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Existence and regularity of solutions of some elliptic system in domains with edges

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CONTENTS

1. Introduction

This paper is devoted to presenting the existence and regularity properties of solutions to the elliptic system

$$\text{rot}\, v = \omega \quad \text{in } \Omega, \tag{1.1}$$

$$\text{div}\, v = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$v_n|_{\partial\Omega} = b \quad \text{on } \partial\Omega, \tag{1.3}$$

where $v_n = v \cdot \bar{n}$ and $\bar{n}$ is the unit outward normal vector to the boundary, in a bounded domain $\Omega \subset \boldsymbol{R}^3$ with edges on its boundary. We assume that the edges do not intersect one another so that there are at most two-surface angles between each two boundary surfaces which intersect along one of the edges. Moreover, the following compatibility conditions are necessary:

$$\int_{\partial\Omega} b(s)ds = 0, \tag{1.4}$$

$$\text{div}\, \omega = 0. \tag{1.5}$$

Although (1.1), (1.2) form an overdetermined elliptic system, it is shown in [15] that the problem (1.1)–(1.3) is well posed in domains with smooth boundary.

In this paper the problem (1.1)–(1.3) itself is not considered, but is replaced by the Neumann problem (see (3.2))

$$\Delta\varphi = 0 \ \text{ in } \Omega, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0 \ \text{ on } \partial\Omega, \tag{1.6}$$

together with the elliptic problem (see (3.9))

$$-\Delta e = \omega \ \text{ in } \Omega, \quad e_\tau|_{\partial\Omega} = 0, \quad \text{div}\, e|_{\partial\Omega} = 0 \ \text{ on } \partial\Omega, \tag{1.7}$$

where $e_\tau = e \cdot \bar{\tau}$, is any tangent vector to $\partial\Omega$, $u = \text{rot}\, e$ (see the transformation (3.6)) and $v = \nabla\varphi + u$. This follows from two reasons. We neither know how to find a weak solution nor how to apply Kondrat'ev's theorems to the problem (1.1)–(1.3) directly. Recently the existence of weak solutions of (1.1)–(1.3) in the smooth boundary case was obtained by geometrical methods (see [19]) which are very far from the methods presented in this paper.

Since the results about the Neumann problem are already available [13], [14], this paper is devoted to examining the problem (1.7) only.

Replacing the problem (1.1)–(1.3) by (1.6) and (1.7) raises the question of equivalence. In the case of smooth boundary and sufficiently smooth solutions (at least of class $C^{2+\alpha}$) these ways of showing the existence and regularity are equivalent. For a nonsmooth boundary Kondrat'ev's relation between the maximal magnitude of dihedral angles and the regularity of solutions is more restrictive for the problems (1.6), (1.7) because they involve second order operators.

The aim of this paper is to prove the existence and regularity of solutions of (1.7) in such spaces that the solutions do not vanish in a neighbourhood of the edges. We use the spaces $W^l_{p,\mu}(\Omega)$, $C^l_s(\Omega)$ (all notation and auxiliary results are introduced in Section 2). We show that these spaces are suitable for the problem (1.7) (and also for the Neumann problem).

The proof of the existence and regularity of solutions of (1.7) in a bounded domain with edges is divided into two main steps. The first step is the proof of the existence of a weak solution to (1.7) in $H^1(\Omega)$ (see Theorem 10.1). The second step is to show the regularity of the weak solution in a neighbourhood of edges (interior regularity and regularity near smooth parts of the boundary are well known). This is formulated in Theorems 10.3, 10.4. To prove these theorems we use a suitable partition of unity (see Section 10 and Theorem 2.7) so that we locally replace the problem (1.7) by a nonhomogeneous problem in a dihedral angle $\mathscr{D}_\vartheta$ (angle of magnitude ϑ between two planes Γ_1 and Γ_2 (see (10.24))

$$
\begin{aligned}
-\Delta e &= \tilde{F} \quad \text{in } \mathscr{D}_\vartheta,\\
e\cdot\bar{\tau}_i|_{\Gamma_i} &= \tilde{\Phi}_i \quad \text{on } \Gamma_i,\\
\operatorname{div} e|_{\Gamma_i} &= \tilde{\Psi}_i \quad \text{on } \Gamma_i,
\end{aligned}
\tag{1.8}
$$

where $i = 1, 2$ and $\bar{\tau}_i$ are vectors tangent to Γ_i, $i = 1, 2$. This problem is divided into two problems: the Dirichlet problem for the Laplace equation (see (10.25))

$$
\begin{aligned}
-\Delta e_3 &= F_3 \quad \text{in } \mathscr{D}_\vartheta,\\
e_3|_{\Gamma_i} &= \bar{\Phi}_i \quad \text{on } \Gamma_i, \quad i = 1, 2,
\end{aligned}
\tag{1.9}
$$

and (see (10.26))

$$
\begin{aligned}
-\Delta e_i &= F_i \quad \text{in } \mathscr{D}_\vartheta, \quad i = 1, 2,\\
e\cdot\bar{\tau}_j|_{\Gamma_j} &= \Phi'_j \quad \text{on } \Gamma_j, \quad j = 1, 2,\\
\bar{n}_j\cdot\nabla(e\cdot n_j)|_{\Gamma_j} &= \Phi_j \quad \text{on } \Gamma_j, \quad j = 1, 2,
\end{aligned}
\tag{1.10}
$$

where $\bar{\tau}_j$ is tangent to Γ_j and normal to the edge, $\bar{n}_j$ is outward normal to Γ_j and also normal to the edge, $j = 1, 2$.

The right-hand side functions of the problems (1.9) and (1.10) do not vanish near the edge of $\mathscr{D}_\vartheta$ which makes it necessary to use the spaces $W^l_{p,\mu}(\mathscr{D}_\vartheta)$ and $C^l_s(\mathscr{D}_\vartheta)$.

Sections 4–9 are devoted to examining the existence and regularity of solutions of the problem (1.10) (for the problem (1.9) these properties mainly follow from the results of Maz'ya–Plamenevskii [10]). First, we have to consider the problem (1.10) in the angle d_ϑ (which is the projection of $\mathscr{D}_\vartheta$ on the plane perpendicular to the edge; denote this new problem by (1.10)′). This follows from the fact that only for conical domains we have the theorems of Kondrat'ev [6], [7] and Maz'ya–Plamenevskii [11] on the solvability of boundary value problems for elliptic equations. These theorems require the data to be either in $V^l_{p,\mu}(d_\vartheta)$ or in $\mathring{C}^l_s(d_\vartheta)$. The solvability theorems for (1.10) are formulated in Section 4 (Theorems 4.1, 4.2). However, the right-hand sides of (1.10)′ belong either to $W^l_{p,\mu}(d_\vartheta)$ or to $C^l_s(d_\vartheta)$. Moreover, considering the homogeneous problem (1.10)′ we see that it has eigenvalue 1 (the Neumann problem has eigenvalue 0), which implies that the spaces $W^l_{p,\mu}(d_\vartheta)$ and $C^l_s(d_\vartheta)$ are natural for showing the existence and regularity of solutions of (1.10)′ also in the case of data in either $V^l_{p,\mu}(d_\vartheta)$ or $\mathring{C}^l_s(d_\vartheta)$ (see the proofs of Theorems 4.3, 6.2, 8.3, 9.3). Therefore there are constructed functions v_i, $i = 1, 2$ (see Lemma 4.3 and relations (4.22), (4.23)) which enable us to reduce (1.10)′ with data in $W^l_{p,\mu}$ to (1.10)′ with data in $V^l_{p,\mu}$. Moreover, the right-hand side functions of (1.10)′ can not be arbitrary but have to satisfy a condition which is implied by the fact that 1 as an eigenvalue (see (4.21)). The construction of the functions v_i, $i = 1, 2$, is divided into three parts. Let $\pi/\vartheta \notin N$. Then for $\mu+2/p \notin Z$ the construction is based on the Hardy inequality (see (2.1), (2.2) and also Lemmas 2.1, 2.3, 4.2). For $\mu+2/p \in Z$ the construction needs additional considerations (see Lemma 2.4, 2.5 which are generalizations and modifications of Lemmas 4.17, 4.19 from [7] and Lemmas 3.2, 3.3, 3.4 from [13] where they are obtained for $\mu = 0$ and $p = 2$). For $\pi/\vartheta \in N$ some compatibility conditions for the derivatives of the right-hand side functions at the vertex of d_ϑ have to be added.

We have to underline that in this paper the compatibility conditions which are necessary for the existence and regularity of solutions of the problems (1.10), (1.10)′ in the case of angles equal to π/m, $m \in N$, $m \geqslant 2$, are not explicitly formulated.

Constructing the functions v_i, $i = 1, 2$, for data in $W^l_{p,\mu}$ (Lemma 4.3) gives that they are determined up to some polynomials, which are not important for data in $L^l_{p,\mu}$ (Lemma 4.4).

The main result of Section 4 states the existence of solutions of the problem (1.10)′ in the spaces $L^l_{p,\mu}(d_\vartheta)$ if the data are also from these spaces and

$$\Lambda(\vartheta) > l+2-\left(\mu+\frac{2}{p}\right) > 0 \tag{1.11}$$

where $\Lambda(\vartheta) = \pi/\vartheta-1$ for $\vartheta < \pi$, $\Lambda(\vartheta) = 1-\pi/\vartheta$ for $\pi < \vartheta \leqslant \frac{3}{2}\pi$ and $\Lambda(\vartheta) = 2\pi/\vartheta-1$ for $\frac{3}{2}\pi \leqslant \vartheta < 2\pi$ (Theorem 4.3 for $l+2-(\mu+2/p) \neq 1$ and

Theorem A.1 for $l+2-(\mu+2/p) = 1$). The proof is based on the existence of a generalized solution, construction of the functions v_i, $i = 1, 2$, and the result about the solvability of the problem (1.10)′ for data in $V^l_{p,\mu}$ (Theorem 4.1). Similar results in the case of weighted Hölder spaces are formulated in Theorem 4.4.

Section 5 is devoted to obtaining results which enable us to replace the problem (1.10) with nonhomogeneous boundary data by the same problem with zero boundary data, and conversely (Lemmas 5.1, 5.2). The most important result is the construction of functions v_i, $i = 1, 2$, which give us a possibility of replacing the problem (1.10) with data in $L^l_{p,\mu}$ by the same problem with data in $V^l_{p,\mu}$ and conversely (see Lemma 5.2). Hence using Lemmas 5.1, 5.2 instead of the problem (1.10) we can consider

$$(1.12)\qquad \begin{aligned} -\Delta e_i = f_i \quad \text{in} \quad \mathscr{D}_\vartheta, \quad i = 1, 2, \\ e\cdot\bar{\tau}_i|_{\Gamma_i} = 0, \quad \bar{n}_i\cdot\nabla(e\cdot\bar{n}_i)|_{\Gamma_i} = 0, \quad i = 1, 2, \end{aligned}$$

and $\bar{\tau}_i$, $\bar{n}_i$, $i = 1, 2$, are described in (1.10).

We have to underline that the eigenvalue 1 imposes the compatibility condition on the data functions (see (5.4)) which implies that the functions v_i, $i = 1, 2$, have to satisfy some restriction (see (5.7)).

For $\pi/\vartheta \in N$ the above construction needs some additional compatibility conditions on data functions at points of the edge of $\mathscr{D}_\vartheta$.

In the case of Hölder spaces similar results are formulated without proofs.

In Section 6 the existence of a weak solution in $H^1(\mathscr{D}_\vartheta)$ for (1.12) is proved (Theorem 6.1). Then using the Fourier transformation with respect to the direction parallel to the edge of $\mathscr{D}_\vartheta$ and the results of Kondrat'ev for general elliptic boundary value problems in conical domains (Theorem 4.1, $p = 2$, with right-hand side functions in $H^l_\mu(d_\vartheta)$) we show that the weak solution belongs to $L^{l+2}(D_\vartheta)$ if the condition (1.11) is satisfied (Theorem 6.2).

By using Theorem 6.2 the existence and estimates for the Green function (Theorem 7.1 and (7.16)) are found in Section 7. Solonnikov's methods [14], [16] are applied here. By means of the Green function a solution of (1.12) can be written in the form

$$(1.13)\qquad e_i(x) = \sum_{j=1}^{2} \int_{\mathscr{D}_\vartheta} G_{ij}(x, y) f_j(y)\,dy, \qquad x \in \mathscr{D}_\vartheta,\ i = 1, 2.$$

In Section 8 using the methods of Solonnikov [14] we find the estimate in $L^2_{p,\mu}(\mathscr{D}_\vartheta)$ for solutions of (1.12) in the integral form (1.13) ((8.5) and Theorem 8.2). The main result of this section states that a solution of (1.12) belongs to $L^{l+2}_{p,\mu}(\mathscr{D}_\vartheta)$ if $f_i \in L^l_{p,\mu}(\mathscr{D}_\vartheta)$, $i = 1, 2$, and (1.11) is satisfied (Theorem 8.3).

In Section 9 by repeating the argument of Solonnikov [14], estimates for solutions of the problem (1.12) in the form (1.13) are found in Hölder spaces (Theorems 9.1, 9.2). Then we show the existence of solutions of (1.12) in Hölder spaces $C_s^{l+2}(\mathscr{D}_\vartheta)$ if $f_i \in C_s^l(\mathscr{D}_\vartheta)$, $i = 1, 2$, and

$$s < \Lambda(\vartheta) \tag{1.14}$$

(see Theorem 9.3).

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2. Notation and auxiliary results

In this section we introduce some notation and results which are also used in [13], [14].

Let r, φ be the polar coordinates in the plane; $d_\vartheta \subset \boldsymbol{R}^2$ the infinite angle $\{r > 0, 0 < \varphi < \vartheta\}$ with magnitude ϑ; γ_1, γ_2 the sides of d_ϑ described by $\varphi = 0$ and $\varphi = \vartheta$, respectively; $\mathscr{D}_\vartheta = d_\vartheta \times \boldsymbol{R}^1$ the dihedral angle in $\boldsymbol{R}^3$ with sides $\Gamma_i = \gamma_i \times \boldsymbol{R}^1$, $i = 1, 2$, and with edge $M = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$.

The points of $\mathscr{D}_\vartheta$ are denoted by $x = (x', z)$, where $x' = (x_1, x_2) \in d_\vartheta$, $z \in \boldsymbol{R}^1$. Moreover, we introduce a coordinate system such that γ_1 is the x_1 axis and the point $x' = 0$ is the vertex of d_ϑ. Finally we write $\nabla = \text{grad}$ and $\nabla' = (\partial_{x_1}, \partial_{x_2})$.

By $\zeta(x) \in C_0^\infty(\boldsymbol{R}^3)$ we denote a monotonic function of $|x|$ equal to one for $|x| \leqslant 1/2$ and to zero for $|x| \geqslant 1$. Moreover, let

$$K_r(z) = \{x \in \mathscr{D}_\vartheta : |x - z| < r\} \quad \text{for } z \in \bar{\mathscr{D}}_\vartheta.$$

For a function $f(x)$, $x \in \mathscr{D}_\vartheta$, and $j = 0, 1, \ldots$, by $f^{(j)}(x)$ we denote the Taylor polynomial of f of degree j with respect to x':

$$f^{(j)}(x) = \sum_{|\alpha| = \alpha_1 + \alpha_2 \leqslant j} D_{x'}^\alpha f(x)|_{x'=0} \frac{x_1^{\alpha_1} x_2^{\alpha_2}}{\alpha_1! \alpha_2!},$$

and for a function Φ defined on Γ_i, $i = 1, 2$,

$$\Phi^{(j)}(r) = \sum_{k=0}^{j} \frac{1}{k!} \frac{\partial^k}{\partial r^k} \Phi|_{r=0} r^k.$$

Now we introduce some function spaces for the domain $\mathscr{D}_\vartheta$ (which for the domain $d_\vartheta \subset \boldsymbol{R}^2$ can be defined similarly) [13], [14]: $\mathscr{H}(\mathscr{D}_\vartheta)$ is the space of functions with finite Dirichlet integral

$$\|u\|_{\mathscr{H}(\mathscr{D}_\vartheta)} = \Big(\int_{\mathscr{D}_\vartheta} |\nabla u|^2 dx \Big)^{1/2};$$

$W^k_{p,\mu}(\mathcal{D}_\vartheta)$, $V^k_{p,\mu}(\mathcal{D}_\vartheta)(p,\mu\in\boldsymbol{R},\ p>1,\ k\in\boldsymbol{N})$ are the spaces with norms

$$\|u\|_{W^k_{p,\mu}(\mathcal{D}_\vartheta)}=\Big(\sum_{|\alpha|\leqslant k}\int_{\mathcal{D}_\vartheta}|x'|^{p\mu}|D^\alpha_x u|^p dx\Big)^{1/p},$$

$$\|u\|_{V^k_{p,\mu}(\mathcal{D}_\vartheta)}=\Big(\sum_{|\alpha|\leqslant k}\int_{\mathcal{D}_\vartheta}|x'|^{p(\mu-(k-|\alpha|))}|D^\alpha_x u|^p dx\Big)^{1/p},$$

$L^k_{p,\mu}(\mathcal{D}_\vartheta)$ is the closure of the set of smooth functions with compact supports in the norm

$$\|u\|_{L^k_{p,\mu}(\mathcal{D}_\vartheta)}=\Big(\sum_{|\alpha|=k}\int_{\mathcal{D}_\vartheta}|x'|^{p\mu}|D^\alpha_x u|^p dx\Big)^{1/p}.$$

For $k=0$ the above spaces coincide and are denoted by $L_{p,\mu}(\mathcal{D}_\vartheta)$. Moreover, we write that $V^k_{2,\mu}(\mathcal{D}_\vartheta)=H^k_\mu(\mathcal{D}_\vartheta)$, $L^k_{2,\mu}(\mathcal{D}_\vartheta)=L^k_\mu(\mathcal{D}_\vartheta)$, $k>0$, and we use the following standard notation: $L_{p,0}(\mathcal{D}_\vartheta)=L_p(\mathcal{D}_\vartheta)$, $W^k_{p,0}(\mathcal{D}_\vartheta)=W^k_p(\mathcal{D}_\vartheta)$, $W^k_{2,0}(\mathcal{D}_\vartheta)=H^k(\mathcal{D}_\vartheta)$.

For $k>0$ the elements of $W^k_{p,\mu}(\mathcal{D}_\vartheta)$, $V^k_{p,\mu}(\mathcal{D}_\vartheta)$, $L^k_{p,\mu}(\mathcal{D}_\vartheta)$ have traces on every two-dimensional plane Γ passing through the edge. These traces belong to the following function spaces: $W^{k-1/p}_{p,\mu}(\Gamma)$, $V^{k-1/p}_{p,\mu}(\Gamma)$, with norms

$$\|u\|^p_{W^{k-1/p}_{p,\mu}(\Gamma)}=\|u\|^p_{L^{k-1/p}_{p,\mu}(\Gamma)}+\sum_{|\alpha|\leqslant k-1}\int_\Gamma|D^\alpha u|^p\xi_1^{p\mu}d\xi,$$

$$\|u\|^p_{V^{k-1/p}_{p,\mu}(\Gamma)}=\|u\|^p_{L^{k-1/p}_{p,\mu}(\Gamma)}+\sum_{|\alpha|\leqslant k-1}\int_\Gamma|D^\alpha u|^p\xi_1^{p(\mu-k+|\alpha|)+1}d\xi,$$

and $L^{k-1/p}_{p,\mu}(\Gamma)$, which is the closure of the set of smooth functions on Γ with compact support in the norm

$$\|u\|_{L^{k-1/p}_{p,\mu}(\Gamma)}=\left(\sum_{|\alpha|=k-1}\int_\Gamma\xi_1^p d\xi\int_{K_+(\xi)}|D^\alpha u(\xi+\eta)-D^\alpha u(\xi)|^p\frac{d\eta}{|\eta|^{1+p}}\right)^{1/p},$$

where $\xi=(\xi_1,\xi_2)$ are the Cartesian coordinates on Γ such that $\Gamma=\{\xi\in\boldsymbol{R}^2:\ \xi_1>0\}$, and $K_+(\xi)=\{\eta\in\Gamma:\ |\eta|<\xi_1\}$.

By $\mathcal{B}^1_p(\boldsymbol{R}^n)$, $l\in\boldsymbol{Z}$, we denote the closure of the smooth functions with compact support in the norm

$$\langle\langle u\rangle\rangle^1_{p,\boldsymbol{R}^n}=\left(\sum_{|\alpha|=[l]}\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}\left|D^\alpha u(x)-2D^\alpha u\left(\frac{x+y}{2}\right)+D^\alpha u(y)\right|^p\frac{dx\,dy}{|x-y|^{n+p(l-[l])}}\right)^{1/p},$$

where $[l]$ is the maximal integer less than l. This norm is the principal part of the norm in the Besov space $B^l_p(\boldsymbol{R}^n)$.

To consider the Fourier transform of functions in $H^k_\mu(\mathcal{D}_\vartheta)$ and $L^k_\mu(\mathcal{D}_\vartheta)$ we define the spaces $\mathcal{E}^k_\mu(d_\vartheta)$ and $\mathcal{L}^k_\mu(d_\vartheta)$ with norms

$$\|u\|_{\mathcal{E}^k_\mu(d_\vartheta)}=\Big(\sum_{i=0}^k\xi^{2i}\|u\|^2_{H^{k-i}_\mu(d_\vartheta)}\Big)^{1/2},$$

$$\|u\|_{\mathscr{L}^k_\mu(d_\vartheta)} = \left(\sum_{i=0}^{k} \xi^{2i}\|u\|^2_{L^{k-i}_\mu(d_\vartheta)}\right)^{1/2}.$$

Now we introduce weighted Hölder spaces $\mathring{C}^l_s(\mathscr{D}_\vartheta)(l > 0,\ l \notin N)$ denotes the space of functions with the norm

$$|u|_{\mathring{C}^l_s(\mathscr{D}_\vartheta)} = \sum_{|\alpha|<l} \sup_{x\in\mathscr{D}_\vartheta} |x'|^{|\alpha|-s}|D^\alpha u(x)| + \sup_{x\in\mathscr{D}_\vartheta} |x'|^{l-s}[u]^{(l)}_{K_{|x'|/2}(x)},$$

where

$$[u]^{(l)}_K = \sum_{|\alpha|=[l]} \sup_{x,y\in K} |x-y|^{[l]-l}|D^\alpha u(x) - D^\alpha u(y)|.$$

$C^l_s(\mathscr{D}_\vartheta)$ $(s\in(0, l],\ l > 0,\ l\in N)$ denotes the space of functions with the norm

$$|u|_{C^l_s(\mathscr{D}_\vartheta)} = \langle u\rangle^l_{s,\mathscr{D}_\vartheta} + \sum_{|\alpha|<s} \sup_{\mathscr{D}_\vartheta} |D^\alpha u(x)|,$$

where

$$\langle u\rangle^l_{s,\mathscr{D}_\vartheta} = \sup_{\mathscr{D}_\vartheta} |x'|^{l-s}[u]^{(l)}_{K_{|x'|/2}(x)} + [u]^{(s)}_{\mathscr{D}_\vartheta}.$$

For $s\in(0, 1]$

$$\mathring{C}^l_s(\mathscr{D}_\vartheta) = \{u = C^l_s(\mathscr{D}_\vartheta)\colon\ D^\alpha u|_{x'=0} = 0,\ 0 \leqslant |\alpha| \leqslant [s]\}.$$

Moreover, for $u\in\mathring{C}^l_s(\mathscr{D}_\vartheta)$, $s\in(0, l]$ the norms $|u|_{\mathring{C}^l_s(\mathscr{D}_\vartheta)}$ and $\langle u\rangle^{(l)}_{s,\mathscr{D}_\vartheta}$ are equivalent.

For $s < 0$, we let

$$\langle u\rangle^{(l)}_{s,\mathscr{D}_\vartheta} = |u|_{\mathring{C}^l_s(\mathscr{D}_\vartheta)}.$$

We denote Ω a bounded domain in $\boldsymbol{R}^3$ with boundary $\partial\Omega = \bigcup_{\nu=1}^{r} S_\nu$, where S_ν, $\nu = 1,.., r$, are smooth manifolds which can intersect only along an edge. Let $L = S_i \cap S_j \neq \emptyset$ be one of the edges. Then at $x\in L$ we have the tangent spaces T_xS_i, T_xS_j and the dihedral angle $\mathscr{D}_{\vartheta(x)}$. We introduce curvilinear coordinates in a neighbourhood of each S_ν, $\nu = 1,\dots, r$. Let $x = (x_1, x_2, x_3)$ be the Cartesian coordinates and τ_1, τ_2, n the orthogonal coordinates such that $n(x) = C$ is the surface S_ν and τ_1, τ_2 are coordinates on S_ν. Let $x = x(\tau_1, \tau_2, n)$. Then $x_{,\tau_\nu} = H_\nu\tau_\nu$, $\nabla\tau_\nu = H_\nu^{-1}\bar{\tau}_\nu$, $\nu = 1, 2$, $x_{,n} = H_n\bar{n}$, $\nabla x = H_n^{-1}\bar{n}$, where $\bar{\tau}_1, \bar{\tau}_2, \bar{n}$ are orthonormal vectors and H_1, H_2, H_n are Lamé's coefficients [5, § 18]. Mainly we shall assume that $H_n = 1$.

Finally we introduce the spaces $W^k_{p,\mu}(\Omega)$ and $C^l_s(\Omega)$ with the following norms

$$\|u\|_{W^k_{p,\mu}(\Omega)} = \Big(\sum_{|\alpha|\leqslant k}\int_\Omega \varrho^{p\mu}(x)|D^\alpha u|^p dx\Big)^{1/p},$$

$$|u|_{C^l_s(\Omega)} = \langle u\rangle^{(l)}_{s,\Omega} + \sum_{|\alpha|<s}\sup_\Omega |D^\alpha u(x)|,$$

where

$$\langle u\rangle^{(l)}_{s,\Omega} = \sup_\Omega \varrho(x)^{l-s}[u]^{(l)}_{K_{\varrho(x)/2}(x)} + [u]^{(s)}_\Omega$$

and $\varrho(x)$ is the distance from x to the nearest of edge.

First we recall some theorems from [13], [14].

THEOREM 2.1. *Let* $u\in L^k_{p,\mu}(\mathscr{D}_\vartheta)$, $|\alpha|<k$. *Then* $D^\alpha u|_\Gamma \in L^{k-|\alpha|-1/p}_{p,\mu}(\Gamma)$ *and*

$$\|D^\alpha u\|_{L^{k-|\alpha|-1/p}_{p,\mu}(\Gamma)} \leqslant c\|u\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)}.$$

Let $u\in W^k_{p,\mu}(\mathscr{D}_\vartheta)$, $\mu>1-1/p$. *Then* $D^\alpha u|_\Gamma \in W^{k-|\alpha|-1/p}_{p,\mu}(\Gamma)$ *and*

$$\|D^\alpha u\|_{W^{k-|\alpha|-1/p}_{p,\mu}(\Gamma)} \leqslant c\|u\|_{W^k_{p,\mu}(\mathscr{D}_\vartheta)}.$$

THEOREM 2.2. *Let on* Γ *be given functions* $\varphi_j\in L^{k-j-1/p}_{p,\mu}(\Gamma)$, $j=0,\ldots,k-1$. *Then there exists a function* $u\in L^k_{p,\mu}(\mathscr{D}_\vartheta)$ *such that*

$$\frac{\partial^j u}{\partial n^j}\Big|_\Gamma = \varphi_j$$

and

$$\|u\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c\sum_{j=0}^{k-1}\|\varphi_j\|_{L^{k-j-1/p}_{p,\mu}(\Gamma)}.$$

If $\varphi_j\in W^{k-j-1/p}_{p,\mu}(\Gamma)$, *then there exists a function* $v\in W^k_{p,\mu}(\mathscr{D}_\vartheta)$, *such that*

$$\frac{\partial^j v}{\partial n^j}\Big|_\Gamma = \varphi_j$$

and

$$\|v\|_{W^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c\sum_{j=0}^{k-1}\|\varphi_j\|_{L^{k-j-1/p}_{p,\mu}(\Gamma)}.$$

THEOREM 2.3. *If* $u\in V^k_{p,\mu}(\mathscr{D}_\vartheta)$, $|\alpha|<k$, *then* $D^\alpha u|_\Gamma \in V^{k-|\alpha|-1/p}_{p,\mu}(\Gamma)$ *and*

$$\|D^\alpha u\|_{V^{k-|\alpha|-1/p}_{p,\mu}(\Gamma)} \leqslant c\|u\|_{V^k_{p,\mu}(\mathscr{D}_\vartheta)}.$$

Let $\varphi^i_j\in V^{k-j-1/p}_{p,\mu}(\Gamma_i)$, $i=1,2$, $j=0,\ldots,k-1$, *then there exists a function*

$u \in V^k_{p,\mu}(\mathcal{D}_\vartheta)$, such that

$$\left.\frac{\partial^j u}{\partial n^j}\right|_{\Gamma_i} = \varphi^i_j$$

and

$$\|u\|_{V^k_{p,\mu}(\mathcal{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \sum_{j=0}^{k-1} \|\varphi^i_j\|_{V^{k-j-1/p}_{p,\mu}(\Gamma_i)}.$$

THEOREM 2.4. *If $u \in L^k_{p,\mu}(\mathcal{D}_\vartheta)$, $\mu > -2/p$ and $|\alpha| < k-\mu-1/p$, then $D^\alpha u|_M \in \mathscr{B}^{k-\mu-|\alpha|-2/p}_p(M)$ and*

$$\langle\langle D^\alpha u\rangle\rangle^{(k-\mu-|\alpha|-2/p)}_{p,M} \leqslant c\|u\|_{L^k_{p,\mu}(\mathcal{D}_\vartheta)}.$$

Let $\varphi_\alpha \in \mathscr{B}^{k-\mu-|\alpha|-2/p}_p(M)$ with compact supports be given for all multiindices $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| < k-\mu-2/p$. Then there exists a function $u \in L^k_{p,\mu}(\mathcal{D}_\vartheta)$ with compact support such that $D^\alpha_{x'} u|_M = \varphi_\alpha$ and

$$\|u\|_{L^k_{p,\mu}(\mathcal{D}_\vartheta)} \leqslant c \sum_\alpha \langle\langle \varphi_\alpha\rangle\rangle^{(k-\mu-|\alpha|-2/p)}_{p,M}.$$

THEOREM 2.5. *For arbitrary $u \in H^l(\mathcal{D}_\vartheta)$ the following interpolation inequality is valid:*

$$\int_{\mathcal{D}_\vartheta} u^2|x'|^{-2\mu}dx \leqslant \varepsilon^{2(l-\mu)}\|u\|^2_{L^l_{2,0}(\mathcal{D}_\vartheta)} + C\varepsilon^{-2\mu}\|u\|^2_{L_2(\mathcal{D}_\vartheta)},$$

for $\mu \in (0, 1)$ and $\varepsilon > 0$.

Now we prove some results based on the Hardy inequality and partly obtained in [13], [14]. We use the following Hardy inequalities [17]:

Let $f \in L^1_{p,\nu}(d_\vartheta)$. Then for $f(0) = 0$, $1-2/p-\nu > 0$ we have

$$(2.1)\qquad \Big(\int_0^\infty |f|^p r^{\nu p-p+1}dr\Big)^{1/p} \leqslant \frac{1}{1-2/p-\nu}\Big(\int_0^\infty |f_r|^p r^{p\nu+1}dr\Big)^{1/p},$$

and for $f(\infty) = 0$, $1-2/p-\nu < 0$, we have

$$(2.2)\qquad \Big(\int_0^\infty |f|^p r^{\nu p-p+1}dr\Big)^{1/p} \leqslant \frac{1}{\nu+2/p-1}\Big(\int_0^\infty |f_r|^p r^{p\nu+1}dr\Big)^{1/p}.$$

LEMMA 2.1. *Let $u \in L^k_{p,\mu}(d_\vartheta)$, $p, \mu \in \boldsymbol{R}$, $k \in \boldsymbol{Z}$, $p \geqslant 1$, $k \geqslant 1$ and $\mu+2/p \geqslant 0$.*

(a) *Let $s = \mu+2/p \in \boldsymbol{Z}$. Then there exists an integer j such that $k-s-1 < j < k-s$ and*

$$(2.3)\qquad \|u-u^{(j)}\|_{L_{p,\mu-k}(d_\vartheta)} \leqslant c\|u\|_{L^k_{p,\mu}(d_\vartheta)},$$

where $u^{(j)}$ is defined at the beginning of this section. For $j < k-s < 0$, $u^{(j)} = 0$.

Moreover, $u-u^{(j)}\in V^k_{p,\mu}(d_\vartheta)$ *and*

$$\|u-u^{(j)}\|_{V^k_{p,\mu}(d_\vartheta)} \leqslant c\|u\|_{L^k_{p,\mu}(d_\vartheta)} \tag{2.4}$$

(b) *Let* $s=\mu+2/p\in \boldsymbol{Z}$. *Let* $D^{k-s}u\in L_{p,\mu-s}(d_\vartheta)$ *and* $j=k-s$. *Then* $u-u^{(j-1)}\in L_{p,\mu-k}(d_\vartheta)$ *and*

$$\|u-u^{(j-1)}\|_{L_{p,\mu-k}(d_\vartheta)} \leqslant c\|D^{k-s}u\|_{L_{p,\mu-s}(d_\vartheta)}. \tag{2.5}$$

Moreover, $u-u^{(j-1)}\in V^k_{p,\mu}(d_\vartheta)$ *and*

$$\|u-u^{(j-1)}\|_{V^k_{p,\mu}(d_\vartheta)} \leqslant c\big(\|u\|_{L^k_{p,\mu}(d_\vartheta)}+\|D^{k-s}u\|_{L_{p,\mu-s}(d_\vartheta)}\big). \tag{2.6}$$

Proof. (a) For $u\in L^k_{p,\mu}(d_\vartheta)$ one has

$$\int\limits_0^\infty |D^k_r u|^p r^{p\mu+1}\,dr<\infty \qquad \text{for a.e. } \varphi\in[0,\vartheta].$$

Let j be an integer such that

$$s-\sigma>0 \qquad \text{for } \sigma\leqslant k-j-1. \tag{2.7}$$

Hence the inequality (2.2) can be used $k-j-1$ times to obtain

$$\int\limits_0^\infty |D^{j+1}_r u|^p r^{p[\mu-(k-j-1)]+1}\,dr \leqslant c\int\limits_0^\infty |D^k_r u|^p r^{p\mu+1}\,dr, \tag{2.8}$$

where $p[\mu-(k-j-1)]+1>-1$, which is equivalent to (2.7) for $\sigma=k-j-1$. From (2.7) and the fact that $s\notin \boldsymbol{Z}$ it follows that j can be so chosen that

$$p[\mu-(k-j)]+1<-1 \qquad \text{and} \qquad p[\mu-(k-j-1)]+1<p-1.$$

The last inequality implies that the Hardy inequality (2.1) can be used for the function $u-u^{(j)}$ because $D^{j+1}_r u^{(j)}=0$ and $D^\sigma_{x'}(u-u^{(j)})|_{x'=0}=0$ for $\sigma\leqslant j$. Applying (2.1) $(j+1)$-times we obtain (2.3).

