t)\big), f\big(t, u(t)\big), \tilde{f}\big(t, u(t)\big)\big),$$
$$\tilde{q}_t = \big(x\big(t^a, u(t)\big), g\big(t, u(t)\big), \tilde{g}\big(t, u(t)\big)\big).$$

Let π_0 be δ-regular with respect to X at every point of $\tilde{V}$.

By assumption, $\tilde{q}_t$ lies in the cone $\{\tilde{p}_t + \mu(\eta, 1)\colon\ \eta \in \boldsymbol{C}^m,\ |\eta| < \delta,\ \mu \in \boldsymbol{C}\}$, so for every t there exists an analytic function $\mu_t(\eta)$ ($|\eta| < \delta$) such that

$$\tilde{q}_t = \tilde{p}_t + \mu_t(\eta_t)(\eta_t, 1),$$

where

$$\eta_t = \big(0, \big[g\big(t, u(t)\big) - f\big(t, u(t)\big)\big]\big) / \big[\tilde{g}\big(t, u(t)\big) - \tilde{f}\big(t, u(t)\big)\big].$$

Clearly $\mu_t(0) = \lambda_i\big(t, u(t)\big)$ for some $i \geqslant 2$, and i is independent of t. Consider the equality

$$\tilde{g}(t, u) = \tilde{f}(t, u) + \varphi_i\big(t, u, g(t, u) - f(t, u)\big).$$

It holds in a neighbourhood of the set $\big\{\big(t, u(t)\big)\colon |t| \neq 0, \text{ sufficiently small}\big\}$. Both sides are analytic and well-defined (since $r_i = b$, and so $|g - f| < (1/2)|\lambda_i|$ for $|t|$ small enough). Thus the equality holds identically. It follows that $\tilde{f}(t, u) \neq \tilde{g}(t, u)$ for all u and $t \neq 0$.

To apply Lemma 5.3, we have to know when a projection is regular at all points of a quasi-wing. Let $X \subset \boldsymbol{C}^m$ be a hypersurface with a reduced equation $F = 0$, where F is a distinguished polynomial with respect to x_m. Let $\Omega \subset \boldsymbol{C}^{m-1}$ be a neighbourhood of the origin such that $F\big(x + \lambda(\xi, 1)\big)$ is equivalent to a distinguished polynomial in λ for $\xi \in \Omega$; let $\Delta_i(x, \xi)$ be its generalized discriminants. Let $\xi_1, \ldots, \xi_p$ be points such that for every i the functions $\Delta_i(x, \xi_\alpha)$ generate the ideal generated by all $\Delta_i(x, \xi)$, $\xi \in \Omega$. Assume for simplicity of notation that one of the ξ_α's is 0.

Lemma 5.4. *Let $V = \big\{\big(x(t^a, u), f(t, u)\big)\big\} \subset X$ be a quasi-wing containing a curve $\Gamma = \big\{\big(x\big(t^a, u(t)\big), f\big(t, u(t)\big)\big)\big\}$. Suppose that*

1° *the standard projection $\pi\colon \boldsymbol{C}^m \to \boldsymbol{C}^{m-1}$ is regular with respect to X at every point of Γ,* 2° *for every* α *there is an* i_α *such that for all* $x \in V$ *we have* $\Delta_i(x, \xi_\alpha) = 0$ *for* $i < i_\alpha$ *and* $\Delta_{i_\alpha}(x, \xi_\alpha) \neq 0$. *Then* π *is regular at every point of* V.

Proof. Let $\Delta_i(x, e_m) = 0$ for $i < i_0$, $\Delta_{i_0}(x, e_m) \neq 0$ for $x \in \Gamma \setminus \{0\}$. We have to show that, for some ε', $\Delta_i(x, \xi) = 0$ for $i < i_0$ and $\Delta_{i_0}(x, \xi) \neq 0$ for all $x = (x(t^a, u), f(t, u))$, $t \neq 0$ and $|\xi| < \varepsilon'$. Note that for all α we have $i_\alpha \geqslant i_0$; in fact, $\Delta_{i_\alpha}(x, \xi_\alpha) \neq 0$ for $x \in \Gamma \setminus \{0\}$, and e_m is not in the tangent cone to X at x. Thus $\Delta_i(x, \xi_\alpha) = 0$ for $i < i_0$ and for all $x \in V$ and α. Therefore $\Delta_i(x, \xi) = 0$ for all $i < i_0$ and for all $x \in V$ and all ξ. Now,

$$\Delta_{i_0}\big((x(t^a, u), f(t, u)), \xi_\alpha\big) = t^{r(\alpha)} \tilde{\Delta}_\alpha(t, u),$$

where $\tilde{\Delta}_\alpha$ is either invertible or identically 0, and

$$\Delta_{i_0}\Big(\big(x(t^a, u(t)), f(t, u(t))\big), 0\Big) = t^{r_0} \tilde{\Delta}_0(t),$$

$\tilde{\Delta}_0(t)$ invertible. As before, $\Delta_{i_0}\Big(\big(x(t^a, u(t)), f(t, u(t))\big), \xi\Big) \neq 0$ for all $t \neq 0$ and $|\xi|$ sufficiently small, and so for all ξ

$$\Delta_{i_0}\Big(\big(x(t^a, u(t)), f(t, u(t))\big), \xi\Big) = t^{r_0} \Delta^*(t, \xi),$$

$\Delta^*(t, 0)$ invertible. Therefore $r(\alpha) \geqslant r_0$ and

$$\Delta_{i_0}\big((x(t^a, u), f(t, u)), \xi\big) = t^{r_0} \Delta^*(t, u, \xi),$$

$\Delta^*(t, u, 0)$ invertible. Thus $\Delta_{i_0}\big((x(t^a, u), f(t, u)), \xi\big) \neq 0$ for all $t \neq 0$ and $|\xi| < \varepsilon'$, for some constant ε'.

D. Tangent spaces to nicely situated quasi-wings. Let $V = \{(x(t^a, u), f(t, u))\}$, $W = \{(x(t^a, u), g(t, u))\}$ be nicely situated quasi-wings in C^m over U_ε. Take a point $p_0 = x(t_0^a, u_0)$, $t_0 \neq 0$, $|u_0| < \varepsilon/2$, construct $D(p_0)$, $D'(p_0)$, ϱ, f_ϱ, g_ϱ as in (5.4) and (5.5). For $q \in V_\varrho = \text{graph } f_\varrho$ (resp. $q \in W_\varrho = \text{graph } g_\varrho$) let P_q be the orthogonal projection $C^m \to T_q V_\varrho$ (resp. $C^m \to T_q W_\varrho$). Let $P_q^\perp = I - P_q$.

LEMMA 5.5. *There is a constant* C, *independent of* p_0, *such that for* $q_1 \in V_\varrho$, $q_2 \in W_\varrho$ *we have*

$$|P_{q_2} - P_{q_1}| \leqslant C|q_1 - q_2|/|p_{01}|^{s/p},$$

provided that $\pi(q_1), \pi(q_2) \in D'(p_0)$ (s *and* p *are defined at the beginning of* § 5).

Proof. Let $q_1' = \pi(q_1)$, $q_2' = \pi(q_2)$, $q_1'' = f_\varrho(q_1')$, $q_2'' = g_\varrho(q_2')$. $T_{q_1} V_\varrho$ is spanned by the vectors $(e_j, D_j f_\varrho(q_1'))$, where $D_j = \partial/\partial x_j$, and $T_{q_2} W_\varrho$ by $(e_j, D_j g_\varrho(q_2'))$. Thus it is enough to show that

$$|Df_\varrho(q_1') - Gg_\varrho(q_2')| \leqslant C|q_1 - q_2|/|p_{01}|^{s/p}.$$

First observe that for $q_1', q_2' \in D'(p_0)$

$$|Df_\varrho(q_1') - Df_\varrho(q_2')| \leqslant C|q_1' - q_2'|/|p_{01}|^{s/p}$$

since $|D^2 f| \leqslant C/|p_{01}|^{s/p}$ in $D'(p_0)$. Now let $i_1, \ldots, i_l$ be the indices for which $f_{i_\beta} \neq g_{i_\beta}$ for $t \neq 0$. Then

$$f_{i_\beta}(t, u) - g_{i_\beta}(t, u) = t^{a(\beta)} h_\beta(t, u),$$

where $|h_\beta| \geqslant \text{const} > 0$, and therefore

$$f_{i_\beta,\varrho}(x) - g_{i_\beta,\varrho}(x) = \varrho^{a(\beta)} x_1^{a(\beta)/pa} h_{\beta,\varrho}(x)$$

for $x \in D(p_0)$. Thus

$$\begin{aligned} |Df_{i_\beta,\varrho}(q_2') - Dg_{i_\beta,\varrho}(q_2')| &\leqslant C|f_{i_\beta,\varrho}(q_2') - g_{i_\beta,\varrho}(q_2')|/|q_{2,1}'|^{s/p} \\ &\leqslant C|f_{i_\beta,\varrho}(q_2') - g_{i_\beta,\varrho}(q_2')|/|p_{01}|^{s/p} \end{aligned}$$

($q_{2,1}'$ is the x_1-coordinate of q_2'). If $i \neq i_\beta$, then

$$f_i(t, u) - g_i(t, u) = t^\alpha h_i(t, u),$$

where $\alpha = \min_\beta a(\beta)$ (h_i not necessarily invertible); so

$$f_{i,\varrho} - g_{i,\varrho} = \varrho^\alpha x_1^{\alpha/pa} h_{i,\varrho}$$

and therefore

$$\begin{aligned} |Df_{i,\varrho}(q_2') - Dg_{i,\varrho}(q_2')| &\leqslant C|q_{2,1}'|^{\alpha/pa}/|q_{2,1}'|^{s/p} \\ &\leqslant C|f_\varrho(q_2') - g_\varrho(q_2')|/|p_{01}|^{s/p}. \end{aligned}$$

Thus, finally,

$$\begin{aligned} |Df_\varrho(q_1') - Dg_\varrho(q_2')| &\leqslant |Df_\varrho(q_1') - Df_\varrho(q_2')| + |Df_\varrho(q_2') - Dg_\varrho(q_2')| \\ &\leqslant C\big(|q_1' - q_2'| + |f_\varrho(q_2') - g_\varrho(q_2')|\big)/|p_{01}|^{s/p} \\ &\leqslant C\big(|q_1' - q_2'| + |f_\varrho(q_2') - f_\varrho(q_1')| + |f_\varrho(q_1') - g_\varrho(q_2')|\big)/|p_{01}|^{s/p} \\ &\leqslant C(|q_1' - q_2'| + |q_1'' - q_2''|)/|p_{01}|^{s/p} = C|q_1 - q_2|/|p_{01}|^{s/p}. \end{aligned}$$

6. Proof of Proposition 1.2

We shall prove that a generalized normal partition $\{X_n^j\}$ of C^n (compatible with a given set $Z \subset C^n$) satisfies the estimates of Proposition 1.2 if the points ξ_i^m are chosen in the following way. For $\xi_i^n \in \Omega_n$ we take points such that:

1° for all s the ideal generated by all $\Delta_s^n(x, \xi)$, $\xi \in \Omega_n$, is generated by $\Delta_s^n(x, \xi_i^n)$, 2° for any $n+1$ curves $q_1(t), \ldots, q_{n+1}(t)$ in C^n there is an i such that

a) the projection $\pi(\xi_i^n)$ is regular with respect to X_n^{n-1} at each of these curves,

b) if $\vec{t}_j(t)$ is the unit tangent vector to $q_j(t)$, then, for some C and every $t \neq 0$ (with $|t|$ sufficiently small),

$$|\pi(\xi_i^n)\vec{t}_j(t)| \geqslant C > 0.$$

For $\xi_i^{n-1} \in \Omega_{n-1}$ we take vectors such that

1° for all s the functions $\Delta_s^{n-1}(x, \xi_i^{n-1})$ generate the ideal generated by all $\Delta_s^{n-1}(x, \xi)$, $\xi \in \Omega_{n-1}$,

2° for any $n+1$ curves $q_1(t), \ldots, q_{n+1}(t)$ in C^{n-1} there is an i such that

a) $\pi(\xi_i^{n-1})$ is regular with respect to X_{n-1}^{n-2} at each of these curves,

b) $|\pi(\xi_i^{n-1})\vec{t}_j(t)| \geqslant C > 0$ for all $|t| \neq 0$, sufficiently small, where $\vec{t}_j(t)$ is the unit tangent vector to $q_j(t)$.

Similarly we choose ξ_i^m.

We now give some preliminary lemmas based on the curve selection lemma.