To show (2.4) we apply the above considerations to the expression $\|u-u^{(j)}\|_{L^k_{p,\mu}(d_\vartheta)}$, using polar coordinates.

(b) In this case the inequality (2.8) for $j\geqslant k-s$ is valid. For $j=k-s$ the left-hand side of (2.8) is

$$\int\limits_0^\infty |D^{k-s+1}_r u|^p r^{p(\mu-s+1)+1}\,dr=\int\limits_0^\infty |D^{k-s+1}_r u|^p r^{p-1}\,dr.$$

Therefore the parameter ν from the inequalities (2.1) and (2.2) is such that $\nu+2/p-1=0$, and so they cannot be used. Hence to estimate the derivatives of order $<k-s+1$ we must assume that $D^{k-s}u\in L_{p,\mu-s}(d_\vartheta)$, so that

$$\|D^{k-s}u\|^p_{L_{p,\mu-s}(d_\vartheta)}=\int\limits_{d_\vartheta}|D^{k-s}u|^p r^{-1}\,dr\,d\varphi.$$

Comparing this expression with the right-hand sides of (2.1), (2.2), we see that $-1 = \nu p+1$, so $\nu = -2/p$ and $\nu+2/p-1 < 0$. Therefore, the Hardy inequality (2.1) for $u-u^{(j-1)}$ can be used. In this way we get (2.5). Using (2.5), we obtain (2.6) just as (2.4) above. This concludes the proof. ■

From Lemma 2.1 we have

LEMMA 2.2. *Suppose* $u \in L^k_{p,\mu}(\mathscr{D}_\vartheta)$, k, p, μ *as in the previous lemma.*

(a) *Let* $s = \mu+2/p < k$, $s \notin \mathbf{Z}$. *Then there exists an integer* j *such that* $k-s-1 < j < k-s$ *and*

$$\|u-u^{(j)}\|_{L_{p,\mu-k}(\mathscr{D}_\vartheta)} \leqslant c\|u\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)}. \tag{2.9}$$

Moreover $u-u^{(j)} \in V^k_{p,\mu}(\mathscr{D}_\vartheta)$ *and*

$$\|u-u^{(j)}\|_{V^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c\|u\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)}. \tag{2.10}$$

(b) *Let* $s = \mu+2/p < k$, $s \in \mathbf{Z}$, $D^{k-s}u \in L_{p,\mu-s}(\mathscr{D}_\vartheta)$ *and* $j = k-s$. *Then* $u-u^{(j-1)} \in L_{p,\mu-k}(\mathscr{D}_\vartheta)$ *and*

$$\|u-u^{(j-1)}\|_{L_{p,\mu-k}(\mathscr{D}_\vartheta)} \leqslant c\|D^{k-s}u\|_{L_{p,\mu-s}(\mathscr{D}_\vartheta)}. \tag{2.11}$$

Moreover $u-u^{(j-1)} \in V^k_{p,\mu}(\mathscr{D}_\vartheta)$ *and*

$$\|u-u^{(j-1)}\|_{V^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c(\|u\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)}+\|D^{k-s}u\|_{L_{p,\mu-s}(\mathscr{D}_\vartheta)}). \tag{2.12}$$

LEMMA 2.3. *Let* $u \in L^k_{p,\mu}(d_\vartheta)$, $\mu+2/p = s \geqslant 1$, $s \in \mathbf{Z}$, $D^j_{x_2}u|_{\gamma_1} \in V^{k-j-1/p}_{p,\mu}(\gamma_1)$, $j = 0,\ldots, k-s$. *Then* $u \in V^k_{p,\mu}(d_\vartheta)$ *and*

$$\|u\|_{V^k_{p,\mu}(d_\vartheta)} \leqslant c\big(\|u\|_{L^k_{p,\mu}(d_\vartheta)}+\sum_{j=0}^{k-s}\|D^j_{x_2}u\|_{V^{k-j-1/p}_{p,\mu}(\gamma_1)}\big). \tag{2.13}$$

Proof. The inequality (2.2) implies

$$\sum_{i=k-s+1}^{k}\|u\|^p_{L^i_{p,\mu+i-k}(d_\vartheta)} \leqslant c\sum_{i=k-s+1}^{k}\int_{d_\vartheta}|D^i u|^p r^{ps+p(i-k)-1}\,dr\,d\varphi$$

$$\leqslant c\int_{d_\vartheta}|D^k u|^p r^{ps-1}\,dr\,d\varphi.$$

On the other hand we have

$$\sum_{i+j\leqslant k-s}\int_{d_\vartheta}|D^i_{x_1}D^j_{x_2}u|^p r^{p(\mu+i+j-k)+1}\,dr\,d\varphi \leqslant c\sum_{j=0}^{k-s}\|D^j_{x_2}u\|^p_{V^{k-j-1/p}_{p,\mu}(\gamma_1)}$$

and $p(\mu+i+j-k)+1 = p(s+i+j-k)-1 \leqslant 1$, so (2.2) yields that all remaining derivatives in the definition of the norm in $V^k_{p,\mu}(d_\vartheta)$ can also be estimated. Hence (2.13) is valid. ■

LEMMA 2.4. *Let* $k, l, s \in \mathbf{Z}$, $p \in \mathbf{R}$, $k > 0$, $\mu + 2/p = s \geqslant 1$, $l \geqslant s-2$, $p > 1$. *Then for any* $\omega \in W_{p,\mu}^{l+1-1/p}(\gamma_1)$ *there exists* $\omega_k \in W_{p,\mu}^{l+k+1-1/p}(\gamma_1) \cap V_{p,\mu}^{k+s-1-1/p}(\gamma_1)$ *such that*

$$\frac{d^k \omega_k}{dx_1^k} - \omega \in V_{p,1-2/p}^{1-1/p}(\gamma_1)$$

and

$$\|\omega_k\|_{W_{p,\mu}^{l+k+1-1/p}(\gamma_1)} \leqslant c\|\omega\|_{W_{p,\mu}^{l+1-1/p}(\gamma_1)}, \tag{2.14}$$

$$\left\|\frac{d^k \omega_k}{dx_1^k} - \omega\right\|_{V_{p,1-2/p}^{1-1/p}(\gamma_1)} \leqslant c\|\omega\|_{W_{p,\mu}^{l+1-1/p}(\gamma_1)}, \tag{2.15}$$

$$\|\omega_k\|_{V_{p,\mu}^{k+s-1-1/p}(\gamma_1)} \leqslant c\|\omega\|_{W_{p,\mu}^{l+1-1/p}(\gamma_1)}. \tag{2.16}$$

Proof. Define

$$\omega_k(x_1) = \frac{1}{(k-1)!}\int_0^{x_1} (x_1-\tau)^{k-1}\omega(\tau)\big(1+K(\tau)\big)d\tau, \tag{2.17}$$

where $K(\tau)$ is a smooth function with compact support such that $\lim_{\tau\to 0} K(\tau)\tau^{-1} < \infty$. From (2.17) it follows that

$$\left.\frac{d^j \omega_k}{dx_1^j}\right|_{x_1=0} = 0, j \leqslant k-1, \quad \frac{d^k \omega_k}{dx_1^k} = \omega(x_1)\big(1+K(x_1)\big), \tag{2.18}$$

so the inequalities (2.14), (2.15) are satisfied. To show (2.16) we use the one-dimensional Hardy inequalities [17]:

$$\Big(\int_0^\infty |f|^p x^{p(\mu-1)}dx\Big)^{1/p} \leqslant \frac{1}{1-1/p-\mu}\Big(\int_0^\infty |f_x|^p x^{\mu p}dx\Big)^{1/p}, \tag{2.19}$$

where $f_x \in L_{p,\mu}(\mathbf{R}_+^1)$, $f(0) = 0$, $1-1/p-\mu > 0$, and

$$\Big(\int_0^\infty |f|^p x^{p(\mu-1)}dx\Big)^{1/p} \leqslant \frac{1}{\mu+1/p-1}\Big(\int_0^\infty |f_x|^p x^{\mu p}dx\Big)^{1/p}, \tag{2.20}$$

where $f_x \in L_{p,\mu}(\mathbf{R}_+^1)$, $f(\infty) = 0$, $1-1/p-\mu < 0$.

From (2.14) and Theorem 2.2 it follows that

$$\|\tilde{\omega}_k\|_{W_{p,\mu}^{l+k+1}(d_{\pi/2})} \leqslant c\|\omega_k\|_{W_{p,\mu}^{l+k+1-1/p}(\gamma_1)} \leqslant c\|\omega\|_{W_{p,\mu}^{l+1-1/p}(\gamma_1)},$$

where $\tilde{\omega}_k$ is an extension of ω_k to $d_{\pi/2}$. Therefore $(2.18)_1$ implies that

$$\int_0^\infty \left|\frac{\partial^{m-r}\tilde{\omega}_k}{\partial x_1^{m-r}}\right|^2 x_1^{p(\mu-r)}dx_1 \leqslant c\int_0^\infty \left|\frac{\partial^m \tilde{\omega}_k}{\partial x_1^m}\right|^p x_1^{p\mu}dx_1,$$

$r \leqslant m$, $m = k+s-1 \leqslant l+k+1$, so $\tilde{\omega}_k \in V^m_{p,\mu}(d_\vartheta)$, where (2.20) for $r \leqslant s-2$ and (2.19) for $r \leqslant s-1$ have been used, because

$$1-\frac{1}{p}-(\mu-(s-2)) < 0 < 1-\frac{1}{p}-(\mu-(s-1)).$$

Hence Theorem 2.3 implies (2.16). ∎

LEMMA 2.5. *Suppose $v_\alpha \in W^{l-j}_{p,\mu}(d_\vartheta)$, where $1 < p$, μ are real, s, l are integer such that $\mu+2/p = s \geqslant 1$, $j+s \leqslant l$, and $\alpha = (\alpha_1, \alpha_2)$ is a multiindex with $|\alpha| = \alpha_1+\alpha_2 = j$. Then there exists a function $v \in W^l_{p,\mu}(d_\vartheta) \cap V^{j+s-1}_{p,\mu}(d_\vartheta)$ such that*

$$\|v\|_{W^l_{p,\mu}(d_\vartheta)} \leqslant c \sum_{|\alpha|=j} \|v_\alpha\|_{W^{l-j}_{p,\mu}(d_\vartheta)}, \tag{2.21}$$

$$\|v\|_{V^{l+s-1}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{|\alpha|=j} \|v_\alpha\|_{W^{l-j}_{p,\mu}(d_\vartheta)}, \tag{2.22}$$

$$\|D^\alpha v - v_\alpha\|_{V^s_{p,\mu}(d_\vartheta)} \leqslant c \sum_{|\alpha|=j} \|v_\alpha\|_{W^{l-j}_{p,\mu}(d_\vartheta)}. \tag{2.23}$$

Proof. Let $v_\alpha = v_{(k,j-k)}$, $k = 0,\ldots,j$. We construct a function $\omega_{(k,j-k)}$

$$\left.\frac{d^k}{dx_1^k}\omega_{(k,j-k)} - v_{(k,j-k)}\right|_{\gamma_1} \in V^{s-1/p}_{p,\mu}(\gamma_1),$$

$$\left.\frac{d^i}{dx_1^i}\omega_{(k,j-k)}\right|_{x_1=0} = 0, \quad i \leqslant k-1,$$

in the form

$$\omega_{(k,j-k)}(x_1) = \frac{1}{(k-1)!}\int_0^{x_1} (x_1-\tau)^{k-1} v_{(k,j-k)}(\tau, 0)\big(1+K(\tau)\big)d\tau, \tag{2.24}$$

where $K(\tau)$ is described in (2.17). Hence Lemma 2.4 implies

$$\begin{aligned} &\|\omega_{(k,j-k)}\|_{W^{l-j+k-1/p}_{p,\mu}(\gamma_1)} + \|\omega_{(k,j-k)}\|_{V^{k+s-1-1/p}_{p,\mu}(\gamma_1)} \\ &\quad + \left\|\frac{d^k\omega_{(k,j-k)}}{dx_1^k} - v_{(k,j-k)}\right\|_{V^{s-1/p}_{p,\mu}(\gamma_1)} \leqslant c\|v_{(k,j-k)}\|_{W^{l-j-1/p}_{p,\mu}(\gamma_1)} \\ &\quad \leqslant c\|v_{(k,j-k)}\|_{W^{l-j}_{p,\mu}(d_\vartheta)}. \end{aligned} \tag{2.25}$$

Now we seek a function v such that

$$\left.\frac{d^{j-k}}{dx_2^{j-k}}v\right|_{\gamma_1} \omega_{(k,j-k)}, \quad k = 0,\ldots,j,$$

which can be constructed similarly to [4, Part 2, Ch. 2, § 6] in the following form

$$v(x_1, x_2) = \sum_{k=0}^{j} \frac{1}{(j-k)!} x_2^{j-k} \int_{\mathbf{R}^1} K_k(t)\omega_{(k,j-k)}(x_1+tx_2)dt, \tag{2.26}$$

where K_k, $k = 0,\ldots, j$, are smooth functions with compact support such that $\int_{\mathbf{R}^1} K_k(t)dt = 1$, $\int_{\mathbf{R}^1} t^m K_k(t)dt = 0$, for $m = 1,\ldots, k$. Then

$$\frac{d^k}{dx_1^k}\frac{d^{j-k}}{dx_2^{j-k}} v\bigg|_{\gamma_1} = v_{(k,j-k)}(x_1, 0)(1+K(x_1)).$$

Therefore for $|\alpha| = j$ we have

$$\begin{aligned}\|D^\alpha v - v_\alpha\|_{V^s_{p,\mu}(d_\vartheta)} &\leqslant c\|D^\alpha v - v_\alpha\|_{V^{s-1/p}_{p,\mu}(\gamma_1)} \leqslant c\|K(\cdot)v_\alpha(\cdot, 0)\|_{V^{s-1/p}_{p,\mu}(\gamma_1)}\\ &\leqslant c\|K(|\cdot|)v_\alpha\|_{V^s_{p,\mu}(d_\vartheta)} \leqslant c\|v_\alpha\|_{W^s_{p,\mu}(d_\vartheta)},\end{aligned}$$

where Theorem 2.3 and the properties of the space $V^s_{p,\mu}(d_\vartheta)$ have been used. Hence (2.23) is proved. The inequality (2.21) follows from (2.26) and (2.25). From (2.25) we see that

$$\frac{d^{j-k}}{dx_2^{j-k}} v\bigg|_{\gamma_1} \in V^{k-1/p}_{p,\mu}(\gamma_1).$$

Therefore Lemma 2.3 gives (2.22). ■

LEMMA 2.6. *Suppose $u \in L^k_{p,\mu}(\mathscr{D}_\vartheta)$, p, $\mu \in \mathbf{R}$, $k \in \mathbf{Z}$, $p > 1$, $\mu+2/p = s \geqslant 1$, $s \in \mathbf{Z}$. Let $D^j_{x_2}u|_{\Gamma_1} \in V^{k-j-1/p}_{p,\mu}(\Gamma_1)$, $j = 0,\ldots, k-s$. Then $u \in V^k_{p,\mu}(\mathscr{D}_\vartheta)$ and*

$$\|u\|_{V^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c\Big(\|u\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)} + \sum_{j=0}^{k-s} \|D^j_{x_2}u\|_{V^{k-j-1/p}_{p,\mu}(\Gamma_1)}\Big). \tag{2.27}$$

Proof. The inequality (2.2) implies

$$\begin{aligned}\sum_{i=k-s+1}^{k} \|u\|^p_{L^i_{p,\mu+i-k}(\mathscr{D}_\vartheta)} &\leqslant c \sum_{i=k-s+1}^{k} \int_{\mathscr{D}_\vartheta} |D^i u|^p r^{p(s+i-k)-1} dr\,d\varphi\,dz\\ &\leqslant c\int_{\mathscr{D}_\vartheta} |D^k u|^p r^{ps-1} dr\,d\varphi\,dz.\end{aligned}$$

On the other hand, we have

$$\sum_{i+j+\sigma\leqslant k-s} \int_{\mathscr{D}_\vartheta} |D^i_{x_1}D^j_{x_2}D^\sigma_z u|^p r^{p(\mu+i+j+\sigma-k)+1} dr\,d\varphi\,dz \leqslant c\sum_{j=0}^{k-s} \|D^j_{x_2}u\|^p_{V^{k-j-1/p}_{p,\mu}(\Gamma_1)},$$

and

$$p(\mu+i+j+\sigma-k)+1 = p(s+i+j+\sigma-k)-1 \leqslant -1,$$

so all derivatives $V^k_{p,\mu}(\mathscr{D}_\vartheta)$ with corresponding wedges appear. Therefore (2.27) is valid. ■

LEMMA 2.7. *Let $l', l \in \boldsymbol{N} \cup \{0\}$, $l' \leqslant l$. Moreover let p, $\mu \in \boldsymbol{R}$, $p > 1$, $\mu + 2/p = s \geqslant 1$, $s \in \boldsymbol{Z}$. Then for arbitrary functions $v_{\alpha'} \in W^{l-j'}_{p,\mu}(\mathscr{D}_\vartheta)$, $\alpha' = (\alpha_1, \alpha_2)$, $\alpha_i \in \boldsymbol{Z}$, $\alpha_i \geqslant 0$, $|\alpha'| = \alpha_1 + \alpha_2 \leqslant l'$, there exists a function $v \in W^l_{p,\mu}(\mathscr{D}_\vartheta)$ such that*

$$\|v\|_{W^l_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{j'=0}^{l'} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|_{W^{l-j'}_{p,\mu}(\mathscr{D}_\vartheta)}, \tag{2.28}$$

$$\sum_{|\beta'|=i'} \int_{\boldsymbol{R}^1} \|D_z^{\alpha_3}(D^{\beta'} v - v_{\beta'})\|^p_{V^s_{p,\mu}(d_\vartheta)} dz \leqslant c \sum_{j'=0}^{l'} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|^p_{W^{l-j'}_{p,\mu}(\mathscr{D}_\vartheta)}, \tag{2.29}$$

where $\beta' = (\beta_1, \beta_2)$, $\beta_i \in \boldsymbol{Z}$, $\beta_i \geqslant 0$, $i' \leqslant l'$, $i' + \alpha_3 \leqslant l - s$,

$$\|D_z^{\alpha_3}(D^{\beta'} v - v_{\beta'})\|_{L_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{j'=0}^{l'} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|_{W^{l-j'}_{p,\mu}(\mathscr{D}_\vartheta)}, \tag{2.30}$$

where $|\beta'| \leqslant k' \leqslant l'$, $\alpha_3 + l' = l$,

$$D^{\alpha'} v|_{x=0} = 0, \quad |\alpha'| \leqslant l' - 1. \tag{2.31}$$

Proof. To prove the statement it is sufficient to consider smooth functions with compact support with respect to z. Hence we can use the results of Lemma 2.5 for derivatives $D_z^{\alpha_3} v_{\alpha'}$. From (2.21) after integrating over z one has

$$\int_{\boldsymbol{R}^1} \|D_z^{\alpha_3} v\|^p_{W^{l-\alpha_3}_{p,\mu}(d_\vartheta)} dz \leqslant c \sum_{|\alpha'|=j'} \int_{\boldsymbol{R}^1} \|D_z^{\alpha_3} v_{\alpha'}\|^p_{W^{l-j'-\alpha_3}_{p,\mu}(d_\vartheta)} dz.$$

The right-hand side is estimated by

$$c \sum_{j'=0}^{l' \leqslant l} \sum_{\alpha_3=0}^{l-j'} \sum_{|\alpha'|=j'} \int_{\boldsymbol{R}^1} \|D_z^{\alpha_3} v_{\alpha'}\|^p_{W^{l-j'-\alpha_3}_{p,\mu}(d_\vartheta)} dz$$

$$= c \sum_{j'=0}^{l' \leqslant l} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|^p_{W^{l-j'}_{p,\mu}(\mathscr{D}_\vartheta)} \equiv cX^p.$$

Therefore one has

$$\sum_{\alpha_3=0}^{l} \int_{\boldsymbol{R}^1} \|D_z^{\alpha_3} v\|^p_{W^{l-\alpha_3}_{p,\mu}(d_\vartheta)} dz \leqslant cX^p,$$

so (2.28) is satisfied.

From (2.23) for $j' + \alpha_3 \leqslant l - s$ we have

$$\sum_{|\alpha'|=j'} \int_{\boldsymbol{R}^1} \|D_z^{\alpha_3} D^{\alpha'} v - D_z^{\alpha_3} v_{\alpha'}\|^p_{V^s_{p,\mu}(d_\vartheta)} dz \leqslant c \sum_{|\alpha'|=j'} \int_{\boldsymbol{R}^1} \|D_z^{\alpha_3} v_{\alpha'}\|^p_{W^{l-j'-\alpha_3}_{p,\mu}(d_\vartheta)} dz \leqslant cX^p,$$

Therefore (2.29) is satisfied. Using (2.28), one gets (2.30). The equalities (2.31) one obtains by (2.22). ■

Remark 2.1 (see [13, p. 21]). Lemma 2.7 will be used in the case $l' = k - s$, $l = k$. In this case after the transformation $x \to \lambda^{-1} x$, $v \to v(\lambda^{-1} x)$,

$v_{\alpha'} \to \lambda^{-|\alpha'|} v_{\alpha'}(\lambda^{-1} x)$ and $\lambda \to 0$, instead of (2.28)–(2.30) we get

(2.32) $$\|v\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{j'=0}^{k-s} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|_{L^{k-j'}_{p,\mu}(\mathscr{D}_\vartheta)},$$

(2.33) $$\sum_{|\beta'|=i'} \int_{R^1} \|D_z^{\alpha_3}(D^{\beta'} v - v_{\beta'})\|^p_{V^s_{p,\mu}(d_\vartheta)} dz \leqslant c \sum_{j'=0}^{k-s} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|^p_{L^{k-j'}_{p,\mu}(\mathscr{D}_\vartheta)}, \quad i'+\alpha_3 = k-s,$$

(2.34) $$\|D_z^{\alpha_3}(D^{\beta'} v - v_{\beta'})\|_{L_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{j'=0}^{k-s} \sum_{|\alpha'|=j'} \|v_{\alpha'}\|_{L^{k-j'}_{p,\mu}(\mathscr{D}_\vartheta)},$$

where $|\beta'| = k'$, $\alpha_3 + k' = k$, $k' \leqslant k-s$.

THEOREM 2.6 ([14]). *Let $l, s \notin Z$, $0 \leqslant s \leqslant l$. Assume that for each α with $|\alpha| = \alpha_1+\alpha_2 < s$ we are given a function $\varphi_\alpha \in C^{s-|\alpha|}(M)$, $|\alpha| = \alpha_1+\alpha_2 < s$, with compact support. Then there exists a function $u \in C^l_s(\mathscr{D}_\vartheta)$ with compact support such that $D^\alpha_{x'} u|_M = \varphi_\alpha$ for all α and*

(2.36) $$\langle u \rangle^l_{s,\mathscr{D}_\vartheta} \leqslant c \sum_\alpha [\varphi_\alpha]_M^{(s-|\alpha|)}.$$

Let L be one of the edges of Ω which is the intersection of two boundary surfaces S_1, S_2 and let $\Omega_d(\xi) = K_d(\xi) \cap \Omega$ for $\xi \in L$.

THEOREM 2.7. *There exists a number d such that $\Omega_d(\xi)$ can be transformed onto the dihedral angle $\mathscr{D}_{\vartheta(\xi)}$ between the half-planes $T_\xi S_1$ and $T_\xi S_2$ ($\vartheta(\xi)$ is the angle between $T_\xi S_1$ and $T_\xi S_2$) by a transformation $T \in C^{l+2}_s(\Omega)$ ($S_\nu \in C^{l+2}$, $\nu = 1, 2$).*

3. Statement of the problem (1.1)–(1.3)

The solutions of the problem (1.1)–(1.3) will be sought in the form

(3.1) $$v = \nabla\varphi + u,$$

where φ is the solution of the Neumann problem

(3.2) $$\Delta\varphi = 0, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = b,$$

where $\partial/\partial n = \bar{n}\cdot\nabla$, and u is a solution of the problem

(3.3) $$\operatorname{rot} u = \omega,$$

(3.4) $$\operatorname{div} u = 0,$$

(3.5) $$u_n|_{\partial\Omega} = 0.$$

By [1, Lemma 1] (3.4) and (3.5) there is a vector e such that

(3.6) $$u = \operatorname{rot} e, \quad \operatorname{div} e = 0, \quad e_\tau|_{\partial\Omega} = 0,$$

where $e_\tau = e\cdot\bar{\tau}$, $\bar{\tau}$ in any tangent vector to $\partial\Omega$.

The vector e is defined as

(3.7) $$e = e^1 + e^2,$$

where

$$e^1(x) = \frac{1}{4\pi}\operatorname{rot}\int_\Omega \frac{u(y)}{|x-y|}dy, \qquad e^2 = \nabla\psi$$

and ψ is the solution of the Dirichlet problem

(3.8) $$\Delta\psi = 0, \quad \psi|_{\partial\Omega} = -\psi_0,$$

where $e^1_\tau|_{\partial\Omega} = \psi_{0,\tau}$. Using (3.6) we can replace the problem (3.3)–(3.5) by

(3.9) $$-\Delta e = \omega, \quad e_\tau|_{\partial\Omega} = 0, \quad \operatorname{div} e|_{\partial\Omega} = 0,$$

where we have taken into account that $\Delta \operatorname{div} e = 0$, $\operatorname{div} e|_{\partial\Omega} = 0$ imply $\operatorname{div} e = 0$.

In a curvilinear system of coordinates (τ_1, τ_2, n) in a neighbourhood of $\partial\Omega$ the vector e can be written in the form $e = \sum_{\mu=1}^{2} e_\mu\bar{\tau}_\mu + e_n\bar{n}$. Therefore (3.9) can be replaced by

(3.10) $$-\Delta e = \omega, \quad e_\tau|_{\partial\Omega} = 0, \quad (\bar{n}\cdot\nabla e_n + e_n \operatorname{div}\bar{n})|_{\partial\Omega} = 0.$$

To consider the problem (3.10) in domains with edges we must examine it in d_ϑ and $\mathscr{D}_\vartheta$. In $\mathscr{D}_\vartheta$, (3.10) reduces to the following two problems:

(3.11) $$\begin{gathered} -\Delta e_i = \omega_i, \quad i = 1, 2, \\ e_1|_{\Gamma_1} = 0, \quad (e_1\cos\vartheta + e_2\sin\vartheta)|_{\Gamma_2} = 0, \\ \left.\frac{\partial e_2}{\partial n}\right|_{\Gamma_1} = 0, \quad \left.\left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\right|_{\Gamma_2} = 0. \end{gathered}$$

and

(3.12) $$-\Delta e_3 = \omega_3, \quad e_3|_{\Gamma_1\cup\Gamma_2} = 0.$$

In the subsequent considerations it will also be necessary to analyse the nonhomogeneous problem corresponding to (3.11):

(3.13) $$\begin{gathered} -\Delta e_i = f_i, \quad i = 1, 2, \\ e_1|_{\Gamma_1} = \Phi_1, \quad (e_1\cos\vartheta + e_2\sin\vartheta)|_{\Gamma_2} = \Phi_2, \\ \left.\frac{\partial e_2}{\partial n}\right|_{\Gamma_1} = \Psi_1, \quad \left.\left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\right|_{\Gamma_2} = \Psi_2. \end{gathered}$$

Moreover, we shall have to consider the problem (3.13) in d_ϑ:

$$(3.14)\qquad \begin{gathered} -\Delta' e_i = h_i, \quad i = 1, 2, \\ e_1|_{\gamma_1} = \varphi_1, \quad (e_1 \cos\vartheta + e_2 \sin\vartheta)|_{\gamma_2} = \varphi_2, \\ \frac{\partial e_2}{\partial n}\bigg|_{\gamma_1} = \psi_1, \quad \left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2} = \psi_2. \end{gathered}$$

Finally, we shall also examine the homogeneous problem corresponding to (3.14) in d_ϑ:

$$(3.15)\qquad \begin{gathered} -\Delta' e_i = f_i, \quad i = 1, 2, \\ e_1|_{\gamma_1} = 0, \quad (e_1 \cos\vartheta + e_2 \sin\vartheta)|_{\gamma_2} = 0, \\ \frac{\partial e_2}{\partial n}\bigg|_{\gamma_1} = 0, \quad \left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2} = 0. \end{gathered}$$

4. The problem (3.14)

In this section we consider the problem (3.14) in d_ϑ. First we calculate the eigenfunctions and eigenvalues of the homogeneous problem (3.14). Secondly we formulate solvability results for this problem in $V^l_{p,\mu}(d_\vartheta)$ and next in $W^l_{p,\mu}(d_\vartheta)$. The first result is a particular case of results from [11], where Kondrat'ev's results [7] are generalized. The second is a modification of Theorem 3.2 of [13] for the Neumann problem to the case of the problem (3.14).

Moreover, by using [11] the existence and smoothness properties of solutions of the problem (3.14) in weighted Hölder spaces $\overset{\circ}{C}{}^l_s(d_\vartheta)$ are formulated.

LEMMA 4.1 *The homogeneous problem* (3.15) *has eigenvalues*

$$(4.1)\qquad \lambda_m^{\mp} = m\frac{\pi}{\vartheta} \mp 1, \quad m = 0, \mp 1, \mp 2, \ldots,$$

and corresponding eigenvectors

$$(4.2)\qquad (e_1^m, e_2^m) = (a_m r^{\lambda_m} \sin\lambda_m\varphi,\ b_m r^{\lambda_m}\cos\lambda_m\varphi),$$

where $a_m = b_m$ *for* $\lambda_m = m\pi/\vartheta - 1$, $a_m = -b_m$ *for* $\lambda_m = m\pi/\vartheta + 1$. *The number* $\lambda = 0$ *is an eigenvalue for* $\vartheta = \pi$ *only, and the corresponding eigenvector has the form:* $(e_1, e_2) = (0, \alpha)$, *where* α *is an arbitrary number.*

For further considerations it is necessary to know the smallest positive eigenvalues which are:

$$(4.3)\qquad \begin{gathered} 1 \ \textit{for}\ \vartheta \leqslant \frac{\pi}{2}, \quad \frac{\pi}{\vartheta} - 1 \ \textit{for}\ \frac{\pi}{2} \leqslant \vartheta \leqslant \pi, \\ 1 - \frac{\pi}{\vartheta} \ \textit{for}\ \pi \leqslant \vartheta \leqslant \frac{3}{2}\pi, \quad \frac{2\pi}{\vartheta} - 1 \ \textit{for}\ \frac{3}{2}\pi \leqslant \vartheta \leqslant 2\pi. \end{gathered}$$

Proof. In polar coordinates, (3.15) with $f_i = 0$, $i = 1, 2$, has the form

$$\frac{\partial^2 e_i}{\partial r^2} + \frac{1}{r}\frac{\partial e_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 e_i}{\partial \varphi^2} = 0, \quad i = 1, 2,$$

$$e_1|_{\gamma_1} = 0, \quad (e_1 \cos\vartheta + e_2 \sin\vartheta)|_{\gamma_2} = 0,$$

$$\left.\frac{\partial e_2}{\partial \varphi}\right|_{\gamma_1} = 0, \quad \left.\left(\frac{\partial e_1}{\partial \varphi}\sin\vartheta - \frac{\partial e_2}{\partial \varphi}\cos\vartheta\right)\right|_{\gamma_2} = 0.$$

Introducing the variable $\tau = \ln 1/r$ we obtain

$$\begin{gathered} e_{i,\tau\tau} + e_{i,\varphi\varphi} = 0, \quad i = 1, 2, \\ e_1|_{\gamma_1} = 0, \quad (e_1 \cos\vartheta + e_2 \sin\vartheta)|_{\gamma_2} = 0, \\ \left.\frac{\partial e_2}{\partial \varphi}\right|_{\gamma_1} = 0, \quad \left.\left(\frac{\partial e_1}{\partial \varphi}\sin\vartheta - \frac{\partial e_2}{\partial \varphi}\cos\vartheta\right)\right|_{\gamma_2} = 0. \end{gathered} \tag{4.4}$$

The Fourier transform of $e_i(\tau, \varphi)$ with respect to τ satisfies

$$e_i(\tau, \varphi) = \frac{1}{2\pi}\int_{R^1} \tilde{e}_i(\lambda, \varphi) e^{i\lambda\tau} d\lambda$$

so instead of (4.4) we have

$$\tilde{e}_{i,\varphi\varphi} - \lambda^2 \tilde{e}_i = 0, \quad i = 1, 2, \tag{4.5}$$

$$\tilde{e}_1|_{\varphi=0} = 0, \quad \left.\frac{\partial \tilde{e}_2}{\partial \varphi}\right|_{\varphi=0} = 0, \tag{4.6}$$

$$(\tilde{e}_1 \cos\vartheta + \tilde{e}_2 \sin\vartheta)|_{\varphi=\vartheta} = 0, \quad \left.\left(\frac{\partial \tilde{e}_1}{\partial \varphi}\sin\vartheta - \frac{\partial \tilde{e}_2}{\partial \varphi}\cos\vartheta\right)\right|_{\varphi=\vartheta} = 0. \tag{4.7}$$

We look for solutions of the problem (4.5)–(4.7) in the form

$$\tilde{e}_j = a_j \sin i\lambda\varphi + b_j \cos i\lambda\varphi, \quad j = 1, 2. \tag{4.8}$$

From (4.6) we obtain $b_1 = 0$, $a_2 = 0$. Therefore

$$\tilde{e}_1 = \alpha_1 \sin i\lambda\varphi, \quad \tilde{e}_2 = \alpha_2 \cos i\lambda\varphi. \tag{4.9}$$

From (4.7) and (4.9) we have

$$\begin{gathered} \alpha_1 \sin i\lambda\vartheta \cos\vartheta + \alpha_2 \cos i\lambda\vartheta \sin\vartheta = 0, \\ i\lambda(\alpha_1 \cos i\lambda\vartheta \sin\vartheta + \alpha_2 \sin i\lambda\vartheta \cos\vartheta) = 0. \end{gathered} \tag{4.10}$$

Now (4.10) implies

$$\sin^2 i\lambda\vartheta \cos^2\vartheta - \cos^2 i\lambda\vartheta \sin^2\vartheta = 0, \tag{4.11}$$

hence $\sin i\lambda\vartheta \cos\vartheta = \mp\cos i\lambda\vartheta \sin\vartheta$, so $\sin(i\lambda \pm 1)\vartheta = 0$. Therefore we have $i\lambda \pm 1 = m\pi/\vartheta$. Thus (4.1) and (4.2) have been obtained.

Finally, consider the case $i\lambda = 0$. From (4.5) we have $\tilde{e}_i = \alpha_i + \beta_i \varphi$, $i = 1, 2$. Hence, from (4.6) we have $\alpha_1 = 0$, $\beta_2 = 0$. Moreover, (4.7) implies $\beta_1 \vartheta \cos\vartheta + \alpha_2 \sin\vartheta = 0$, $\beta_1 \sin\vartheta = 0$. Therefore for $\vartheta \neq \pi$ we have $\alpha_2 = \beta_1 = 0$, but for $\vartheta = \pi$ we have $\beta_1 = 0$, $\alpha_2 \neq 0$. This finishes the proof. ■

THEOREM 4.1 ([11]). *Let $p \in (1, \infty)$, $k, \tilde{k} \in \boldsymbol{N} \cup \{0\}$, $\mu, \tilde{\mu} \in \boldsymbol{R}$ and*

$$\frac{2}{p'} + k - \mu \neq \lambda_i, \tag{4.12}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \lambda_i = \frac{i\pi}{\vartheta} - 1 \quad or \quad \lambda_i = \frac{i\pi}{\vartheta} + 1, \quad i = 0, \mp 1, \mp 2, \ldots$$

Then, for arbitrary $h_i \in V^k_{p,\mu}(d_\vartheta)$, $\varphi_i \in V^{k+2-1/p}_{p,\mu}(\gamma_i)$, $\psi_i \in V^{k+1-1/p}_{p,\mu}(\gamma_i)$, $i = 1, 2$, there exists a unique solution $e_i \in V^{k+2}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, of the problem (3.14), *and*

$$\sum_{i=1}^{2} \|e_i\|_{V^{k+2}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \left(\|f_i\|_{V^k_{p,\mu}(d_\vartheta)} + \|\varphi_i\|_{V^{k+2-1/p}_{p,\mu}(\gamma_i)} + \|\psi_i\|_{V^{k+2-1/p}_{p,\mu}(\gamma_i)} \right). \tag{4.13}$$

Let $\bar{e} \in V^{k+2}_{p,\mu}(d_\vartheta)$, $\tilde{\bar{e}} \in V^{\tilde{k}+2}_{p,\tilde{\mu}}(d_\vartheta)$, $\bar{e} = (e_1, e_2)$, $\tilde{\bar{e}} = (\tilde{e}_1, \tilde{e}_2)$ be two solutions of the problem (3.14) *with the same right-hand sides such that $\bar{h} \in V^k_{p,\mu}(d_\vartheta) \cap V^{\tilde{k}}_{p,\tilde{\mu}}(d_\vartheta)$, $\varphi_i \in V^{k+2-1/p}_{p,\mu}(\gamma_i) \cap V^{\tilde{k}+2-1/p}_{p,\tilde{\mu}}(\gamma_i)$, $\psi_i \in V^{k+1-1/p}_{p,\mu}(\gamma_i) \cap V^{\tilde{k}+1-1/p}_{p,\tilde{\mu}}(\gamma_i)$, $i = 1, 2$, $\bar{h} = (h_1, h_2)$ and $k - \mu < \tilde{k} - \tilde{\mu}$, $2/p' + k - \mu \neq \lambda_i$, $2/p' + \tilde{k} - \tilde{\mu} \neq \lambda_i$, then*

$$\begin{aligned} e_1 &= \tilde{e}_1 + \sum_q a_q r^{\lambda_q} \sin \lambda_q \varphi, \\ e_2 &= \tilde{e}_2 + \sum_q b_q r^{\lambda_q} \cos \lambda_q \varphi, \end{aligned} \tag{4.14}$$

where

$$a_q = b_q \ for \ \lambda_q = \frac{q\pi}{\vartheta} - 1, \quad a_q = -b_q \ for \ \lambda_q = \frac{q\pi}{\vartheta} + 1$$

and the summation in (4.14) *is taken over all q such that $2/p' + k - \mu < \lambda_q < 2/p' + \tilde{k} - \tilde{\mu}$.*

Using [12], the parameters a_q, b_q can be calculated.

THEOREM 4.2 (11]). *Let $l, \tilde{l}, s, \tilde{s} \notin \boldsymbol{Z}$ and*

$$s \neq \lambda_i, \tag{4.15}$$

where $\lambda_i = i\pi/\vartheta - 1$ or $\lambda_i = i\pi/\vartheta + 1$, $i = 0, \mp 1, \mp 2, \ldots$ Then for arbitrary $h_i \in \mathring{C}^l_{s-2}(d_\vartheta)$, $\varphi_i \in \mathring{C}^{l+2}_s(\gamma_i)$, $\psi_i \in \mathring{C}^{l+1}_{s-1}(\gamma_i)$, $i = 1, 2$, there exists a unique solution *of the problem* (3.14) *such that* $e_i \in \mathring{C}_s^{l+2}(d_\vartheta)$, $i = 1, 2$, *and*

(4.16) $$\sum_{i=1}^{2} |e_i|_{\mathring{C}_s^{l+2}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} (|h_i|_{\mathring{C}_{s-2}^{l}(d_\vartheta)} + |\varphi_i|_{\mathring{C}_s^{l+2}(\gamma_i)} + |\psi_i|_{\mathring{C}_{s-1}^{l+1}(\gamma_i)}).$$

Let $e_i \in \mathring{C}_s^{l+2}(d_\vartheta)$, $\tilde{e}_i \in \mathring{C}_{\tilde{s}}^{\tilde{l}+2}(d_\vartheta)$, $i = 1, 2$, *be two solutions of the problem* (3.14) *with the same right-hand sides such that* $h_i \in \mathring{C}_{s-2}^{l}(d_\vartheta) \cap \mathring{C}_{\tilde{s}-2}^{\tilde{l}}(d_\vartheta)$, $\varphi_i \in \mathring{C}_s^{l+2}(\gamma_i) \cap \mathring{C}_{\tilde{s}}^{\tilde{l}+2}(\gamma_i)$, $\psi_i \in \mathring{C}_{s-1}^{l+1}(\gamma_i) \cap \mathring{C}_{\tilde{s}-1}^{\tilde{l}+1}(\gamma_i)$, $i = 1, 2$, $s < \tilde{s}$, s, $\tilde{s}$ *satisfy* (4.15). *Then* e_i, $\tilde{e}_i$, $i = 1, 2$, *satisfy* (4.14), *where the summation is taken over all* q *such that* $s < \lambda_q < \tilde{s}$.

In the case $\vartheta = \pi/m$, $2 \leqslant m$ natural to guarantee the existence of solutions of the problem (3.14) in the cases of Theorems 4.1 and 4.2 some compatibility conditions on the right-hand side functions of (3.14) at $x' = 0$ must be imposed.

LEMMA 4.2. *Let* $h_i(x')$, $\psi_i(x')$, $\varphi_i(x')$, $i = 1, 2$, *be homogeneous polynomials of degree* $l-2$, $l-1$, l, *respectively, i.e.*:

$$h_i(x') = \sum_{i_1+i_2=l-2} h_{i_1 i_2}^{i} x_1^{i_1} x_2^{i_2}, \qquad \psi_i(x') = b_{il-1}|x'|^{l-1}, \qquad \varphi_i(x') = a_{il}|x'|^{l},$$

where $h_{i_1 i_2}^{i}$, b_{il-1}, a_{il} *are constants. If* $l \neq s\pi/\vartheta \mp 1$, $l \geqslant 2$, $s = 0, \mp 1, \mp 2, \ldots$, *then the problem* (3.14) *has a unique solution in the form of a homogeneous polynomial of degree* l:

$$e_i = \sum_{i_1+i_2=l} e_{i_1 i_2}^{i} x_1^{i_1} x_2^{i_2}, \qquad i = 1, 2.$$

For $l = 0$ *we get a unique polynomial of order zero determined by the Dirichlet type conditions of* (3.14) *only. In this case for* $\vartheta \neq \pi$ *we have* $e_{00}^1 = a_{10}$, $e_{00}^2 = [a_{20} - a_{10}\cos\vartheta]/\sin\vartheta$. *For* $l = 1$ *the following compatibility condition must be satisfied*

(4.17) $$a_{11} + b_{20} = a_{21} + b_{10}$$

and then for $\vartheta \neq k\pi/2$, $k = 1, \ldots, 4$, *one has*

(4.18) $$e_{10}^1 = a_{11}, \quad e_{01}^2 = -b_{10}, \quad e_{01}^1 + e_{10}^2 = \frac{1}{\sin 2\vartheta}[a_{21} + b_{20} + \cos 2\vartheta(a_{11} + b_{10})]$$

In the case $\vartheta = \pi/m$, $m \in N$, $m \geqslant 2$, *to ensure the existence of solutions of the problem* (4.19) *below some compatibility conditions are necessary. Then a solution generally depends on arbitrary parameters.*

Proof. Any homogeneous polynomial of degree m has the form

$$q_m(x') = r^m \sum_{j=0}^{[m/2]} [a_j \cos(m-2j)\varphi + b_j \sin(m-2j)\varphi],$$

where a_j, b_j are constants. Let $e_i = r^l p_{il}(\varphi)$, $i = 1, 2$, $l \geqslant 2$, where $p_{il}(\varphi)$ are solutions of the problem

$$\frac{d^2 p_{il}}{d\varphi^2} + l^2 p_{il} = \sum_{j=0}^{[(l-2)/2]} (\alpha_{ij}\cos(l-2-2j)\varphi + \beta_{ij}\sin(l-2-2j)\varphi), \qquad i = 1, 2,$$

$$\text{(4.19)} \qquad p_{1l}|_{\varphi=0} = a_{1l}, \qquad (p_{1l}\cos\vartheta + p_{2l}\sin\vartheta)|_{\varphi=\vartheta} = a_{2l},$$

$$\left.\frac{\partial p_{2l}}{\partial\varphi}\right|_{\varphi=0} = -b_{1l-1}, \qquad \left.\left(\frac{\partial p_{1l}}{\partial\varphi}\sin\vartheta - \frac{\partial p_{2l}}{\partial\varphi}\cos\vartheta\right)\right|_{\varphi=\vartheta} = b_{2l-1},$$

where α_{ij}, β_{ij}, $j = 0, \ldots, [(l-2)/2]$, $i = 1, 2$, are constants. We look for a solution of (4.19) in the form

$$p_{il} = \sum_{j=0}^{[(l-2)/2]} (\alpha'_{ij}\cos(l-2-2j)\varphi + \beta'_{ij}\sin(l-2-2j)\varphi) + \gamma_i\cos l\varphi + \delta_i\sin l\varphi, \qquad i = 1, 2.$$

Inserting this into $(4.19)_1$, we obtain

$$\alpha'_{ij}[l^2-(l-2-2j)^2] = \alpha_{ij}, \qquad \beta'_{ij}[l^2-(l-2-2j)^2] = \beta_{ij},$$

where $i = 1, 2, j = 0, \ldots, [(l-2/2]$, so all coefficients α'_{ij}, β'_{ij} can be calculated. Now the boundary conditions $(4.19)_{2,3}$ determine γ_i, δ_i, $i = 1, 2$, because $l \neq m\pi/\vartheta \mp 1$, for any integer m.

For $l = 0$ the proof is trivial. For $l = 1$ from the boundary conditions (4.19) one gets

$$\text{(4.20)} \qquad AZ \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos^2\vartheta & \sin\vartheta\cos\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ 0 & 0 & 0 & -1 \\ -\sin^2\vartheta & \sin\vartheta\cos\vartheta & \sin\vartheta\cos\vartheta & -\cos^2\vartheta \end{bmatrix} Z = Y,$$

where $Z = (e^1_{10}, e^1_{01}, e^2_{10}, e^2_{01})$, $Y = (a_{11}, a_{21}, b_{10}, b_{20})$. Since the vector $(1, -1, -1, 1)$ is a null vector of the matrix in (4.20), one has the compatibility condition (4.17), and a first order polynomial satisfying (4.18) can be found. From (4.18) it follows that its coefficients depend on the arbitrary parameter. This concludes the proof. ∎

LEMMA 4.3. *Let* $g_i \in W^k_{p,\mu}(d_\vartheta)$, $\varphi_i \in W^{k+2-1/p}_{p,\mu}(\gamma_i)$, $\psi_i \in W^{k+1-1/p}_{p,\mu}(\gamma_i)$, $i = 1, 2$, *be such that*

$$\text{(4.21)} \qquad \left.\left[\frac{\partial}{\partial x_1}\varphi_1 - \left(\cos\vartheta\frac{\partial}{\partial x_1} + \sin\vartheta\frac{\partial}{\partial x_2}\right)\varphi_2 - \psi_1 + \psi_2\right]\right|_{x'=0} = 0.$$

Then there exist functions v_i *such that* $D^2 v_i \in W^k_{p,\mu}(d_\vartheta)$, $i = 1, 2$,

$$
\begin{aligned}
f_i &= \Delta' v_i + g_i \in V^k_{p,\mu}(d_\vartheta),\\
\Phi_1 &= \varphi_1 - v_1|_{\gamma_1} \in V^{k+2-1/p}_{p,\mu}(\gamma_1),\\
\Phi_2 &= \varphi_2 - (v_1 \cos\vartheta + v_2 \sin\vartheta)|_{\gamma_2} \in V^{k+2-1/p}_{p,\mu}(\gamma_2),\\
\Psi_1 &= \psi_1 - \frac{\partial v_2}{\partial n}\bigg|_{\gamma_1} \in V^{k+1-1/p}_{p,\mu}(\gamma_1),\\
\Psi_2 &= \psi_2 - \left(\sin\vartheta \frac{\partial v_1}{\partial n} - \cos\vartheta \frac{\partial v_2}{\partial n}\right)\bigg|_{\gamma_2} \in V^{k+1-1/p}_{p,\mu}(\gamma_2)
\end{aligned} \tag{4.22}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{2} \left(\|D^2 v_i\|_{W^k_{p,\mu}(d_\vartheta)} + \|f_i\|_{V^k_{p,\mu}(d_\vartheta)} + \|\Phi_i\|_{V^{k+2-1/p}_{p,\mu}(\gamma_i)} + \|\Psi_i\|_{V^{k+1-1/p}_{p,\mu}(\gamma_i)}\right)\\
&\quad \le c \sum_{i=1}^{2} \left(\|g_i\|_{W^k_{p,\mu}(d_\vartheta)} + \|\varphi_i\|_{W^{k+2-1/p}_{p,\mu}(\gamma_i)} + \|\psi_i\|_{W^{k+1-1/p}_{p,\mu}(\gamma_i)}\right) \equiv cX.
\end{aligned} \tag{4.23}
$$

If $\varphi_i = \psi_i = 0$, $i = 1, 2$, we assume that v_i, $i = 1, 2$, satisfy the homogeneous boundary conditions (3.15).

In the case $\pi/\vartheta = m$, $m \in \mathbf{Z}$, to obtain the existence of solutions of the problems (4.25), (4.29) *below some relations among g_i, φ_i, ψ_i, $i = 1, 2$, at $x' = 0$, must be imposed additionally. The functions v_i, $i = 1, 2$, can be constructed in such a way that they have compact supports.*

Proof. (a) Let $\mu + 2/p = s \notin \mathbf{Z}$. Let us introduce the homogeneous polynomials

$$
\varphi_{iq} = \frac{r^q}{q!}\left(\frac{\partial}{\partial r}\right)^q \varphi_i|_{r=0}, \quad q < k+2-s,\ i = 1, 2,
$$

$$
\psi_{iq} = \frac{r^q}{q!}\left(\frac{\partial}{\partial r}\right)^q \psi_i|_{r=0}, \quad q < k+1-s,\ i = 1, 2, \tag{4.24}
$$

$$
g_{iq} = \sum_{|\alpha|=q} D^\alpha_{x'} g_i|_{x'=0} \frac{x_1^{\alpha_1} x_2^{\alpha_2}}{\alpha_1! \alpha_2!}, \quad q < k-s,\ |\alpha| = \alpha_1 + \alpha_2,\ i = 1, 2.
$$

Then using (4.24) we construct homogeneous polynomials v_{iq}, $i = 1, 2$, satisfying

$$
\begin{aligned}
&-\Delta' v_{iq+2} = g_{iq},\\
&v_{1q}|_{\gamma_1} = \varphi_{1q}, \quad (v_{1q}\cos\vartheta + v_{2q}\sin\vartheta)|_{\gamma_2} = \varphi_{2q},\\
&\frac{\partial v_{2q+1}}{\partial n}\bigg|_{\gamma_1} = \psi_{1q}, \quad \left(\frac{\partial v_{1q+1}}{\partial n}\sin\vartheta - \frac{\partial v_{2q+1}}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2} = \psi_{2q},
\end{aligned} \tag{4.25}
$$

where $q < k-s$. By Lemma 4.2 solutions of the problem (4.25) in the form of homogeneous polynomials exist. To calculate the first order polynomials the compatibility condition (4.21) must be imposed. Moreover the first order polynomials are determined up to an arbitrary parameter.

Let

$$v_i = \sum_{q<k+2-s} v_{iq}\zeta(x'), \qquad i = 1, 2.$$

Then by the Hardy inequality (2.4) and its version for traces of functions the properties (4.22) are satisfied. In (4.22) the arbitrary parameter disappears. The inequality (4.23) for the last three terms on the left-hand side also follows from the Hardy inequalities. Finally, Lemma 4.2, (4.24), (4.25) and embedding theorems [20] imply the estimate for the first term in (4.23).

(b) Let $\mu+2/p = s \in \boldsymbol{Z}$. In this case the construction of v is in two step. First we take polynomials v_{iq}, $i = 1, 2$, satisfying (4.25) for $q \leqslant k-s-1$. Then we define

$$v_i^1 = \sum_{q\leqslant k-s-1} v_{iq}\zeta(x'), \qquad i = 1, 2. \tag{4.26}$$

Introducing

$$f_i^1 = \Delta' v_i^1 + g_i, \qquad i = 1, 2,$$
$$\Phi_1^1 = \varphi_1 - v_1^1|_{\gamma_1}, \qquad \Phi_2^1 = \varphi_2 - (v_1^1\cos\vartheta + v_2^1\sin\vartheta)|_{\gamma_2},$$
$$\Psi_1^1 = \psi_1 - \frac{\partial v_1^1}{\partial n}\bigg|_{\gamma_1}, \qquad \Psi_2^1 = \psi_2 - \left(\frac{\partial v_1^1}{\partial n}\sin\vartheta - \frac{\partial v_2^1}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2}$$

from (4.25) we have

$$\begin{aligned} D_{x'}^j f_i^1|_{x'=0} &= 0, \quad j \leqslant k-s-1,\ i = 1, 2,\\ D_r^j \Phi_i^1|_{r=0} &= 0, \quad j \leqslant k+1-s,\ i = 1, 2,\\ D_r^j \Psi_i^1|_{r=0} &= 0, \quad j \leqslant k-s,\ i = 1, 2. \end{aligned} \tag{4.27}$$

Therefore from (4.24), (4.25) and the embedding theorem from [20] we derive that $D^2 v_i^1 \in W_{p,\mu}^k(d_\vartheta)$, $f_i^1 \in W_{p,\mu}^k(d_\vartheta)$, $\Phi_i^1 \in W_{p,\mu}^{k+2-1/p}(\gamma_i)$, $\Psi_i^1 \in W_{p,\mu}^{k+1-1/p}(\gamma_i)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \left(\|D^2 v_i^1\|_{W_{p,\mu}^k(d_\vartheta)} + \|f_i^1\|_{W_{p,\mu}^k(d_\vartheta)} + \|\Phi_i^1\|_{W_{p,\mu}^{k+2-1/p}(\gamma_i)} + \|\Psi_i^1\|_{W_{p,\mu}^{k+1-1/p}(\gamma_i)}\right) \leqslant cX. \tag{4.28}$$

Now we shall construct functions v_i^2, $i = 1, 2$, such that $v_i = v_i^1 + v_i^2$, $i = 1, 2$, satisfy the assertion of the lemma. First, we introduce functions $v_{i(\alpha)}$,

$(\alpha) = (\alpha_1 \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2 = l$, by the relations

$$
(4.29)\quad
\begin{aligned}
&-\partial_{x_1}^s \partial_2^{l-2-s}(v_{i(20)} + v_{i(02)}) = \partial_{x_1}^s \partial_{x_2}^{l-2-s} f^1, \quad s = 0, \dots, l-2,\\
&\partial_{x_1}^l v_{1(00)} = \partial_{x_1}^l \tilde{\Phi}_1^1,\\
&(\cos\vartheta\partial_{x_1} + \sin\vartheta\partial_{x_2})^l (v_{1(00)}\cos\vartheta + v_{2(00)}\sin\vartheta) = (\cos\vartheta\partial_{x_1} + \sin\vartheta\partial_{x_2})^l \tilde{\Phi}_2^1,\\
&-\partial_{x_1}^{l-1} v_{2(01)} = \partial_{x_1}^{l-1} \tilde{\Psi}_1^1,\\
&(\cos\vartheta\partial_{x_1} + \sin\vartheta\partial_{x_2})^{l-1}[-\sin^2\vartheta v_{1(10)} + \sin\vartheta\cos\vartheta(v_{1(01)} + v_{2(10)}) - \cos^2\vartheta v_{2(01)}]\\
&\qquad = (\cos\vartheta\partial_{x_1} + \sin\vartheta\partial_{x_2})^{l-1} \tilde{\Psi}_2^1,
\end{aligned}
$$

where $\tilde{\Phi}_i^1$, $\tilde{\Psi}_i^1$, $i = 1, 2$, are extensions of Φ_i^1, Ψ_i^1, $i = 1, 2$, to d_ϑ. These extensions are such that (4.27) are satisfied and

$$
(4.30)\quad
\begin{aligned}
&\|\tilde{\Phi}_i^1\|_{W_{p,\mu}^{k+2}(d_\vartheta)} \leqslant c\|\Phi_i^1\|_{W_{p,\mu}^{k+2-1/p}(\gamma_i)}, \quad i = 1, 2\\
&\|\tilde{\Psi}_i^1\|_{W_{p,\mu}^{k+1}(d_\vartheta)} \leqslant c\|\Psi_i^1\|_{W_{p,\mu}^{k+1-1/p}(\gamma_i)}, \quad i = 1, 2.
\end{aligned}
$$

Moreover, we write

$$
\partial_{x_1} v_{i(\alpha\beta)} = v_{i(\alpha+1\beta)}, \quad \partial_{x_2} v_{i(\alpha\beta)} = v_{i(\alpha\beta+1)}, \quad i = 1, 2.
$$

Similarly to [13, p. 20] we can show that the functions $v_{i(\alpha)}$, $|\alpha| = l \geqslant 2$, $i = 1, 2$, can be uniquely calculated from the system of algebraic equations (4.29).

By Lemma 2.5 for $l = |\alpha| = k+2-s$ there exist functions v_i^2, $i = 1, 2$, such that

$$
(4.31)\quad
\begin{aligned}
&\|D^\alpha v_i^2 - v_{i(\alpha)}\|_{V_{p,\mu}^s(d_\vartheta)} \leqslant c \sum_{|\alpha|=k+2-s} \|v_{i(\alpha)}\|_{W_{p,\mu}^s(d_\vartheta)},\\
&\|v_i^2\|_{V_{p,\mu}^{k+1}(d_\vartheta)} \leqslant c \sum_{|\alpha|=k+2-s} \|v_{i(\alpha)}\|_{W_{p,\mu}^s(d_\vartheta)},\\
&\|v_i^2\|_{W_{p,\mu}^{k+2}(d_\vartheta)} \leqslant c \sum_{|\alpha|=k+2-s} \|v_{i(\alpha)}\|_{W_{p,\mu}^s(d_\vartheta)},
\end{aligned}
$$

where $i = 1, 2$. Expressing the functions $v_{i(\alpha)}$, $i = 1, 2$, $|\alpha| = k+2-s$, by the right-hand sides of (4.29) and using (4.30) one gets

$$
(4.32)\quad \sum_{|\alpha|=k+2-s} \|v_{i(\alpha)}\|_{W_{p,\mu}^s(d_\vartheta)} \leqslant cX, \quad i = 1, 2.
$$

By $(4.31)_2$,

$$
(4.33)\quad D_{x'}^\alpha v_{i(\alpha)}^2|_{x'=0} = 0 \quad \text{for } |\alpha| \leqslant k+1-s.
$$

On the other hand, $(4.31)_1$ and the trace theorems imply that

(4.34)
$$\|D^{k-s}(\Delta' v_i^2 + f_i^1)\|_{V^s_{p,\mu}(d_\vartheta)} \leqslant cX, \qquad i = 1, 2$$

(4.35)
$$\|D_r^{k+2-s}(v_1^2 - \Phi_1^1)\|_{V^{s-1/p}_{p,\mu}(\gamma_1)} \leqslant cX,$$
$$\|D_r^{k+2-s}(v_1^2 \cos\vartheta + v_2^2 \sin\vartheta - \Phi_2^1)\|_{V^{s-1/p}_{p,\mu}(\gamma_2)} \leqslant cX,$$
$$\left\|D_r^{k+1-s}\left(\frac{\partial v_1^2}{\partial n} - \Psi_1^1\right)\right\|_{V^{s-1/p}_{p,\mu}(\gamma_1)} \leqslant cX,$$
$$\left\|D_r^{k+1-s}\left(\frac{\partial v_1^2}{\partial n}\sin\vartheta - \frac{\partial v_2^2}{\partial n}\cos\vartheta - \Psi_2^2\right)\right\|_{V^{s-1/p}_{p,\mu}(\gamma_2)} \leqslant cX.$$

Now (4.27), (4.33) Lemma 2.1 and the embedding theorem in [20] imply (4.22) and (4.23), where $v = v^1 + v^2$.

(c) Let $\vartheta = \pi/m$, $m \in \boldsymbol{N}$, $m \geqslant 2$. Let $\mu + 2/p = s \notin \boldsymbol{Z}$. In this case we have solutions of the homogeneous problem corresponding to (4.25) in the form

(4.36)
$$\omega_{1,m}^{(j)\pm} = A_{j,m}^{(\pm)} r^{jm\pm1}\sin(jm\pm1)\varphi,$$
$$\omega_{2,m}^{(j)\pm} = B_{j,m}^{(\pm)} r^{jm\pm1}\cos(jm\pm1)\varphi,$$

where relations between $A_{j,m}^{(\pm)}$ and $B_{j,m}^{(\pm)}$ are described in (4.2) and $j \geqslant 0$ for $+$ and $j \geqslant 1$ for $-$, because we are interested in solutions of the problem (3.14) in a space to which a generalized solution also belongs (see the proof of Theorem 4.3 below). In this case to obtain the functions v_i, $i = 1, 2$, we repeat the considerations from part (a). However, to solve the problem (4.25) some compatibility conditions on the right-hand side functions must be imposed and a solution is determined up to an arbitrary polynomial (4.36) for $jm+1 \leqslant q < k+2-s$ (because we look for solutions in the form of polynomials of degree not greater than q). Let v'_{iq}, $i = 1, 2$, $q < k+2-s$ be a solution of (4.25). Then

$$v_i = \sum_{q<k+2-s} v'_{iq}\zeta(x'), \ i = 1, 2.$$

We not continue as in (a).

Let $\mu + 2/p = s \in \boldsymbol{Z}$. We first find a solution $v_{iq}^{1'}$, $i = 1, 2$, $q \leqslant k+1-s$ of the problem (4.25) (with compatibility conditions on the right-hand side functions) up to an arbitrary polynomial (4.36) for $jm+1 \leqslant q \leqslant k+1-s$. Let

$$v_i^{1'} = \sum_{q\leqslant k+1-s} v_{iq}^{1'}\zeta(x'), \qquad i = 1, 2.$$

Then we find solutions $v_{i(\alpha)}$, $i = 1, 2$, $|\alpha| = k+2-s$, of the problem (4.29) with right-hand side functions satisfying some additional compatibility conditions.

They are determined up to an arbitrary polynomial (4.36) of degree $jm+1 \leq k+2-s$. Constructing a function v_i^2, $i = 1, 2$, as in (b) and letting $v_i = v_i^1 + v_i^2$, $i = 1, 2$, proves the lemma.

To obtain the estimate (4.23) the arbitrary polynomials appearing in v_i, $i = 1, 2$, must be suitably chosen. However, as we shall see in the next lemma this is not important. This concludes the proof. ■

LEMMA 4.4. *Let* $g_i \in L^k_{p,\mu}(d_\vartheta)$, $\varphi_i \in L^{k+2-1/p}_{p,\mu}(\gamma_i)$, $\psi_i \in L^{k+1-1/p}_{p,\mu}(\gamma_i)$, $i = 1, 2$, *be such that* (4.21) *is satisfied. For* $\vartheta = \pi/m$, $m \geq 2$ *natural, some additional compatibility conditions are needed* (see Lemma 4.3 (c)). *Then there exist functions* $v_i \in L^{k+2}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, *such that* (4.22) *is satisfied and*

$$\sum_{i=1}^{2} \left(\|v_i\|_{L^{k+2}_{p,\mu}(d_\vartheta)} + \|f_i\|_{V^k_{p,\mu}(d_\vartheta)} + \|\Phi_i\|_{V^{k+2-1/p}_{p,\mu}(\gamma_i)} + \|\Psi_i\|_{V^{k+1-1/p}_{p,\mu}(\gamma_i)}\right) \leq c \sum_{i=1}^{2} \left(\|g_i\|_{L^k_{p,\mu}(d_\vartheta)} + \|\varphi_i\|_{L^{k+2-1/p}_{p,\mu}(\gamma_i)} + \|\psi_i\|_{L^{k+1-1/p}_{p,\mu}(\gamma_i)}\right). \tag{4.37}$$

Proof. Using the Remark at the end of Chapter 3 in [13] we consider the transformation $v_i(x) \to \lambda^2 v_i(\lambda^{-1} x)$, $g_i(x) \to g_i(\lambda^{-1} x)$, $\varphi_i(x) \to \lambda^2 \varphi_i(\lambda^{-1} x)$, $\psi_i(x) \to \lambda \psi_i(\lambda^{-1} x)$, $\Phi_i(x) \to \lambda^2 \Phi_i(\lambda^{-1} x)$, $\Psi_i(x) \to \lambda \Psi_i(\lambda^{-1} x)$. Then after letting $\lambda \to 0$ Lemma 4.3 implies (4.37). ■

Remark 4.1. The arbitrary polynomials which appeared in the proof of Lemma 4.3 are not important in Lemma 4.4.

LEMMA 4.5. *Let* $u_i \in \mathscr{H}(d_\vartheta)$, $i = 1, 2$, *be such that*

$$u_1|_{\gamma_1} = 0, \quad (u_1 \cos\vartheta + u_2 \sin\vartheta)|_{\gamma_2} = 0, \tag{4.38}$$

and $\sin\vartheta \neq 0$. *Then*

$$\sum_{i=1}^{2} \|u_i\|^2_{L_{2,-1}(d_\vartheta)} \leq c \sum_{i=1}^{2} \|u_i\|_{\mathscr{H}(d_\vartheta)}. \tag{4.39}$$

Proof. From the assumptions of the lemma and [6] one has

$$\|u_1\|_{L_{2,-1}(d_\vartheta)} \leq c \|u_1\|_{\mathscr{H}(d_\vartheta)},$$

$$\|u_1 \cos\vartheta + u_2 \sin\vartheta\|_{L_{2,-1}(d_\vartheta)} \leq c \sum_{i=1}^{2} \|u_i\|_{\mathscr{H}(d_\vartheta)}.$$

Therefore for $\sin\vartheta \neq 0$, (4.39) is satisfied. ■

THEOREM 4.3. *Let* $p, \mu \in \mathbf{R}$, $p > 1$, $\mu \geq 0$, $k \in \mathbf{Z}$. *Let* $f_i \in L^k_{p,\mu}(d_\vartheta)$, $i = 1, 2$, *have compact support and*

$$\Lambda(\vartheta) > k+2-\left(\mu + \frac{2}{p}\right) > 0, \tag{4.40}$$

where

$$\Lambda(\vartheta) = \frac{\pi}{\vartheta} - 1 \text{ for } \vartheta < \pi, \quad \Lambda(\vartheta) = 1 - \frac{\pi}{\vartheta} \text{ for } \pi < \vartheta \leqslant \frac{3}{2}\pi,$$

$$\Lambda(\vartheta) = \frac{2\pi}{\vartheta} - 1 \text{ for } \frac{3}{2}\pi \leqslant \vartheta \leqslant 2\pi.$$

Assuming that

$$k+2-\left(\mu+\frac{2}{p}\right) \neq 1 \quad \text{for } p \neq 2, \tag{4.41}$$

$$k+1-\mu = 1 \quad \text{for } p = 2. \tag{4.42}$$

Then there exist a unique solution $e_i \in L^{k+2}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, *of the problem* (3.15) *such that*

$$\sum_{i=1}^{2} \|e_i\|_{L^{k+2}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L^{k}_{p,\mu}(d_\vartheta)}. \tag{4.43}$$

In the case $\vartheta = \pi/m$, $m \in N$, $m \geqslant 2$, *the right-hand side functions of* (3.15) *have to satisfy some compatibility conditions at* $x' = 0$ *which are implied by Lemma* 4.4 *and Theorem* 4.1.