If $X \subset \boldsymbol{R}^n$ is semianalytic, u, f, g are nonnegative functions defined on X, u semianalytic, f, g real-analytic, $g \not\equiv 0$, and there is a sequence $x_\nu \to x_0$, $x_\nu \in X$, such that $(uf/g)(x_\nu) \to 0$, then there is a real-analytic map $\varphi(r)$ such that for $r > 0$ $\varphi(r) \in X$, $g(\varphi(r)) \neq 0$ and $u(\varphi(r))f(\varphi(r))/g(\varphi(r)) \to 0$ as $r \to 0$ and $\varphi(0) = x_0$.

In fact, let A be the graph of u. Then $(0, u(x_0), x_0)$ belongs to the closure of the semianalytic set

$$B = \{(t, z, x)\colon tzf(x) = g(x),\ x \in X,\ (x, z) \in A,\ g(x) \neq 0\}.$$

Applying the curve selection lemma we get a real-analytic curve $(t(r), z(r), x(r))$ lying in B for $r > 0$, $t(0) = 0$. So we can put $\varphi(r) = x(r)$.

Similarly, if uf/g is unbounded in every neighbourhood of x_0, then there exists a real-analytic curve $\varphi(r)$ such that $\varphi(0) = x_0$, $\varphi(r) \in X$, $g(\varphi(r)) \neq 0$ for $r > 0$ and $u(\varphi(r))f(\varphi(r))/g(\varphi(r)) \to \infty$.

If $X \subset \boldsymbol{R}^n$ is semianalytic and $f(r)$ is a real-analytic map, $r \in \boldsymbol{R}$, $f(r) \in \boldsymbol{R}^n$, then there exists a real-analytic map $g(r)$ such that $g(r) \in X$ for $r > 0$ and $|f(r) - g(r^{1/p})| \leqslant C \operatorname{dist}(f(r), X)$, for some $p \in \boldsymbol{N}$ and $C > 0$.

In fact, let ϱ_X be a semianalytic function such that $(1/2)\operatorname{dist}(x, X) \leqslant \varrho_X(x) \leqslant 2\operatorname{dist}(x, X)$; apply the curve selection lemma to the set $\{(r, x)\colon x \in X,\ |f(r) - x| \leqslant 2\varrho_X(x)\}$.

LEMMA 6.1. *Let $X \subset C^n$ be an analytic set and $f(r) \in C^n$ a real-analytic map. If* $\operatorname{dist}(f(r), X)$ *is of order r^a, then for complex t we have dist $(f(t), X) \geqslant C|t|^a$ (by $f(t)$ we mean here the complexification of f: $\boldsymbol{R} \to C^n$).*

Proof. Assume that the conclusion is false. Then, by our previous remarks, there exists a real-analytic map $\varphi(r) \in C$, $\varphi(0) = 0$ such that

$\varrho_X(f(\varphi(r))) \leqslant C|\varphi(r)|^{a+\varepsilon}$ for some C, ε. Let $h(r)$ be a real-analytic map such that $h(r)\in X$, $|f(\varphi(r))-h(r^{1/p})| \leqslant C|\varphi(r)|^{a+\varepsilon}$. Replacing everywhere r by r^p we can assume that $p=1$. For $t\in C$ we have $h(t)\in X$, and therefore $\operatorname{dist}(f(\varphi(t)), X) \leqslant C|\varphi(t)|^{a+\varepsilon}$ for all $t\in C$. But in the complex domain φ is surjective, and we have a contradiction.

LEMMA 6.2. *If* (1.6, k) *fails to hold, then there exist real-analytic maps* $q_{j_1}(r), \ldots, q_{j_k}(r)$ *such that for* $r>0$ *we have*:

$$q_{j_l}(r)\in \mathring{X}_n^{j_l},$$

$$(6.1)\qquad |q_{j_1}(r)-q_{j_s}(r)| \leqslant c_0\varrho_{j_s}(q_{j_1}(r)), \qquad s=2,\ldots,k$$

(c_0 *is the constant appearing in the definition of a chain*),

$$(6.2)\qquad |P^{\perp}_{q_{j_1}}(r)P_{q_{j_2}}(r)\ldots P_{q_{j_k}}(r)| \geqslant |q_{j_1}(r)-q_{j_2}(r)|/\varrho_{j_k-1}(q_{j_1}(r))^{1+\eta} \quad \textit{for some } \eta>0.$$

Proof. Observe that for fixed j the functions

$$\mathring{X}_n^j \ni x \mapsto P_x \in C^{n^2}$$

and

$$\mathring{X}_n^j \ni x \mapsto P_x^{\perp} \in C^{n^2}$$

are restrictions of real-meromorphic functions, well defined on $\mathring{X}_n^j$ (cf. e. g. [1]). Write

$$P_x = A_j(x)/R_j(x), \qquad P_x^{\perp} = B_j(x)/R_j(x),$$

where A_j, B_j, R_j are analytic and $R_j(x)\neq 0$ for $x\in \mathring{X}_n^j$ (A_j and B_j are matrix-valued). Let

$$G(q_{j_1},\ldots,q_{j_k}) = \varrho_{j_k-1}(q_{j_1})|B_{j_1}(q_{j_1})A_{j_2}(q_{j_2})\ldots A_{j_k}(q_{j_k})|/|q_{j_1}-q_{j_2}|\prod_l |R_{j_l}(q_{j_l})|.$$

G is unbounded on the semianalytic set

$$\{(q_{j_1},\ldots,q_{j_k})\colon\ q_{j_i}\in \mathring{X}_n^{j_i},\ \varrho_l(q_{j_1}) \geqslant (1/2c_0)\varrho_{j_s}(q_{j_1}) \ \text{(for all } l<j_s),\ |q_{j_1}-q_{j_s}| \leqslant c_0\varrho_{j_s}(q_{j_1})\},$$

and so there are real-analytic maps $q_{j_1}(r), \ldots, q_{j_k}(r)$ such that

$$G(q_{j_1}(r),\ldots,q_{j_k}(r)) \to \infty \qquad \text{as} \qquad r\to 0.$$

Necessarily $q_{j_1}(0)=\ldots=q_{j_k}(0)\in X_n^{j_k-1}$, so $\varrho_{j_k-1}(q_{j_l}(r))$ is of order r^{μ}, $\mu>0$. If $G(q_{j_1}(r),\ldots,q_{j_k}(r))$ is of order $r^{-\nu}$, $\nu>0$, then it is enough to take $\eta=\nu/2\mu$.

LEMMA 6.3. *Let* $q_t \in X_n^j$ *be a curve. Choose admissible projections* $\pi^i = \pi(\xi_{m_i}^i)$: $C^i \to C^{i-1}$ *for* $j < i \leqslant n$; *let* π^{pq}: $C^p \to C^q$ *be their composition. For simplicity of notation assume that every* π^i *is the standard projection. Assume that every* π^i *is* δ*-regular with respect to* X_i^{i-1} *at* $\pi^{n,i+1}(q_t)$. *Then*

$$\operatorname{dist}(q_t, X_n^{j-1}) \leqslant (1/\delta)^{n-j} \operatorname{dist}\big(\pi^{n,j}(q_t), X_j'^{j-1}\big).$$

Proof. Fix a t and let y be the closest point of $X_j'^{j-1}$ to $\pi^{n,j}(q_t)$. Let l_j be the real interval joining these points. We lift l_j via π^{j+1} to a real-analytic curve l_{j+1} in $\pi^{n,j+1}(X_n^j)$ starting at $\pi^{n,j+1}(q_t)$. Clearly, it lies outside the cone $S_\delta\big(\pi^{n,j+1}(q_t), 0\big)$. Again we lift l_{j+1} via π^{j+2} to a real-analytic curve l_{j+2} in $\pi^{n,j+2}(X_n^j)$ starting at $\pi^{n,j+2}(q_t)$; it lies outside the cone $S_\delta\big(\pi^{n,j+2}(q_t), 0\big)$. Finally we get l_n. If z is its end, then $z \in X_n^{j-1}$ (by the definition of X_n^{j-1}) and

$$|q_t - z| \leqslant (1/\delta)^{n-j} |\pi^{n,j}(q_t) - y|.$$

We shall now prove (1.7,0); the proof of this case (which is the first step of induction in the proof of (1.7, k)) is simpler than the general case, since we need not use essentially quasi-wings (except the definition and Lemma 5.5).

Suppose that (1.7,0) fails for some j. Then there exist real-analytic maps $q(r)$, $q'(r)$ such that $q(r), q'(r) \in \mathring{X}_n^j$ for $r > 0$, $|q(r) - q'(r)| \leqslant (1/2c_0)\varrho_{j-1}\big(q(r)\big)$ and for some $\eta > 0$

$$|P_{q(r)} - P_{q'(r)}| \geqslant C |q(r) - q'(r)| / \operatorname{dist}\big(q(r), X_n^{j-1}\big)^{1+\eta}.$$

After replacing r by its power we can assume that the order of $\operatorname{dist}\big(q(r), X_n^{j-1}\big)$ is an integer s. Let $q(t)$ ($t \in C$) be the complexification of $q(r)$; then the order of dist $\big(q(t), X_n^{j-1}\big)$ is $\leqslant s$. Necessarily

$$|q(r) - q'(r)| = o(r^s). \tag{6.3}$$

Now choose admissible projections $\pi^i = \pi(\xi_{m_i}^i)$:$C^i \to C^{i-1}$ (assumed to be the standard projections, for simplicity of notation), $j < i \leqslant n$, such that every π^i is regular with respect to X_i^{i-1} at $\pi^{n,i+1}\big(q(t)\big)$ and the angle between the tangent vector to $\pi^{n,i}\big(q(t)\big)$ and the kernel of π^i is $\geqslant \delta_0 > 0$ for some δ_0 and all $t \neq 0$. Then, by Lemma 6.3, $\operatorname{dist}\big(\pi^{n,j}(q(t)), X_j^{j-1}\big)$ is of order $\leqslant |t|^s$. In C^j we take coordinates $x_1, \ldots, x_j$ so that the equations of $\pi^{n,j}\big(q(t)\big)$ are: $x_1 = t^p$, $x_i = x_i(t)$, $i > 1$, ord $x_i(t) \geqslant p$, and define U_ε by (5.1) and (5.2), taking $\pi^{n,j}\big(q(t)\big)$ for Γ. Using (6.3) it is easy to show that in the (t, u)-"coordinates" in U_ε the equations of $\pi^{n,j}\big(q'(r)\big)$ are:

$$t = r + \varphi(r), \quad u_i = u_i(r), \quad \text{where } \varphi(r) = o(r),\ u_i(r) = o(1).$$

For ε sufficiently small $(\pi^{n,j})^{-1}(U_\varepsilon) \cap X_n^j$ is a sum of quasi-wings

$$V_\alpha = \big\{\big(x(t^a, u), f^\alpha(t, u)\big)\big\},$$

and $f_j^\alpha(t, u) \neq f_j^\beta(t, u)$ for all $\alpha \neq \beta$, all j and $t \neq 0$. Let $q(r)$ lie on V_α and

$q'(r)$ on V_β. If $\alpha \neq \beta$, we can use Lemma 5.5 directly. If $\alpha = \beta$, we must take more care of different branches of V_α. For simplicity we omit the index α. For any two a-th roots of unity ϱ, ϱ', $f_i(\varrho t, u) - f_i(\varrho' t, u)$ is either identically 0 or $\neq 0$ for $t \neq 0$; in the latter case

$$f_i(\varrho t, u) - f_i(\varrho' t, u) = t^{\mu_i} G_i(t, u) \tag{6.4}$$

with G_i invertible.

Let $p_0 = \pi^{n,j}(q(r_0))$ for $r_0 \neq 0$; construct $D(p_0)$, $D'(p_0)$ as before and let f_ϱ, $f_{\varrho'}$ be the branches in $D(p_0)$ of $V(=V_\alpha)$ containing $q(r)$ and $q'(r)$, respectively. If $f_{i,\varrho} = f_{i,\varrho'}$ for all i, we can again use Lemma 5.5. If $f_{i,\varrho} \neq f_{i,\varrho'}$ for some i, then it follows from (6.4) that in $D'(p_0)$

$$|Df_{i,\varrho} - Df_{i,\varrho'}| \leqslant C\,|f_{i,\varrho} - f_{i,\varrho'}|/\mathrm{dist}(p_0, X_j^{j-1})$$

(C independent of p_0) and to finish the proof we proceed as in Lemma 5.5.