Proof. Lemma 4.4 implies that there exist functions $v_i \in L^{k+2}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, satisfying the homogeneous boundary conditions (3.15), that f_i, $i = 1, 2$, are replaced by $h_i = \Delta' v_i + f_i \in V^k_{p,\mu}(d_\vartheta)$, $i = 1, 2$, (which also have compact supports), and

$$\sum_{i=1}^{2} \|h_i\|_{V^{k}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L^{k}_{p,\mu}(d_\vartheta)}. \tag{4.44}$$

Then instead of (3.15) we have

$$\begin{gathered} -\Delta' u_i = h_i, \quad i = 1, 2, \\ u_i|_{\gamma_1} = 0, \quad (u_1 \cos\vartheta + u_2 \sin\vartheta)|_{\gamma_2} = 0, \\ \left.\frac{\partial u_2}{\partial n}\right|_{\gamma_1} = 0, \quad \left.\left(\frac{\partial u_1}{\partial n}\sin\vartheta - \frac{\partial u_2}{\partial n}\cos\vartheta\right)\right|_{\gamma_2} = 0, \end{gathered} \tag{4.45}$$

and

$$e_i = u_i + v_i, \quad i = 1, 2. \tag{4.46}$$

We define a generalized (or weak) solution of (4.45) to be functions $u_i \in \mathscr{H}(d_\vartheta)$, $i = 1, 2$, such that (4.38) is satisfied and

$$\sum_{i=1}^{2} \int_{d_\vartheta} \nabla' u_i \nabla' \eta_i \, dx' = \sum_{i=1}^{2} \int_{d_\vartheta} h_i \eta_i \, dx' \tag{4.47}$$

for all $\eta_i \in \mathscr{H}(d_\vartheta)$, $i = 1, 2$, satisfying (4.38).

From $h_i \in L^k_{p,\mu}(d_\vartheta)$ and the compactness of supp h_i, $i = 1, 2$, one gets $h_i \in L_{2,1}(d_\vartheta)$, $i = 1, 2$. Then by Lemma 4.5 the right-hand side of (4.47) is a continuous linear functional on $\mathscr{H}(d_\vartheta)$, because

$$\Big|\sum_{i=1}^{2} \int_{d_\vartheta} h_i \eta_i \, dx'\Big| \leqslant c \sum_{i=1}^{2} \|\eta_i\|_{\mathscr{H}(d_\vartheta)}.$$

Hence by the Riesz Theorem there exists a unique weak solution u_i, $i = 1, 2$, of (4.47) such that

$$\sum_{i=1}^{2} \|u_i\|_{\mathscr{H}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|h_i\|_{L_{2,1}(d_\vartheta)}. \tag{4.48}$$

Let (4.41) be satisfied. Then Theorem 4.1 implies that the weak solution belongs to $V^{k+2}_{p,\mu}(d_\vartheta)$ and

$$\sum_{i=1}^{2} \|u_i\|_{V^{k+2}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|h_i\|_{V^{k}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L^{k}_{p,\mu}(d_\vartheta)}. \tag{4.49}$$

Let (4.42) be satisfied. In this case $k+1-\mu$ is equal to 1 which is an eigenvalue of the problem (4.45). The corresponding eigenfunctions are αx_2, $-\alpha x_1$, where α is an arbitrary parameter. Hence they belong to ker $D^2_{x'}$. Therefore Remark 1.1 of [7] implies that the weak solution satisfies $D^2_{x'} u_i \in V^k_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|D^2_{x'} u_i\|_{V^{k}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|h_i\|_{V^{k}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L^{k}_{p,\mu}(d_\vartheta)}. \tag{4.50}$$

Using (4.46), (4.49) and (4.50) one has (4.43).

The necessity of the condition (4.40) follows by arguments similar to those in the proof of Theorem 5.1 in [7, p. 279], where the eigenfunctions (4.2) are used. Moreover, in order to have a weak solution, (4.40) must be assumed. This concludes the proof. ■

Remark 4.2. The eigenvalue $\lambda = 1$ does not intervene in the proof of Theorem 4.3 because the compatibility condition (4.21) is obviously satisfied here.

Using the anologue of Lemma 4.3 in the case of Hölder spaces, we can obtain instead of Theorem 4.3

THEOREM 4.4 *Let* $f_i \in C^l_{s-2}(d_\vartheta)$, supp f_i *compact*, $i = 1, 2$, l, $s \in \boldsymbol{R}$. *Let*

$$\Lambda(\vartheta) > s. \tag{4.51}$$

Then the generalized solution of the problem (3.15) *belongs to* $C^{l+2}_s(d_\vartheta)$ *and*

$$\sum_{i=1}^{2} \|e_i\|_{C^{l+2}_s(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{C^{l}_{s-2}(d_\vartheta)}. \tag{4.52}$$

5. Auxiliary results in $\mathscr{D}_\vartheta$

This section is devoted to obtaining results which enable us to pass from nonhomogeneous boundary value problems to homogeneous ones and conversely, in the case of weighted Sobolev as well as Hölder spaces.

LEMMA 5.1. *Let* $\varphi_i \in V_{p,\mu}^{k+2-1/p}(\Gamma_i)$, $\psi_i \in V_{p,\mu}^{k+1-1/p}(\Gamma_i)$, $i = 1, 2$. *Then there exist functions* $v_i \in V_{p,\mu}^{k+2}(d_\vartheta)$, $i = 1, 2$, *such that*

$$(5.1)\qquad \begin{gathered} v_1|_{\Gamma_1} = \varphi_1, \quad (v_1 \cos\vartheta + v_2 \sin\vartheta)|_{\Gamma_2} = \varphi_2, \\ -\left.\frac{\partial v_2}{\partial n}\right|_{\Gamma_1} = \psi_1, \quad \left.\left(\frac{\partial v_1}{\partial n}\sin\vartheta - \frac{\partial v_2}{\partial n}\cos\vartheta\right)\right|_{\Gamma_2} = \psi_2 \end{gathered}$$

and

$$(5.2)\qquad \sum_{i=1}^{2} \|v_i\|_{V_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \left(\|\varphi_i\|_{V_{p,\mu}^{k+2-1/p}(\Gamma_i)} + \|\psi_i\|_{V_{p,\mu}^{k+1-1/p}(\Gamma_i)}\right).$$

Conversely, let $v_i \in V_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)$, $i = 1, 2$, *be given. Then*

$$\begin{aligned} \varphi_1 &= v_1|_{\Gamma_1} \in V_{p,\mu}^{k+2-1/p}(\Gamma_1), \\ \varphi_2 &= (v_1 \cos\vartheta + v_2 \sin\vartheta)|_{\Gamma_2} \in V_{p,\mu}^{k+2-1/p}(\Gamma_2), \\ \psi_1 &= \left.\frac{\partial v_2}{\partial n}\right|_{\Gamma_1} \in V_{p,\mu}^{k+1-1/p}(\Gamma_1), \\ \psi_2 &= \left.\left(\frac{\partial v_1}{\partial n}\sin\vartheta - \frac{\partial v_2}{\partial n}\cos\vartheta\right)\right|_{\Gamma_2} \in V_{p,\mu}^{k+1-1/p}(\Gamma_i) \end{aligned}$$

and

$$(5.3)\qquad \sum_{i=1}^{2} \left(\|\varphi_i\|_{V_{p,\mu}^{k+2-1/p}(\Gamma_i)} + \|\psi_i\|_{V_{p,\mu}^{k+1-1/p}(\Gamma_i)}\right) \leqslant c \sum_{i=1}^{2} \|v_i\|_{V_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)}.$$

Proof. To prove the first part of the lemma we replace the relations (5.1) by

$$\left.\begin{aligned} (v_1 \cos\varphi + v_2 \sin\varphi)|_{\Gamma_i} &= \varphi_i, \\ \frac{\partial}{\partial n}(v_1 \sin\varphi - v_2 \cos\varphi)|_{\Gamma_i} &= \varphi_i + \psi_i, \end{aligned}\right. \qquad i = 1, 2.$$

Now the results from Section 1 of [10] can be used. The second part of the lemma also follows directly from [10]. ∎

LEMMA 5.2. *Let* $g_i \in L_{p,\mu}^{k}(\mathscr{D}_\vartheta)$, $\varphi_i \in L_{p,\mu}^{k+2-1/p}(\Gamma_i)$, $\psi_i \in L_{p,\mu}^{k+1-1/p}(\Gamma_i)$, $i = 1, 2$, *be such that*

$$(5.4)\qquad \left(\frac{\partial}{\partial x_1}\varphi_1-\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)\varphi_2-\psi_1+\psi_2\right)\Big|_{x=0}=0.$$

For $\vartheta=\pi m$, $m\geqslant 2$ natural, additional compatibility conditions on the functions φ_i, ψ_i, g_i, $i=1,2$, at $x\in M$, must be imposed in order that the problems (5.8), (5.20)–(5.21) *below be solvable.*

Then there exist functions $v_i\in L^{k+2}_{p,\mu}(\mathscr{D}_\vartheta)$, $i=1,2$, such that

$$(5.5)\qquad \begin{aligned} f_i&=\Delta v_i+g_i\in V^k_{p,\mu}(\mathscr{D}_\vartheta),\\ \Phi_1&=\varphi_1-v_1|_{\Gamma_1}\in V^{k+2-1/p}_{p,\mu}(\Gamma_1)\\ \Phi_2&=\varphi_2-(v_1\cos\vartheta+v_2\sin\vartheta)|_{\Gamma_2}\in V^{k+2-1/p}_{p,\mu}(\Gamma_2),\\ \Psi_1&=\psi_1-\frac{\partial v_2}{\partial n}\Big|_{\Gamma_1}\in V^{k+1-1/p}_{p,\mu}(\Gamma_1),\\ \Psi_2&=\psi_2-\left(\frac{\partial v_1}{\partial n}\sin\vartheta-\frac{\partial v_2}{\partial n}\cos\vartheta\right)\Big|_{\Gamma_2}\in V^{k+1-1/p}_{p,\mu}(\Gamma_2),\end{aligned}$$

and

$$(5.6)\qquad \sum_{i=1}^{2}\left(\|v_i\|_{L^{k+2}_{p,\mu}(\mathscr{D}_\vartheta)}+\|f_i\|_{V^k_{p,\mu}(\mathscr{D}_\vartheta)}+\|\Phi_i\|_{V^{k+2-1/p}_{p,\mu}(\Gamma_i)}+\|\Psi_i\|_{V^{k+1-1/p}_{p,\mu}(\Gamma_i)}\right)$$
$$\leqslant c\sum_{i=1}^{2}\left(\|g_i\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)}+\|\varphi_i\|_{L^{k+2-1/p}_{p,\mu}(\Gamma_i)}+\|\psi_i\|_{L^{k+1-1/p}_{p,\mu}(\Gamma_i)}\right)\equiv cX,$$

and moreover,

$$(5.7)\qquad \Bigg[\frac{\partial v_1}{\partial x_1}-\left(\sin\vartheta\frac{\partial}{\partial x_1}+\cos\vartheta\frac{\partial}{\partial x_2}\right)(v_1\cos\vartheta+v_2\sin\vartheta)$$
$$+\frac{\partial v_2}{\partial x_2}+\left(-\sin\vartheta\frac{\partial}{\partial x_1}+\cos\vartheta\frac{\partial}{\partial x_2}\right)(v_1\sin\vartheta-v_2\cos\vartheta)\Bigg]\Bigg|_{x'=0}=0.$$

Proof. (a) Let $\mu+2/p=s\notin \mathbf{Z}$. We first find homogeneous polynomials R_{iq}, $i=1,2$, of degree q with respect to the variables $x'=(x_1,x_2)$ satisfying the following inductive conditions [14]:

$$(5.8)\qquad \begin{aligned} &-\Delta' R_{iq+2}=D_z^2R_{iq}+g_{iq},\quad i=1,2,\quad q<k-s,\\ &R_{1q}|_{\gamma_1}=\varphi_{1q},\quad (R_{1q}\cos\vartheta+R_{2q}\sin\vartheta)|_{\gamma_2}=\varphi_{2q},\quad q<k+2-s,\\ &\frac{\partial R_{2q}}{\partial n}\Big|_{\gamma_1}=\psi_{1q-1},\quad \left(\frac{\partial R_{1q}}{\partial n}\sin\vartheta-\frac{\partial R_{2q}}{\partial n}\cos\vartheta\right)\Big|_{\gamma_2}=\psi_{2q-1},\quad q<k+2-s,\end{aligned}$$

where

$$\varphi_{iq} = \frac{r^q}{q!}\left(\frac{\partial}{\partial r}\right)^q \varphi_i\Big|_{r=0}, \quad \psi_{iq} = \frac{r^q}{q!}\left(\frac{\partial}{\partial r}\right)^q \psi_i\Big|_{r=0},$$

$$g_{iq} = \sum_{|\alpha|=q} D^\alpha_{x'} g|_{x'=0} \frac{x_1^{\alpha_1} x_2^{\alpha_2}}{\alpha_1!\alpha_2!}, \quad \alpha = (\alpha_1, \alpha_2), \quad i = 1, 2.$$

The polynomials R_{i0}, $i = 1, 2$, are calculated from the Dirichlet conditions $(5.8)_2$ only in the form $R_{10} = \varphi_{10}$, $R_{20} = [\varphi_{20} - \cos\vartheta\varphi_{10}]/\sin\vartheta$. The polynomials of the first degree are determined by the boundary conditions also. Let $R_{11} = \alpha_1 x_1 + \alpha_2 x_2$, $R_{21} = \beta_1 x_1 + \beta_2 x_2$. Then Lemma 4.2 implies that the coefficients α_1, α_2, β_1, β_2 are not uniquely determined. They are solutions of (4.20) with $Z = (\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $Y = (\varphi_{11,r}, \varphi_{21,r}, \psi_{10}, \psi_{20})$. Hence the compatibility condition (4.17) has the form

$$\varphi_{11,r} + \psi_{20} = \varphi_{21,r} + \psi_{10}, \tag{5.9}$$

and for $\vartheta \neq k\pi/2$, $k = 1, 2, 3$, we have

$$\alpha_1 = \varphi_{11,r}, \quad \beta_2 = -\psi_{10}, \tag{5.10}$$
$$\beta_1 + \alpha_2 = \frac{1}{\sin 2\vartheta}[\varphi_{21,r} + \psi_{20} + \cos 2\vartheta(\varphi_{11,r} + \psi_{11})],$$

so the polynomials of the first degree are determined up to one arbitrary parameter. Therefore (5.8) implies that all odd degree polynomials are also nonunique. By choosing the arbitrary parameters appropriately Theorem 2.4 implies that all above polynomials make sense and

$$\sum_{|\alpha| \leqslant [k+2-s]} \langle\langle D^\alpha_{x'} R_{i|\alpha|} \rangle\rangle^{(k+2-s-|\alpha|)}_{p,M} \leqslant cX. \tag{5.11}$$

Now by Theorem 2.4 we find functions $v_i \in L^{k+2}_{p,\mu}(\mathscr{D}_\vartheta)$, $i = 1, 2$, such that

$$D^\alpha_{x'} v_i|_{x'=0} = D^\alpha_{x'} R_{i|\alpha|}, \quad |\alpha| \leqslant [k+2-s], \; i = 1, 2, \tag{5.12}$$

and

$$\sum_{i=1}^2 \|v_i\|_{L^{k+2}_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^2 \sum_{|\alpha| \leqslant [k+2-s]} \langle\langle D^\alpha_x R_{i|\alpha|} \rangle\rangle^{(k+2-s-|\alpha|)}_{p,M} \leqslant cX. \tag{5.13}$$

We shall show that the functions v_i, $i = 1, 2$, satisfy (5.5) and (5.6). Considering the expressions

$$D^\beta_{x'}(\Delta v_i + g_i) = D^2_z D^\beta_{x'}(v_i - R_{i|\beta|}) + \Delta' D^\beta_{x'}(v_i - R_{i|\beta|}) + D^\beta_{x'}(g_i - g_{i|\beta|})$$

for $|\beta| < k-s$, from (5.12) we see that $D^\beta_{x'} f_i|_{x'=0} = 0$ for $|\beta| < k-s$, $i = 1, 2$. Therefore Lemma 2.2 implies that $f_i \in V^k_{p,\mu}(\mathscr{D}_\vartheta)$, $i = 1, 2$, and

$$\|f_i\|_{V^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c\|f_i\|_{L^k_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant cX, \quad i = 1, 2. \tag{5.14}$$

Moreover, from (5.8) and (5.12) one has

$$\begin{aligned} D_r^\beta \Phi_i|_{r=0} &= 0, \quad |\beta| < k+2-s, \ i = 1, 2, \\ D_r^\beta \Psi_i|_{r=0} &= 0, \quad |\beta| < k+1-s, \ i = 1, 2. \end{aligned} \tag{5.15}$$

Hence Lemma 2.2 and Theorems 2.2, 2.3 imply (5.5) and (5.6) for Φ_i and Ψ_i, $i = 1, 2$.

The compatibility condition (5.7) follows from (5.9), (5.10) and (5.12) because

$$\begin{aligned} \frac{\partial R_{11}}{\partial x_1} &- \left(\cos\vartheta \frac{\partial}{\partial x_1} + \sin\vartheta \frac{\partial}{\partial x_2}\right)(R_{11}\cos\vartheta + R_{21}\sin\vartheta) \\ &+ \frac{\partial R_{21}}{\partial x_2} - \left(-\sin\vartheta \frac{\partial}{\partial x_1} + \cos\vartheta \frac{\partial}{\partial x_2}\right)(R_{11}\sin\vartheta - R_{21}\cos\vartheta) = 0. \end{aligned} \tag{5.16}$$

Hence (5.16) and (5.8) imply some relations for odd degree polynomials R_{iq}, $i = 1, 2$. From these relations and (5.12) we obtain (5.7) and some relations for odd order derivatives of v_i, $i = 1, 2$, at $x' = 0$. Moreover (5.9) is equivalent to (5.4).

(b) Let $\mu + 2/p = s \geqslant 1$, $s \in \mathbf{Z}$. In this case the derivatives $D_{x'}^\alpha g_i$, $|\alpha| = k-s$, $D_r^{k+2-s}\varphi_i$, $D_r^{k+1-s}\psi_i$, $i = 1, 2$, do not have traces on M. Hence we construct the functions v_i, $i = 1, 2$, in the form $v_i = v_i^1 + v_i^2$, $i = 1, 2$, where v_i^1, $i = 1, 2$, satisfy the relations (5.12) for $|\alpha| \leqslant k+1-s$.

Then Theorem 2.4 implies that $v_i^1 \in L_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|v_i^1\|_{L_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \sum_{|\alpha| \leqslant k+1-s} \langle\langle D_{x'}^\alpha R_{i|\alpha|}\rangle\rangle_{p,M}^{(k+2-s-|\alpha|)} \leqslant cX. \tag{5.17}$$

Let

$$\begin{gathered} f_i^1 = \Delta' v_i^1 + g_i, \quad i = 1, 2, \\ \Phi_1^1 = \varphi_1 - v_1^1|_{\Gamma_1}, \quad \Phi_2^1 = \varphi_2 - (v_1^1\cos\vartheta + v_2^1\sin\vartheta)|_{\Gamma_2}, \\ \Psi_1^1 = \psi_1 - \frac{\partial v_2^1}{\partial n}\bigg|_{\Gamma_1}, \quad \Psi_2^1 = \psi_2 - \left(\frac{\partial v_1^1}{\partial n}\sin\vartheta - \frac{\partial v_2^1}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2}. \end{gathered}$$

Then from (5.8) and (5.12) we have

$$\begin{aligned} D_{x'}^\alpha f_i^1|_{x'=0} &= 0, \quad |\alpha| \leqslant k-1-s, \ i = 1, 2, \\ D_r^\alpha \Phi_i^1|_{r=0} &= 0, \quad \alpha \leqslant k+1-s, \ i = 1, 2, \\ D_r^\alpha \Psi_i^1|_{r=0} &= 0, \quad \alpha \leqslant k-s, \ i = 1, 2, \end{aligned} \tag{5.18}$$

and the compatibility conditions described in (a) are satisfied. Moreover, (5.13) implies that

$$f_i^1 \in L_{p,\mu}^k(\mathscr{D}_\vartheta), \quad \Phi_i^1 \in L_{p,\mu}^{k+2-1/p}(\Gamma_i), \quad \Psi_i^1 \in L_{p,\mu}^{k+1-1/p}(\Gamma_i), \quad i = 1, 2,$$

and

$$(5.19)\qquad \sum_{i=1}^{2}(\|f_i^1\|_{L_{p,\mu}^k(\mathcal{D}_\vartheta)}+\|\Phi_i^1\|_{L_{p,\mu}^{k+2-1/p}(\Gamma_i)}+\|\Psi_i^1\|_{L_{p,\mu}^{k+1-1/p}(\Gamma_i)})\leqslant cX.$$

To construct the functions v_i^2, $i=1,2$, we must calculate the functions $v_{i(\alpha)}$, $|\alpha|=k+2-s$, considered in Lemma 2.7. To calculate them we consider the following system of equations

$$(5.20)\qquad -D_{x_1}^{\nu}D_{x_2}^{j-2-\nu}D_z^l(v_{i(2,0)}+v_{i(0,2)})=D_{x_1}^{\nu}D_{x_2}^{j-2-\nu}D_z^l(D_z^2v_{i(0,0)}+f_i^1),$$

where $i=1,2$, $2\leqslant j\leqslant k+2-s$, $\nu\leqslant j-2$, $j+1\leqslant k+2$,

$$(5.21)\qquad \begin{aligned} &D_{x_1}^jD_z^lv_{1(0,0)}=D_{x_1}^jD_z^l\tilde{\Phi}_1^1,\\ &\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)^jD_z^l(v_{1(0,0)}\cos\vartheta+v_{2(0,0)}\sin\vartheta)\\ &\qquad=\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)^jD_z^l\tilde{\Phi}_2^1,\\ &-D_{x_1}^{j-1}D_z^lv_{2(0,1)}=D_{x_1}^{j-1}D_2^l\tilde{\Psi}_1^1,\\ &\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)^{j-1}D_z^l[-\sin^2\vartheta v_{1(1,0)}+\sin\vartheta\cos\vartheta(v_{1(0,1)}+v_{2(1,0)})\\ &\qquad-\cos^2\vartheta v_{2(0,1)}]=\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)^{j-1}D_z^l\tilde{\Psi}_2^1,\end{aligned}$$

where $j\leqslant k+2-s$, $j+l\leqslant k+2$,

$$\frac{\partial}{\partial x_1}v_{i(\alpha_1,\alpha_2)}=v_{i(\alpha_1+1,\alpha_2)},\qquad \frac{\partial}{\partial x_2}v_{i(\alpha_1,\alpha_2)}=v_{i(\alpha_1,\alpha_2+1)},\qquad i=1,2,$$

and $\tilde{\Phi}_i^1$, $\tilde{\Psi}_i^1$, $i=1,2$, are extensions of Φ_i^1, Ψ_i^1, $i=1,2$, into $\mathcal{D}_\vartheta$ such that

$$(5.22)\qquad \begin{aligned}&\|\tilde{\Phi}_i^1\|_{L_{p,\mu}^{k+2}(\mathcal{D}_\vartheta)}\leqslant c\|\Phi_i^1\|_{L_{p,\mu}^{k+2-1/p}(\Gamma_i)},\qquad i=1,2,\\ &\|\tilde{\Psi}_i^1\|_{L_{p,\mu}^{k+1}(\mathcal{D}_\vartheta)}\leqslant c\|\Psi_i^1\|_{L_{p,\mu}^{k+1-1/p}(\Gamma_i)},\qquad i=1,2,\end{aligned}$$

and $(5.18)_{2,3}$ are satisfied.

The functions $D_z^lv_{i(0,0)}$, $i=1,2$, are calculated explicitly from the first two equations of (5.21) for $j=0$, $l\leqslant k+2$. To calculate $D_z^lv_{i(\alpha)}$, $i=1,2$, $|\alpha|=1$, $l\leqslant k+1$, from (5.21) we have to consider the algebraic system (4.20) with

$$Z=(D_z^lv_{1(1,0)},D_z^lv_{1(0,1)},D_z^lv_{2(1,0)},D_z^lv_{2(0,1)}),$$

$$Y=\left(D_z^l\frac{\partial}{\partial x_1}\tilde{\Phi}_1^1,\ D_z^l\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)\tilde{\Phi}_2^1,\ D_z^l\tilde{\Psi}_1^1,\ D_z^l\tilde{\Psi}_2^1\right).$$

Therefore to solve the system the following condition, similar to (6.9),

$$\frac{\partial}{\partial x_1}\tilde{\Phi}_1^1-\left(\cos\vartheta\frac{\partial}{\partial x_1}+\sin\vartheta\frac{\partial}{\partial x_2}\right)\tilde{\Phi}_2^1-\tilde{\Psi}_1^1+\tilde{\Psi}_2^1=0 \tag{5.23}$$

must be satisfied. Then solutions have the form (5.10), so they depend on one arbitrary parameter.

The condition (5.23) is stronger than (5.9) because it must be satisfied for all $x\in\mathscr{D}_\vartheta$ while (5.9) for $x\in M$ only. But on the other hand (5.23) describes a relation among extensions of Φ_i^1, Ψ_i^1, $i=1,2$. Hence to obtain the extensions such that (5.23) is satisfied we must know that (5.23) is satisfied for Φ_i^1, Ψ_i^1, $i=1,2$, and for $x\in M$, which follows from $(5.18)_{2,3}$. Hence (5.23) can be satisfied without any additional assumptions.

To calculate the higher derivatives we use induction. For $j=2$, $l\leqslant k$, $D_z^l v_{i(\alpha)}$, $i=1,2$, $|\alpha|=j$, can be calculated from (5.20) and (5.21). Assume $D_z^l v_{i(\alpha)}$ $|\alpha|\leqslant j-1$, $l\leqslant k+2-(j-1)$, are found. Then inserting $v_{i(\alpha-1)}$, $i=1,2$, where $(\alpha-1)$ is either (α_1-1,α_2) or (α_1,α_2-1), into (5.20), from (5.20), (5.21) we obtain $v_{i(\alpha+1)}$, $i=1,2$, and its derivatives with respect to z. In this way we calculate all $D_z^l v_{i(\alpha)}$, $i=1,2$, $|\alpha|\leqslant k+2-s$, $l+|\alpha|\leqslant k+2$.

Now by Lemma 2.7 and Remark 2.1 there exist v_i^2; $i=1,2$, such that $v_i^2\in L_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)$, $i=1,2$, and

$$D_{x'}^\alpha v_i^2|_{x'=0}=0,\qquad i=1,2,\ |\alpha|\leqslant k-s-1. \tag{5.24}$$

Let

$$f_i^2=\Delta v_i^2+f_i^1,\qquad i=1,2,$$

$$\Phi_1^2=\Phi_1^1-v_1^2|_{\Gamma_1},\qquad \Phi_2^2=\Phi_2^1-(v_1^2\cos\vartheta+v_2^2\sin\vartheta)|_{\Gamma_2},$$

$$\Psi_1^2=\Psi_1^1-\frac{\partial v_2^2}{\partial n}\bigg|_{\Gamma_1},\qquad \Psi_2^2=\Psi_2^1-\left(\frac{\partial v_1^2}{\partial n}\sin\vartheta-\frac{\partial v_2^2}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2}.$$

Then Remark 2.1 implies

$$\begin{aligned}\sum_{i=1}^{2}\Big(\|v_i^2\|_{L_{p,\mu}^{k+2}(\mathscr{D}_\vartheta)}&+\|f_i^2\|_{L_{p,\mu}^{k}(\mathscr{D}_\vartheta)}+\|\Phi_i^2\|_{L_{p,\mu}^{k+2-1/p}(\Gamma_i)}+\|\Psi_i^2\|_{L_{p,\mu}^{k+1-1/p}(\Gamma_i)}\\&+\sum_{|\alpha|=k-s}\big(\int_{\mathbf{R}^1}\|D^\alpha f_i^2\|^p_{V_{p,\mu}^{s}(d_\vartheta)}dz\big)^{1/p}+\sum_{|\alpha|=k+2-s}\big(\int_{\mathbf{R}^1}\|D^\alpha\Phi_i^2\|^p_{V_{p,\mu}^{s-1/p}(\gamma_i)}dz\big)^{1/p}\\&+\sum_{|\alpha|=k+1-s}\big(\int_{\mathbf{R}^1}\|D^\alpha\Psi_i^2\|^p_{V_{p,\mu}^{s-1/p}(\gamma_i)}dz\big)^{1/p}\Big)\leqslant cX.\end{aligned} \tag{5.25}$$

From (5.25) we have

$$\sum_{i=1}^{2}\big(\|f_i^2\|_{L_{p,\mu-s}^{k-s}(\mathscr{D}_\vartheta)}+\|\Phi_i^2\|_{L_{p,\mu-s}^{k+2-s-1/p}(\Gamma_i)}+\|\Psi_i^2\|_{L_{p,\mu-s}^{k+1-s-1/p}(\Gamma_i)}\big)\leqslant cX. \tag{5.26}$$

Therefore using Lemma 2.2, (5.18) and (5.24) we see that (5.5) and (5.6) are satisfied.

In the case $\vartheta = \pi/m$, $m \in N$, $m \geqslant 2$, in addition to the previous considerations, first polynomials of the type (4.36) appear as solutions of the problems (5.8) and (5.21) if suitable restrictions on their right-hand sides are imposed (see Lemma 4.3); secondly, since solutions of problems (5.8) and (5.21) are constructed inductively and the polynomials of the type (4.36) of lowest degree are of the form

$$\begin{gathered} \omega_{1,m}^{(0)+} = A_{0,m}^{(+)} r\sin\varphi, \quad \omega_{2,m}^{(0)+} = B_{0,m}^{(+)} r\cos\varphi, \\ \omega_{1,2}^{(0)-} = A_{1,2}^{(-)} r\sin\varphi, \quad \omega_{1,2}^{(0)-} = B_{1,2}^{(-)} r\cos\varphi, \quad \text{for } m = 2, \end{gathered} \tag{5.27}$$

we are able to get the form of polynomials generated by (5.27) which appear in the solution we are looking for (these polynomials are arbitrary in such a way that they depend on arbitrary functions of z). Now to obtain the estimate (5.6) the arbitrary functions have to be properly chosen. This concludes the proof. ∎

LEMMA 5.3. *Let $\varphi_i \in \mathring{C}_s^l(\Gamma_i)$, $\psi_i \in \mathring{C}_{s-1}^{l-1}(\Gamma_i)$, with compact supports, $i = 1, 2$, $l \notin \boldsymbol{Z}$. Then there exist functions $u_i \in \mathring{C}_s^l(\mathscr{D}_\vartheta)$, $i = 1, 2$, such that*

$$\begin{gathered} u_1|_{\Gamma_1} = \varphi_1, \quad (u_1\cos\vartheta + u_2\sin\vartheta)|_{\Gamma_2} = \varphi_2, \\ \frac{\partial u_2}{\partial n}\bigg|_{\Gamma_2} = \psi_1, \quad \left(\frac{\partial u_1}{\partial n}\sin\vartheta - \frac{\partial u_2}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2} = \psi_2 \end{gathered} \tag{5.28}$$

and

$$\sum_{i=1}^{2} |u_i|_{\mathring{C}_s^l(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \left(|\varphi_i|_{\mathring{C}_s^l(\Gamma_i)} + |\psi_i|_{\mathring{C}_{s-1}^{l-1}(\Gamma_i)}\right). \tag{5.29}$$

The inverse statement is trivially valid.

LEMMA 5.4. *Let $g_i \in C_{s-2}^l(\mathscr{D}_\vartheta)$, $\varphi_i \in C_s^{l+2}(\Gamma_i)$, $\psi_i \in C_{s-1}^{l+1}(\Gamma_i)$, with compact supports $i = 1, 2$, $s > 1$. Then there exist functions $v_i \in C_s^{l+2}(\mathscr{D}_\vartheta)$, $i = 1, 2$, such that*

$$f_i = g_i + \Delta v_i \in \mathring{C}_{s-2}^l(\mathscr{D}_\vartheta), \quad i = 1, 2,$$

$$\Phi_1 = \varphi_1 - v_1|_{\Gamma_1} \in \mathring{C}_s^{l+2}(\Gamma_1), \quad \Phi_2 = \varphi_2 - (v_1\cos\vartheta + v_2\sin\vartheta)|_{\Gamma_2} \in \mathring{C}_s^{l+2}(\Gamma_2),$$

$$\Psi_1 = \psi_1 - \frac{\partial v_2}{\partial n}\bigg|_{\Gamma_1} \in \mathring{C}_{s-1}^{l+1}(\Gamma_1), \quad \Psi_2 = \psi_2 - \left(\frac{\partial v_1}{\partial n}\sin\vartheta - \frac{\partial v_2}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2} \in \mathring{C}_{s-1}^{l+1}(\Gamma_2),$$

and

$$\begin{aligned} \sum_{i=1}^{2} \left(\langle v_i\rangle_{s,\mathscr{D}_\vartheta}^{(l+2)} + |f_i|_{\mathring{C}_{s-2}^l(\mathscr{D}_\vartheta)} + |\Phi_i|_{\mathring{C}_s^{l+2}(\Gamma_i)} + |\Psi_i|_{\mathring{C}_{s-1}^{l+1}(\Gamma_i)}\right) \\ \leqslant c \sum_{i=1}^{2} \left(\langle g_i\rangle_{s-2,\mathscr{D}_\vartheta}^{(l)} + \langle\varphi_i\rangle_{s,\Gamma_i}^{(l+2)} + \langle\psi_i\rangle_{s-1,\Gamma_i}^{(l+1)}\right). \end{aligned} \tag{5.30}$$

The given functions have to satisfy some compatibility conditions both for arbitrary angles and equal to π/m.