We now prove (1.6, k) by increasing induction on k. The case $k = 2$ is covered by the induction step. Assume that (1.6, k) does not hold and let k be the smallest integer with this property. Take real-analytic maps $q_{j_1}(r), \ldots, q_{j_k}(r)$ as in Lemma 6.2. Then

$$\mathrm{ord}\,\varrho_{j_k-1}(q_{j_1}(r)) < \mathrm{ord}\,\varrho_{j_{k-1}-1}(q_{j_1}(r)) \quad (= \mathrm{ord}\,\varrho_{j_k}(q_{j_1}(r))),$$

for otherwise (1.6, k) follows from (1.6, $k-1$). It follows that for all s

$$\mathrm{ord}\,|q_{j_s}(r) - q_{j_k}(r)| > \mathrm{ord}\,\varrho_{j_k-1}(q_{j_k}(r)). \tag{6.5}$$

Fix a $\pi^n = \pi(\xi^n_{i_n})$: $C^n \to C^{n-1}$ which is ε-regular (for some ε) with respect to X^{n-1}_n at every $q_{j_s}(t)$ and $|\pi^n \vec{t}_{j_s}| > C > 0$ for some C and all s, where $\vec{t}_{j_s}$ is the unit tangent vector to $q_{j_s}(t)$. Then fix a $\pi^{n-1} = \pi(\xi^{n-1}_{i_{n-1}})$ which is ε-regular with respect to X^{n-2}_{n-1} at every $\pi^n q_{j_s}(t)$ and $|\pi^{n-1} \vec{t}'_{j_s}| > C > 0$ for all s, where $\vec{t}'_{j_s}$ is the unit tangent vector to $\pi^n q_{j_s}(t)$, etc., until we fix a π^{j_k+1}. For simplicity of notation assume that every π^i: $C^i \to C^{i-1}$ is the standard projection. Let π^{pq}: $C^p \to C^q$ be their composition, $p \geqslant q \geqslant j_k$.

Consider $\pi^{n,j_k}(q_{j_k}(r))$; replacing r by its power we can assume that $\mathrm{dist}(\pi^{n,j_k}(q_{j_k}(r)), X'^{j_k-1}_{j_k})$ is of order r^s, $s \in N$; thus s is also the order of $\mathrm{dist}(q_{j_k}(r), X^{j_k-1}_n)$ by Lemma 6.3, and, by Lemma 6.1,

$$\mathrm{dist}(\pi^{n,j_k}(q_{j_k}(t)), X'^{j_k-1}_{j_k}) \geqslant C\,|t|^s, \; t \in C.$$

By a proper choice of coordinates $x_1, \ldots, x_{j_k}$ in C^{j_k} we can assume that $\pi^{n,j_k}(q_{j_k}(0)) = 0$ and that $\pi^{n,j_k}(q_{j_k}(t))$ is not tangent at $t = 0$ to the direction of the x_1-axis. Then the x_1-coordinate of $\pi^{n,j_k}(q_{j_k}(t))$ can be written as $(\pi^{n,j_k}(q_{j_k}(t)))_1 = t^p + o(t^p)$. Let Γ of § 5 be $\pi^{n,j_k}(q_{j_k}(t))$ and define $x(t, u)$ and

U_ε by (5.1) and (5.2). By (6.5), $\pi^{n,j_k}(q_{j_\alpha}(r))$ can be written as $x\,(t_\alpha(r), u_\alpha(r))$, $\alpha = 1, \ldots, k$, where $t_\alpha\colon \boldsymbol{R} \to \boldsymbol{C}$ and $u_\alpha\colon \boldsymbol{R} \to \boldsymbol{C}^{j_k-1}$ satisfy $t_\alpha(r) = r + o(r)$, $u_\alpha(r) = o(1)$.

LEMMA 6.4. *There exist an integer q, arbitrarily small perturbations $\tilde{q}_{j_\alpha}(r)$ of $q_{j_\alpha}(r^q)$ and quasi-wings $V_\alpha = \{(x(t^a, u), f_\alpha(t, u))\}$ over $U_{\varepsilon'}$ (for some $\varepsilon' \leqslant \varepsilon$) such that:*

$$V_\alpha \subset \mathring{X}_n^{j_\alpha},$$
$$\tilde{q}_{j_\alpha}(r) = \big(x(t_\alpha(r)^a, u_\alpha(r)), f_\alpha(t_\alpha(r), u_\alpha(r))\big),$$
every pair V_α, $V_{\alpha+1}$ is nicely situated.

Proof. We construct $\tilde{q}_{j_\alpha}$ and V_α by decreasing induction on α. To get V_k, put first $V_k^{j_k} = U_\varepsilon$. We lift $V_k^{j_k}$ to $X_{j_k+1}^{j_k}$, i.e. we consider $(\pi^{j_k+1})^{-1}(V_k^{j_k}) \cap X_{j_k+1}^{j_k}$. By Lemma 5.2, this is a sum of quasi-wings over $U_{\varepsilon/2}$. From among them we select that one (call it $V_k^{j_k+1}$) which contains $\pi^{n,j_k+1}(q_{j_k}(r))$. Now we lift $V_k^{j_k+1}$ to $X_{j_k+2}^{j_k+1}$ and select $V_k^{j_k+2}$; we repeat this procedure until we get $V_k^n = V_k$.

Suppose we have already $\tilde{q}_{j_\alpha}$ and $V_\alpha = \{(x(t^a, u), f_\alpha(t, u))\}$, where $f_\alpha = (f_{\alpha,1}, \ldots, f_{\alpha,n-j_k})$. We shall construct $\tilde{q}_{j_{\alpha-1}}$ and $V_{\alpha-1}$. We perturb $q_{j_{\alpha-1}}$ so that the perturbed map $\tilde{q}_{j_{\alpha-1}}$ lies in $X_n^{j_{\alpha-1}}$ and satisfies for every $r \neq 0$ and every β, $j_k \leqslant \beta \leqslant j_{\alpha-1}$: $\pi^{n,j_k}(\tilde{q}_{j_{\alpha-1}}(r)) = \pi^{n,j_k}(q_{j_{\alpha-1}}(r))$ and $\pi^{n,\beta}(\tilde{q}_{j_{\alpha-1}}(r)) \notin V_\alpha^\beta \cup X_\beta^{\beta-1}$ for $r \neq 0$, $\beta \leqslant j_{\alpha-1}$, where

$$V_\alpha^\beta = \{(x(t^a, u), (f_{\alpha,1}(t, u), \ldots, f_{\alpha,\beta-j_k}(t, u))\}.$$

Then the same holds for the complexification of $\tilde{q}_{j_{\alpha-1}}$. Using Lemma 5.1 (after maybe replacing r by its power) we construct (inductively on β) quasi-wings $V_{\alpha-1}^\beta$ (for $\beta \leqslant j_{\alpha-1}$) containing $\pi^{n,\beta}(\tilde{q}_{j_{\alpha-1}}(r))$ such that V_α^β and $V_{\alpha-1}^\beta$ are nicely situated, $\pi^\beta(V_{\alpha-1}^\beta) = V_{\alpha-1}^{\beta-1}$ and $V_{\alpha-1}^\beta \cap X_\beta^{\beta-1} = \emptyset$. We lift $V_{\alpha-1}^{j_{\alpha-1}}$ to $X_{j_{\alpha-1}+1}^{j_{\alpha-1}}$ and, as before, we select a quasi-wing $V_{\alpha-1}^{j_{\alpha-1}+1}$ containing $\pi^{n,j_{\alpha-1}+1}(\tilde{q}_{j_{\alpha-1}}(r))$; by Lemmas 5.3 and 5.4 the quasi-wings $V_{\alpha-1}^{j_{\alpha-1}+1}$ and $V_\alpha^{j_{\alpha-1}+1} = \{(x(t^a, u), (f_{\alpha,1}(t, u), \ldots, f_{\alpha,j_{\alpha-1}+1-j_k}(t, u))\}$ are nicely situated. We repeat this procedure until we get $V_{\alpha-1}^n = V_{\alpha-1}$.

Now we show that (1.6, k) holds for $\tilde{q}_{j_1}(r), \ldots, \tilde{q}_{j_k}(r)$, $r > 0$. Observe that $x_1(t^a, 0) = t^{pa} + o(t^{pa})$. Let $\varphi(t) = t + \ldots$ so that $x_1(t^a, 0) = \varphi(t)^a$. In $D\,(\pi^{n,j_k}(\tilde{q}_{j_k}(r)))$ (for $r > 0$) we choose the branch of $x_1^{1/pa}$ which takes the value $\varphi(t)$ for $u = 0$. Put $\varrho = 1$; then for all α we have $\tilde{q}_{j_\alpha}(r) \in V_{\alpha,\varrho}$. For $x \in V_{\alpha,\varrho}$ let

$P'_x\colon C^n \to T_x V_{\alpha,\varrho}$ be the orthogonal projection and let $P'^{\perp}_x = I - P'_x$. Then, by Lemmas 5.5 and 6.3 we have

$$(6.6)\qquad |P'^{\perp}_{\tilde q_{j_\alpha}(r)} P'_{\tilde q_{j_{\alpha+1}}(r)}| \leqslant C|\tilde q_{j_\alpha}(r) - \tilde q_{j_{\alpha+1}}(r)|/\operatorname{dist}(\tilde q_{j_k}(r),\, X_n^{j_k-1}).$$

Further, we have

$$(6.7)\quad |(P_{\tilde q_{j_\alpha}(r)} - P'_{\tilde q_{j_\alpha}(r)}) P'_{\tilde q_{j_{\alpha+1}}(r)}| \leqslant C|\tilde q_{j_\alpha}(r) - \tilde q_{j_{\alpha+1}}(r)|/\operatorname{dist}(\tilde q_{j_k}(r),\, X_n^{j_k-1}),$$

for

$$|(P_{\tilde q_{j_\alpha}(r)} - P'_{\tilde q_{j_\alpha}(r)}) P'_{\tilde q_{j_{\alpha+1}}(r)}| = |(P_{\tilde q_{j_\alpha}(r)} P'^{\perp}_{\tilde q_{j_\alpha}(r)} - P^{\perp}_{\tilde q_{j_\alpha}(r)} P'_{\tilde q_{j_\alpha}(r)}) P'_{\tilde q_{j_{\alpha+1}}(r)}|$$
$$= |P_{\tilde q_{j_\alpha}(r)} P'^{\perp}_{\tilde q_{j_\alpha}(r)} P'_{\tilde q_{j_{\alpha+1}}(r)}| \leqslant |P'^{\perp}_{\tilde q_{j_\alpha}(r)} P_{\tilde q_{j_{\alpha+1}}(r)}|.$$

Now it is enough to write

$$|P^{\perp}_{\tilde q_{j_1}(r)} P_{\tilde q_{j_2}(r)} \ldots P_{\tilde q_{j_k}(r)}| \leqslant |P'^{\perp}_{\tilde q_{j_1}(r)} P'_{\tilde q_{j_2}(r)} \ldots P'_{\tilde q_{j_k}(r)}| + \sum_\alpha |P^{\perp}_{\tilde q_{j_1}(r)} P_{\tilde q_{j_2}(r)} \ldots$$
$$\ldots P_{\tilde q_{j_\alpha}(r)} (P_{\tilde q_{j_\alpha}(r)} - P'_{\tilde q_{j_\alpha}(r)}) P'_{\tilde q_{j_{\alpha+1}}(r)} \ldots P'_{\tilde q_{j_k}(r)}|$$
$$\leqslant |P'^{\perp}_{\tilde q_{j_1}(r)} P'_{\tilde q_{j_2}(r)} \ldots P'_{\tilde q_{j_k}(r)}| + C\sum_\alpha \frac{|P^{\perp}_{\tilde q_{j_1}(r)} P_{\tilde q_{j_2}(r)} \ldots P_{\tilde q_{j_\alpha}(r)}|\,|\tilde q_{j_\alpha}(r) - \tilde q_{j_{\alpha+1}}(r)|}{\operatorname{dist}(\tilde q_{j_k}(r),\, X_n^{j_k-1})}$$

and use (1.6, l) for $l < k$, (6.6) and (6.1).

To get a contradiction it is enough to remark that if $\tilde q_{j_i}(r)$ are sufficiently small perturbations of $q_{j_i}(r^q)$, then they also satisfy (6.2).

To prove (1.7, k) we use induction on k. First we prove, as in the proof of (1.6, k), that

$$|P^{\perp}_{q'} P_{q_{j_1}} \ldots P_{q_{j_k}}| \leqslant C|q' - q_{j_1}|/\operatorname{dist}(q_{j_1},\, X_n^{j_k-1}).$$

Then we note that

$$|(P_{q'} - P_{q_{j_1}}) P_{q_{j_2}} \ldots P_{q_{j_k}}| \leqslant |P^{\perp}_{q'} P_{q_{j_1}} \ldots P_{q_{j_k}}| + |P_{q'} P^{\perp}_{q_{j_1}} P^{\perp}_{q_{j_1}} P_{q_{j_2}} \ldots P_{q_{j_k}}|$$

and use (1.6, k).

7. Whitney's conditions

Let $q \in \mathring{X}_n^j$, $q' \in \mathring{X}_n^k$, $k < j$; let $2\alpha(q, q')$ be the angle between $T_q \mathring{X}_n^j$ and $T_{q'} \mathring{X}_n^k$; then $|2\sin\alpha(q, q')| = |P_q^{\perp} P_{q'}|$.

Proposition 7.1. $|\alpha(q, q') \leqslant C|q - q'|/\operatorname{dist}(\{q, q'\},\, X_n^{k-1})$.

Remark. This inequality is much stronger than Whitney's condition A, which is equivalent, by Łojasiewicz's inequality, to the estimate

$$(7.1)\qquad |\alpha(q, q')| \leqslant C|q-q'|^{\lambda}/\mathrm{dist}(\{q, q'\}, X_n^{k-1})^{\mu}$$

for some $\lambda, \mu > 0$. The existence of a stratification satisfying (7.1) with $\lambda = 1$ is proved in [8] (in the real case).

Of course we are assuming that the vectors ξ_i^m are chosen as in § 5.

Proof. Our estimate is trivial if $|q-q'| \geqslant (1/2c_0)\varrho_{k-1}(q)$. Assume that $|q-q'| \leqslant (1/2c_0)\varrho_{k-1}(q)$ and take a chain $q_{j_1} = q, \ldots, q_{j_l}$. Let m be the smallest integer such that $j_m \geqslant k$. Then

$$|P_{q_{j_1}}^{\perp} P_{q_{j_m}}| = |P_{q_{j_1}}^{\perp}(P_{q_{j_2}} + P_{q_{j_2}}^{\perp}) \ldots (P_{q_{j_{m-1}}} + P_{q_{j_{m-1}}}^{\perp}) P_{q_{j_m}}|$$
$$\leqslant C|q_{j_1} - q_{j_m}|/\mathrm{dist}(q, X_n^{k-1}),$$

i.e. $|\alpha(q, q_{j_m})| \leqslant C|q-q_{j_m}|/\mathrm{dist}(q, X_n^{k-1})$. Further, we have

$$|P_{q_{j_m}} - P_{q'}| \leqslant C|q_{j_m} - q'|/\mathrm{dist}(\{q_{j_m}, q'\}, X_n^{k-1}),$$

which finishes the proof.

In [8] it is proved that condition (7.1) with $\lambda = 1$ implies Whitney's condition B (in the complex case it is actually equivalent to it, as was proved by B. Teissier). However, we shall prove it directly for the case of generalized normal partitions.

PROPOSITION 7.2. *The stratification $\{X_n^j\}$ satisfies Whitney's condition* B.

Proof. Suppose that this is not the case; then for some $k < j$ there exist real-analytic maps $q(r)$, $q'(r)$ such that $q(0) = q'(0) \in \mathring{X}_n^k$, $q(r) \in \mathring{X}_n^j$, $q'(r) \in \mathring{X}_n^k$ (for $r > 0$) and

$$\measuredangle\,(q(r) - q'(r), T_{q(r)}\mathring{X}_n^j) \nrightarrow 0.$$

Then the same holds for a sufficiently small perturbation $\tilde{q}(r)$ of $q(r)$. We fix projections $\pi^n = \pi(\xi_i^n)$, $\pi^{n-1} = \pi(\xi_j^{n-1}), \ldots, \pi^{k+2} = \pi(\xi_l^{k+2})$ such that every π^i is ε-regular with respect to X_i^{i-1} at $\pi^{ni}(q(0))$. Again assume for simplicity that every π^i is the standard projection. Let $y_1, \ldots, y_{k+1}$ be coordinates in C^{k+1} such that X_{k+1}^k has equation $y_{k+1} = 0$ in a neighbourhood of $\pi^{n,k+1}(q(0))$. Then every X_m^k in a neighbourhood of $\pi^{nm}(q(0))$ has equations of the form $y_{k+1} = 0$, $x_s = \varphi_s(y_1, \ldots, y_k)$, φ_s analytic, $s = k+1, \ldots, m$. We perturb $q(r)$ a little, so that the perturbed map $\tilde{q}(r)$ satisfies

$$\pi^{nm}(\tilde{q}(r)) \notin X_m^{m-1} \quad \text{for } r > 0 \text{ and } m = k+1, \ldots, j+1.$$

We select a quasi-wing $V_j \subset C^j$ containing $\pi^{nj}(\tilde{q}(r))$ such that $V_j \cap (X_j^{j-1} \setminus X_j^k) = \emptyset$. Then we lift it succesively to a quasi-wing $V_s \subset X_s^j$ containing $\pi^{ns}(\tilde{q}(r))$, $s = j+1, \ldots, n$. The quasi-wings V_n is of the form $x_s = f_s(y_1, \ldots, y_k, y_{k+1}^{1/r})$

$= \varphi_s(y_1, \ldots, y_k) + \psi_s(y_1, \ldots, y_k, y_{k+1}^{1/r}) y_{k+1}^a$, ψ_s invertible, $a \geq 1$. Then (see e. g. [5]) $\measuredangle(\tilde{q}(r) - q'(r), T_{\tilde{q}(r)} V_n) \to 0$ and we have a contradiction.

Remark. One could ask if an estimate of the form

$$|\measuredangle(q-q', T_q \mathring{X}_n^j)| \leq C|q-q'|/\mathrm{dist}(\{q, q'\}, X_n^{k-1})$$

holds for $q \in \mathring{X}_n^j$, $q' \in \mathring{X}_n^k$, $k < j$. This is not the case: there is no stratification of $\{(x^2+y^2-z^3)(x^2-z^3) = 0\} \subset C^3$ having this property.

8. Extended quasi-wings

These will be objects needed in the proof of the second part of Proposition 1.3.

Fix a splitting $C^m = C^k \oplus C^1 \oplus C^{m-k-1}$; we shall denote points in C^m by (x, y, z), where $x = (x_1, \ldots, x_k) \in C^k$, $z \in C^1$, $y = (y_1, \ldots, y_{m-k-1}) \in C^{m-k-1}$. Let $\Gamma \subset C^k$ be a (germ of a) curve in C^k and define $x(t, u)$ by (5.1) and $U_\varepsilon \subset C^k$ by (5.2). An extended quasi-wing W over U_ε is the set of points of the form

$$x = x(t^a, u), \quad z = f_1(t, u) + t^c \zeta^d, \quad y = \varphi(t, \zeta, u), \quad t \neq 0, \quad \zeta \neq 0,$$

where c, d are non-negative integers and $f_1(t, u)$, $\varphi(t, \zeta, u)$ are analytic functions satisfying

$$(8.1)\qquad \begin{gathered} |D_u \varphi| \leq C|t|^{sa}, \\ |D_\zeta \varphi| \leq C|t|^c|\zeta|^{d-1} \text{ if } d \geq 1 \quad \text{and} \quad |D_\zeta \varphi| \leq C|t|^c \text{ if } d = 0. \end{gathered}$$

Put $\Phi(t, \zeta, u) = \big(x(t^a, u), f_1(t, u) + t^c \zeta^d, \varphi(t, \zeta, u)\big)$; then $W = \Phi(\{t \neq 0, \zeta \neq 0\})$. Let $\dot{W} = \Phi(\{t \neq 0, \zeta = 0\})$.

We shall not distinguish between W and its germ at $\Phi(0, 0, 0)$.

If we replace t, ζ by t^α, ζ^β ($\alpha, \beta \in N$), we obtain an equivalent representation of W. The numbers s/a, c/a are of course the same for equivalent representations.

Consider a point $p_0 = \Phi(t_0, \zeta_0, u_0) \in W$. As in the case of quasi-wings, in a neighbourhood of p_0 we can solve the equations $x = x(t^a, u)$, $z = f_1(t, u) + t^c \zeta^d$ with respect to t, ζ, u; if we insert the solutions into φ, we obtain the decomposition of W into branches:

$$W_{\varrho\varrho'}\colon\ y = \varphi_{\varrho\varrho'}(x, z),$$

where ϱ is a pa-th root of unity and ϱ' a c-th root of unity; the estimates (8.1) are equivalent to the condition

$$(8.2)\qquad |D_z \varphi_{\varrho\varrho'}| \leq C, \quad |D_{x_i} \varphi_{\varrho\varrho'}| \leq C \text{ for } i > 1, \quad \text{where } C \text{ is independent of } p_0.$$

We can define similarly the branches of $\dot{W}_\varrho$ of $\dot{W}$.

Put $f_2(t, u) = \varphi(t, 0, u)$. If $\dot{W}$: $z = f_1(t, u)$, $y = f_2(t, u)$ is a quasi-wing, we shall say that W is an extension of $\dot{W}$. Note that necessarily we have $|D_u f_2| \leqslant C|t|^{sa}$.

The following selection lemma generalizes Lemma 5.1 and, applied to every y_i, will enable us to select extended quasi-wings.

LEMMA 8.1. *Suppose we are given the following objects:*

1) *a function* $f(t, u)$ *(with values in* **C***) such that* $|D_u f| \leqslant C|t|^{\sigma}$,

2) l *functions* $\varphi_i(t, \zeta, u)$ *(also* **C***-valued) such that:*

a) $\varphi_i(t, 0, u) = f(t, u)$,

b) $\varphi_i(t, \zeta, u) \neq \varphi_j(t, \zeta, u)$ *for* $i \neq j$ *and* $t \neq 0$, $\zeta \neq 0$,

c) $|D_u \varphi_i| \leqslant C|t|^{\sigma}$, $|D_\zeta \varphi_i| \leqslant C|t|^c |\zeta|^{d-1}$,

3) *functions* $u(t)$, $\zeta(t)$, $y(t)$ *such that:*

a) $y(t) \neq \varphi_i(t, \zeta(t), u(t))$ *for all* i *and* $t \neq 0$,

b) $\operatorname{ord} \zeta(t) = 1$,

c) $|y(t) - f(t, u(t))| \leqslant C|t|^c |\zeta(t)|^d$ *(equivalently,* $|y(t) - f(t, u(t))| \leqslant C|t|^{c+d}$*).*

Then there exists a function $\varphi(t, \zeta, u)$ *such that:*

(1) $\varphi(t, \zeta, u) \neq \varphi_i(t, \zeta, u)$ *for all* i *and* $t \neq 0$, $\zeta \neq 0$,

(2) $\varphi(t, 0, u) = f(t, u)$,

(3) $y(t) = \varphi(t, \zeta(t), u(t))$,

(4) $|D_u \varphi| \leqslant C|t|^{\sigma}$, $\quad |D_\zeta \varphi| \leqslant C|t|^c |\zeta|^{d-1}$.

Proof. If $l = 1$, then we put

$$\varphi(t, \zeta, u) = \left(y(t)/\zeta(t)^d\right)\zeta^d + \varphi_1(t, \zeta, u).$$

Now we proceed by induction on l. Clearly, we can assume that $\varphi_1 = 0$. Then

$$\varphi_i(t, \zeta, u) = t^{r_i} \zeta^{s_i} \tilde{\varphi}_i(t, \zeta, u), \qquad \tilde{\varphi}_i \text{ invertible.}$$

Necessarily $r_i \geqslant c$, $s_i \geqslant d$ and $|D_u \tilde{\varphi}_i| \leqslant C|t|^{\sigma - r_i}$. Suppose first that there are r_i, s_i (suppose that r_1, s_1 are among them) such that $r_i + s_i = \operatorname{ord} y(t) = \nu$. Then φ will be in the form $\varphi = t^{r_1} \zeta^{s_1} \tilde{\varphi}$, $\tilde{\varphi}$ invertible. Then we must have

$$\begin{aligned} &\tilde{\varphi}(t, \zeta(t), u(t)) = \tilde{y}(t), \quad \text{where } \tilde{y}(t) = y(t)/t^{\nu}, \\ &\tilde{\varphi} \neq \tilde{\varphi}_i \quad \text{for } t \neq 0, \ \zeta \neq 0, \\ &|D_u \tilde{\varphi}| \leqslant C|t|^{\sigma - r_1}. \end{aligned}$$

These conditions are not automatically satisfied only for those $\tilde{\varphi}_i$ for which $\tilde{\varphi}_i(0, 0, 0) = y(0)$, and there are less than l such functions. Thus we can use the induction assumption.