Proof. The proof is similar to that of Lemma 5.2. We underline differences only.

The functions v_i, $i = 1, 2$, are determined by Theorem 2.6 by using (5.13) for $|\alpha| \leqslant [s]$ and $s < \pi/\vartheta$.

For $\pi/\vartheta = m \leqslant [s]$ the polynomials R_{i_q} are determined up to polynomials described by (5.27). Therefore the solutions of the problem (5.8) and (5.21) exist if some compatibility conditions are satisfied. This concludes the proof. ■

6. Existence of solutions of (3.14) in $H^k_\mu(\mathcal{D}_\vartheta)$

In this section we prove the existence of generalized solutions of (3.14) and next we show that the generalized solution belongs to $H^k_\mu(\mathcal{D}_\vartheta)$ for sufficiently smooth right-hand sides.

Using Lemma 5.1 and 5.2 we can reduce the problem (3.14) to

$$-\Delta e_i = f_i, \quad i = 1, 2, \tag{6.1}$$

$$\begin{gathered} e_1|_{\Gamma_1} = 0, \quad (e_1\cos\vartheta + e_2\sin\vartheta)|_{\Gamma_2} = 0, \\ \frac{\partial e_2}{\partial n}\bigg|_{\Gamma_1} = 0, \quad \left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2} = 0. \end{gathered} \tag{6.2}$$

Lemma 6.1. *If* $u_i \in \mathcal{H}(\mathcal{D}_\vartheta)$, $i = 1, 2$, *satisfy*

$$u_1|_{\Gamma_1} = 0, \quad (u_1\cos\vartheta + u_2\sin\vartheta)|_{\Gamma_2} = 0, \tag{6.3}$$

and $\sin\vartheta \neq 0$, *then*

$$\sum_{i=1}^{2} \|u_i\|^2_{L_{2,-1}(\mathcal{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|u_i\|^2_{\mathcal{H}(\mathcal{D}_\vartheta)}. \tag{6.4}$$

Proof. From the assumptions of the lemma and [6] we have

$$\|u_1\|_{L_{2,-1}(\mathcal{D}_\vartheta)} \leqslant c\|u_1\|_{\mathcal{H}(\mathcal{D}_\vartheta)},$$

$$\|u_1\cos\vartheta + u_2\sin\vartheta\|_{L_{2,-1}(\mathcal{D}_\vartheta)} \leqslant c\|u_1\cos\vartheta + u_2\sin\vartheta\|_{\mathcal{H}(\mathcal{D}_\vartheta)} \leqslant c\sum_{i=1}^{2} \|u_i\|_{\mathcal{H}(\mathcal{D}_\vartheta)}.$$

Therefore for $\sin\vartheta \neq 0$ we get (6.4) ■

Definition 6.1. A *generalized* (*or weak*) *solution* of the problem (6.1), (6.2) is defined to be functions $e_i \in \mathcal{H}(\mathcal{D}_\vartheta)$, $i = 1, 2$, satisfying (6.3) and

$$\sum_{i=1}^{2} \int_{\mathcal{D}_\vartheta} \nabla e_i \nabla \eta_i = \sum_{i=1}^{2} \int_{\mathcal{D}_\vartheta} f_i \eta_i, \tag{6.5}$$

for all $\eta_i \in \mathscr{H}(\mathscr{D}_\vartheta)$, $i = 1, 2$, satisfying (6.3). A motivation for (6.5) comes from

$$\sum_{i=1}^{2} \int_{\Gamma_\vartheta} \bar{n}\cdot\nabla e_i \eta_i = \int_{\Gamma_1} \bar{n}\cdot\nabla e_1 \eta_1 + \int_{\Gamma_2} \bar{n}\cdot\nabla e_2 \frac{1}{\sin\vartheta}(\eta_1\cos\vartheta + \eta_2 \sin\vartheta) = 0.$$

THEOREM 6.1. *Let* $\sum_{i=1}^{2} \|f_i\|_{L_{2,1}(\mathscr{D}_\vartheta)} = N < \infty$. *Then there exists a generalized solution of* (6.1), (6.2) *and*

$$\sum_{i=1}^{2} \|e_i\|_{\mathscr{H}(\mathscr{D}_\vartheta)} \leqslant cN. \tag{6.6}$$

Proof. By Lemma 6.1 the right-hand side of (6.5) is a linear continuous functional on $\mathscr{H}(\mathscr{D}_\vartheta)$, because

$$\Big|\sum_{i=1}^{2} \int_{\mathscr{D}_\vartheta} f_i\eta_i\Big| \leqslant cN \sum_{i=1}^{2} \|\eta_i\|_{\mathscr{H}(\mathscr{D}_\vartheta)}.$$

Therefore using the Riesz theorem concludes the proof. ∎

THEOREM 6.2. *Let* $k \in \boldsymbol{N} \cup \{0\}$, $\mu \in \boldsymbol{R}$, $\mu \geqslant 0$,

$$\Lambda(\vartheta) > k+1-\mu \geqslant 0 \tag{6.7}$$

where

$$\Lambda(\vartheta) = \frac{\pi}{\vartheta} - 1 \text{ for } \vartheta \leqslant \pi, \quad \Lambda(\vartheta) = 1 - \frac{\pi}{\vartheta} \text{ for } \pi < \vartheta \leqslant \frac{3}{2}\pi,$$
$$\Lambda(\vartheta) = \frac{2\pi}{\vartheta} - 1 \text{ for } \frac{3}{2}\pi \leqslant \vartheta \leqslant 2\pi. \tag{6.8}$$

Let $f_i \in L_\mu^k(\mathscr{D}_\vartheta)$, $i = 1, 2$, *have compact supports. Then the problem* (6.1), (6.2) *has a unique solution* $e_i \in L_\mu^{k+2}(\mathscr{D}_\vartheta) \cap \mathscr{H}(\mathscr{D}_\vartheta)$, $i = 1, 2$, *and*

$$\sum_{i=1}^{2} \|e_i\|_{L_\mu^{k+2}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L_\mu^k(\mathscr{D}_\vartheta)}. \tag{6.9}$$

In the case $\vartheta = \pi/m$, $m \in \boldsymbol{N}$, $m \geqslant 2$, *some compatibility conditions on the right-hand side functions of* (6.1), (6.2) *have to be imposed at* $x \in M$; *they follow from Theorems* 4.1, 4.3 *and Lemma* 5.2.

Proof. Since $f_i \in L_\mu^k(\mathscr{D}_\vartheta)$ and $\operatorname{supp} f_i$ is compact, $i = 1, 2$, it follows that $f_i(x') \in L_1(\boldsymbol{R}^1)$, $x' \in d_\vartheta$, $i = 1, 2$. Thus the Fourier transform is $\tilde{f}(x', \xi) = (2\pi)^{-1/2} \int_{\boldsymbol{R}^1} f(x) e^{-iz\xi} dz$. After taking the Fourier transforms the problem (6.1), (6.2) has the form

$$(6.10)\qquad \begin{gathered} -\Delta'\tilde{e}_i+\xi^2\tilde{e}_i=\tilde{f}_i, \quad i=1,2,\\ \tilde{e}_1|_{\gamma_1}=0, \quad (\tilde{e}_1\cos\vartheta+\tilde{e}_2\sin\vartheta)|_{\gamma_2}=0,\\ \frac{\partial\tilde{e}_2}{\partial n}\bigg|_{\gamma_1}=0, \quad \left(\frac{\partial\tilde{e}_1}{\partial n}\sin\vartheta-\frac{\partial\tilde{e}_2}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2}=0. \end{gathered}$$

A generalized solution of the problem (6.10) is defined to be functions $\tilde{e}_i\in H^1(d_\vartheta)$, $i=1,2$, such that

$$(6.11)\qquad \tilde{e}_1|_{\gamma_1}=0, \quad (\tilde{e}_1\cos\vartheta+\tilde{e}_2\sin\vartheta)|_{\gamma_2}=0,$$

and

$$(6.12)\qquad \sum_{i=1}^{2}\int_{d_\vartheta}(\nabla'\tilde{e}_i\cdot\nabla'\bar{\eta}_i+\xi^2\tilde{e}_i\bar{\eta}_i)dx'=\sum_{i=1}^{2}\int_{d_\vartheta}\tilde{f}_i\bar{\eta}_i\,dx',$$

for all $\eta_i\in H^1(d_\vartheta)$, $i=1,2$, satisfying (6.11). In the above expressions all functions are complex-valued and $\bar{\eta}$ denotes the complex conjugate function.

The identity (6.12) implies that for almost all ξ there exists a unique weak solution of (6.10) in $H^1(d_\vartheta)$.

(a) Let $k=0$, $\mu\in(0,1]$. Then (6.7), (6.8) imply that

$$\vartheta<\frac{\pi}{2-\mu}\quad \text{for } \vartheta<\pi \text{ and } \mu\in(0,1],$$

$$\vartheta>\frac{\pi}{\mu}\quad \text{for } \pi<\vartheta\leqslant\tfrac{3}{2}\pi \text{ and } \mu\in(\tfrac{2}{3},1],$$

$$\vartheta<\frac{2\pi}{2-\mu}\quad \text{for } \tfrac{3}{2}\pi\leqslant\vartheta<2\pi \text{ and } \mu\in[\tfrac{2}{3},1].$$

Putting $\eta_i=\tilde{e}_i\xi^{2-2\mu}$, $i=1,2$, in (6.12) one obtains

$$(6.13)\qquad \sum_{i=1}^{2}\xi^{2-2\mu}\int_{d_\vartheta}(|\nabla'\tilde{e}_i|^2+\xi^2|\tilde{e}_i|^2)dx' \leqslant \xi^{2-2\mu}\Big(\sum_{i=1}^{2}\int_{d_\vartheta}|\tilde{f}_i|^2|x'|^2dx'\Big)^{1/2}\Big(\sum_{i=1}^{2}\int_{d_\vartheta}|\tilde{e}_i|^2|x'|^{-2\mu}dx'\Big)^{1/2}$$

Theorem 2.5 for $\mu\in(0,1)$, $l=1$ and $\varepsilon=|\xi|^{-1}$ gives

$$(6.14)\qquad \xi^{2-2\mu}\sum_{i=1}^{2}\int_{d_\vartheta}|\tilde{e}_i|^2|x'|^{-2\mu}dx'\leqslant c\sum_{i=1}^{2}\int_{d_\vartheta}(|\nabla'\tilde{e}_i|^2+\xi^2|\tilde{e}_i|^2)dx'$$

so using it in (6.13) one gets

$$(6.15)\qquad \xi^{2-2\mu}\sum_{i=1}^{2}\int_{d_\vartheta}(|\nabla'\tilde{e}_i|^2+\xi^2|\tilde{e}_i|^2)dx'\leqslant c\sum_{i=1}^{2}\int_{d_\vartheta}|\tilde{f}_i|^2|x'|^{2\mu}dx'$$

for $\mu \in (0, 1)$. Putting $\mu = 1$ in (6.13), then using

$$\sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{e}_i|^2 |x'|^{-2} dx' \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\nabla' \tilde{e}_i|^2 dx' \tag{6.16}$$

which is obtained similarly to (6.4), one has (6.15) for $\mu = 1$. To show the estimate

$$\sum_{i=1}^{2} \xi^2 \int_{d_\vartheta} (|\nabla' \tilde{e}_i|^2 + \xi^2 |\tilde{e}_i|^2) |x'|^{2\mu} dx' \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx' \tag{6.17}$$

we repeat some considerations from the proof of Theorem 4.2 (a) in [13]. We put $\eta_i = \tilde{e}_i V_s(x', \xi)$ in (6.12), where

$$V_s(x', \xi) = \min\left(s|\xi|^{-2\mu}, \max(|x'|^{2\mu}, |\xi|^{-2\mu})\right)\xi^2, \ s \geqslant 1.$$

The function V_s is bounded and continuous in x'. Therefore

$$\begin{aligned} J_s \equiv & \sum_{i=1}^{2} \int_{d_\vartheta} (|\nabla' \tilde{e}_i|^2 + \xi^2 |\tilde{e}_i|^2) V_s dx' \\ = & \sum_{i=1}^{2} \int_{d_\vartheta} (\tilde{f}_i \tilde{e}_i V_s - \nabla' \tilde{e}_i \nabla' V_s \tilde{e}_i) dx' \\ \leqslant & \Big(\sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'\Big)^{1/2} \Big(\sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{e}_i|^2 V_s^2 |x'|^{-2\mu} dx'\Big)^{1/2} \\ & + \Big(\sum_{i=1}^{2} \int_{d_\vartheta} |\nabla' \tilde{e}_i|^2 V_s dx'\Big)^{1/2} \Big(\sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{e}_i|^2 |\nabla' V_s|^2 V_s^{-1} dx'\Big)^{1/2}. \end{aligned} \tag{6.18}$$

Since $\tilde{e}_i \in H^1(d_\vartheta)$, $i = 1, 2$, J_s makes sense. From

$$|\nabla' V_s|^2 V_s^{-1} \leqslant (2\mu)^2 \xi^{4-2\mu}$$

we have

$$\sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{e}_i|^2 |\nabla' V_s|^2 V_s^{-1} dx' \leqslant (2\mu)^2 \xi^{4-2\mu} \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{e}_i|^2 dx'.$$

Hence (6.15) implies that the right-hand side of the above inequality is bounded by $c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'$. Moreover,

$$\sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{e}_i|^2 V_s^2 |x'|^{-2\mu} dx' \leqslant \sum_{i=1}^{2} \Big[\xi^2 \int_{d'_\vartheta} |\tilde{e}_i|^2 V_s dx' + \int_{d_\vartheta \setminus d'_\vartheta} |\tilde{e}_i|^2 |x'|^{-2\mu} \xi^{4-4\mu} dx'\Big],$$

where $d'_\vartheta = \{x' \in d_\vartheta \colon |x'| \geqslant |\xi|^{-1}\}$. From (6.14) and (6.15) we see that the last term in the last inequality is estimated by $c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'$. Therefore using the above estimates in (6.18) gives

$$J_s \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx',$$

and letting $s \to \infty$ we obtain (6.17).

From (6.17) and Theorem 4.1 it follows that there exists a unique solution of the problem (6.10) such that $\tilde{e}_i \in H^2_\mu(d_\vartheta)$, $i = 1, 2$, and

$$(6.19) \quad \sum_{i=1}^{2} \int_{d_\vartheta} \big((|\nabla'\nabla'\tilde{e}_i|^2 + \xi^2 |\nabla'\tilde{e}_i|^2 + \xi^4 |\tilde{e}_i|^2)|x'|^{2\mu} + |\nabla'\tilde{e}_i|^2 |x'|^{2\mu-2} + |\tilde{e}_i|^2 |x'|^{2\mu-4}\big) dx' \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'.$$

Putting $\eta_i = \tilde{e}_i |x'|^{2\mu-2}$ in (6.12) (which can be done because a weak solution satisfies (6.19)) we obtain

$$(6.20) \quad \sum_{i=1}^{2} \int_{d_\vartheta} (|\nabla'\tilde{e}_i|^2 |x'|^{2\mu-2} + \xi^2 |\tilde{e}_i|^2 |x'|^{2\mu-2}) dx' \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} (|\tilde{f}_i|^2 |x'|^{2\mu} + |\tilde{e}_i|^2 |x'|^{2\mu-4}) dx'.$$

From (6.19), (6.20) and the Parseval identity we get

$$(6.21) \quad \sum_{i=1}^{2} \|e_i\|_{H^2_\mu(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L_{2,\mu}(\mathscr{D}_\vartheta)},$$

which implies (6.9) for $k = 0$, $\mu \in (0, 1]$.

(b) Let $k > 0$, $\mu \in (0, 1]$. Then we can only consider $\vartheta < \pi$.

Lemma 4.4 implies the existence of functions $v_i \in L^{k+2}_\mu(d_\vartheta)$, $i = 1, 2$, satisfying homogeneous boundary conditions in (6.10) and such that $\tilde{f}_i - \Delta v_i \in H^k_\mu(d_\vartheta)$, $i = 1, 2$. Therefore in this section we can restrict our considerations to $\tilde{f}_i \in H^k_\mu(d_\vartheta)$, $i = 1, 2$. From (6.19) and (6.20) we have

$$(6.22) \quad \sum_{i=1}^{2} \int_{d_\vartheta} (|\nabla'\tilde{e}_i|^2 |x'|^{2\mu-2} + \xi^2 |\tilde{e}_i|^2 |x'|^{2\mu-2}) dx' \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'$$

and from (6.19) and (6.22) we get

$$(6.23) \quad \sum_{i=1}^{2} \xi^4 \int_{d_\vartheta} (|\nabla'\tilde{e}_i|^2 |x'|^{2\mu} + |\tilde{e}_i|^2 |x'|^{2\mu-2}) dx' \leqslant c \sum_{i=1}^{2} \xi^2 \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx',$$

so $\xi^2 \tilde{e}_i \in H^1_\mu(d_\vartheta)$, $i = 1, 2$. Knowing that $\tilde{f}_i \in \mathscr{E}^1_\mu(d_\vartheta)$ and $\operatorname{supp} f_i$ is compact, $i = 1, 2$, we shall show that $\xi^2 \tilde{e}_i \in H^1_{\mu+1}(d_\vartheta)$. To do this we consider the expression

$$\sum_{i=1}^{2} \xi^4 \int_{d_\vartheta} (|\nabla' \tilde{e}_i|^2 + \xi^2 |\tilde{e}_i|^2)|x'|^{2\mu+2} dx'$$

$$= \sum_{i=1}^{2} \xi^4 \Big(\int_{d_\vartheta''} + \int_{d_\vartheta \setminus d_\vartheta''}\Big)(|\nabla' \tilde{e}_i|^2 + \xi^2 |\tilde{e}_i|^2)|x'|^{2\mu+2} dx' = I_1 + I_2,$$

where $d_\vartheta'' = \{x' \in d_\vartheta \colon |\xi| \cdot |x'| \leqslant 1\}$. Using (6.17) we obtain

$$I_1 \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'. \tag{6.24}$$

The previous estimates only guarantee that I_2 makes sence for functions $\tilde{e}_i$, $i = 1, 2$, with compact support. Therefore to estimate I_2 we introduce the functions

$$\tilde{\omega}_i = \tilde{e}_i \zeta\left(\frac{|\xi| x'}{r}\right), \quad i = 1, 2, \; r \gg 1, \tag{6.25}$$

which are generalized solutions of the problem

$$\begin{gathered} -\Delta' \tilde{\omega}_i + \xi^2 \tilde{\omega}_i = \tilde{f}_i \zeta - 2\nabla' e_i \nabla' \zeta - \tilde{e}_i \Delta' \zeta \equiv \tilde{g}_i, \quad i = 1, 2, \\ \tilde{\omega}_1|_{\gamma_1} = 0, \quad (\tilde{\omega}_1 \cos\vartheta + \tilde{\omega}_2 \sin\vartheta)|_{\gamma_2} = 0, \\ \frac{\partial \tilde{\omega}_2}{\partial n}\bigg|_{\gamma_1} = \left(\tilde{e}_2 \frac{\partial \zeta}{\partial n}\right)\bigg|_{\gamma_1} = 0, \\ \left(\frac{\partial \tilde{\omega}_1}{\partial n} \sin\vartheta - \frac{\partial \tilde{\omega}_2}{\partial n} \cos\vartheta\right)\bigg|_{\gamma_2} = (\tilde{e}_1 \sin\vartheta - \tilde{e}_2 \cos\vartheta) \frac{\partial \zeta}{\partial n}\bigg|_{\gamma_2} = 0 \end{gathered} \tag{6.26}$$

because $\zeta(x')$ is a function of $|x'|$. From (6.19) one has

$$\sum_{i=1}^{2} \|g_i\|_{L_{2,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} (\|f_i\|_{L_{2,\mu}(d_\vartheta)} + \|\nabla' \tilde{e}_i\|_{L_{2,\mu-1}(d_\vartheta)} + \|\tilde{e}_i\|_{L_{2,\mu-2}(d_\vartheta)}) \leqslant c \sum_{i=1}^{2} \|f_i\|_{L_{2,\mu}(d_\vartheta)}, \tag{6.27}$$

where

$$|\nabla' \zeta| \leqslant c \frac{|\xi|}{r} |\dot{\zeta}| = \frac{c}{|x'|} \frac{|\xi||x'|}{r} |\dot{\zeta}| \leqslant \frac{c}{|x'|}$$

and the dot in $\dot{\zeta}$ denotes the derivative. Therefore $\tilde{\omega}_i$, $i = 1, 2$, are solutions of the integral inequality (6.12) where $\tilde{f}_i$ is replaced by $\tilde{g}_i$, $i = 1, 2$, and thus they also satisfy (6.19) and (6.20).

Now we shall estimate I_2. Since the functions $\eta_i = \tilde{\omega}_i |x'|^{2\mu+2} \zeta_1(|\xi||x'|)$, $i = 1, 2$, where $\zeta_i(r_0) = 0$ for $r_0 \leqslant \frac{1}{2}$ and $\zeta_1(r_0) = 1$ for $r_0 \geqslant 1$, belong to $H^1(d_\vartheta)$

for every bounded r, we can put them into (6.12), where $\tilde{e}_i, \tilde{f}_i$ are replaced by $\tilde{\omega}_i$, $\tilde{g}_i$, $i = 1, 2$, respectively. Hence

$$I_2'(r) \equiv \xi^4 \sum_{i=1}^{2} \int_{d_\vartheta} (|\nabla'\tilde{\omega}_i|^2 + \xi^2|\tilde{\omega}_i|^2)|x'|^{2\mu+2}\zeta_1(|\xi||x'|)dx'$$
$$= -\xi^4 \sum_{i=1}^{2} \int_{d_\vartheta} \nabla'\tilde{\omega}_i\bar{\tilde{\omega}}_i((2\mu+2)|x'|^{2\mu+1}\nabla'|x'|\zeta_1 + |x'|^{2\mu+2}\nabla'\zeta_1)dx'$$
$$+\xi^4 \sum_{i=1}^{2} \int_{d_\vartheta} \tilde{g}_i\cdot\bar{\tilde{\omega}}_i|x'|^{2\mu+2}\zeta_1(|\xi||x'|)dx'.$$

Using the Young inequality one has

$$(6.28)\qquad \sum_{i=1}^{2} \xi^4 \int_{d_\vartheta} (|\nabla'\tilde{\omega}_i|^2 + \xi^2|\tilde{\omega}_i|^2)|x'|^{2\mu+2}\zeta_1(|\xi||x'|)dx'$$
$$\leqslant \sum_{i=1}^{2} \int_{d_\vartheta} \Big[\varepsilon|\tilde{\omega}_i|^2|\xi|^6|x'|^{2\mu+2}\zeta_1 + \frac{c}{\varepsilon}|\nabla'\tilde{\omega}_i|^2|\xi|^2|x'|^{2\mu}$$
$$+c|\nabla'\omega_i|^2|\xi|^2|x'|^{2\mu+2} + c|\tilde{\omega}_i|^2|\xi|^8|x'|^{2\mu+4}|\zeta_1'|$$
$$+\frac{c}{\varepsilon}|\tilde{g}_i|^2|\xi|^2|x'|^{2\mu+2} + \varepsilon|\tilde{\omega}_i|^2|\xi|^6|x'|^{2\mu+2}\zeta_1\Big]dx'.$$

We use $\nabla'\zeta_1 = \zeta_1'|\xi|\nabla'|x'|$ and that $\zeta_1' \neq 0$ for $\frac{1}{2} \leqslant |\xi||x'| \leqslant 1$. Since $\tilde{\omega}_i$, $i = 1, 2$, satisfy 6.19, from (6.28) we obtain for sufficiently small ε

$$(6.29)\qquad I_2'(r) \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} \xi^2(|\nabla'\tilde{\omega}_i|^2 + \xi^2|\tilde{\omega}_i|^2)|x'|^{2\mu}dx'$$
$$+c\xi^2 \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{g}_i|^2|x'|^{2\mu+2}dx'$$
$$\leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} [\xi^2|\tilde{g}_i|^2|x'|^{2\mu+2} + |\tilde{g}_i|^2|x'|^{2\mu}]dx'.$$

The first term on the right-hand side of (6.29) is estimated by

$$\xi^2 \sum_{i=1}^{2} \int_{d_\vartheta} (|\tilde{f}_i|^2|x'|^{2\mu+2} + |\nabla'\tilde{\omega}_i|^2|x'|^{2\mu} + |\tilde{\omega}_i|^2|x'|^{2\mu-2})dx'.$$

Hence using (6.19), (6.20) with $\tilde{\omega}_i$, $\tilde{g}_i$, $i = 1, 2$, instead of $\tilde{e}_i$, $\tilde{f}_i$, $i = 1, 2$, and (6.27) we obtain the following estimate

$$(6.30)\qquad I_2'(r) \leqslant c(\xi^2+1) \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2|x'|^{2\mu}dx',$$

where the compactness of supp f_i, $i = 1, 2$, has been used. Letting $r \to \infty$ in the left-hand side gives

$$I_2 \leqslant c(\xi^2+1) \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx'. \tag{6.31}$$

From (6.24) and (6.31) we get

$$\sum_{i=1}^{2} \xi^4 \int_{d_\vartheta} (|\nabla' \tilde{e}_i|^2 + \xi^2 |\tilde{e}_i|^2) |x'|^{2\mu+2} dx' \leqslant c \sum_{i=1}^{2} \Big[\xi^2 \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu} dx' + \int_{d_\vartheta} |\tilde{f}_i|^2 |x'|^{2\mu-2} dx'\Big]. \tag{6.32}$$

From the first term of (6.32) and the Hardy inequality (2.20) we have

$$\sum_{i=1}^{2} \|\xi^2 e_i\|_{H^1_{1+\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{\mathscr{E}^1_\mu(d_\vartheta)}. \tag{6.33}$$

From (6.23), (6.33) and the compactness of $\operatorname{supp} \tilde{f}_i$, $i = 1, 2$, we see that $\tilde{f}_i - \zeta^2 \tilde{e}_i \in H^1_\mu(d_\vartheta) \cap H^1_{\mu+1}(d_\vartheta)$. Therefore, for $\mu \in (0, 1)$ Theorem 4.1 implies the existence of two different solutions $\tilde{e}_i$ and $\tilde{e}_i^1$, $i = 1, 2$, of the problem (6.10) with the same right-hand sides $\tilde{f}_i$, $i = 1, 2$, such that $\tilde{e}_i \in H^3_{1+\mu}(d_\vartheta) \cap \mathscr{H}(d_\vartheta)$ and $\tilde{e}_i^1 \in H^3_\mu(d_\vartheta) \cap \mathscr{H}(d_\vartheta)$, $i = 1, 2$. Since 1 is an eigenvalue for the problem (6.10) between $1+1-(1+\mu) = 1-\mu$ and $1+1-\mu = 2-\mu$, the second part of Theorem 4.1 gives

$$\tilde{e}_1^1 = \tilde{e}_i + a x_2, \quad \tilde{e}_2^1 = \tilde{e}_2 - a x_1. \tag{6.34}$$

Moreover, imbedding theorems imply that

$$a = -\tilde{e}_{1,x_2}|_{x'=0} = \tilde{e}_{2,x_1}|_{x'=0}.$$

Hence (6.23) gives

$$\sum_{i=1}^{2} \|\tilde{e}_i^1\|^2_{H^3_\mu(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{H^1_\mu(d_\vartheta)}. \tag{6.35}$$

To show higher smoothness we use induction. Assuming that $\tilde{e}_i^0 = \tilde{e}_i$, $i = 1, 2$, from (6.19), (6.23) and (6.35) we have

$$\sum_{i=1}^{2} \|\tilde{e}_i^0\|^2_{\mathscr{E}^2_\mu(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{L_{2,\mu}(d_\vartheta)}, \tag{6.36}$$

$$\sum_{i=1}^{2} \|\tilde{e}_i^1\|^2_{\mathscr{E}^3_\mu(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{\mathscr{E}^1_\mu(d_\vartheta)}. \tag{6.37}$$

Using the considerations from [13], [14] we shall prove (6.9) for $k > 1$. Therefore we shall find solutions of (6.10) in the form

$$\tilde{e}_i = \tilde{e}_i^\sigma + \sum_{j=0}^{\sigma} P_i^j(x'), \quad i = 1, 2, \tag{6.38}$$

where $\sigma \leqslant k$, $P^j_i(x')$ are homogeneous polynomials of degree j with respect to x', $\tilde{e}^\sigma_i \in \mathscr{E}^{\sigma+2}_\mu(d_\vartheta)$ and

$$\sum_{i=1}^{2} \|\tilde{e}^\sigma_i\|^2_{\mathscr{E}^{\sigma+2}_\mu(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{\mathscr{E}^\sigma_\mu(d_\vartheta)}. \tag{6.39}$$

Hence by the Parseval identity we get $e^\sigma_i \in H^{\sigma+2}_\mu(\mathcal{D}_\vartheta)$ and

$$\sum_{i=1}^{2} \|e^\sigma_i\|_{H^{\sigma+2}_\mu(\mathcal{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{H^\sigma_\mu(\mathcal{D}_\vartheta)}. \tag{6.40}$$

On the other hand, using the decomposition (6.38) and the form of the homogeneous polynomials one gets

$$\|\tilde{e}_i\|_{\mathscr{L}^{\sigma+2}_\mu(d_\vartheta)} = \|\tilde{e}^\sigma_i\|_{\mathscr{L}^{\sigma+2}_\mu(d_\vartheta)} \leqslant \|e^\sigma_i\|_{\mathscr{E}^{\sigma+2}_\mu(d_\vartheta)}. \tag{6.41}$$

Now we prove formulas (6.38)–(6.41) inductively, but in a constructive way. Let $\tilde{e}^\sigma_i(x')$, $i = 1, 2$, $\sigma \leqslant k$, be determined by the following system of problems

$$\begin{gathered} -\Delta' \tilde{e}^\sigma_i = -\xi^2 \tilde{e}^{\sigma-2}_i + \tilde{f}_i, \quad i = 1, 2, \\ \tilde{e}^\sigma_1|_{\gamma_1} = 0, \quad (\tilde{e}^\sigma_1 \cos\vartheta + \tilde{e}^\sigma_2 \sin\vartheta)|_{\gamma_2} = 0, \\ \left.\frac{\partial \tilde{e}^\sigma_2}{\partial n}\right|_{\gamma_1} = 0, \quad \left.\left(\frac{\partial \tilde{e}^\sigma_1}{\partial n}\sin\vartheta - \frac{\partial \tilde{e}^\sigma_2}{\partial n}\cos\vartheta\right)\right|_{\gamma_2} = 0, \end{gathered} \tag{6.42}$$

where $\sigma \geqslant 2$, $\tilde{e}^0_i = \tilde{e}_i$, $i = 1, 2$, and $\tilde{e}^1_i$, $i = 1, 2$, are defined by (6.34). Similarly the homogeneous polynomials P^j_i are determined by

$$\begin{gathered} -\Delta' P^j_i = -\xi^2 P^{j-2}_i, \quad i = 1, 2, \\ P^j_1|_{\gamma_1} = 0, \quad (P^j_1 \cos\vartheta + P^j_2 \sin\vartheta)|_{\gamma_2} = 0, \\ \left.\frac{\partial P^j_2}{\partial n}\right|_{\gamma_1} = 0, \quad \left.\left(\frac{\partial P^j_1}{\partial n}\sin\vartheta - \frac{\partial P^j_2}{\partial n}\cos\vartheta\right)\right|_{\gamma_2} = 0, \end{gathered} \tag{6.43}$$

where $j \geqslant 2$, and

$$P^0_i(x') = 0,\ i = 1, 2, \quad P^1_1(x') = -a x_2, \quad P^1_2(x') = a x_1, \tag{6.44}$$

where a appears in (6.34). By (6.43) and (6.44), the homogeneous polynomials of even degree are equal to zero.

Theorem 4.1 and Lemma 4.3 guarantee the existence of solutions of the problems (6.42) and (6.43), respectively.

Since (6.38) is valid for $\sigma = 1$ (see (6.34)), assume that (6.38) and (6.39) are satisfied for $\sigma \leqslant s-1$. Then using Theorem 4.1 to (6.42) for $\sigma = s$ we have the existence of solutions $\tilde{e}^s_i \in \mathscr{E}^{s+2}_\mu(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|\tilde{e}^s_i\|^2_{\mathscr{E}^{s+2}_\mu(d_\vartheta)} \leqslant c \sum_{i=1}^{2} (\xi^2 \|\tilde{e}^{s-2}_i\|^2_{\mathscr{E}^s_\mu(d_\vartheta)} + \|\tilde{f}_i\|^2_{\mathscr{E}^s_\mu(d_\vartheta)}) \tag{6.45}$$

$$\leqslant c \sum_{i=1}^{2} (\xi^2 \|\tilde{f}_i\|^2_{\mathscr{E}_\mu^{s-2}(d_\vartheta)} + \|\tilde{f}_i\|^2_{\mathscr{E}_\mu^{s}(d_\vartheta)})$$

$$\leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{\mathscr{E}_\mu^{s}(d_\vartheta)},$$

where (6.39) for $\sigma = s-2$ and the definition of the space $\mathscr{E}^s_\mu(d_\vartheta)$ have been used. Now we prove (6.38). To do this we prove that $\tilde{e}_i^s + P_i^s = \tilde{e}_i^{s-1}$, $i = 1, 2$, by observing that $\tilde{e}_i^s + (1-\zeta(x'))P_i^s \equiv v_i' \in H_\mu^{s+2}(d_\vartheta)$ and $\tilde{e}_i^{s-1} - \zeta(x')P_i^s \equiv v_i'' \in H_\mu^{s+1}(d_\vartheta)$, $i = 1, 2$, are solutions of the same problem

$$-\Delta' v_i = h_i, \quad i = 1, 2,$$

$$\text{(6.46)} \qquad v_1|_{\gamma_1} = 0, \quad (v_1 \cos\vartheta + v_2 \sin\vartheta)|_{\gamma_2} = 0,$$

$$\left.\frac{\partial v_2}{\partial n}\right|_{\gamma_1} = 0, \quad \left.\left(\frac{\partial v_1}{\partial n}\sin\vartheta - \frac{\partial v_2}{\partial n}\cos\vartheta\right)\right|_{\gamma_2} = 0,$$

where

$$h_i = -\xi^2 \tilde{e}_i^{s-2} - \xi^2(1-\zeta(x'))P_i^{s-2} + 2\nabla'\zeta\nabla' P_i^s + P_i^s \Delta'\zeta$$

$$= -\xi^2 \tilde{e}_i^{s-3} + \xi^2 \zeta(x') P_i^{s-2} + 2\nabla'\zeta\nabla' P_i^s + P_i^s\Delta'\zeta \in H_\mu^s(d_\vartheta) \cap H_\mu^{s-1}(d_\vartheta).$$

Therefore Theorem 4.1 implies that $v' = v''$, so $\tilde{e}_i^s + P_i^s = \tilde{e}_i^{s-1}$. Hence (6.38)-(6.41) are shown. From (6.41) using Lemma 4.4 and the Parserval identity we obtain (6.9).