Now suppose that $r_i + s_i \neq \nu$ for all i. We note that then the set of all pairs (r_i, s_i) is ordered by the relation $(r, s) \leqslant (r', s') \Leftrightarrow r \leqslant r'$, $s \leqslant s'$. It is easy to show that there exists a pair (r_0, s_0) such that $r_0 + s_0 = \nu$, $r_0 \geqslant c$, $s_0 \geqslant d$

and the set of pairs $\{(r_0, s_0), (r_1, s_1), \ldots, (r_l, s_l)\}$ is ordered by $\leqslant$. We put

$$\varphi = t^{r_0}\zeta^{s_0}\tilde{\varphi}(\zeta), \quad \text{where} \quad \tilde{\varphi}(\zeta(t)) = \tilde{y}(t).$$

We shall need also the following generalization of Lemma 5.2.

LEMMA 8.2. *Let $X \subset C^{m+1} = (C^k \oplus C^1 \oplus C^{m-k-1}) \oplus C^1$ be a hypersurface with a reduced equation $F = 0$. Let Ω be a neighbourhood of 0 in C^m such that for all $\xi \in \Omega$ we have the decomposition*

$$F(x+\lambda(\xi, 1)) = Q(x, \xi, \lambda)P(x, \xi, \lambda), \tag{8.3}$$

where Q is invertible and P is a distinguished polynomial with respect to λ. Denote by $\Delta_\alpha(x, \xi)$ its generalized, discriminants.

Let $\pi\colon C^{m+1} \to C^m$ be the standard projection. Let $W = \Phi(\{t \neq 0, \zeta \neq 0\}) \subset C^m$ be an extended quasi-wing. Assume that $\pi\colon \pi^{-1}(W_{\varrho\varrho'}) \cap X \to W_{\varrho\varrho'}$ and $\pi\colon \pi^{-1}(\mathring{W}_\varrho) \cap X \to \mathring{W}_\varrho$ are local isomorphisms for all branches of W and $\mathring{W}$. Thus, after replacing t and ζ by their powers, we can write $\pi^{-1}(W) \cap X = \bigcup_l \tilde{W}_l$, where $\tilde{W}_l$ is the image of $\{t \neq 0, \zeta \neq 0\}$ under an analytic map of the form $\tilde{\Phi}_l = (\Phi, \psi_l)$.

Let $\varepsilon > 0$ be a given number. Let $\{\xi_i\}$ be a finite set of points of Ω, $\xi_1 = 0$. Put

$$B_j = \{i\colon |\xi_i - \xi_j| < \varepsilon\}$$

and assume that for all j and α

$$\sum_{i \in B_j} \Delta_\alpha(x, \xi_i)\mathcal{O}_x = \sum_{\xi \in \Omega} \Delta_\alpha(x, \xi)\mathcal{O}_x. \tag{8.4}$$

Assume that for every i and l there is an $\alpha(i, l)$ such that for all $x \in \tilde{W}_l$

$$\Delta_\beta(x, \xi_i) = 0 \text{ for } \beta < \alpha(i, l), \quad \Delta_{\alpha(i,l)}(x, \xi_i) \neq 0.$$

Assume we are given an l_0 and a curve $\Gamma \subset \pi^{-1}(\mathring{W}) \cap (\text{closure of } W_{l_0})$ given by $\zeta = 0$, $u = u(t)$, such that $\pi = \pi(\xi_1)$ is ε-regular with respect to X at a cusp-like neighbourhood S of Γ in $\tilde{W}_{l_0}$.

Then, for some $\varepsilon' \leqslant \varepsilon$, every point of $\tilde{W}_{l_0}$ is ε'-regular with respect to X. In particular, $\tilde{W}_{l_0}$ is an extended quasi-wing.

Proof. It is a modification of the proof of Lemma 5.4.

To prove that for all i we have $\alpha(i, l_0) \geqslant \alpha(1, l_0)$, we take a curve $\Gamma' \subset S$; its points are ε-regular with respect to X and we repeat the first part of the proof of Lemma 5.4. Now put $\Delta = \Delta_{\alpha(1,l_0)}$. Then $\Delta(\tilde{\Phi}_{l_0}(t, \zeta, u), \xi_i)$ is either identically 0 or $\neq 0$ for $t \neq 0$, $\zeta \neq 0$; thus in the latter case

$$\Delta(\tilde{\Phi}_{l_0}(t, \zeta, u), \xi_i) = t^{r_i}\zeta^{s_i}\tilde{\Delta}_i(t, \zeta, u),$$

$\tilde{\Delta}_i$ invertible. We claim that $r_i \geqslant r_1$, $s_i \geqslant s_1$ for all i.

$\tilde{\Phi}_{l_0}^{-1}(S)$ contains for some N the set

$$S' = \{(t, \zeta, u)\colon\ 0 < |\zeta| < |t|^N,\ |u - u(t)| < |t|^N\}.$$

It follows from the regularity of π that for some M and all $(t, \zeta, u) \in S'$

$$0 < M^{-1} \leqslant |\Delta(\tilde{\Phi}_{l_0}(t, \zeta, u), \xi)/\Delta(\tilde{\Phi}_{l_0}(t, \zeta, u), \xi_1)| \leqslant M$$

for $|\xi - \xi_1| = |\xi| < \varepsilon$. Put $\zeta = t^N \zeta$. Then for all $i \in B_1$

$$t^{r_i + Ns_i} \zeta^{s_i} = t^{r_1 + Ns_1} \zeta^{s_1} \delta_i(t, \xi, u),$$

where δ_i are defined in S' and $M^{-1} \leqslant |\delta_i| \leqslant M$. Then $r_i = r_1$, $s_i = s_1$. The claim now follows from (8.3).

We thus have for all $\xi \in \Omega$

$$\Delta(\tilde{\Phi}_{l_0}(t, \zeta, u), \xi) = t^{r_1} \zeta^{s_1} \Delta^*(t, \zeta, u, \xi),$$

and so $\Delta(\tilde{\Phi}_{l_0}(t, \zeta, u), \xi) \neq 0$ for $|\xi| < \varepsilon'$ for some $\varepsilon' \leqslant \varepsilon$, which finishes the proof.

To apply Lemma 8.2 we shall have to find points $\{\xi_i\}$ satisfying (8.4). This will be done with the help of the following lemma, which is proved as Hilbert's Basissatz in [6].

LEMMA 8.3. *Let* $B = \{x \in \boldsymbol{C}^n\colon\ x = (x_1, \ldots, x_n),\ |x_i| < \varepsilon$ *for all* $i\}$, *where* ε *is a fixed number, let* V_0 *be an open connected set in* $\boldsymbol{C}^p$ *and let* I *be an ideal in* $\mathcal{O}(B \times V_0)$. *For any open set* $V \subset V_0$ *there exist a finite number of points* $\xi_i \in V$ *and a finite number of functions* $F_s \in I$ *such that*

$$\sum_{i,s} F_s(.,\xi_i)\,\mathcal{O}_x = \sum_{F \in I, \xi \in V} F(.,\xi)\,\mathcal{O}_x.$$

Proof. Since $\boldsymbol{C}[[x_1, \ldots, x_n]]$ is flat over $\mathcal{O}_x$, it is enough to prove the lemma after replacing $\mathcal{O}_x$ by $\boldsymbol{C}[[x_1, \ldots, x_n]]$. We proceed by induction on n.

If $n = 1$, we consider the Taylor expansion of every $F \in I$:

$$F(x, \xi) = \sum_{j=0}^{\infty} f_j^F(\xi)\, x_1^j.$$

Let j_0 be the smallest index such that $f_{j_0}^{F_0} \neq 0$ for some $F_0 \in I$. Choose a $\xi_1 \in V$ such that $f_{j_0}^{F_0}(\xi_1) \neq 0$. Then every $F(.,\xi)$ is divisible by $F_0(.,\xi_1)$, i.e. $F_0(.,\xi_1)$ generates $\sum_{F \in I, \xi \in V} F(.,\xi)\,\boldsymbol{C}[[x_1]]$.

For the induction step we consider the expansion of every $F \in I$ with respect to x_n

$$F(x, \xi) = \sum_{j=0}^{\infty} f_j^F(x', \xi)\, x_n^j,$$

where $x' = (x_1, \ldots, x_{n-1})$. Put $B' = \{(x_1, \ldots, x_{n-1}): |x_i| < \varepsilon \text{ for all } i\}$. Let $j(F)$ be the smallest integer for which $f^F_{j(F)} \neq 0$. Clearly, $\{f^F_{j(F)}: F \in I\}$ is an ideal in $\mathcal{O}(B' \times V_0)$. Select points $\xi_i \in V$ and functions $F_s \in I$ such that $f^{F_s}_{j(F_s)}(., \xi_i)$ generate the ideal $\sum_{F \in I, \xi \in V_0} f^F_{j(F)}(., \xi) C[[x_1, \ldots, x_{n-1}]]$. Then it is easy to prove, as in [6], that if $\xi \in V_0$ and $j(F) \geqslant \max_s j(F_s)$, then $F(., \xi) \in \sum_{i,s} F_s(., \xi_i) C[[x_1, \ldots, x_n]]$. To treat the case of $j(F) < \max_s j(F_s)$, we repeat the argument of [6].

9. Proof of Proposition 1.3

The proof of the first part of this proposition is simple; we shall prove (by increasing induction on k) that if ξ_i^m are chosen as at the beginning of § 6, then the generalized normal partition $\{X_n^j\}$ satisfies (1.8, k).

Let $k = 1$; we have to prove that if $v \in T_q \mathring{X}_n^j$, then

$$|D_v P_q| \leqslant C|v|/\operatorname{dist}(q, X_n^{j-1}).$$

Let $\gamma(r)$ be a real curve in $\mathring{X}_n^j$ such that $\gamma(0) = q$, $(d\gamma/dr)(0) = v$. Then, by (1.7,0),

$$\begin{aligned} |D_v P_q| &= \lim_{r \to 0} |P_{\gamma(r)} - P_{\gamma(0)}|/r \\ &\leqslant \overline{\lim_{r \to 0}} |P_{\gamma(r)} - P_{\gamma(0)}|/|r| \\ &\leqslant \overline{\lim_{r \to 0}} C|\gamma(r) - \gamma(0)|/|r| \operatorname{dist}(\gamma(0), X_n^{j-1}) \leqslant C|v|/\operatorname{dist}(q, X_n^{j-1}). \end{aligned}$$

For the induction step suppose that (1.8, k) does not hold; let k be the smallest integer with this property. Then there exist real-analytic curves $q_{j_1}(r) = q(r), q_{j_2}(r), \ldots, q_{j_l}(r)$ such that $q_{j_s}(r) \in \mathring{X}_n^{j_s}$ (for $r > 0$) are a chain for $q(r)$ and for some $v = v(r) \in P_{q_{j_1}}(r) \ldots P_{q_{j_k}}(r) C^n$ and $\eta > 0$

$$|D_v P_{q(r)}| \geqslant C|v|/\operatorname{dist}(q(r), X_n^{j_k - 1})^{1+\eta}. \tag{9.1}$$

We choose admissible projections π^i (for $i > j_k$) as on p. 36. By Lemma 6.4 there exist perturbations $\tilde{q}_{j_s}(r)$ of $q_{j_s}(r^q)$ (which can be chosen to be arbitrarily. small) and pairwise nicely-situated quasi-wings $V_s = \{(x(t^a, u), f_s(t, u))\} \subset \mathring{X}_n^j$ containing $\{q_{j_s}(r): r \neq 0\}$.

Assuming that π^i are the standard projections, we choose (as in [7]) for every $i > j_k$ an analytic function $P_i(x)$ such that P_i depends only on $x_1, \ldots, x_i$, is a distinguished polynomial with respect to x_i, its discriminant is

$\neq 0$ outside of X_{i-1}^{i-2} and $X_n^j \subset \{P_i(x) = 0 \text{ for all } i > j\}$, $j \geqslant j_k$. If $z \in \mathring{X}_n^j$, then $T_z \mathring{X}_n^j$ is spanned by the vectors

$$w_\alpha(z) = e_\alpha - \sum_{i > j} \left(\frac{\partial P_i}{\partial x_\alpha}(z) \Big/ \frac{\partial P_i}{\partial x_i}(z) \right) e_i, \quad \alpha \leqslant j.$$

The vector fields $w_\alpha\big((x(t^a, u), f_1(t, u))\big)$ are analytic in t, u, since (obviously) they are meromorphic and are bounded (by the regularity of projections). Thus if $z \in V_1$, $v' \in T_z V_1$, then

$$|D_{v'} w_\alpha(z)| \leqslant C |v'|/\mathrm{dist}(z, X_n^{j_k - 1}),$$

and so

$$|D_{v'} P_z| \leqslant C |v'|/\mathrm{dist}(z, X_n^{j_k - 1}). \tag{9.2}$$

Now let $v = P_{\tilde{q}_{j_1}(r)} \dots P_{\tilde{q}_{j_k}(r)} v^*$, $v^* \in C^n$. Put

$$v' = P'_{\tilde{q}_{j_1}(r)} \dots P'_{\tilde{q}_{j_k}(r)} v^*,$$

where P'_z is (as in § 6) the orthogonal projection onto $T_z V_s$ for $z \in V_s$. Then

$$|v - v'| \leqslant C |\tilde{q}_{j_1}(r) - \tilde{q}_{j_k}(r)| \, |v^*|/\mathrm{dist}(\tilde{q}_{j_1}(r), X_n^{j_k - 1}), \tag{9.3}$$

for

$$v - v' = \sum_{s=1}^{k-1} P_{\tilde{q}_{j_1}(r)} \dots P_{\tilde{q}_{j_{s-1}}(r)} \big(P_{\tilde{q}_{j_s}(r)} - P'_{\tilde{q}_{j_s}(r)}\big) P'_{\tilde{q}_{j_{s+1}}(r)} \dots P'_{\tilde{q}_{j_k}(r)} v^*$$

and we can use (6.7). Writing $D_v P = D_{v'} P + D_{v-v'} P$ and using (9.2), (9.3) and the inductive assumption we get

$$|D_v P_{\tilde{q}_{j_1}(r)}| \leqslant C |v|/\mathrm{dist}(\tilde{q}_{j_1}(r), X_n^{j_k - 1}).$$

To get a contradiction, it is enough to remark that (9.1) holds after replacing $q(r)$ by $\tilde{q}(r)$, provided that the perturbation is small enough.

For the proof of the second part of Proposition 1.3 we first specify admissible projections $\pi(\xi_i^m)$. We choose them (using Lemma 8.3) in such a way that, besides the conditions on p. 31, the following two are satisfied: 1) there exists an ε such that for every $m \leqslant n$ and for every curve $q(t) \in C^m$ there exists a ξ_i^m such that $\pi(\xi_i^m)$ is ε-regular with respect to X_m^{m-1} in a cusp-like neighbourhood of $q(t)$, 2) if $F = 0$ is a reduced equation of X_m^{m-1}, P is defined by (8.3), $\Delta_\alpha(x, \xi)$ are the generalized discriminants of P, $\Omega = \Omega_m$, then (8.4) holds.

Now we shall define an integer N_0; we start with a lemma.

LEMMA 9.1. *Let $P(x, t)$ ($x \in C^n$, $t \in C$) be a distinguished polynomial with respect to t of degree p without multiple factors, $P(0, t) = t^p$. Let $\Delta_0(x)$ be its discriminant. Put $B = \{(x, t): |P(x, t)| < |x|^p |\Delta_0(x)|^p\}$. If $(x, t) \in B$, then there* *exists exactly one* t_i *such that* $P(x, t_i) = 0$ *and* $|t-t_i| < |x|\,|\Delta_0(x)|$, *while for all other roots* t_j *of the equation* $P(x, t) = 0$ *we have* $|t-t_j| \geqslant (1-|x|)\;|\Delta_0(x)|$. *Thus the function* $B \ni (x, t) \mapsto (x, t_i)$ *is a retraction* $B \to B \cap \{P = Q)$ *preserving fibres of the standard projection* $B \to C^n$.

Proof. If t_j are all the roots of $P(x, t) = 0$ counted with their multiplicities, then $P(x, t) = \prod (t-t_j)$. Thus if $|P(x, t)| < |x|^p |\Delta_0(x)|^p$, then for some $i.|t-t_i| < |x|\,|\Delta_0(x)|$. Since $\Delta_0(x) = \prod_{j \neq k} (t_j - t_k)$, we have $|t_j - t_k| \geqslant |\Delta_0(x)|$ for $j \neq k$, and so, for $j \neq i$, $|t-t_j| \geqslant |t_i - t_j| - |t-t_i| \geqslant (1-|x|)|\,\Delta_0(x)|$.

Now let l, k be integers, $l \geqslant k$, and let $\underline{\pi} = (\pi^l_{i_l}, \ldots, \pi^{k+1}_{i_{k+1}})$ be any sequence of admissible projections. Put

$$\underline{\pi}' = (\pi^{l-1}_{i_{l-1}}, \ldots, \pi^{k+1}_{i_{k+1}}).\ \text{Let}\ \mathring{X}^k_l(\underline{\pi}) = X^k_l \setminus (\pi^{k+1}_{i_{k+1}} \circ \ldots \circ \pi^l_{i_l})^{-1}(X'^{k-1}_k)$$

where X'^{k-1}_k is defined on p. 12. Using Lemma 9.1 we select for every $\underline{\pi}$ a semianalytic neighbourhood $V_{\underline{\pi}}$ of $\mathring{X}^k_l(\underline{\pi})$ and a retraction $r_{\underline{\pi}}\colon V_{\underline{\pi}} \to \mathring{X}^k_l$ such that $\pi^l_{i_k} \circ r_{\underline{\pi}} = r_{\underline{\pi}'} \circ \pi^l_{i_l}$.

Let $L(\underline{\pi})$ be an integer such that for all $p \in V_{\underline{\pi}}$

$$|p - r_{\underline{\pi}}(p)| \leqslant \operatorname{dist}(p, X^k_l)/\operatorname{dist}(p, X^{k-1}_l)^{L(\underline{\pi})}.$$

There exist a semianalytic neighbourhood $V'_{\underline{\pi}}$ of $\mathring{X}^k_l(\underline{\pi})$, $V'_{\underline{\pi}} \subset V_{\underline{\pi}}$, and an integer $M(\underline{\pi})$ such that for all $p = (p_1, \ldots, p_l)$, $q = (q_1, \ldots, q_l) \in V'_{\underline{\pi}}$:

$$\text{if}\quad r_{\underline{\pi}}(p) \neq r_{\underline{\pi}}(q),\quad \pi^l_{i_l}(p) = \pi^l_{i_l}(q),\quad \text{then}$$
$$|p_l - q_l| \geqslant C \operatorname{dist}\big(\pi^l_{i_l}(p), X^{k-1}_{l-1}\big)^{M(\underline{\pi})}.$$

We take for N_0 any integer such that the following two conditions hold: 1) $N_0 > L(\underline{\pi})+1$, $N_0 > M(\underline{\pi})+1$ for all k, l and $\underline{\pi}$, 2) if $\tilde{\pi}\colon C^n \to C^l$ is any composition of admissible projections, then $\tilde{\pi}(U_{N_0,k}) \subset V'_{\underline{\pi}}$, for every k, l and $\underline{\pi}$ ($U_{N_0,k}$ is defined on p. 7).

We now pass to the proof of (1.9, k).

Lemma 9.2. *There exists an* $\alpha < 1$ *and a* $\delta > 0$ *such that, for every* j *and* k: *if* $q \in \mathring{X}^j_n$, $q = q_{j_1}, q_{j_2}, \ldots$ *is a chain for* q *and* $q \in U_{N_0,j_k}$, *then there exists a* $w \in T_q \mathring{X}^j_n$, $|w| = 1$, *such that* $|\measuredangle(w, P_{q_{j_1}} \ldots P_{q_{j_k}}(C^n)| \geqslant \delta$ *and*

$$(9.4)\qquad |D_w P_q| \leqslant C/[\operatorname{dist}(q, X^{j_k-1}_n)^{N_0-1} \operatorname{dist}(q, X^{j_k}_n)^{\alpha}].$$

This of course implies (1.9, k), for writing $\alpha = 1-\eta$, $\eta > 0$; it is easy to check that (9.4) implies that (1.9, k) is satisfied in U_{N,j_k}, where $N \geqslant 2(N_0-1)/\eta$.

Lemma 9.3. *Assume that Lemma 9.2 is false. Then there exist positive numbers* $\varkappa$, μ *and real-analytic curves* $q(r) = q_{j_1}(r), q_{j_2}(r), \ldots$ *such that* $q_{j_l}(r)$

are a chain for $q(r)$ and for every $r \neq 0$ and every $w \in C^n$, $|w| = 1$, we have

(9.5) *if* $|\measuredangle(w, P_{q_{j_1}(r)} \ldots P_{q_{j_k}(r)}(C^n)| \geqslant r^{\varkappa}$, *then*

$$|D_w P_{q(r)}| \geqslant \mu/[\operatorname{dist}(q(r), X_n^{j_k-1})^{N_0-1} \operatorname{dist}(q(r), X_n^{j_k})].$$

(9.5) *holds also for sufficiently small perturbations $\tilde{q}_{j_i}(r)$ of $q_{j_i}(r)$.*

This lemma is of course a direct consequence of the curve selection lemma.

LEMMA 9.4. *If $q(r) \in \mathring{X}_n^j \cap U_{N_0, j_k}$ for $r \neq 0$, then there exist an arbitrarily small perturbation $\tilde{q}(r)$ of $q(r)$ and an extended (j_k+1)-dimensional quasi-wing $W \subset \mathring{X}_n^j$ (in a suitable splitting $C^n = C^{j_k} \oplus C^1 \oplus C^{n-j_k-1}$) containing $q(r)$ and $\dot{W} \subset X_n^{j_k}$.*

Proof. Let $q'(r) \in X_n^{j_k}$ be a curve such that $|q(r) - q'(r)| = \operatorname{dist}(q(r), X_n^{j_k})$. Let $q(t)$, $q'(t)$ be the complexifications of $q(r)$, $q'(r)$. We choose admissible projections $\pi^m = \pi(\xi_{i_m}^m)$ (for $m > j_k$) such that every π^m is ε-regular with respect to X_m^{m-1} at $(\pi^{m+1} \circ \ldots \circ \pi^n) q'(t)$. As before, assume that every π^m is the standard projection $C^m \to C^{m-1}$ and let $\pi^{pq}\colon C^p \to C^q$ be their composition. We take $\pi^{n,j_k} q'(t)$ for Γ of § 5 and define U_ε, $x(t, u)$ by (5.2) and (5.1), where $s = \operatorname{ord}\operatorname{dist}(\pi^{n,j_k} q(r), X_{j_k}^{j_k-1})$, which can be assumed to be an integer, maybe after replacing r by its power. We take a perturbation $\tilde{q}(t)$ of $q(t)$ such that $\pi^{n,i} \tilde{q}(t) \notin X_i^{i-1}$ for $t \neq 0$ and $j_k < i \leqslant j_1 = j$. Let c (appearing in the definition of an extended quasi-wing) be $(N_0 - 1)s$.

By increasing induction on m we shall construct (for $m \leqslant j$) a (j_k+1)-dimensional extended quasi-wing $W_m \subset C^m$ containing $\pi^{nm} \tilde{q}(t)$ such that $\dot{W}_m \subset X_m^{j_k}$, $W_m \cap X_m^{m-1} = \emptyset$ and $\pi^{m+1,m}(W_{m+1}) = W_m$.

To start induction we observe that, by Lemma 5.2, $X_{j_k+1}^{j_k}$ is a sum of quasi-wings over U_ε. Select this one that contains $\pi^{n,j_k+1} q'(t)$ and denote it by V_{j_k+1}; we can describe it by $x_{j_k+1} = f_1(t, u)$. We put

$$W_{j_k+1}\colon x_{j_k+1} = f_1(t, u) + t^c \zeta^d,$$

where d is an integer such that the equations of $\pi^{n,j_k+1}(\tilde{q}(t))$ are: $u = u(t)$, $\zeta = \zeta(t)$, and $\operatorname{ord} \zeta(t) = 1$.

Now consider $X_{j_k+2}^{j_k} \cap (\pi^{j_k+2})^{-1}(V_{j_k+1})$; this is a sum of quasi-wings which can be described (after replacing t by its power) as

$$V_{j_k+2,i}\colon x_{j_k+1} = f_1(t, u), \qquad x_{j_k+2} = f_{2,i}(t, u).$$

Suppose that $V_{j_k+2,1}$ contains $\pi^{n,j_k+2}(q'(t))$. The open set

$$D\colon |x_{j_k+1} - f_1(t, u)| < |t|^c, \qquad |x_{j_k+2} - f_{2,1}(t, u)| < |t|^c$$

does not contain the graphs $x_{j_k+1} = f_1(t, u)$, $x_{j_k+2} = f_{2,i}(t, u)$ for $i > 1$. By Lemma 8.2, $D \cap X_{j_k+2}^{j_k+1}$ is a sum of extended quasi-wings. Let $u = u(t)$, $\zeta = \zeta(t)$, $x_{j_k+2} = x_{j_k+2}(t)$ be the equations of $\pi^{n,j_k+2}(\tilde{q}(t))$. If

$$\operatorname{ord}\big(x_{j_k+2}(t)-f_{2,1}(t, u(t))\big) \geqslant \operatorname{ord}\big(x_{j_k+1}(t)-f_1(t, u(t))\big)$$

we can use Lemma 8.1 to find W_{j_k+2}.