Now we consider the case $\mu = 1$. By (6.23), we have $\xi\tilde{e}_i \in H_1^1(d_\vartheta)$, $i = 1, 2$. Hence the right-hand side functions in (6.10) equal to $\tilde{f}_i - \xi^2\tilde{e}_i$, $i = 1, 2$, belong to $H_1^1(d_\vartheta)$. However, in this case Theorem 4.1 cannot be used for the problem (6.10) because $1+k-\mu = 1+1-1 = 1$ is its eigenvalue. Therefore Remark 1.1 on p. 220 of [7] implies that the weak solution of the problem (6.10) is such that $D_{x'}^2\tilde{e}_i \in H_1^1(d_\vartheta)$, $i = 1, 2$, and

$$\text{(6.47)} \qquad \sum_{i=1}^{2} \|D_{x'}^2 \tilde{e}_i\|_{H_1^1(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{H_1^1(d_\vartheta)}.$$

Using (6.19) we have

$$\text{(6.48)} \qquad \sum_{i=1}^{2} \|\tilde{e}_i\|_{\mathscr{L}_1^2(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{L_{2,1}(d_\vartheta)}$$

so (6.47) and (6.48) imply

$$\text{(6.49)} \qquad \sum_{i=1}^{2} \|\tilde{e}_i\|_{\mathscr{L}_1^3(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{\mathscr{L}_1^1(d_\vartheta)}.$$

To obtain the further estimates we use induction. Let $\tilde{e}_i \in \mathscr{L}_1^{\sigma+2}(d_\vartheta)$, $i = 1, 2$, for

$\sigma \leqslant s-1$, $s \leqslant k$, and

$$(6.50)\qquad \sum_{i=1}^{2} \|\tilde{e}_i\|_{\mathscr{L}_1^{\sigma+2}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{\mathscr{L}_1^{\sigma}(d_\vartheta)}.$$

Then using (6.50) for $\sigma = s-2$ we get $\xi^2\tilde{e}_i \in \mathscr{L}_1^s(d_\vartheta)$, $i = 1, 2$, and

$$(6.51)\qquad \|\xi^2\tilde{e}_i\|_{\mathscr{L}_1^s(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|_{\mathscr{L}_1^s(d_\vartheta)}.$$

Hence by (6.51) and Theorem 4.3 the weak solution of the problem (6.10) belongs to $\mathscr{L}_1^{s+2}(d_\vartheta)$ and (6.50) is satisfied for $\sigma = s$. Therefore, by induction, (6.50) is valid for all $\sigma \leqslant k$, and by the Parseval identity $e_i \in L_\mu^{k+2}(\mathscr{D}_\vartheta)$, $i = 1, 2$, and (6.9) is valid too.

(c) Let $k \geqslant 0$, $\mu = 0$, $\vartheta \leqslant \pi/2$. Putting $\eta_i = \tilde{e}_i$, $i = 1, 2$, into (6.12) we obtain

$$(6.52)\qquad \sum_{i=1}^{2} \xi^2 \int_{d_\vartheta} (|\nabla'\tilde{e}_i|^2 + \xi^2|\tilde{e}_i|^2)dx' \leqslant c \sum_{i=1}^{2} \int_{d_\vartheta} |\tilde{f}_i|^2 dx',$$

therefore $\xi^2\tilde{e}_i \in L_{2,0}(d_\vartheta)$. Since $1-\mu = 1$ is an eigenevalue of the problem (6.10), from (6.10) and Remark 1.1 on p. 220 of [7] it follows that $D_{x'}^2\tilde{e}_i \in L_{2,0}(d_\vartheta)$ and

$$(6.53)\qquad \sum_{i=1}^{2} \|D_{x'}^2\tilde{e}_i\|^2_{L_{2,0}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{L_{2,0}(d_\vartheta)}.$$

Therefore we have proved (6.9) for $k = 0$. For $k > 0$ and $\tilde{f}_i \in \mathscr{L}_0^k(d_\vartheta)$ we shall prove (6.9) inductively. Let $k = 1$; then from (6.52) it follows that $-\xi^2\tilde{e}_i + \tilde{f}_i \in \mathscr{L}_0^1(d_\vartheta)$. Therefore from Theorem 4.3 it follows that $\tilde{e}_i \in \mathscr{L}_0^3(d_\vartheta)$ and

$$(6.54)\qquad \sum_{i=1}^{2} \|\tilde{e}_i\|^2_{\mathscr{L}_0^3(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{\mathscr{L}_0^1(d_\vartheta)}.$$

By induction, $\tilde{e}_i \in \mathscr{L}_0^{s+2}(d_\vartheta)$, $s \leqslant k$, and

$$(6.55)\qquad \sum_{i=1}^{2} \|\tilde{e}_i\|^2_{\mathscr{L}_0^{s+2}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\tilde{f}_i\|^2_{\mathscr{L}_0^s(d_\vartheta)}, \qquad s \leqslant k.$$

From (6.55) using the Parseval identity we obtain (6.9).

(d) Let $k \geqslant 0$, $\mu \in [0, 1]$, $\pi/2 < \vartheta \leqslant \pi$. In this case the considerations are similar, except that 1 is not an eigenvalue. Moreover, in this case we only have $k = 0$, because $1 > \pi/\vartheta - 1 > 1-\mu$.

(e) $k \geqslant 0$, $\mu \in [0, 1]$, $\pi \leqslant \vartheta < 2\pi$ and $2\pi/\vartheta - 1 < 1$, so (6.7) holds for $k = 0$ only. This case can be considered in the same way as (d), because $1-\pi/\vartheta < 1$.

(f) $k \geqslant 0$, $\mu > 0$. This case can be proved using the local estimates. See the proof of Theorem 4.2 on p. 30 of [13].

Remark 6.1. In cases (a), (d), (e) where the eigenvalue 1 of the problem (6.10) is greater than $k+1-\mu$ the above results follow from the general methods of Maz'ya and Plamenevskii [10], [11].

7. Green function

In this section we construct and obtain estimates for the Green function of the problem (6.1), (6.2). In our considerations we use the results and methods from [14], [16]. First we introduce half-spaces R_1, R_2 with boundaries Γ_1, Γ_2, respectively such that $\mathscr{D}_\vartheta = R_1 \cap R_2$ for $\vartheta \leqslant \pi$ and $\mathscr{D}_\vartheta = R_1 \cup R_2$ for $\vartheta > \pi$.

The Green function for (6.1), (6.2) has a matrix form $G_{ij}(x, y)$, $i, j = 1, 2$, and is a solution of the following system

(7.1)
$$\begin{gathered} -\Delta G_{ij}(x, y) = \delta_{ij}\delta(x-y), \quad i, j = 1, 2, \\ G_{1j}|_{x\in\Gamma_1} = 0, \quad (G_{1j}\cos\vartheta + G_{2j}\sin\vartheta)|_{x\in\Gamma_2} = 0, \\ \left.\frac{\partial G_{2j}}{\partial n}\right|_{x\in\Gamma_1} = 0, \quad \left.\left(\frac{\partial G_{1j}}{\partial n}\sin\vartheta - \frac{\partial G_{2j}}{\partial n}\cos\vartheta\right)\right|_{x\in\Gamma_2} = 0. \end{gathered}$$

Following [14] let

(7.2)
$$G_{ij}(x, y) = \psi(x, y)\mathscr{E}(x, y)\delta_{ij} + G'_{ij}(x, y), \quad i, j = 1, 2,$$

where $\mathscr{E}(x-y)$ is the fundamental solution for the Laplace equation, $\psi(x, y)$ is a smooth function such that $\psi(x, y) = 1$ for x, y sufficiently close to each other and $\psi(x, y) = 0$ for x, y sufficiently distant and for x close to M. Then for any $y \in \mathscr{D}_\vartheta$ the functions $G'_{ij}(x, y)$, $i, j = 1, 2$, are solutions of the following problem:

(7.3)
$$\begin{gathered} -\Delta G'_{ij} = (2\nabla\psi\nabla\mathscr{E} + \mathscr{E}\Delta\psi)\delta_{ij}, \quad i, j = 1, 2, \\ G'_{1j}|_{\Gamma_1} = -(\psi\mathscr{E})|_{\Gamma_1}\delta_{1j}, \quad \left.\frac{\partial G'_{2j}}{\partial n}\right|_{\Gamma_1} = -\frac{\partial}{\partial n}(\psi\mathscr{E})|_{\Gamma_1}\delta_{2j}, \\ (G'_{1j}\cos\vartheta + G'_{2j}\sin\vartheta)|_{\Gamma_2} = -(\psi\mathscr{E})|_{\Gamma_2}(\delta_{1j}\cos\vartheta + \delta_{2j}\sin\vartheta), \\ \left.\left(\frac{\partial G'_{1j}}{\partial n}\sin\vartheta - \frac{\partial G'_{2j}}{\partial n}\cos\vartheta\right)\right|_{\Gamma_2} = -\frac{\partial}{\partial n}(\psi\mathscr{E})|_{\Gamma_2}(\delta_{1j}\sin\vartheta - \delta_{2j}\cos\vartheta), \quad j = 1, 2. \end{gathered}$$

The Green function has the following properties (see [14], [16]).

(1) $G_{ij}(x, y)$, $i, j = 1, 2$, do not depend on the mollifier function $\psi(x, y)$.

(2) $G_{ij}(\lambda x, \lambda y) = \lambda^{-1}G(x, y)$, $\forall\, \lambda > 0$.

(3) $G_{ij}(x, y)$ are infinitely differentiable with respect to x, y if $x \neq y$ and x is sufficiently far from M, $\forall y \in \mathscr{D}_\vartheta$.

(4)

(7.4)
$$G_{ij}(x, y) = G_{ji}(y, x).$$

(5) The generalized solution $e_i \in \mathscr{H}(\mathscr{D}_\vartheta)$, $i = 1, 2$, of the problem (6.1), (6.2),

where f_i, $i = 1, 2$, are integrable functions with compact supports, can be expressed by

$$(7.5)\qquad e_i(x) = \sum_{j=1}^{2} \int_{\mathscr{D}_\vartheta} G_{ij}(x, y) f_j(y) dy, \quad i = 1, 2.$$

The first three properties follow from the definition (7.2), (7.3). To show (4), consider the expression [16]:

$$(7.6)\ \sum_{i=1}^{2} \int_{\mathscr{D}_\vartheta \backslash K_\varepsilon(y) \cup K_\varepsilon(z)} [G_{ij}(x, y) \Delta G_{ik}(x, z) - G_{ik}(x, z) \Delta G_{ij}(x, y)] \varphi_R(x) dx = 0,$$

where $y \neq z$, $K_\varepsilon(y)$, $K_\varepsilon(z) \subset \mathscr{D}_\vartheta$, $K_\varepsilon(y) \cap K_\varepsilon(z) = \emptyset$, $\varphi_R(x) = \zeta(x/R)$. Integrating by parts we get

$$(7.7)\qquad \sum_{i=1}^{2} \int_{\Gamma_1 \cup \Gamma_2} [G_{ij}(x, y) \frac{\partial}{\partial n} G_{ik}(x, z) - G_{ik}(x, z) \frac{\partial}{\partial n} G_{ij}(x, y)] \varphi_R(x) dx$$
$$+ \sum_{i=1}^{2} \int_{\partial K_\varepsilon(y) \cup \partial K_\varepsilon(z)} [G_{ij}(s, y) \frac{\partial}{\partial n} G_{ik}(s, z)$$
$$- G_{ik}(s, z) \frac{\partial}{\partial n} G_{ij}(s, y)] \varphi_R(s) ds$$
$$- \sum_{i=1}^{2} \int_{\mathscr{D}_\vartheta \backslash K_\varepsilon(y) \cup K_\varepsilon(z)} [G_{ij}(x, y) \nabla G_{ik}(x, z)$$
$$- G_{ik}(x, z) \nabla G_{ij}(x, y)] \nabla \varphi_R(x) dx = 0,$$

where the compactness of $\operatorname{supp} \varphi_R(x)$ implies that the other boundary terms disappear. From (7.1) it follows that the first term in (7.7) also vanishes.

We shall use

$$(7.8)\qquad \int_{\partial K_\varepsilon(y)} \frac{\partial G_{ij}(s, y)}{\partial n} ds = -4\pi \delta_{ij}, \qquad \int_{\partial K_\varepsilon(y)} \frac{\partial G_{ij}(s, z)}{\partial n} ds = 0.$$

Let us consider the behaviour as $\varepsilon \to 0$ of the first part of the second integral in (7.7). The second part can be examined analogously. Let

$$I_1 \equiv \sum_{i=1}^{2} \int_{\partial K_\varepsilon(y)} G_{ij}(s, y) \frac{\partial}{\partial n} G_{ik}(s, z) ds.$$

Since $s \notin K_\varepsilon(z)$, from (7.2), (7.3) and Theorem 6.2 we have $|\partial G_{ik}(s, z)/\partial n| \leqslant c$. Moreover, Theorem 6.2 implies $|G_{ij}(s, y)| \leqslant c/\varepsilon + c$, where $\varepsilon = |s - y|$, $s \in \partial K_\varepsilon(y)$, $s \notin \Gamma_i$, $i = 1, 2$. Hence

$$|I_1| \leqslant c\varepsilon^2 \left(1 + \frac{1}{\varepsilon}\right) \to 0 \quad \text{as } \varepsilon \to 0.$$

Let

$$I_2 \equiv \sum_{i=1}^{2} \int_{\partial K_\varepsilon(z)} G_{ij}(s, y)\frac{\partial}{\partial n} G_{ik}(s, z)dz$$
$$= \sum_{i=1}^{2} G_{ij}(z, y) \int_{\partial K_\varepsilon(z)} \frac{\partial}{\partial n} G_{ik}(s, z)ds$$
$$+ \sum_{i=1}^{2} \int_{\partial K_\varepsilon(z)} [G_{ij}(s, y)-G_{ij}(z, y)]\frac{\partial}{\partial n} G_{ik}(s, z)ds.$$

Theorem 6.2 implies the continuity of $G(s, y)$ with respect to s, $s \in \partial K_\varepsilon(z)$, so using (7.8) gives

$$I_2 \to -4\pi G_{kj} \quad \text{as } \varepsilon \to 0.$$

Therefore after letting $R\to\infty$ and $\varepsilon\to 0$ the second integral in (7.7) is equal to $-4\pi\big(G_{kj}(z, y)-G_{jk}(y, z)\big)$.

Let y, z be given. Since $\operatorname{supp}\nabla\varphi_R = \{x\colon \frac{1}{2}R < |x| < R\}$, for R sufficiently large $\psi(x, y) = \psi(x, z) = 0$ if $x \in \operatorname{supp}\nabla\varphi_R$. Therefore Theorems 6.1, 6.2 and (7.2), (7.3) yield that the last term in (7.7) vanishes. Hence after letting $\varepsilon\to 0$, $R\to\infty$ we see that property (4) holds.

To show property (5) we put $\eta_i(x) = \big[1-\zeta\big((x-y)/\varepsilon\big)\big]G_{ij}(x, y)$ into (6.5) and then let $\varepsilon\to 0$.

Together with the construction (7.2), (7.3) we shall use the following constructions of the Green function. Assume

(7.9) $G_{ij}(x, y) = \psi(x, y)G^k_{ij}(x, y)+G^{k'}_{ij}(x, y), \quad y \in \mathscr{D}^{(k)}_\vartheta,\ k = 0, 1;\ i, j = 1, 2,$

where $\psi(x, y) = \zeta(2|x-y|/|y'|)$, $\mathscr{D}^{(0)}_\vartheta = \mathscr{D}_{\vartheta/3}$, $\mathscr{D}^{(1)}_\vartheta = \mathscr{D}_\vartheta\backslash\mathscr{D}_{2\vartheta/3}$ and $G^0_{ij}(x, y)$ are solutions of

(7.10) $$-\Delta G^0_{ij}(x, y) = \delta_{ij}\delta(x-y), \quad G^0_{1j}|_{\Gamma_1} = 0,$$
$$\frac{\partial G^0_{2j}}{\partial n}\bigg|_{\Gamma_1} = 0, \quad i, j = 1, 2,\ x \in R_1.$$

Moreover, the $G^1_{ij}(x, y)$ are solutions of

(7.11) $$-\Delta G^1_{ij}(x, y) = \delta_{ij}\delta(x-y),$$
$$(G^1_{ij}\cos\vartheta+G^1_{2j}\sin\vartheta)|_{\Gamma_2} = 0, \quad \left(\frac{\partial G^1_{1j}}{\partial n}\sin\vartheta-\frac{\partial G^1_{2j}}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2} = 0,$$

where $i, j = 1, 2$, $x \in R_2$. Finally, for $y \in \mathscr{D}^{(2)}_\vartheta$, $\mathscr{D}^{(2)}_\vartheta = \mathscr{D}_\vartheta\backslash(\mathscr{D}^{(0)}_\vartheta \cup \mathscr{D}^{(1)}_\vartheta)$, we assume that the Green function has the form (7.2). Solving (7.10) we get

(7.12) $$G^0_{11}(x, y) = \mathscr{E}(x-y)-\mathscr{E}(x-y^*),$$
$$G^0_{22}(x, y) = \mathscr{E}(x-y)+\mathscr{E}(x-y^*),$$
$$G^0_{12} = G^0_{21} = 0,$$

where y^* is the point symmetrical to y with respect to Γ_1. Solving (7.11) we get
$G^1_{1j}\cos\vartheta + G^1_{2j}\sin\vartheta = [\mathscr{E}(x-y) - \mathscr{E}(x-\bar{y}^*)]a_j$, $j = 1, 2$, $a_1 = \cos\vartheta$, $a_2 = \sin\vartheta$,
$G^1_{1j}\sin\vartheta - G^1_{2j}\cos\vartheta = [\mathscr{E}(x-y) + \mathscr{E}(x-\bar{y}^*)]b_j$, $j = 1, 2$, $b_1 = \sin\vartheta$, $b_2 = -\cos\vartheta$,
therefore

$$G^1_{12} = G^1_{21} = -\mathscr{E}(x-\bar{y}^*)\sin 2\vartheta,$$
$$G^1_{11} = \mathscr{E}(x-y) - \mathscr{E}(x-\bar{y}^*)\cos 2\vartheta, \tag{7.13}$$
$$G^1_{22} = \mathscr{E}(x-y) + \mathscr{E}(x-\bar{y}^*)\cos 2\vartheta,$$

where $\bar{y}^*$ is the point symmetrical to y with respect to Γ_2.

Now we shall formulate the problems for $G^{k'}_{ij}(x, y)$, $k = 0, 1, 2$. From (7.9) and (7.10) we have

$$-\Delta G^{0'}_{ii} = 2\nabla\psi\nabla G^0_{ii} + G^0_{ii}\Delta\psi,\ i = 1, 2, \quad \Delta G^{0'}_{12} = \Delta G^{0'}_{21} = 0,$$
$$G^{0'}_{1j}|_{\Gamma_1} = 0, \quad \frac{\partial G^{0'}_{21}}{\partial n}\bigg|_{\Gamma_1} = 0, \quad \frac{\partial G^{0'}_{22}}{\partial n}\bigg|_{\Gamma_1} = -G^0_{22}\frac{\partial\psi}{\partial n}\bigg|_{\Gamma_1},$$
$$(G^{0'}_{1j}\cos\vartheta + G^{0'}_{2j}\sin\vartheta)|_{\Gamma_2} = -\psi(G^0_{1j}\cos\vartheta + G^0_{2j}\sin\vartheta)|_{\Gamma_2}, \tag{7.14}$$
$$\left(\frac{\partial G^{0'}_{1j}}{\partial n}\sin\vartheta - \frac{\partial G^{0'}_{2j}}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2} = -\left(\frac{\partial(\psi G^0_{1j})}{\partial n}\sin\vartheta - \frac{\partial(\psi G^0_{2j})}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2},$$

and $y \in \mathscr{D}^{(0)}_\vartheta$. From (7.9) and (7.13) we obtain

$$-\Delta G^{1'}_{ij} = 2\nabla\psi\nabla G^1_{ij} + G^1_{ij}\Delta\psi, \quad i, j = 1, 2,$$
$$G^{1'}_{1j}|_{\Gamma_1} = -\psi G^1_{1j}|_{\Gamma_1}, \quad \frac{\partial G^{1'}_{2j}}{\partial n}\bigg|_{\Gamma_1} = -\frac{\partial(\psi G^1_{2j})}{\partial n}\bigg|_{\Gamma_1},$$
$$(G^{1'}_{1j}\cos\vartheta + G^{1'}_{2j}\sin\vartheta)|_{\Gamma_2} = 0, \tag{7.15}$$
$$\left(\frac{\partial G^{1'}_{1j}}{\partial n}\sin\vartheta - \frac{\partial G^{1'}_{2j}}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2} = (G^1_{1j}\sin\vartheta - G^1_{2j}\cos\vartheta)\frac{\partial\psi}{\partial n}\bigg|_{\Gamma_2}, \quad j = 1, 2,$$

and $y \in \mathscr{D}^{(1)}_\vartheta$. Finally, let $G^{2'}_{ij} = G'_{ij}$, $i, j = 1, 2$.

Using the above considerations we obtain

THEOREM 7.1. *For arbitrary* $x, y \in \mathscr{D}_\vartheta$ *and multiindices* α, β

$$|D^\alpha_x D^\beta_y G_{ij}(x, y)| \tag{7.16}$$
$$\leqslant c|x-y|^{-1-|\alpha|-|\beta|}\left(\frac{|x'|}{|x'|+|x-y|}\right)^{\lambda_1-|\alpha'|}\left(\frac{|y'|}{|y'|+|x-y|}\right)^{\lambda_2-|\beta'|},$$

where $\alpha' = (\alpha_1, \alpha_2)$, $\beta' = (\beta_1, \beta_2)$, $\lambda_1 = \lambda_1(|\alpha'|) = \min(|\alpha'|, \Lambda-\varepsilon_1)$, $\lambda_2 = \lambda_2(|\beta'|) = \min(|\beta'|; \Lambda-\varepsilon_2)$, $\forall\ \varepsilon_i \in (0, \Lambda)$, $i = 1, 2$ *and* Λ *is described by* (6.8).

For $\vartheta < \frac{2}{3}\pi$ *we have* $\lambda_i < \frac{1}{2}$, $i = 1, 2$.

Proof. We start with the estimates of $G_{ij}^{k'}(x, y)$. Let $H_{ij}^{k(\beta)}(x, y) = D_y^\beta G_{ij}^{k'}(x, y)$. We first consider the problem (7.14) for $H_{ij}^{0(\beta)}(x, y)$, $y \in \mathscr{D}_\vartheta^0$; it has the form

$$-\Delta H_{ij}^{0(\beta)} = D_y^\beta[2\nabla\psi\nabla G_{ij}^0 + G_{ij}^0\Delta\psi] \equiv F_{ij}^{0(\beta)},$$

$$H_{1j}^{0(\beta)} = 0, \qquad \frac{\partial H_{2j}^{0(\beta)}}{\partial n}\bigg|_{\Gamma_1} = -D_y^\beta\left(G_{2j}^0\frac{\partial\psi}{\partial n}\right)\bigg|_{\Gamma_1} \equiv \Psi_1^{0(\beta)},$$

$$(H_{1j}^{0(\beta)}\cos\vartheta + H_{2j}^{0(\beta)}\sin\vartheta)|_{\Gamma_2} = -D_y^\beta[\psi(G_{1j}^0\cos\vartheta + G_{2j}^0\sin\vartheta)]|_{\Gamma_2} = \Phi_2^{0(\beta)},$$

$$\left(\frac{\partial H_{1j}^{0(\beta)}}{\partial n}\sin\vartheta - \frac{\partial H_{2j}^{0(\beta)}}{\partial n}\cos\vartheta\right)\bigg|_{\Gamma_2}$$

$$= -D_y^\beta\left[\frac{\partial}{\partial n}(\psi G_{1j}^0)\sin\vartheta - \frac{\partial}{\partial n}(\psi G_{2j}^0)\cos\vartheta\right]\bigg|_{\Gamma_2} \equiv \Psi_2^{0(\beta)}.$$

For $|y'| = 1$, $y \in \mathscr{D}_\vartheta^{(0)}$ we have

(7.17)
$$\|F_{ij}^{0(\beta)}\|^2_{L^k_{2,\mu}(\mathscr{D}_\vartheta)} + \|\Psi_1^{0(\beta)}\|^2_{L^{k+1}_{2,\mu}(\Gamma_1)} + \|\Psi_2^{0(\beta)}\|^2_{L^{k+1/2}_{2,\mu}(\Gamma_2)} + \|\Phi_2^{0(\beta)}\|^2_{L^{k+3/2}_{2,\mu}(\Gamma_2)} + \int_{\mathscr{D}_\vartheta}|F_{ij}^{0(\beta)}|^2|x'|^2dx + \int_{\Gamma_1}|\Psi_1^{0(\beta)}|^2|x'|ds + \int_{\Gamma_2}|\Psi_2^{0(\beta)}|^2|x'|ds + \int_{\Gamma_2}|\Phi_2^{0(\beta)}|^2|x'|ds \leqslant c.$$

Therefore, using Sobolev's imbedding theorems and inequalities (6.6), (6.9) for nonhomogeneous problems, for $|x-y| \leqslant \frac{1}{2}$, $k+2 > |\alpha|+\frac{3}{2}$, from (7.17) we get

$$|D_x^\alpha H_{ij}^{0(\beta)}(x, y)| \leqslant c\Big(\sum_{|\gamma|=k+2}\|D_x^\gamma H_{ij}^{0(\beta)}\|^2_{L_2(K_{1/2}(y))} + \|H_{ij}^{0(\beta)}\|^2_{L_2(K_{1/2}(y))}\Big)^{1/2}$$

$$\leqslant c\Big(\sum_{|\gamma|=k+2}\|D_x^\gamma H_{ij}^{0(\beta)}\|^2_{L_{2,\mu}(\mathscr{D})} + \int_{\mathscr{D}_\vartheta}|H_{ij}^{0(\beta)}|^2|x'|^{-2}dx\Big)^{1/2} \leqslant c.$$

Therefore, from the homogeneity of $H_{ij}^{0(\beta)}$ and G_{ij} we have

(7.18)
$$|D_x^\alpha H_{ij}^{0(\beta)}(x, y)| \leqslant c|y'|^{-1-|\alpha|-|\beta|}, \qquad y \in \mathscr{D}_\vartheta^{(0)},\ |x-y| \leqslant \tfrac{1}{2}|y'|,$$

hence

(7.19)
$$|D_x^\alpha D_y^\beta G_{ij}(x, y)| \leqslant c|x-y|^{-1-|\alpha|-|\beta|}, \qquad y \in \mathscr{D}_\vartheta^{(0)},\ |x-y| \leqslant \tfrac{1}{2}|y'|.$$

The same inequality holds for $y \in \mathscr{D}_\vartheta^{(1)}$ and $y \in \mathscr{D}_\vartheta^{(2)}$.

Now we estimate $D_x^\alpha D_y^\beta G$ for $1 = |x-y| \geqslant \frac{1}{2}|y'|$. For $\vartheta < \frac{2}{3}\pi$ we use the estimate [16]

(7.20)
$$|u(x)||x'|^{-\alpha} \leqslant c\|u\|_{H^2_\mu(\mathscr{D}_\vartheta)},$$

which is valid for $u \in H_\mu^2(\mathscr{D}_\vartheta)$, $\alpha = \frac{1}{2}-\mu$, $\mu < \frac{1}{2}$. For an arbitrary ϑ we can use the estimate [14]

$$(7.21)\qquad |D^{\alpha}_{x'}u(x)||x'|^{\varkappa} \leqslant c\,\|u\|_{H^k_\mu(d_\vartheta)},$$

which is valid for $u \in H^k_\mu(d_\vartheta)$, $k > 0$, $\mu \geqslant 0$, $|\alpha| < k-1$, $\varkappa \geqslant 0$, $|\alpha|-\varkappa \leqslant k-1-\mu$. In the following considerations we use (7.21).

Using the estimate (7.21) for $\zeta(4|x-z|)D^{\beta}_y G_{ij}(z, y)$, $z \in \mathscr{D}_\vartheta$, and then Sobolev's imbedding theorems we get

$$(7.22)\quad |D^{\alpha}_x D^{\beta}_y G_{ij}(x, y)||x'|^{\nu_1} \leqslant c\Big(\sum_{|\gamma|\leqslant k+2}\int_{K_{1/2}(x)} |D^{\gamma}_z D^{\beta}_y G_{ij}(z, y)|^2 |z'|^{2\mu} dz\Big)^{1/2},$$

where $k+2 \leqslant |\gamma'|$, $|\alpha''|+1 \leqslant |\gamma''|$, $|\alpha'| < k+1$, $1+k-\mu \geqslant |\alpha'|-\nu_1 \geqslant 0$, $\nu_1 > 0$ and α' corresponds to x', α'' to $x'' = x_3$.

For functions $e_i \in H^{k+2}_\mu(\mathscr{D}_\vartheta)$ which satisfy (6.1), (6.2) in $K_1(y) \cap \mathscr{D}_\vartheta$ for $f_i = 0$, $i = 1, 2$, the following local estimate is valid

$$(7.23)\qquad \sum_{|\alpha|=k+2}\sum_{i=1}^{2}\int_{K_{1/2}(y)} |D^{\gamma}_z D^{\alpha} e_i||x'|^{2\mu} dx < c\sum_{i=1}^{2}\int_{K_1(y)} |e_i|^2 dx.$$

For $\Lambda > 1+k-\mu$, from (7.22) and (7.23) we obtain

$$(7.24)\qquad |D^{\alpha}_x D^{\beta}_y G_{ij}(x, y)||x'|^{\nu_1} \leqslant c\Big(\int_{K_{1/2}(x)} |D^{\beta}_y G_{ij}(z, y)|^2 dz\Big)^{1/2}.$$

The solution of the problem

$$(7.25)\qquad \begin{gathered} -\Delta_z v_{ij}(z, y) = D^{\beta}_y G_{ij}(z, y)\zeta(2|z-x|),\\ v_{1j}|_{\Gamma_1} = 0, \qquad (v_{1j}\cos\vartheta + v_{2j}\sin\vartheta)|_{\Gamma_2} = 0,\\ \left.\frac{\partial v_{2j}}{\partial n_z}\right|_{\Gamma_1} = 0, \qquad \left.\left(\frac{\partial v_{1j}}{\partial n_z}\sin\vartheta - \frac{\partial v_{2j}}{\partial n_z}\cos\vartheta\right)\right|_{\Gamma_2} = 0, \end{gathered}$$

from $L^{k+2}_{2,\mu}(\mathscr{D}_\vartheta) \cap \mathscr{H}'(\mathscr{D}_\vartheta)$ in the form (7.5) satisfies (7.23). Therefore

$$(7.26)\quad \sum_{i=1}^{2} |D^{\beta}_z v_{ij}(z, y)|_{z=y}||y'|^{\nu_2} \leqslant c\Big(\sum_{i=1}^{2}\int_{K_{1/2}(y)} |v_{ij}(z, y)|^2 dz\Big)^{1/2}, \qquad z \in \mathscr{D}_\vartheta,$$

where $\Lambda > |\beta'|-\nu_2 > 0$, $\nu_2 \geqslant 0$. From (7.26) we have

$$(7.27)\qquad \begin{aligned} \sum_{i=1}^{2} |D^{\beta}_z v_{ii}(z, y)|_{z=y}||y'|^{\nu_2} &\leqslant \sum_{j=1}^{2}\sum_{i=1}^{2} |D^{\beta}_z v_{ij}(z, y)|_{z=y}||y'|^{\nu_2}\\ &\leqslant c\Big(\sum_{j=1}^{2}\sum_{i=1}^{2}\int_{K_{1/2}(y)} |v_{ij}(z, y)|^2 dz\Big)^{1/2}. \end{aligned}$$

The right-hand side of (7.27) is estimated by

$$(7.28)\quad \Big(\sum_{i=1}^{2}\int_{\mathscr{D}_\vartheta} |\nabla v_{ij}(z, y)|^2 dz\Big)^{1/2} \leqslant c\Big(\sum_{i=1}^{2}\int_{\mathscr{D}_\vartheta} |D^{\beta}_y G_{ij}(z, y)|^2 \zeta^2(2|x-z|)dz\Big)^{1/2}.$$

From (7.25) and (7.5) we have

$$D_z^\beta v_{ij}(z, y) = \sum_{k=1}^{2} \int_{\mathscr{D}_\vartheta} D_z^\beta G_{ik}(z, s) D_y^\beta G_{kj}(s, y)\zeta(2|s-x|)ds,$$

so using (7.4) gives

$$D_z^\beta v_{ij}(z, y)|_{z=y, i=j} = \sum_{k=1}^{2} \int_{\mathscr{D}_\vartheta} |D_y^\beta G_{kj}(s, y)|^2 \zeta(2|s-y|)ds. \tag{7.29}$$

Therefore from (7.27), (7.28) and (7.29) we get

$$\Big(\sum_{i,j=1}^{2} \int_{\mathscr{D}_\vartheta} |D_y^\beta G_{ij}(z, y)|^2 \zeta(2|x-z|)dz\Big)^{1/2} \leqslant c|y'|^{-\nu_2}. \tag{7.30}$$

Using (7.30) in (7.24) we obtain

$$|D_x^\alpha D_y^\beta G_{ij}(x, y)| \leqslant c|x'|^{-\nu_1}|y'|^{-\nu_2}. \tag{7.31}$$

Since $|\alpha'| \geqslant \Lambda$ and $\Lambda > |\alpha'| - \nu_1 \geqslant 0$, it follows that $|\alpha'| - \nu_1$ can be arbitrarily close to Λ. Therefore we can assume that $\nu_1 = |\alpha'| - \lambda_1(|\alpha'|)$, where $\lambda_1(|\alpha'|) = \min(|\alpha'|, \Lambda - \varepsilon_1)$, for all $\varepsilon_1 \in (0, \Lambda)$. Similarly for ν_2.