Now suppose that

$$c+\alpha = \operatorname{ord}\big(x_{j_k+2}(t)-f_{2,1}(t, u(t))\big)$$
$$< \operatorname{ord}\big(x_{j_k+1}(t)-f_1(t, u(t))\big) = c+\beta,$$

where $x_{j_k+1} = x_{j_k+1}(t)$ on $\tilde{q}(t)$. We take x_{j_k+2} for the z-coordinate of § 8. Let $\zeta(t)$, $\psi(t)$ be functions such that $\operatorname{ord}\zeta(t) = 1$, $\operatorname{ord}\psi(t) > 1$, and

$$x_{j_k+1}(t) = f_1(t, u(t))+t^c\psi(t),$$
$$x_{j_k+2}(t) = f_{2,1}(t, u(t))+t^c\zeta(t)^\alpha.$$

Put $\psi(t) = \tilde{\psi}(\zeta(t))$ and define W_{j_k+2} by

$$x_{j_k+1} = f_1(t, u)+t^c\tilde{\psi}(\zeta), \qquad x_{j_k+2} = f_{2,1}(t, u)+t^c\zeta^\alpha.$$

The induction step is similar. Assume we have already constructed $W_{m-1} = \Phi(\{t \neq 0, \zeta \neq 0\}) \subset C^{m-1}$. Let x_l be the z-coordinate of § 8 ($j_k < l < m$). Let $\dot{W}_{m-1}$ be given by $x_i = f_i(t, u)$, $i = j_k+1, \ldots, m-1$. $X_m^{m-1} \cap (\pi^m)^{-1}(\dot{W}_{m-1})$ is a sum of quasi-wings V_σ over U_ε; after replacing t by its power we can assume that each V_σ is given by $x_i = f_i(t, u)$ ($i < m$), $x_m = f_{m,\sigma}(t, u)$. From among all V_σ's select this one (call it V_1) which contains $\pi^{nm}(q'(t))$. The set

$$D = \{(x_1, \ldots, x_m)\colon (x_1, \ldots, x_{m-1}) \in W_{m-1}, |x_m-f_{m,1}(t, u)| < |t|^c\}$$

has empty intersection with each of the sets $\{(x_1, \ldots, x_m)\colon (x_1, \ldots, x_{m-1}) \in W_{m-1}, x_m = f_{m,\sigma}(t, u)\}$ for $\sigma \neq 1$. By Lemma 8.2 $(\pi^m)^{-1}(W_{m-1} \cap D) \cap X_m^{m-1}$ is a sum of extended quasi-wings.

Let $x_l = x_l(t)$ be the equations of $\pi^{nm}(\tilde{q}(t))$, $i \leqslant m$. Then $\operatorname{ord}[x_m(t)-f_{m,1}(t, u(t))] > c$. If

$$\operatorname{ord}[x_m(t)-f_{m,1}(t, u(t))] \geqslant \operatorname{ord}[x_l(t)-f_l(t, u(t))],$$

we can use Lemma 8.1 to find an extended quasi-wing W_m in C^m containing $\pi^{nm}(\tilde{q}(t))$ such that $W_m \cap X_m^{m-1} = \emptyset$, $\pi^m W_m = W_{m-1}$, $\dot{W}_m = V_1$.

Now suppose that

$$c+\alpha = \operatorname{ord}[x_m(t)-f_{m,1}(t, u(t))]$$
$$< \operatorname{ord}[x_l(t)-f_l(t, u(t))] = c+\beta.$$

The z-coordinate of § 8 for W_m will be x_m. Let $\zeta(t)$, $\psi_i(t)$ ($i = j_k+1, \ldots, m-1$) be functions such that $\operatorname{ord}\zeta(t) = 1$, $\operatorname{ord}\psi_i(t) > 1$ for all i and

$$x_m(t) = f_{m,1}(t, u(t)) + t^c \zeta(t)^\alpha$$
$$x_i(t) = f_i(t, u(t)) + t^c \psi_i(t), \quad i = j_k+1, \ldots, m-1.$$

Put $\psi_i(t) = \tilde{\psi}_i(\zeta(t))$ and define W_m by

$$x_m = f_{m,1}(t, u) + t^c \zeta^\alpha,$$
$$x_i = f_i(t, u) + t^c \tilde{\psi}_i(\zeta), \quad i = j_k+1, \ldots, m-1.$$

In this way we get $W_j \subset C^j$. Using Lemma 8.2 we can lift it via $\pi^{j+1}, \ldots, \pi^n$ to an extended quasi-wing $W \subset C^n$ satisfying all the requirements of the lemma.

We can now prove Lemma 9.3. Let $W = \Phi(\{t \neq 0, \zeta \neq 0\})$ be an extended quasi-wing containing $\tilde{q}(r)$. As in the proof of the first part of Proposition 1.3 we construct vector fields $w_\alpha(t, \zeta, u)$, analytic with respect to t, ζ, u, which span $T_q \mathring{X}^j$ at every point $q = \Phi(t, \zeta, u) \in W$. Put $w' = \Phi_*(\partial/\partial\zeta)$, $w = w'/|w'|$. Then $|w'| \geqslant C|t|^c|\zeta|^{d-1}$ and so

$$|D_{w'} w_\alpha| \leqslant C|\partial w_\alpha/\partial\zeta|/|t|^c|\zeta|^{d-1} \leqslant C/|t|^c|\zeta|^{d-1}.$$

But $|t|^c \geqslant C \operatorname{dist}(q, X_n^{j_k-1})^{N_0}$, $|t|^c|\zeta|^d \geqslant \operatorname{dist}(q, X_n^{j_k})$, so $|t|^c|\zeta|^{d-1} \geqslant C \operatorname{dist}(q, X_n^{j_k-1})^{N_0} \operatorname{dist}(q, X_n^{j_k})^{1-(1/d)}$, which finishes the proof.

10. Proof of Proposition 1.4

Let $\{X_m^j\}$ be a generalized normal partition satisfying the estimates of Proposition 1.2. Let j, l be integers, $j < l \leqslant n$, and let c_0 be a number $\geqslant 2n$. For every $l' \leqslant l$ and every sequences of integers $\underline{j} = (j_1, \ldots, j_{l'})$ and $\underline{m} = (m_2, \ldots, m_{l'})$ we put

$X_l^j(c_0; \underline{j}, \underline{m}) = \{q \in X_l^j$: there exists a c_0-chain

$q = q_{j_1}, q_{j_2}, \ldots, q_{j_{l'}}$ for q in the stratification $\{X_l^r\}$ of

X_l^{l-1} such that $\varrho_{j_s}(q) \leqslant 2^{-m_s}$, $s = 2, \ldots, l'\}$.

For every sequence of integers $\underline{m}^* = (m_{j-1}^*, \ldots, m_0^*)$ we put

$$A_l^j(\underline{m}^*) = \{q \in X_l^j\colon \varrho_i(q) \leqslant 2^{-m_i^*},\ i = j-1, j-2, \ldots, 0\}.$$

We shall estimate the $2j$-dimensional Hausdorff measure $|A_l^j(\underline{m}^*)|_{2j}$ of $A_l^j(\underline{m}^*)$ (induced by the usual metric on C^l). We need a preliminary lemma.

Let k, m be fixed integers, $k < m$. For every compact $K \subset \mathring{X}_m^k$ we put

$$K(r) = \{x \in X_m^k\colon \operatorname{dist}(x, K) < r\},$$
$$B(K, r) = \{x \in C^m\colon \operatorname{dist}(x, K) < r\}.$$

For every $A \subset C^m$ let $|A|_s$ be the s-dimensional Hausdorff measure of A.

LEMMA 10.1. *There exists a constant C, independent of K, such that for every $r < \operatorname{dist}(K, X_m^{k-1})$*

$$|B(K, r)|_{2m} \leqslant C|K(r)|_{2k} r^{2(m-k)}.$$

Proof. We choose a projection $\pi\colon C^m \to C^k$ being a composition of arbitrarily chosen admissible projections $\pi_{i_s}^s$, $s = k+1, \ldots, m$. Since $\pi\colon \mathring{X}_m^k \to C^k$ is a local isomorphism, there exist vectors e_μ^0 $(\mu = k+1, \ldots, m)$ which, considered as vector fields on C^m, are transversal to $\mathring{X}_m^k$ at every point of $\mathring{X}_m^k$. Putting $e'_\mu(x) = P_x^\perp e_\mu^0$ and applying the Gramm–Schmidt orthonormalization, we obtain orthonormal vector fields $e_\mu(x)$, defined on $\mathring{X}_m^k$, satisfying

$$\text{(10.1)} \qquad |De_\mu(x)| \leqslant C/\operatorname{dist}(x, X_m^{k-1}).$$

Clearly, for every point y of $B(K, r)$ there is a point $x \in K(r)$ such that $|y-x|$ is the distance from y to $K(r)$ and therefore the interval $y-x$ is normal to $T_x\mathring{X}_m^k$. Thus

$$y = x + \sum t_\mu e_\mu(x), \qquad |t| = |(t_{k+1}, \ldots, t_m)| \leqslant r.$$

We repeat the calculation from [5]. We have

$$dy = dx + \sum dt_\mu e_\mu + \sum t_\mu de_\mu$$

and $dx = \sum \omega_\alpha e_\alpha$ (where e_α is an orthonormal base of $T_x\mathring{X}_m^k$), $|\omega_\alpha| = 1$, $de_\mu = \sum \omega_{i\mu} e_i$, where the 1-forms $\omega_{i\mu}$ satisfy $|\omega_{i\mu}(x)| \leqslant C/\operatorname{dist}(x, X_m^{k-1})$ by (10.1). Thus

$$dy_1 \wedge \ldots \wedge dy_m = \bigwedge_{\alpha=1}^{k} \left(\omega_\alpha + \sum t_\mu \omega_{\alpha\mu}\right) \wedge \bigwedge_\mu dt_\mu,$$

and since

$$|(\omega_\alpha + t_\mu \omega_{\alpha\mu})(x)| \leqslant 1 + rC/\operatorname{dist}(x, X_m^{k-1}) \leqslant C,$$

we get the desired estimate.

LEMMA 10.2. *For every l there exist constants K_l and $C_l = C$ such that for every c_0, assumed (for simplicity) to be a power of 2, $c_0 = 2^b$,* and *$c_0 \geqslant K_l$, and for every $\underline{j}$, $\underline{m}$, $\underline{m}^*$ we have the estimates*

$$|X_l^j(c_0; \underline{j}, \underline{m})|_{2j} \leqslant C2^{-2[m_2(j_1-j_2)+m_3(j_2-j_3)+\ldots+m_{l'}j_{l'}]},$$

$$|A_l^j(\underline{m}^*)|_{2j} \leqslant C2^{-2\sum m_i^*}.$$

Proof. First we prove that the first estimate implies the second one. Put

$$B_l^j(\underline{m}^*) = \{q \in X_l^j\colon \varrho_i(q) \in [2^{-m_i^*-1}, 2^{-m_i^*}] \text{ for all } i\};$$

since $A_l^j(\underline{m}^*)$ is the sum of $B_l^j(\underline{n}^*)$ over $\underline{n}^* = (n_{j-1}^*, \ldots, n_0^*)$ such that $n_i^* \geqslant m_i^*$ for all i, it is enough to prove that $|B_l^j(\underline{m}^*)|_{2j} \leqslant C2^{-2\Sigma m_i^*}$. This inequality follows from the inclusion $B_l^j(\underline{m}^*) \subset X_l^j(\underline{j}, \underline{m})$, where $\underline{m} = (m_1, m_2, \ldots)$ is defined by $m_s = m_{j_s}^*$ and $\underline{j}$ is defined (by decreasing induction on s) by:

$$j_1 = j, \quad j_{s+1} \text{ is the smallest index } < j_s \quad \text{for which} \quad m_{j_{s+1}}^* \geqslant m_{j_s-1}^* + 1 - b.$$

Now, assuming the first estimate to be correct for some l, we prove it for $l+1$. We distinguish two cases.

1° $j < l+1$. We observe that $X_{l+1}^j(\underline{j}, \underline{m}) \subset A_{l+1}^j(\underline{m}^*)$, where $m_i^* = m_{j_{s+1}}$ for $i \in [j_{s+1}, j_s)$. Let a be the smallest integer greater than $\lg_2(4n^2)$ and let $\underline{m}^{**}$ be given by $m_i^{**} = m_i^* + a$. Since for all s and all i_{l+1} we have $\pi_{i_{l+1}}^{l+1}(X_{l+1}^s) \subset X_l^s$ and because of (1.3), $\pi_{i_{l+1}}^{l+1}(A_{l+1}^j(m^*)) \subset A_l^j(\underline{m}^{**})$. By the inductive assumption, $|A_l^j(\underline{m}^{**})|_{2j} \leqslant C2^{-2\Sigma m_i^{**}}$, so it is enough to prove that for some C, independent of $\underline{m}^*$, $|A_{l+1}^j(\underline{m}^*)|_{2j} \leqslant C|A_l^j(\underline{m}^{**})|_{2j}$. Take a finite cover $\{X_\alpha\colon \alpha = 1, \ldots, A\}$ of $\mathring{X}_l^j$ by measurable subsets X_α such that for every X_α there exists an admissible projection $\pi_{i_{l+1}(\alpha)}^{l+1}$ inducing a homeomorphism of X_α onto its image and being ε-regular with respect to X_{l+1}^l at every point of X_α. Then for every $Y \subset X_\alpha$ we have $|Y|_{2j} \leqslant C|\pi_{i_{l+1}(\alpha)}^{l+1}(Y)|_{2j}$, where C depends only on ε. Thus

$$|A_{l+1}^j(\underline{m}^*)|_{2j} \leqslant \sum_\alpha |A_{l+1}^j(\underline{m}^*) \cap X_\alpha|_{2j}$$
$$\leqslant C\sum_\alpha |\pi_{i_{l+1}(\alpha)}^{l+1}(A_{l+1}^j(\underline{m}^*) \cap X_\alpha)|_{2j} \leqslant CA|A_l^j(\underline{m}^{**})|_{2j}.$$

2° $j = l+1$. It is enough to compute the measure of

$$A_\mu = X_{l+1}^l(c_0; \underline{j}, \underline{m}) \cap \{x \in \mathbf{C}^{l+1}\colon \varrho_{j_2}(x) \in [2^{-(\mu+1)}, 2^{-\mu}]\}.$$

On A_μ we have $\varrho_{j_2-1} \geqslant 2c_0^2/2^{\mu+1}$. We take for K of Lemma 10.1 the set of all possible second terms q_{j_2} of chains for points $q \in A_\mu$, and for r we take $c_0/2^\mu$. Then for a suitable K_{l+1} (independent of $\underline{j}$, $\underline{m}$, μ) and $c_0 \geqslant K_{l+1}$ we have

$$K(r) \subset X_{l+1}^{j_2}(c_0/2; \underline{j}, \underline{m})$$

and we can use the estimate for the measure of $X_{l+1}^{j_2}(c_0/2; \underline{j}, \underline{m})$ (since $j_2 < l+1$) and Lemma 10.1.

Now we observe that Proposition 1.4 follows from the following local statement:

Let $g_x(v, w)$ ($x \in C^n$, $v, w \in T_x C^n$) be a smooth hermitian metric, $\{X_m^j\}$ a generalized normal partition satisfying the estimates of Proposition 1.2 and 1.3, defined in a neighbourhood of 0. If P is an invariant polynomial of degree j and Ω is the curvature form of the induced hermitian metric on $\mathring{X}_n^j$, then $P(\Omega)$ is integrable in a neighbourhood of 0.

We shall estimate $|P(\Omega)|$ on the sets $X_n^j(c_0; j, \underline{m})$. As before, $|\ |$ denotes the standard euclidean norm on TC^n. Let N, α, δ have the same meaning as in Proposition 1.3.

If $q(r) \in \mathring{X}_n^j$ is a real-analytic curve and $q(r) = q_{j_1}(r), q_{j_2}(r), \ldots$ is a chain for $q(r)$, then we put

$$V_s(r) = P_{q_{j_1}(r)} \ldots P_{q_{j_s}(r)}(C^n).$$

LEMMA 10.3. *Let $q(r)$ be a curve in $\mathring{X}_n^j$, $q(r) = q_{j_1}(r), q_{j_2}(r), \ldots$ a chain for $q(r)$, $r > 0$. Then there exist vector fields $e_1(r), \ldots, e_n(r) \in T_{q(r)} C^n$ such that:*

1° for some p every $e_i(r)$ is analytic in $r^{1/p}$ and, for some C independent of r,

$$C^{-1} \leqslant |e_1(r) \wedge \ldots \wedge e_n(r)| \leqslant C,$$

2° $e_1(r), \ldots, e_j(r) \in T_{q(r)} \mathring{X}_n^j$, for $r > 0$,

3° for every $r > 0$ there exist real-analytic vector fields $\tilde{e}_1, \ldots, \tilde{e}_j$, defined in a neighbourhood of $q(r)$, such that $\tilde{e}_i(q(r)) = e_i(r)$ $(i = 1, \ldots, j)$ and for every s

$$(10.2) \qquad |D_v \tilde{e}_i(q(r))| \leqslant C/\operatorname{dist}(q(r), X_n^{j_s - 1}), \quad \text{for } v \in V_s(r),\ |v| = 1,$$

for some C, independent of r.

If, further, $q(r) \in U_{N, j_k}$, then there exists a line-field $l(r) \subset T_{q(r)} \mathring{X}_n^j$ such that $|\measuredangle(l(r), V_k(r))| \geqslant \delta$ and

$$(10.3) \qquad |D_v \tilde{e}_i(q(r))| \leqslant C/\operatorname{dist}(q(r), X_n^{j_k})^\alpha,\ i = 1, \ldots, n,\ v \in l(r),\ |v| = 1.$$

Proof. We note that the limit $\lim_{r \to 0} V_s(r)$ exists in the Grassmannian of j_s-planes in C^n. Using the curve-selection lemma we find $e_1(r), \ldots, e_n(r)$ satisfying 1°, 2°. The fields $\tilde{e}_i$ are defined by $\tilde{e}_i(q) = P_q e_i(r)$ for $i \leqslant j$, $\tilde{e}_i(q) = e_i(r)$, $i > j$. In a sufficiently small neighbourhood of $q(r)$ (10.2) is satisfied by (1.8, k). The proof of (10.3) is similar.

COROLLARY: *If $q \in \mathring{X}_n^j$ and $q = q_{j_1}, q_{j_2}, \ldots, q_{j_l}$ is a chain for q, then*

$$|P(\Omega)| \leqslant C / \prod_{s=2}^{l} \operatorname{dist}(q, X_n^{j_s})^{2(j_s - j_{s+1})},$$

where $j_{l+1} = 0$. If, further, $q \in U_{N, j_k}$, $\gamma = 1 - \alpha$, then

$$|P(\Omega)| \leqslant C \operatorname{dist}(q, X_n^{j_k})^{2\gamma} / \prod_{s=2}^{l} \operatorname{dist}(q, X_n^{j_s})^{2(j_s - j_{s+1})}.$$

Proof. We apply the Gramm–Schmidt orthonormalization in the metric g to the vector fields $e_1(r), \ldots, e_n(r)$; we obtain g-orthonormal vector fields $e_1^*(r), \ldots, e_n^*(r)$ which satisfy, as is easy check, the conditions 1°, 2° of Lemma 10.3, and every $e_i^*(r)$ $(i \leqslant j)$ extends in a neighbourhood of $q(r)$, $r > 0$, to a vector field $\tilde{e}_i^*$ satisfying 10.3. For every $r > 0$ we choose a g-orthonormal base $E(r)$ of $T_{q(r)} \mathring{X}_n^j$ such that $E(r) \cap V_s(r)$ is a base of $V_s(r)$ for all s and r. Then for all $i \leqslant j$ and for all s

$$|D_v \tilde{e}_i^*(q(r))| \leqslant C/\mathrm{dist}(q(r), X_n^{j_s - 1}), \qquad v \in E(r) \cap V_s(r)$$

which proves that the connection matrix in the frame $\tilde{e}_1^*, \ldots, \tilde{e}_j^*$ restricted to $\mathring{X}_n^j$ satisfies

$$|\omega_{i_1 i_2}(q(r))(v)| \leqslant C/\mathrm{dist}(q(r), X_n^{j_s - 1})$$

for all i_1, i_2, s and $v \in E(r) \cap V_s(r)$. From the structural equation

$$\Omega_{\alpha'\alpha} = -\sum_{\mu = j+1}^{n} \omega_{\alpha'\mu} \wedge \omega_{\mu\alpha} + \mathring{\Omega}_{\alpha'\alpha},$$

where $\mathring{\Omega}_{\alpha'\alpha}$ is the curvature form of g on C^n, we get

$$|\Omega_{\alpha'\alpha}(q(r))(v, w)| \leqslant C/[\mathrm{dist}(q(r), X_n^{j_s - 1})\ \mathrm{dist}(q(r), X_n^{j_t - 1})]$$

for $v \in E(r) \cap V_s(r)$, $w \in E(r) \cap V_t(r)$. This implies the first part of the corollary.

To prove the second part, we replace $E(r)$ by another base $E'(r)$. Applying Lemma 10.3 to the degenerate curve $\tilde{q}(r) = q(0)$ we get a vector v_0 such that $|v_0| = 1$, $|\sphericalangle(v_0, V_k(r))| \geqslant \delta$ and (10.3) is satisfied. Let $E(r)$ be the base previously constructed. Let $v_0 = \sum_{v \in E(r)} \lambda_v(r) v$; let $v_1(r) \in E(r)$ satisfy: $v_1(r) \notin V_k(r)$, $|\lambda_{v_1(r)}(r)| \geqslant \max\{|\lambda_v(r)|: v \in E(r) \setminus V_k(r)\}$. We take for $E'(r)$ the set $(E(r) \setminus \{v_1(r)\}) \cup \{v_0\}$.

The corollary is strengthened by the following lemma.

LEMMA 10.4. *There exists a constant* $\beta > 0$ *such that, in the notation of the corollary,*

$$|P(\Omega)(q)| \leqslant C\,\mathrm{dist}(q, X_n^{j_l})^{\beta} / \prod_{s=2}^{l} \mathrm{dist}(q, X_n^{j_s})^{2(j_s - j_s + 1)},$$

and if $q \in U_{N, j_k}$, *then*

$$|P(\Omega)(q)| \leqslant C\,\mathrm{dist}(q, X_n^{j_l})^{\beta}\,\mathrm{dist}(q, X_n^{j_k})^{2\gamma} / \prod_{s=2}^{l} \mathrm{dist}(q, X_n^{j_s})^{2(j_s - j_s + 1)}.$$

Proof. Using the curve selection lemma we see that, in order to prove the first estimate, it is enough to show that for every real-analytic curve $q(r)$, $q(r) \in \mathring{X}_n^j$ for $r > 0$, with a chain $q_{j_1}(r) = q(r)$, $q_{j_2}(r), \ldots, q_{j_l}(r)$ we have

$$|P(\Omega)(q(r))| \prod_{s=2}^{l} \mathrm{dist}(q(r), X_n^{j_s})^{2(j_s - j_s + 1)} \to 0 \quad \text{as } r \to 0. \tag{10.4}$$

Let $e_1^*(r), \ldots, e_n^*(r)$ and $\tilde{e}_i^*$ $(i \leqslant j)$ have the same meaning as in the proof of the first part of the corollary. Let $A(r) = \{a_1(r), \ldots, a_n(r)\}$ be a g-orthonormal base of $T_{q(r)} C^n$ such that $A(r) \cap V_s(r)$ is a base of $V_s(r)$ (for every r and s) and $a_1(r)$ is the unit tangent vector to $q(r)$. Now (10.4) follows from (10.2), by calculations as in the proof of the first part of the corollary, and from the estimate

$$\left|D_{a_1(r)} \tilde{e}_i^* (q(r))\right| \leqslant C|r|^{-1+\tau} \quad \text{for all } i,$$

for some $\tau > 0$, which implies that $\left|\omega_{i_1 i_2}(a_1(r))\right| \leqslant C|r|^{-1+\tau}$.

We can now finish the proof of Proposition 1.4. Fix a $c_0 = 2^b \geqslant K_n$. If we define

$$u(p, q) = \begin{cases} \gamma/n & \text{if} \quad p > Nq, \\ 0 & \text{if} \quad p \leqslant Nq, \end{cases}$$

then Lemmas 10.2 and 10.4 imply that on $X_n^j(c_0; \underline{j}, \underline{m})$

$$|X_n^j(c_0; \underline{j}, \underline{m})|_{2j} |P(\Omega)| \leqslant C 2^{-2 \sum\limits_{s=0}^{l-1} m_{s+1} u(m_{s+1}, m_{s+2})(j_s - j_{s+1}) - \beta m_l}.$$

Proposition 1.4. follows from the convergence of the series

$$\sum_{\underline{j}, \underline{m}} 2^{-2 \sum\limits_{s=0}^{l-1} m_{s+1} u(m_{s+1}, m_{s+2})(j_s - j_{s+1}) - \beta m_l},$$

which is easy to establish.

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