For $\vartheta < \frac{2}{3}\pi$, instead of (7.31) we have

$$|D_x^\alpha D_y^\beta G_{ij}(x, y)| \leqslant c|x'|^{\alpha_1}|y'|^{\alpha_2}, \tag{7.32}$$

where $\alpha_i < 1/2$, $i = 1, 2$. This concludes the proof. ■

Similarly to [14] we have

THEOREM 7.2.

$$\begin{aligned} G_{ij}(x, y) &= G_{ij}^0(x, y) + H_{ij}^0(x, y), \quad && y \in \mathscr{D}_\vartheta^{(0)},\ x \in \mathscr{D}_\vartheta \cap R_0, \\ G_{ij}(x, y) &= G_{ij}^1(x, y) + H_{ij}^1(x, y), && y \in \mathscr{D}_\vartheta^{(1)},\ x \in \mathscr{D}_\vartheta \cap R_1, \\ G_{ij}(x, y) &= \mathscr{E}(x, y)\delta_{ij} + H_{ij}^2(x, y), && y \in \mathscr{D}_\vartheta^{(2)},\ x \in \mathscr{D}_\vartheta, \end{aligned} \tag{7.33}$$

where the H_{ij}^k satisfy

$$\begin{aligned} &|D_x^\alpha D_y^\beta H_{ij}^k(x, y)| \\ &\leqslant c|x'|^{\lambda_1(|\alpha'|)-|\alpha'|}|y'|^{\lambda_2(|\beta'|)-|\beta'|}(|x-y|+|x'|+|y'|)^{-1-|\alpha''|-|\beta''|-\lambda_1(|\alpha'|)-\lambda_2(|\beta'|)}, \\ &\qquad\qquad i, j = 1, 2. \end{aligned} \tag{7.34}$$

Proof. For $y \in \mathscr{D}_\vartheta^{(0)}$, we put

$$H_{ij}^0(x, y) = G_{ij}^{0'}(x, y) + [\psi(x, y) - 1] G_{ij}^0(x, y).$$

Therefore from (7.18) it follows (7.34) for $G_{ij}^{0'}(x, y)$. Moreover,

$$|D_x^\alpha D_x^\beta(\psi - 1) G_{ij}^0| \leqslant c(|x-y|+|x'|+|y'|)^{-1-|\alpha|-|\beta|}.$$

and similarly in the other cases. This concludes the proof. ■

Remark. Let $\pi/\vartheta = m \in \boldsymbol{Z}$. Then we can look for the Green function in the form

$$G_{ij}(x, y) = \mathscr{E}(x-y)\delta_{ij} + \sum_{k=1}^{2m-1} C_{ij}^k \mathscr{E}(x-y_k),$$

where y_k is the point symmetrical to y_{k-1}, $y_0 = y$, with respect to the half-plane L_k obtained by rotating the half-plane $L_0 = \{x\colon x_1 = 0\}$ around M_0 by the angle $k\pi/m$. The constants C_{ij}^k are calculated from the boundary conditions (6.2).

In this case we get

$$|D_x^\alpha D_y^\beta H_{ij}(x, y)| \leqslant c_{ij}(|x-y| + |x'| + |y'|)^{-1-|\alpha|-|\beta|}. \tag{7.35}$$

8. The problem (3.13) in $L_{p,\mu}^k(\mathscr{D}_\vartheta)$ spaces

To estimate the solutions of the problem (3.13) in $L_{p,\mu}^k(\mathscr{D}_\vartheta)$ we shall use the following theorem [14], [2]:

THEOREM 8.1. *Let $K(z)$ be a singular kernel and*

$$u(x) = \int_{\boldsymbol{R}^n} K(x-y) f(y) dy = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y) dy.$$

Then

$$\int_{\boldsymbol{R}^n} |u|^p \omega dx \leqslant c \int_{\boldsymbol{R}^n} |f|^p \omega dx, \quad p > 1, \tag{8.1}$$

for all f such that $\int_{\boldsymbol{R}^n} |f|^p \omega dx < \infty$ with a nonnegative weight function $\omega(x)$ if and only if $\omega(x)$ satisfies the condition

$$|Q|^{-1} \int_Q \omega dx \left(\frac{1}{|Q|} \int_Q \omega^{-1/(p-1)} dx\right)^{p-1} \leqslant c, \quad c > 0, \ \forall Q \subset \boldsymbol{R}^n, \tag{8.2}$$

where Q is a cube.

It is easy to see that $\omega(x) = |x'|^{p\mu}$ satisfies (8.2) *if $-2/p < \mu < 2/p$, where* $1/p + 1/p' = 1$.

First we prove the estimate formulated in the proof of Theorem 3.2 in [14].

LEMMA 8.1. *Let a_1, a_2 be real such that $a_1 > -2$, $a_2 - a_1 - n > 0$, $\mathscr{D}_\vartheta \subset \boldsymbol{R}^n$. Then*

$$\int_{\mathscr{D}_\vartheta} \frac{|x'|^{a_1}}{(|x-y| + |x'| + |y'|)^{a_2}} dx \leqslant \frac{c}{|y'|^{a_2 - a_1 - n}}. \tag{8.3}$$

Proof. We decompose the integral on the left-hand side of (8.3) into the sum

$$\int\limits_{K_{2|y'|}(y)} + \int\limits_{\mathscr{D}_\vartheta \setminus K_{2|y'|}(y)} \equiv I_1 + I_2.$$

Since $|x-y| \leqslant 2|y'|$ implies $|x''-y''| \leqslant 2|y'|$ and $a_2+2 > 0$, we have

$$\begin{aligned} I_1 &\leqslant |y'|^{-a_2} \int\limits_{K_{2|y'|}(y)} |x'|^{a_1}\, dx \\ &\leqslant c|y'|^{-a_2} \int\limits_{K_{2|y'|}(y)} |x'|^{a_1+1}\, d|x'|\, dx'' \\ &\leqslant c|y'|^{-a_2} |x'|^{a_1+2}\Big|_0^{2|y'|} \int\limits_0^{2|y'|} |x''-y''|^{n-3}\, d|x''-y''| \\ &\leqslant c|y'|^{-(a_2-a_1-n)}. \end{aligned}$$

To estimate I_2 we use

$$2|x'| \leqslant 2|y'| + 2|x'-y'| \leqslant 3|x'-y'| \leqslant 3|x-y|.$$

Therefore

$$\begin{aligned} I_2 &\leqslant \int\limits_{\mathscr{D}_\vartheta \setminus K_{2|y'|}(y)} \frac{|x'|^{a_1}}{|x-y|^{a_2}}\, dx \\ &\leqslant c \int\limits_{2|y'|}^{\infty} d|x-y| \int\limits_0^{3|x-y|} \frac{|x'|^{a_1+1}}{|x-y|^{a_2}}\, d|x'|' \\ &= \int\limits_{2|y'|}^{\infty} \frac{dz}{z^{a_2}} \int\limits_0^{3z} \varrho^{a_1+1}\, d\varrho \leqslant c \int\limits_{2|y'|}^{\infty} z^{-a_2+a_1+2}\, dz \\ &\leqslant \frac{c}{|y'|^{a_2-a_1-3}}, \end{aligned}$$

since $a_1+2 > 0$ and $a_2-a_1-3 > 0$. This finishes the proof. ■

Similarly to Theorem 3.2 in [14] we can prove the following result

THEOREM 8.2. *Let $f_i \in L_{p,\mu}(\mathscr{D}_\vartheta)$, $i = 1, 2$, have compact supports, and let p, μ satisfy*

$$(8.4) \qquad \frac{2}{p'} > \mu > \frac{2}{p'} - \min\{\Lambda(\vartheta), 2\}, \qquad \min\{\Lambda(\vartheta), 2\} > 0,$$

where $\Lambda(\vartheta)$ is determined by (6.8). *The second condition of* (8.4) *means that ϑ must be different from π. Then for the functions* (7.5) *the following estimate is valid*

$$(8.5) \qquad \sum_{i=1}^{2} \|e_i\|_{L^2_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|f_i\|_{L_{p,\mu}(\mathscr{D}_\vartheta)}.$$

Moreover, if

$$\frac{2}{p'}-1>\mu>\frac{2}{p}-\min\{\Lambda,2\},\qquad \Lambda>1, \tag{8.6}$$

which means that $\vartheta<\pi/2$, *we have*

$$\sum_{i=1}^{2}\int_{\mathcal{D}_\vartheta}|\nabla e_i(x)-\nabla e_i(x)|_{x=0}|^p|x'|^{p\mu-p}dx\leqslant c\sum_{i=1}^{2}\|f_i\|^p_{L_{p,\mu}(\mathcal{D}_\vartheta)}. \tag{8.7}$$

For $\pi/\vartheta\in\mathbf{Z}$ *instead of* (8.4) *and* (8.6) *we have* $2/p'>\mu>-2/p+1$ *and* $2/p'-1>\mu>-2/p+1$, *respectively.*

Proof. The proof is almost the same as the proof of Theorem 3.2 in [14]. ■

THEOREM 8.3. *Let* $p,k\in\mathbf{Z}$ $p>1,k\geqslant 0$, *and* μ *satisfy*

$$0\leqslant k+2-\left(\mu+\frac{2}{p}\right)<\Lambda(\vartheta),\qquad \mu>-\frac{2}{p}, \tag{8.8}$$

where $\Lambda(\vartheta)$ *is determined in Theorem* 8.2. *Then for functions* $f_i\in L^k_{p,\mu}(\mathcal{D}_\vartheta)$, $\Phi_i\in L^{k+2-1/p}_{p,\mu}(\Gamma_i)$, $\Psi_i\in L^{k+1-1/p}_{p,\mu}(\Gamma_i)$, $i=1,2$, *with compact supports, satisfying the compatibility conditions described by Lemma* 5.2, *the problem* (3.13) *has a unique solution* $e_i\in L^{k+2}_{p,\mu}(\mathcal{D}_\vartheta)$, $i=1,2$, *and*

$$\sum_{i=1}^{2}\|e_i\|_{L^{k+2}_{p,\mu}(\mathcal{D}_\vartheta)}\leqslant c\sum_{i=1}^{2}\left(\|f_i\|_{L^k_{p,\mu}(\mathcal{D}_\vartheta)}+\|\Phi_i\|_{L^{k+2-1/p}_{p,\mu}(\Gamma_i)}+\|\Psi_i\|_{L^{k+1-1/p}_{p,\mu}(\Gamma_i)}\right). \tag{8.9}$$

For $\pi/\vartheta\in\mathbf{Z}$ *the theorem is also valid for* $\Lambda(\vartheta)\leqslant k+2-(\mu+2/p)$, *but in this case some compatibility conditions on the right-hand side functions of* (3.13) *at* $x\in M$ *are needed which guarantee the existence of solutions of the problems* (8.16) *and* (8.17) *below (see also Theorem* 4.3, *Lemma* 5.2).

Proof. Using Lemmas 5.1, 5.2 we replace the problem (3.13) by the homogeneous problem (3.11) with $\omega_i\in V^k_{p,\mu}(\mathcal{D}_\vartheta)$, $i=1,2$, with compact support. We consider several cases.

(a) $\dfrac{2}{p'}>\mu>\dfrac{2}{p'}-1$.

In this case we need only consider $\vartheta<\pi$. For $k=0$ the solutions which we are looking for are functions (7.5) for which the inequality (8.5) is valid. We consider the problem

$$\begin{gathered}-\Delta' e_i=e_{i,zz}+\omega_i,\ i=1,2,\qquad \text{for a.e. } z\in\mathbf{R}^1,\\ e_1|_{\gamma_1}=0,\qquad (e_1\cos\vartheta+e_2\sin\vartheta)|_{\gamma_2}=0,\\ \left.\frac{\partial e_2}{\partial n}\right|_{\gamma_1}=0,\qquad \left.\left(\frac{\partial e_1}{\partial n}\sin\vartheta-\frac{\partial e_2}{\partial n}\cos\vartheta\right)\right|_{\gamma_2}=0.\end{gathered} \tag{8.10}$$

From Theorem 4.1 we have $e_i \in V^2_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|e_i\|_{V^2_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\omega_i\|_{L_{p,\mu}(\mathscr{D}_\vartheta)}, \tag{8.11}$$

which gives (8.9) for $k = 0$. For $k = 1$ from (7.5) and (8.5) we have

$$\sum_{i=1}^{2} \int_{\mathscr{D}_\vartheta} |e_{i,xzz}|^p |x'|^{p\mu} dx \leqslant c \sum_{i=1}^{2} \int_{\mathscr{D}_\vartheta} |\omega_{i,z}|^p |x'|^{p\mu} dx. \tag{8.12}$$

Just as in Lemma 6.1, for functions satisfying (6.3) we obtain

$$\sum_{i=1}^{2} \int_{\Omega} |e_i|^p |x'|^{\mu p - p} dx \leqslant c \sum_{i=1}^{2} \int_{\Omega} |\nabla e_i|^p |x'|^{\mu p} dx, \qquad \mu \geqslant 0, \tag{8.13}$$

where $\Omega = d_\vartheta$ or $\Omega = \mathscr{D}_\vartheta$. From (8.12), (8.13) it follows that $e_{i,zz} \in V^1_{p,\mu}(d_\vartheta)$. The eigenvalue 1 satisfies

$$\frac{2}{p'} + 1 - \mu > \frac{2}{p'} + 1 - \frac{2}{p'} = 1 > \frac{2}{p'} - \mu,$$

so $e_i - Q_{i,1} \in V^3_{p,\mu}(d_\vartheta)$ and

$$\sum_{i=1}^{2} \|e_i - Q_{i,1}\|_{V^3_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\omega_i\|_{V^1_{p,\mu}(d_\vartheta)}, \tag{8.14}$$

where the $Q_{i,1}$, $i = 1, 2$, are homogeneous polynomials of degree one such that

$$Q_{1,1} = -e_{1,x_2}|_{x'=0} x_2, \qquad Q_{2,1} = e_{2,x_1}|_{x'=0} x_1. \tag{8.15}$$

Let $\pi/\vartheta - 1 > k + 2 - (\mu + 2/p)$, $k > 1$. Then show the higher regularity of solutions by induction. For $k > 1$ we introduce homogeneous polynomials $Q_{i,j}$, $i = 1, 2$, of degree j with respect to x_1, x_2 and functions $e_{i,j}$, $i = 1, 2$, which are defined in a recurrent way as the solutions of the following problems

$$\begin{gathered} -\Delta' Q_{i,j} = \nabla_z^2 Q_{i,j-2}, \qquad i = 1, 2, \\ Q_{1,j}|_{\gamma_1} = 0, \qquad (Q_{1,j}\cos\vartheta + Q_{2,j}\sin\vartheta)|_{\gamma_2} = 0, \\ \frac{\partial}{\partial n} Q_{2,j}|_{\gamma_1} = 0, \qquad \left(\frac{\partial Q_{1,j}}{\partial n}\sin\vartheta - \frac{\partial Q_{2,j}}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2} = 0, \end{gathered} \tag{8.16}$$

and

$$\begin{gathered} -\Delta' e_{i,j} = \nabla_z^2 e_{i,j-2} + \omega_i, \qquad i = 1, 2, \\ e_{1,j}|_{\gamma_1} = 0, \qquad (e_{1,j}\cos\vartheta + e_{2,j}\sin\vartheta)|_{\gamma_2} = 0, \\ \frac{\partial}{\partial n} e_{2,j}|_{\gamma_1} = 0, \qquad \left(\frac{\partial e_{1,j}}{\partial n}\sin\vartheta - \frac{\partial e_{2,j}}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2} = 0, \end{gathered} \tag{8.17}$$

where $Q_{i,0} = 0$, $e_{i,0} = e_i$, $Q_{i,1}$ are determined by (8.15) and $e_{i,1} = e_i - Q_{i,1}$,

$i=1,2$. It follows that $Q_{i,j}=0$, $i=1,2$, for j even and

$$Q_{1,j}=\sum_{|\alpha|=j} A_{1\alpha}x'^{\alpha}\nabla_z^{j-1}e_{1,x_2}|_{x'=0},\qquad Q_{2,j}=\sum_{|\alpha|=j} A_{2\alpha}x'^{\alpha}\nabla_z^{j-1}e_{2,x_1}|_{x'=0},$$

for j odd.

Moreover, for a.e. $z\in\mathbf{R}^1$, $e_{i,j}\in V^{j+2}_{p,\mu}(d_\vartheta)$ and

$$\sum_{i=1}^{2}\|e_{i,j}\|_{V^{j+2}_{p,\mu}(d_\vartheta)}\leqslant c\sum_{i=1}^{2}\big(\|\nabla_z^2 e_{i,j-2}\|_{V^j_{p,\mu}(d_\vartheta)}+\|\omega_i\|_{V^j_{p,\mu}(d_\vartheta)}\big)$$
$$\leqslant\ldots\leqslant c\sum_{i=1}^{2}\sum_{k=0}^{j}\|\nabla_z^{j-k}\omega_i\|_{V^k_{p,\mu}(d_\vartheta)}. \tag{8.18}$$

Finally, using induction with respect to j, we show that $e_{i,j}+Q_{i,j}=e_{i,j-1}$. Assuming that this formula is valid for $j<q$, we have

$$e_{i,q}+\big(1-\zeta(x')\big)Q_{i,q}\equiv v_i'\in V^{q+2}_{p,\mu}(d_\vartheta),\qquad e_{i,q-1}-\zeta(x')Q_{i,q}\equiv v_i''\in V^{q+1}_{p,\mu}(d_\vartheta)$$

are solutions of the same problem

$$-\Delta' v_i=h_i,\quad i=1,2,$$
$$v_1|_{\gamma_1}=0,\quad (v_1\cos\vartheta+v_2\sin\vartheta)|_{\gamma_2}=0, \tag{8.19}$$
$$\frac{\partial v_2}{\partial n}\Big|_{\gamma_1}=0,\quad \Big(\frac{\partial v_1}{\partial n}\sin\vartheta-\frac{\partial v_2}{\partial n}\cos\vartheta\Big)\Big|_{\gamma_2}=0,$$

where

$$\begin{aligned}h_i&=\nabla_z^2 e_{i,q-2}+\omega_i+\big(1-\zeta(x')\big)\nabla_z^2 Q_{i,q-2}\\&\quad+2\nabla'\zeta\nabla' Q_{i,q}+Q_{i,q}\Delta'\zeta\\&=\omega_i+\nabla_z^2 e_{i,q-3}-\zeta(x')\nabla_z^2 Q_{i,q-2}+2\nabla'\zeta\cdot\nabla' Q_{i,q}+Q_{i,q}\Delta'\zeta\\&\quad\in V^q_{p,\mu}(d_\vartheta)\cap V^{q-1}_{p,\mu}(d_\vartheta).\end{aligned}$$

Therefore from Theorem 4.4 it follows that $v'=v''$, so $e_{i,q}+Q_{i,q}=e_{i,q-1}$. Hence

$$e_i=Q_{i,1}+\ldots+Q_{i,j}+e_{i,j},\quad i=1,2,\ j\leqslant k. \tag{8.20}$$

Using (8.5) and (8.18) we obtain

$$\begin{aligned}\sum_{i=1}^{2}\|e_i\|^p_{L^{k+2}_{p,\mu}(\mathscr{D}_\vartheta)}&=\sum_{j=1}^{2}\sum_{j=1}^{k}\int_{\mathbf{R}^1}\|\nabla_z^{k-j}e_i\|^p_{L^{j+2}_{p,\mu}(d_\vartheta)}dz+\|\nabla_z^k e_i\|^p_{L^2_{p,\mu}(\mathscr{D}_\vartheta)}\\&=\sum_{i=1}^{2}\sum_{j=1}^{k}\int_{\mathbf{R}^1}\|\nabla_z^{k-j}e_{i,j}\|^p_{L^{j+2}_{p,\mu}(d_\vartheta)}dz+\|\nabla_z^k e_i\|^p_{L^2_{p,\mu}(\mathscr{D}_\vartheta)}\\&\leqslant c\sum_{i=1}^{2}\|\omega_i\|^p_{V^k_{p,\mu}(\mathscr{D}_\vartheta)},\end{aligned} \tag{8.21}$$

so we have proved (8.9).

(b) $\frac{2}{p'}-1>\mu>\frac{2}{p'}-\min\left(\frac{\pi}{\vartheta}-1-k,\ 2\right),\ \frac{\pi}{\vartheta}-1>k+1.$

In this case we need only consider $\vartheta<\pi$. The eigenvalue 1 is less than $2/p'-\mu$, so considering the solvability of (8.10) in $V^2_{p,\mu}(d_\vartheta)$ we see that $e_i-Q_{i,1}=e_{i,0}\in V^2_{p,\mu}(d_\vartheta)$ and

$$\sum_{i=1}^{2}\|e_i-Q_{i,1}\|_{V^2_{p,\mu}(\mathcal{D}_\vartheta)}\leqslant c\sum_{i=1}^{2}\|\omega_i\|_{L^2_{p,\mu}(\mathcal{D}_\vartheta)}. \tag{8.22}$$

The following considerations are the same as in (a) with the only difference that instead of (8.20) we have

$$e_i=Q_{i,1}+\ldots+Q_{i,j+1}+e_{i,j},\qquad i=1,2,\ j\leqslant k. \tag{8.23}$$

(c) $\mu=\frac{2}{p'}-1.$

In this case we need only consider $\vartheta<\pi/2$, because $\mu+2/p=1$ and $k+2-(\mu+2/p)=k+1$. Let $k=0$. Then $\Lambda(\vartheta)>1$ so Theorem 8.2 implies that $e_{i,z}\in L^1_{p,\mu}(d_\vartheta)$, $e_{i,zz}\in L_{p,\mu}(d_\vartheta)$, $i=1,2$. Moreover,

$$\sum_{i=1}^{2}\left(\|e_{i,z}\|_{L^1_{p,\mu}(d_\vartheta)}+\|e_{i,zz}\|_{L_{p,\mu}(d_\vartheta)}\right)\leqslant c\sum_{i=1}^{2}\|\omega_i\|_{L_{p,\mu}(d_\vartheta)} \tag{8.24}$$

for a.e. $z\in\boldsymbol{R}^1$.

Therefore Theorem A.1, see Appendix, implies the existence of solutions of (8.10) such that $D^2_{x'}e_i\in L_{p,\mu}(d_\vartheta)$, $i=1,2$, and

$$\sum_{i=1}^{2}\|D^2_{x'}e_i\|_{L_{p,\mu}(d_\vartheta)}\leqslant c\sum_{i=1}^{2}\|\omega_i\|_{L_{p,\mu}(d_\vartheta)} \tag{8.25}$$

for a.e. $z\in\boldsymbol{R}^1$. From (8.24), (8.25) after integration with respect to z one gets $e_i\in L^2_{p,\mu}(\mathcal{D}_\vartheta)$, $i=1,2$, and

$$\sum_{i=1}^{2}\|e_i\|_{L^2_{p,\mu}(\mathcal{D}_\vartheta)}\leqslant c\sum_{i=1}^{2}\|\omega_i\|_{L_{p,\mu}(\mathcal{D}_\vartheta)}. \tag{8.26}$$

Let $k=1$ and let the z-derivative of (8.10) and (3.11) be denoted by $(8.10)^1$ and $(3.11)^1$, respectively. Applying Theorem 8.2 to $(3.11)^1$ one has $e_{i,zz}\in L^1_{p,\mu}(d_\vartheta)$, $e_{i,zzz}\in L_{p,\mu}(d_\vartheta)$ and

$$\sum_{i=1}^{2}\left(\|e_{i,zz}\|_{L^1_{p,\mu}(d_\vartheta)}+\|e_{i,zzz}\|_{L_{p,\mu}(d_\vartheta)}\right)\leqslant c\sum_{i=1}^{2}\|\omega_{i,z}\|_{L_{p,\mu}(d_\vartheta)} \tag{8.27}$$

for a.e. $z\in\boldsymbol{R}^1$. Now using Theorem A.1 to $(8.10)^1$ one obtains $e_{i,z}\in L^2_{p,\mu}(d_\vartheta)$, $i=1,2$ and

$$\sum_{i=1}^{2}\|e_{i,z}\|_{L^2_{p,\mu}(d_\vartheta)}\leqslant c\sum_{i=1}^{2}\|\omega_{i,z}\|_{L_{p,\mu}(d_\vartheta)} \tag{8.28}$$

for a.e. $z \in \mathbf{R}^1$. Finally, applying Theorem 4.3 to (8.10) gives $e_i \in L^3_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|e_i\|_{L^3_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\omega_i\|_{L^1_{p,\mu}(d_\vartheta)} \tag{8.29}$$

for a.e. $z \in \mathbf{R}^1$. From (8.27)–(8.29) after integrating with respect to z we get that $e_i \in L^3_{p,\mu}(\mathscr{D}_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|e_i\|_{L^3_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\omega_i\|_{L^1_{p,\mu}(\mathscr{D}_\vartheta)}. \tag{8.30}$$

Let $k > 1$. In this case we denote by $(8.10)^\sigma$, $(3.11)^\sigma$, $\sigma \leqslant k$, the D^σ_z-derivatives of (8.10) and (3.11), respectively. Then applying Theorem 8.2 to $(3.11)^k$ gives $D^{k+2}_z e_i \in L_{p,\mu}(d_\vartheta)$, $D^{k+1}_z e_i \in L^1_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} (\|D^{k+1}_z e_i\|_{L^1_{p,\mu}(d_\vartheta)} + \|D^{k+2}_z e_i\|_{L_{p,\mu}(d_\vartheta)}) \leqslant c \sum_{i=1}^{2} \|D^k_z \omega_i\|_{L_{p,\mu}(d_\vartheta)}. \tag{8.31}$$

for a.e. $z \in \mathbf{R}^1$. Hence from $(8.10)^k$ using Theorem A.1. we deduce that $D^k_z e_i \in L^2_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|D^k_z e_i\|_{L^2_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \|D^k_z \omega_i\|_{L_{p,\mu}(d_\vartheta)} \tag{8.32}$$

for a.e. $z \in \mathbf{R}^1$. Now using (8.31) and Theorem 4.3 to $(8.10)^{k-1}$ we see that $D^{k-1}_z e_i \in L^3_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|D^{k-1}_z e_i\|_{L^3_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} (\|D^{k-1}_z \omega_i\|_{L^1_{p,\mu}(d_\vartheta)} + \|D^k_z \omega_i\|_{L_{p,\mu}(d_\vartheta)}). \tag{8.33}$$

Hence in this case we have to use induction. Assume we have shown that $D^{k-\varrho}_z e_i \in L^{\varrho+2}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and that

$$\sum_{i=1}^{2} \|D^{k-\varrho}_z e_i\|_{L^{\varrho+2}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \sum_{\sigma=0}^{\varrho} \|D^{k-\sigma}_z \omega_i\|_{L^\sigma_{p,\mu}(d_\vartheta)} \tag{8.34}$$

for $0 \leqslant \varrho \leqslant s < k$. Putting $\varrho = s-1$ we get $D^{k-(s-1)}_z e_i \in L^{s+1}_{p,\mu}(d_\vartheta)$, $i = 1, 2$. Therefore using $D^{k-(s+1)}_z \omega_i \in L^{s+1}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, from $(8.10)^{k-(s+1)}$ and Theorem 4.3 we obtain $D^{k-(s+1)}_z e_i \in L^{s+3}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, and

$$\sum_{i=1}^{2} \|D^{k-(s+1)}_z e_i\|_{L^{s+3}_{p,\mu}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} \sum_{\sigma=0}^{s+1} \|D^{k-\sigma}_z \omega_i\|_{L^\sigma_{p,\mu}(d_\vartheta)}, \tag{8.35}$$

so we get (8.34) for $\varrho = s+1$. Proceeding in this way we show that $D^{k-s}_z e_i \in L^{s+2}_{p,\mu}(d_\vartheta)$, $i = 1, 2$, for each $s \leqslant k$ and then, finally, we get the estimate (8.34) for $\varrho = k$. Now after integration with respect to z we obtain $e_i \in$

$L^{k+2}_{p,\mu}(\mathcal{D}_\vartheta)$, $i = 1, 2$, and

(8.36) $$\sum_{i=1}^{2} \|e_i\|_{L^{k+2}_{p,\mu}(\mathcal{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} \|\omega_i\|_{L^{k}_{p,\mu}(\mathcal{D}_\vartheta)}.$$

(d) $\mu \geqslant \frac{2}{p'}$.

In this case combining Theorems 4.3, A.1 and the method of local estimates from Theorem 4.2 of [13] we prove the assertion. Theorem A.1 is used only for $\mu = 2/p'+s$, where $s \geqslant 0$, is an integer.

(e) $\frac{\pi}{\vartheta} = m \in \boldsymbol{N}$, $\vartheta < \pi$, $k+\frac{2}{p'}-\mu \neq j\Lambda(\vartheta)$, $k+\frac{2}{p'}-\mu > \Lambda(\vartheta)$, $\mu > -\frac{2}{p}$.

In this case the considerations are similar to those in (a) and (b). For $2/p' > \mu > 2/p'-1$ the functions

$$v_i' = e_{i,m-1}+\big(1-\zeta(x')\big)Q_{i,m-1} \in V^{m+1}_{p,\mu}(d_\vartheta),$$
$$v_i'' = e_{i,m-2}-\zeta(x')Q_{i,m-1} \in V^{m}_{p,\mu}(d_\vartheta), \qquad i = 1, 2,$$

are solutions of the same problem (8.19) with right-hand sides $h_i \in V^{m+1}_{p,\mu}(d_\vartheta) \cap V^{m}_{p,\mu}(d_\vartheta)$, $i = 1, 2$.

Now Theorem 4.1 implies that $v_i'-v_i'' = -W^{(1)}_{i,m-1}(x)$, $i = 1, 2$, where

$$W^{(1)}_{1,m-1}(x) = A_{m-1}(z)r^{m-1}\sin(m-1)\varphi,$$
$$W^{(1)}_{2,m-1}(x) = A_{m-1}(z)r^{m-1}\cos(m-1)\varphi$$

because

$$m-1+\frac{2}{p'}-\mu > m-1 = \Lambda\left(\frac{\pi}{m}\right) > m-2+\frac{2}{p'}-\mu,$$

so

$$e_{i,m-1}+Q_{i,m-1}+W^{(1)}_{i,m-1} = e_{i,m-2}, \qquad i = 1, 2.$$

Moreover, we introduce the functions

$$W^{(a)}_{1,am-1} = A_{am-1}(z)r^{am-1}\sin(am-1)\varphi,$$
$$W^{(a)}_{2,am-1} = A_{am-1}(z)r^{am-1}\cos(am-1)\varphi,$$

where a is a natural number, which are eigenfunctions of the homogeneous problem (3.15) (with right-hand sides equal to zero). Finally, we introduce the functions

$$W^{(a)}_{1,am-1+q} = A_{am-1+q}(z)r^{am-1+q}\sin(am-1)\varphi,$$
$$W^{(a)}_{2,am-1+q} = A_{am-1+q}(z)r^{am-1+q}\cos(am-1)\varphi,$$

determined by

(8.37)
$$\begin{gathered} -\Delta' W^{(a)}_{i,am-1+q} = D_z^2 W^{(a)}_{i,am-1+q-2}, \quad i = 1, 2, \\ W^{(a)}_{1,am-1+q}\big|_{\gamma_1} = 0, \quad \left.\frac{\partial W^{(a)}_{2,am-1+q}}{\partial n}\right|_{\gamma_1} = 0, \\ \left.\left(W^{(a)}_{1,am-1+q}\cos\frac{\pi}{m} + W^{(a)}_{2,am-1+q}\sin\frac{\pi}{m}\right)\right|_{\gamma_2} = 0, \\ \left.\left(\frac{\partial W^{(a)}_{1,am-1+q}}{\partial n}\sin\frac{\pi}{m} - \frac{\partial W^{(a)}_{2,am-1+q}}{\partial n}\cos\frac{\pi}{m}\right)\right|_{\gamma_2} = 0. \end{gathered}$$

Hence (8.37) implies that $W^{(a)}_{i,am-1+q} = 0$ for q odd. Therefore instead of (8.20) and (8.23) we have

(8.38) $$e_i = e_{i,k} + \sum_{j=1}^{k} Q_{i,j} + \sum_{am-1+q\leqslant k} W^{(a)}_{i,am-1+q}, \quad i = 1, 2, \ \frac{2}{p'} > \mu > \frac{2}{p'} - 1,$$

(8.39) $$e_i = e_{i,k} + \sum_{j=1}^{k+1} Q_{i,j} + \sum_{am-1+q\leqslant k+1} W^{(a)}_{i,am-1+q},$$
$$i = 1, 2, \ \frac{2}{p'} - 1 > \mu > -\frac{2}{p'}$$

and the inequality (8.9) is valid. This concludes the proof. ■

9. The problem (3.13) in weighted Hölder spaces

In this section we apply the results of [14]. First we shall consider the functions (7.5) assuming that the f_i, $i = 1, 2$, sufficiently rapidly decrease at infinity.

THEOREM 9.1. *For functions satisfying* (3.11) *the following estimate is valid*

(9.1) $$\sum_{i=1}^{2} [e_i]^{(s)}_{\mathcal{D}_\vartheta} \leqslant c \sum_{i=1}^{2} \sup_{x\in\mathcal{D}_\vartheta} |x'|^{2-s} |f_i(x)| \equiv c \sum_{i=1}^{2} |f_i|_s$$

for $s \in (0, 1)$ *and* $s \in (1, 2)$, *and* $s < \Lambda(\vartheta)$. *For* $\pi/\vartheta \in \mathbf{Z}$ *the last inequality is not necessary.*

Moreover,

(9.2) $$|\Delta_j^{k+q}(h) e_i(x)| \leqslant c h^2 \sup_{x\in\mathcal{D}_\vartheta} \sum_{i=1}^{2} |\Delta_j^q f_i(x)|,$$

where $j > 2$, $q > s$, $k > 2$, *and* $\Delta_j^q(h) f$ *is the difference of order* q *with respect to* x_j *with step* h.

Proof. For $s\in(0,1)$ and $x, y\in\mathscr{D}_\vartheta$ we have

$$\sum_{i=1}^{2}|e_i(x)-e_i(z)|\leqslant\sum_{i=1}^{2}|f_i|_s\int_{\mathscr{D}_\vartheta}|G_{jk}(x,y)-G_{jk}(z,y)||y'|^{s-2}dy.$$

Writing $r=2|x-z|$, from (7.17) we get

$$\begin{aligned}
&\int_{K_r(x)}|G_{jk}(x,y)-G_{jk}(z,y)||y'|^{s-2}dy\\
&\quad\leqslant\int_{K_r(x)}(|x-y|^{-1}+|z-y|^{-1})|y'|^{s-2}dy\\
&\quad\leqslant\int_{K_r(x)}|x-y|^{-1}|y'|^{s-2}dy+\int_{K_{2r}(z)}|z-y|^{-1}|y'|^{s-2}dy\\
&\quad\leqslant c\int_{K_{2r}(x)}|x-y|^{-1}|y'|^{s-2}dy\\
&\quad= c\int_{K_{2r}(x)\cap\{y:\,|y'|\leqslant|x-y|/2\}}|x-y|^{-1}|y'|^{s-2}dy\\
&\qquad\qquad+c\int_{K_{2r}(x)\cap\{y:\,|y'|\geqslant|x-y|/2\}}|x-y|^{-1}|y'|^{s-2}dy\\
&\quad\leqslant c\int_0^{2r}d|x-y|\int_0^{|x-y|/2}\frac{|y'|^{s-2}|y'|d|y'|}{|x-y|}+c\int_{K_{2r}(x)}|x-y|^{-3}dy\leqslant cr^s.
\end{aligned}$$

Now we consider

$$\int_{\mathscr{D}_\vartheta\setminus K_r(x)}|G_{jk}(x,y)-G_{jk}(z,y)||y'|^{s-2}dy, \tag{9.4}$$

where $|x-z|=\frac{1}{2}r$. First we consider the difference of Green functions

$$G_{jk}(x,y)-G_{jk}(z,y)=G_{jk}(x,y)-G_{jk}(\xi,y)+G_{jk}(\xi,y)-G_{jk}(z,y),$$

where ξ is such that $\xi''=x''$, $\xi'=z'$, $|z-\xi|\leqslant r$. Using (7.16) we have

$$\begin{aligned}
|G_{jk}(z,y)-G_{jk}(\xi,y)|&=\left|\int_0^1\frac{d}{ds}G_{jk}(\xi+s(z-\xi),y)ds\right| \qquad (9.5)\\
&=\left|\int_0^1\frac{\partial G_{jk}}{\partial\eta_i}(\eta,y)(z_i-\xi_i)ds\right|,
\end{aligned}$$

where $\eta=\xi+s(z-\xi)$, so (9.5) is less than $c|z-\xi|\int_0^1\frac{ds}{|\xi+s(z-\xi)-y|^2}$. We know that

$$|\xi-y+s(z''-\xi'')|=|x-y+\xi-x+s(z''-\xi'')|\geqslant|x-y|-|\xi-x+s(z''-\xi'')|.$$

Since

$$|\xi'-x'+s(z''-\xi'')| = |z'-x'+s(z''-x'')| \leqslant |z-x|$$

and

$$|z-x| \leqslant \tfrac{1}{2}|x-y| \quad \text{for } y \in \mathscr{D}_\vartheta \setminus K_r(x)$$

we have $|\xi-y+s(z''-\xi'')| \geqslant \frac{1}{2}|x-y|$ and

$$\text{(9.6)} \quad |G_{jk}(z, y)-G_{jk}(\xi, y)| \leqslant c\frac{|z-\xi|}{|x-y|^2} = c\frac{|z-\xi||x-y|^{\lambda-1}}{|x-y|^{1+\lambda}} \leqslant c\frac{r^\lambda}{|x-y|^{1+\lambda}}$$

because $|z-\xi| \leqslant r$, $|x-y| \geqslant r$, $\lambda-1 \leqslant 0$.

Now we consider

$$\text{(9.7)} \quad |G_{jk}(x, y)-G_{jk}(\xi, y)| = \left|\int_0^1 \frac{d}{ds} G(\xi+s(x-\xi), y)\,ds\right| = \left|\int_0^1 \frac{\partial G(\eta, y)}{\partial \eta_i}(x_i'-\xi_i')\,ds\right|$$

(by (7.16), this is less than)

$$\leqslant |x'-\xi'|\int_0^1 \frac{1}{|\eta-y|^2}\left(\frac{|\eta'|}{|\eta'|+|\eta-y|}\right)^{\lambda-1} ds, \quad \text{for } \lambda < \Lambda(\vartheta),$$

where $\eta = \xi+s(x-\xi)$. By using

$$|\eta-y| \geqslant |x-y|-|\eta-x| \geqslant |x-y|-|z-x| \geqslant \tfrac{1}{2}|x-y|$$

and

$$|x'-\xi'| = |x'-z'| \leqslant r,$$

the above integral is estimated by

$$\text{(9.8)} \quad c\frac{|x'-\xi'|}{|x-y|^{1+\lambda}}\int_0^1 |\eta-y|^{\lambda-1}\left(\frac{|\eta'|}{|\eta'|+|\eta-y|}\right)^{\lambda-1} ds$$

$$\leqslant c\frac{|x'-\xi'|}{|x-y|^{1+\lambda}}\int_0^1 (|\eta-y|^{\lambda-1}+|\eta'|^{\lambda-1})\,ds \leqslant c\frac{r^\lambda}{|x-y|^{1+\lambda}}+c\frac{|x'-\xi'|}{|x-y|^{1+\lambda}}\int_0^1 |\eta'|^{\lambda-1}\,ds.$$

Now we estimate the last integral in (9.8).

Let $|\xi'| \leqslant |x'-\xi'|$. Then

$$|x'-\xi'|\int_0^1 |\eta'|^{\lambda-1}\,ds$$

$$\leqslant \int_0^{|\xi'|/|x'-\xi'|} |x'-\xi'|(|\xi'|-s|x'-\xi'|)^{\lambda-1}\,ds+\int_{|\xi'|/|x'-\xi'|}^1 |x'-\xi'|(s|x'-\xi'|-|\xi'|)^{\lambda-1}\,ds$$

$$= \frac{1}{\lambda}|\xi'|^\lambda+\frac{1}{\lambda}(|x'-\xi'|-|\xi'|)^\lambda \leqslant cr^\lambda,$$

since

$$|\xi'+s(x'-\xi')| \geqslant |\xi'|-s|x'-\xi'| \quad \text{for } s \leqslant \frac{|\xi'|}{|x'-\xi'|},$$

$$|\xi'+s(x'-\xi')| \geqslant s|x'-\xi'|-|\xi'| \quad \text{for } s \geqslant \frac{|\xi'|}{|x'-\xi'|},$$

$$|\xi'| \leqslant |x'-\xi'| \leqslant r.$$

Let $|\xi'| \geqslant |x'-\xi'|$. Then

$$|\xi'+s(x'-\xi')| \geqslant |\xi'|-s|x'-\xi'|,$$

$$|x'-\xi'|\int_0^1 |\eta'|^{\lambda-1}ds \leqslant \int_0^1 |x'-\xi'|(|\xi'|-s|x'-\xi'|)^{\lambda-1}ds$$

$$= \frac{1}{\lambda}[|\xi'|^\lambda-|x'-\xi'|^\lambda] \leqslant \frac{1}{\lambda}|x'-\xi'|^\lambda \leqslant cr^\lambda.$$

From the above inequalities we obtain for (9.7) the same estimate as in (9.6). Therefore (9.4) is estimated by

$$(9.9)\qquad \int_{\mathscr{D}_\vartheta\setminus K_r(x)} |G_{jk}(x,y)-G_{jk}(z,y)||y'|^{s-2}dy \leqslant cr^\lambda \int_{\mathscr{D}_\vartheta\setminus K_r(x)} \frac{|y'|^{s-2}}{|x-y|^{1+\lambda}}dy.$$

Now,

$$\int_{\mathscr{D}_\vartheta\setminus K_r(x)} \frac{|y'|^{s-2}}{|x-y|^{1+\lambda}}dy$$

$$= \int_{\mathscr{D}_\vartheta\setminus K_r(x)\cap\{y:\,|y'|\leqslant|x-y|/2\}} \frac{|y'|^{s-2}}{|x-y|^{1+\lambda}} + \int_{\mathscr{D}_\vartheta\setminus K_r(x)\cap\{y:\,|y'|\geqslant|x-y|/2\}} \frac{|y'|^{s-2}}{|x-y|^{1+\lambda}}dy$$

$$\leqslant c\int_r^\infty d|x-y| \int_0^{|x-y|/2} \frac{|y'|^{s-2}|y'|d|y'|}{|x-y|^{1+\lambda}} + c\int_r^\infty \frac{1}{|x-y|^{\lambda-s+1}}d|x-y|$$

$$\leqslant c\int_r^\infty t^{s-\lambda-1}dt \leqslant cr^\lambda \quad \text{for } \lambda > s.$$

Therefore we have proved (9.1) for $s\in(0,1)$. For $s\in(1,2)$ we have

$$\sum_{i=1}^2 |\nabla e_i(x)-\nabla e_i(z)| \leqslant \sum_{i=1}^2 |f_i|_s \int_{\mathscr{D}_\vartheta} |\nabla G_{ij}(x,y)-\nabla G_{ij}(z,y)||y'|^{s-2}dy$$

$$\leqslant cr^{s-1}\sum_{i=1}^2 |f_i|_s,$$

because

(9.10) $$\int_{\mathcal{D}_\vartheta} |\nabla G_{ij}(x, y)-\nabla G_{ij}(z, y)||y'|^{s-2}dy$$

$$\leqslant c \int_{K_r(x)} (|x-y|^{-2}+|z-y|^{-2})|y'|^{s-2}dy+r^{\lambda} \int_{\mathcal{D}_\vartheta\setminus K_r(x)} |x-y|^{-2-\lambda'}|y'|^{s-2}dy$$

$$\leqslant cr^{s-1},$$

where the considerations are almost the same as in the case $s\in(0, 1)$ and $\lambda' < \Lambda(\vartheta)-1$, $\lambda'\in(s-1, 1)$. The inequality (9.2) can be proved in the same way as in Theorem 4.1 of [14]. ■

From (7.1) and (7.12) we have

$$e_{i,kl} = \sum_{j=1}^{2} \Big[\int_{\mathcal{D}_\vartheta} G_{ij,kl}(x, y) f_j(y)dy-\tfrac{1}{3}\delta_{ij}\delta_{kl} f_j(x)\Big],$$

where the integral is singular and $e_{i,kl} = e_{i,x_k x_l}$.

THEOREM 9.2. *Suppose $\vartheta < \pi/2$ or $\pi/\vartheta\in N$. Then for the functions*

$$e_{i,kl}(x) = \sum_{j=1}^{2} \int_{\mathcal{D}_\vartheta} G_{ij,x_k x_l}(x, y) f_j(y)dy, \qquad k = 3,$$

the following estimates are valid:

(9.11) $$[e_{i,kl}]^{(\sigma)}_{\mathcal{D}_\vartheta} \leqslant c \sum_{j=1}^{2} [f_j]^{(\sigma)}_{\mathcal{D}_\vartheta},$$

(9.12) $$|e_{i,kl}|_\sigma \leqslant c \sum_{j=1}^{2} |f_j|_{\mathring{C}^{\sigma}_{\sigma-2}(\mathcal{D}_\vartheta)},$$

where $\sigma\in(0, \min\{1, \pi/\vartheta-2\})$ and for $\pi/\vartheta\in N$, $\sigma\in(0, 1)$.

Proof. Let $x, z\in\mathcal{D}_\vartheta$, $r = 2|x-z|$. By considerations similar to those in Chapter 3, § 2 of [9] and in the proof of Theorem 4.2 in [14] we have

$$\begin{aligned} e_{i,kl}(x)-e_{i,kl}(z) = &\sum_{j=1}^{2} \int_{K_r(x)} G_{ij,kl}(x, y)[f_j(y)-f_j(x)]dy \\ &-\sum_{j=1}^{2} \int_{K_r(x)} G_{ij,kl}(z, y)[f_j(y)-f_j(z)]dy \\ &+\sum_{j=1}^{2} [f_j(x)-f_j(z)]\cdot\lim_{\varepsilon\to 0} \int_{K_r(x)\setminus K_\varepsilon(x)} G_{ij,kl}(x, y)dy \\ &+\sum_{j=1}^{2} \int_{\mathcal{D}_\vartheta\setminus K_r(x)} [G_{ij,kl}(x, y)-G_{ij,kl}(z, y)][f_j(y)-f_j(z)]dy, \end{aligned}$$

where, by (7.16), the integral

$$\lim_{\varepsilon\to 0}\int_{K_r(x)\setminus K_\varepsilon(x)} G_{ij,kl}(x, y)dy$$

$$= -\int_{\partial K_r(x)} G_{ij,k}(x, y)n_l(y)ds+\lim_{\varepsilon\to 0}\int_{\partial K_\varepsilon(x)} G_{ij,k}(x, y)n_l(y)ds$$

is estimated by a constant. Now, we use

$$\begin{aligned}
&|G_{ij,kl}(x, y)-G_{ij,kl}(z, y)|\\
&\leqslant |G_{ij,kl}(x, y)-G_{ij,kl}(\xi, y)|+|G_{ij,kl}(\xi, y)-G_{ij,kl}(z, y)| (\xi' = z',\ \xi'' = x'')\\
&\leqslant \left|\int_0^1 \frac{d}{ds} G_{ij,kl}(\xi+s(z-\xi), y)ds\right|+\left|\int_0^1 \frac{d}{ds} G_{ij,kl}(\xi+s(x-\xi), y)ds\right|\\
&= \left|\int_0^1 \frac{\partial}{\partial\eta_r} G_{ij,kl}(\eta, y)(z_r-\xi_r)ds\right|+\left|\int_0^1 \frac{\partial}{\partial\vartheta_r} G_{ij,kl}(\vartheta, y)(x_r-\xi_r)ds\right|\\
&= I^1_{kl}+I^2_{kl}
\end{aligned}$$

and $\eta = \xi+s(z-\xi)$, $\vartheta = \xi+s(x-\xi)$, where from (7.16) for $l = 1, 2$, $k = 3$ we have

$$I^1_{kl} \leqslant |z-\xi|\int_0^1 \frac{1}{|\eta-y|^4}\left(\frac{|\eta'|}{|\eta'|+|\eta-y|}\right)^{\lambda(1)-1} ds \leqslant c\frac{r^{\lambda(1)}}{|x-y|^{3+\lambda(1)}},$$

$$I^2_{kl} \leqslant |x-\xi|\int_0^1 \frac{1}{|\vartheta-y|^4}\left(\frac{|\vartheta'|}{|\vartheta'|+|\vartheta-y|}\right)^{\lambda(2)-2} ds \leqslant c\frac{r^{\lambda(2)-1}}{|x-y|^{2+\lambda(2)}},$$

and for $l = k = 3$ we have

$$I^1_{kl} \leqslant c\frac{|z-\xi|}{|\eta-y|^4} \leqslant c\frac{r^{\lambda(1)}}{|x-y|^{3+\lambda(1)}},$$

$$I^2_{kl} \leqslant |x-\xi|\int_0^1 \frac{1}{|\vartheta-y|^4}\left(\frac{|\vartheta'|}{|\vartheta'|+|\vartheta-y|}\right)^{\lambda(1)-1} ds \leqslant c\frac{r^{\lambda(1)}}{|x-y|^{3+\lambda(1)}},$$

Hence we obtain the following estimate

$$\begin{aligned}
|e_{i,kl}(x)-e_{i,kl}(z)| &\leqslant c\sum_{j=1}^{2} [f_j]^{(\sigma)}_{\mathscr{D}_\vartheta}\Big[\int_{K_r(x)} (|x-y|^{-3+\sigma}+|z-y|^{-3+\sigma})dy\\
&+cr^\sigma+r^{\lambda(1)}\int_{\mathscr{D}_\vartheta\setminus K_r(x)} |x-y|^{-3-\lambda(1)+\sigma}dy+r^{\lambda(2)}\int_{\mathscr{D}_\vartheta\setminus K_r(x)} |x-y|^{-3-\lambda(2)+\sigma}dy\Big]\\
&\leqslant cr^\sigma\sum_{l=1}^{2} [f_l]^{(\sigma)}_{\mathscr{D}_\vartheta}, \quad \text{for } \sigma < \lambda(1) \text{ and } \sigma < \lambda(2).
\end{aligned}$$

This implies (9.11).

The proof of (9.12) is similar to that of Theorem 4.2 in [14]. ∎

THEOREM 9.3. *Let* $f_i \in C^l_{s-2}(\mathcal{D}_\vartheta)$, $\Phi_i \in C^{l+2}_s(\Gamma_i)$, $\Psi_i \in C^{l+1}_{s-1}(\Gamma_i)$, $i = 1, 2$, *have compact supports and*

$$s < \Lambda(\vartheta). \tag{9.13}$$

Then the problem (3.13) *has a unique solution* $e_i \in C^{l+2}_s(\mathcal{D}_\vartheta) \cap \mathcal{H}(\mathcal{D}_\vartheta)$ *such that*

$$\sum_{i=1}^{2} \langle e_i \rangle^{(l+2)}_{s,\mathcal{D}_\vartheta} \leqslant c \sum_{i=1}^{2} (\langle f_i \rangle^{(l)}_{s-2,\mathcal{D}_\vartheta} + \langle \Phi_i \rangle^{(l+2)}_{s,\Gamma_i} + \langle \Psi_i \rangle^{(l+1)}_{s-1,\Gamma_i}). \tag{9.14}$$

For $\pi/\vartheta = m \in N$ *we can omit the restriction* (9.13), *but some compatibility conditions between* f_i, Φ_i, Ψ_i, $i = 1, 2$, *described by Lemma* 5.4 *must be satisfied.*

Proof. Using Lemmas 5.3, 5.4 we can assume that $f_i \in \mathring{C}^l_{s-2}(\mathcal{D}_\vartheta)$, $\Phi_i = 0$, $\Psi_i = 0$, $i = 1, 2$.

For $s < 2$ the inequality (9.1) is valid. Moreover, from local estimates of solutions of the following problems in the half-space

$$\Delta e_i = f_i,\ i = 1, 2, \quad e_i|_\Gamma = 0, \quad \left.\frac{\partial e_2}{\partial n}\right|_\Gamma = 0, \tag{9.15}$$

and

$$\Delta e_i = f_i, \quad i = 1, 2, \tag{9.16}$$

$$(e_1 \cos\vartheta + e_2 \sin\vartheta)|_\Gamma = 0, \quad \left.\left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\right|_\Gamma = 0,$$

we obtain

$$\sum_{i=1}^{2} [e_i]^{(l+2)}_{K_r(z)} \leqslant c \sum_{i=1}^{2} ([f_i]^{(l)}_{K_{3r/2}(z)} + r^{-l-2+s}[e_i]^{(s)}_{K_{3r/2}(z)}), \tag{9.17}$$

where $r = \frac{1}{2}|z'|$. Therefore

$$\sup_{z\in\mathcal{D}_\vartheta} |z'|^{l+2-s} \sum_{i=1}^{2} [e_i]^{(l+2)}_{K_r(z)} \leqslant c \sum_{i=1}^{2} (|f_i|_{C^l_{s-2}(\mathcal{D}_\vartheta)} + [e_i]^{(s)}_{\mathcal{D}_\vartheta}) \tag{9.18}$$

and using (9.1) we get

$$\sum_{i=1}^{2} \langle e_i \rangle^{(l+2)}_{s,\mathcal{D}_\vartheta} \leqslant c \sum_{i=1}^{2} |f_i|_{C^l_{s-2}(\mathcal{D}_\vartheta)}. \tag{9.19}$$

Let $s > 2$. Now we estimate $[e_i]^{(s)}_{\mathcal{D}_\vartheta}$. From (9.2) we have

$$\sum_{i=1}^{2} [e_i(x', \cdot)]^{(s)}_{R^1} \leqslant c \sum_{i=1}^{2} [f_i(x', \cdot)]^{(s-2)}_{R^1}, \quad \forall\, x' \in d_\vartheta. \tag{9.20}$$

Let $\sigma = s-[s]$, $s_j = j+\sigma$, $j \leqslant [s]$. To estimate $[e_i(\cdot, z)]^{(s)}_{d_\vartheta}$ we use $Q_{i,j}$ and $e_{i,j}$,

$i = 1, 2$, described by (8.16) and (8.17). From the boundary conditions (3.11) we have $e_i|_{x'=0} = 0$, $i = 1, 2$. Therefore (9.19) implies that

$$(9.21) \qquad \sum_{i=1}^{2} |e_i|_{\mathring{C}^{s_2}_{\sigma}(\mathcal{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} |f_i|_{\mathring{C}^{\sigma}_{\sigma-2}(\mathcal{D}_\vartheta)}.$$

Let $f_i \in \mathring{C}^{\sigma}_{\sigma-1}(\mathcal{D}_\vartheta)$, then since $\sigma < 1 < s_1$, from Theorem 4.2 and $\nabla' e_{i,1}|_{x'=0} = 0$ it follows that

$$(9.22) \qquad \sum_{i=1}^{2} |e_{i,1}|_{\mathring{C}^{s_2}_{s_1}(\mathcal{D}_\vartheta)} \leqslant c \sum_{i=1}^{2} |f_i|_{\mathring{C}^{\sigma}_{\sigma-1}(\mathcal{D}_\vartheta)}.$$

The following considerations are the same as in the proof of Theorem 4.3 in [14]. We repeat them in our notation.

Since

$$|\nabla_z^2 e_i|_{\mathring{C}^{\sigma}_{\sigma}(d_\vartheta)} \leqslant 2[\nabla_z^2 e_i]^{(\sigma)}_{d_\vartheta} \leqslant c \sum_{i=1}^{2} [f_i]^{(\sigma)}_{\mathcal{D}_\vartheta},$$

the problem (8.17) has a solution $e_{i,2} \in \mathring{C}^{s_2}_{s_2}(d_\vartheta)$, $i = 1, 2$, and

$$|e_{i,2}|_{\mathring{C}^{s_2}_{s_2}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} |\nabla_z^2 e_i + f_i|_{\mathring{C}^{\sigma}_{\sigma}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} |f_i|_{\mathring{C}^{\sigma}_{\sigma}(\mathcal{D}_\vartheta)}.$$

The functions $e_{i,2} + (1-\zeta(x'))Q_{i,2} \in \mathring{C}^{s_2}_{s_2}(d_\vartheta)$ and $e_{i,1} - \zeta(x')Q_{i,2} \in \mathring{C}^{s_2}_{s_1}(d_\vartheta)$, $i = 1, 2$, are solutions of the same problem (8.19) with $h_i \in \mathring{C}^{\sigma}_{\sigma}(d_\vartheta) \cap \mathring{C}^{\sigma}_{\sigma-1}(d_\vartheta)$, so from Theorem 4.2 it follows that $e_{i,2} + Q_{i,2} = e_{i,1}$, $i = 1, 2$. Repeating the considerations of Theorem 8.3 we obtain (8.20) and the estimate

$$|e_{i,j}|_{\mathring{C}^{s_j}_{s_j}(d_\vartheta)} \leqslant c \sum_{i=1}^{2} |f_i|_{\mathring{C}^{s_j-2}_{s_j-2}(\mathcal{D}_\vartheta)}, \quad j \geqslant 2.$$

Therefore for arbitrary $z \in \boldsymbol{R}^1$

$$[e_i(\cdot, z)]^{(s)}_{d_\vartheta} = [e_{i,[s]}]^{(s)}_{d_\vartheta} \leqslant c \sum_{i=1}^{2} |f_i|_{\mathring{C}^{s-2}_{s-2}(\mathcal{D}_\vartheta)},$$

and using (9.20) we obtain

$$[e_i]^{(s)}_{\mathcal{D}_\vartheta} \leqslant c \sum_{i=1}^{2} |f_i|_{\mathring{C}^{s-2}_{s-2}(\mathcal{D}_\vartheta)}.$$

Now from (9.17) we obtain (9.19) for $s > 2$. In the case $\pi/\vartheta = m \in \boldsymbol{N}$ the construction of a solution can be made in the way described in part (e) of the proof of Theorem 8.3. This concludes the proof. ■

10. The problem (1.1) - (1.3) in a bounded domain Ω

The aim of this section is to show the existence and regularity properties of solutions to the problem (1.1)–(1.3) in a bounded domain Ω. Using the decomposition (3.1) we need only consider the problem (3.3)–(3.5) because the results for the Neumann problem are obtained in [13], [14]. Moreover, by using the transformation (3.6) the problem (1.1)–(1.3) is replaced by the problem (3.9). Therefore in this section the definition, of a weak solution of (3.9) is given, and its existence and regularity in neighbourhoods of edges are shown. To prove the last property the results of Sections 8 and 9 (see Theorems 8.3 and 9.3) are used.

A weak (or generalized) solution of the problem (3.9) is defined to be a function $e \in H^1(\Omega)$, $e_\tau|_{\partial\Omega} = 0$, such that

$$\sum_{i=1}^{3} \int_{\Omega} \nabla e_i \nabla \eta_i + \int_{\partial\Omega} e_n \eta_n \operatorname{div} \bar{n} = \sum_{i=1}^{3} \int_{\Omega} \omega_i \eta_i, \tag{10.1}$$

for all $\eta \in H^1(\Omega)$, $\eta_\tau|_{\partial\Omega} = 0$, where $\operatorname{div} \omega = 0$.

LEMMA 10.1.* *Any weak solution e of* (3.9) *satisfies* $\operatorname{div} e = 0$.

Proof. Put $\eta = \nabla\varphi$, where $\varphi|_{\partial\Omega} = 0$, into (10.1). Then $\int_\Omega \omega\eta = 0$ and instead of (10.1) we have

$$\sum_{i,j=1}^{3} \int_{\Omega} \nabla_i e_j \nabla_i \nabla_j \varphi + \int_{\partial\Omega} \operatorname{div} \bar{n} e_n \bar{n} \cdot \nabla\varphi = 0.$$

Integrating by parts in the first term of the above equality and using $e_\tau|_{\partial\Omega} = 0$ one has

$$\int_{\partial\Omega} e_n \Big[\sum_{i,j=1}^{3} n_i n_j \nabla_i \nabla_j \varphi - \Delta\varphi + \operatorname{div} \bar{n} \bar{n} \cdot \nabla\varphi \Big] - \int_{\Omega} \operatorname{div} e \Delta\varphi = 0. \tag{10.2}$$

Using the curvilinear coordinates (see Section 2) and the assumption $\varphi|_{\partial\Omega} = 0$ one gets

$$\Delta\varphi|_{\partial\Omega} = \operatorname{div}(H_n^{-1} \bar{n}) \frac{\partial\varphi}{\partial n} + H_n^{-2} \frac{\partial^2 \varphi}{\partial n^2},$$

$$n_i n_j \nabla_i \nabla_j \varphi = H_n^{-2} \frac{\partial^2}{\partial n^2} \varphi + \bar{n} \cdot \nabla (H_n^{-1}) \frac{\partial\varphi}{\partial n}.$$

The above expressions imply that the boundary term in (10.2) vanishes so that

$$\int_{\Omega} \operatorname{div} e \Delta\varphi = 0$$

for all $\varphi \in H^2(\Omega)$, $\varphi|_{\partial\Omega} = 0$. Hence $\operatorname{div} e = 0$. This concludes the proof. ∎

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* This lemma was shown to the author by V. A. Solonnikov (private communication)

LEMMA 10.2. *Let* $e \in C^1(\bar{\Omega})$, $\operatorname{div} e = 0$, $e_\tau|_{\partial\Omega} = 0$. *Then*

$$e = \frac{1}{4\pi} \operatorname{rot} \int_\Omega \frac{\operatorname{rot} e(y)}{|x-y|} dy + \nabla\psi, \tag{10.3}$$

where ψ *is a solution of the problem* (3.8) *such that*

$$\frac{1}{4\pi} \operatorname{rot} \int_\Omega \frac{\operatorname{rot} e(y)}{|x-y|} dy \cdot \bar{\tau}|_{\partial\Omega} = \bar{\tau} \cdot \nabla\psi_0, \tag{10.4}$$

where $\bar{\tau}$ *is an arbitrary tangent vector to* $\partial\Omega$.

Proof. The proof is almost the same as in the case of smooth boundary (see the proof of Lemma 1 from [1]). Let e^1 be the first term on the right-hand side of (10.3). Then $\operatorname{rot} e^1 = \operatorname{rot} e$ because

$$\operatorname{div} \int_\Omega \frac{\operatorname{rot} e(y)}{|x-y|} dy = 0,$$

which follows from

$$\operatorname{rot} e \cdot \bar{n}|_{\partial\Omega} = \frac{1}{H_1 H_2} \left[\frac{\partial}{\partial\tau_1}(e_{\tau_2} H_2) - \frac{\partial}{\partial\tau_2}(e_{\tau_1} H_1) \right] \Bigg|_{\partial\Omega} = 0 \tag{10.5}$$

(see the definition of curvilinear coordinates in Section 2 and [5, §18]).

Let l be a closed curve on $\partial\Omega$ which encloses a surface S. From (10.5) it follows that

$$\int_l e_\tau^1 \cdot d\bar{l} = \int_S \operatorname{rot} e^1 \cdot \bar{n} = 0,$$

hence $e_\tau^1|_{S \cap S_\nu}$, $\nu = 1, \ldots, r$ ($\partial\Omega = \bigcup_{\nu=1}^r S_\nu$, see Section 2), is the gradient of some function. Therefore (10.4) can be satisfied and the decomposition (10.3) is possible. ∎

LEMMA 10.3. *Let* e *be a weak solution of* (3.9) *and* $\operatorname{rot} e \in L_2(\Omega)$. *Then* $e \in L_2(\Omega)$ *and*

$$\|e\|_{L_2(\Omega)} \leqslant c \|\operatorname{rot} e\|_{L_2(\Omega)}. \tag{10.6}$$

Proof. From the form of e^1 we have

$$\|e^1\|_{L_2(\Omega)} \leqslant c \|\operatorname{rot} e\|_{L_2(\Omega)}. \tag{10.7}$$

Let

$$\bar{\psi}_0 = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \psi_0,$$

where $|\partial\Omega| = \operatorname{meas} \partial\Omega$. Then instead of (3.8) we consider the problem

$$\Delta\chi = 0, \quad \chi|_{\partial\Omega} = -\chi_0, \tag{10.8}$$

where $\chi = \psi + \bar{\psi}_0$, $\chi_0 = \psi_0 - \bar{\psi}_0$. Hence $\int_{\partial\Omega}\chi_0 = 0$, so

$$\chi_0(s) = \int_{x_0}^{s} \nabla_\tau \chi_0 \cdot \bar{l}\, d\sigma, \tag{10.9}$$

where $s \in \partial\Omega$, $x_0 \in \{x \in \partial\Omega:\ \chi_0(x) = 0\}$, $\nabla_\tau \chi_0 = (\partial\chi_0/\partial\tau_1, \partial\chi_0/\partial\tau_2)$ and the integral is taken along an arbitrary curve l on $\partial\Omega$ with tangent vector $\bar{l}$ from x_0 to s.

Let $\chi_0 \in H^{1/2}(\partial\Omega)$ and let $\tilde{\chi}_0$ be an extension of χ_0 such that $\tilde{\chi}_0|_{\partial\Omega} = \chi_0$ and

$$\|\tilde{\chi}_0\|_{H^1(\Omega)} \leqslant c\,\|\chi_0\|_{H^{1/2}(\Omega)}. \tag{10.10}$$

Let $\varphi = \chi + \tilde{\chi}_0$. Then the problem (10.8) is replaced by

$$\Delta\varphi = \Delta\tilde{\chi}_0, \qquad \varphi|_{\partial\Omega} = 0. \tag{10.11}$$

From (10.11) one gets

$$\|\nabla\varphi\|_{L_2(\Omega)} \leqslant c\,\|\nabla\tilde{\chi}_0\|_{L_2(\Omega)}. \tag{10.12}$$

Using the form of φ one has

$$\|\nabla\chi\|_{L_2(\Omega)} \leqslant c\,\|\nabla\tilde{\chi}_0\|_{L_2(\Omega)} \leqslant c\,\|\tilde{\chi}_0\|_{H^1(\Omega)} \leqslant c\,\|\chi_0\|_{H^{1/2}(\partial\Omega)} \leqslant c\,\|\nabla_\tau\chi_0\|_{L_2(\partial\Omega)}. \tag{10.13}$$

Finally, from (10.4) we get

$$\|\nabla\chi\|_{L_2(\Omega)} \leqslant c\,\|e^1\cdot\bar{\tau}\|_{L_2(\partial\Omega)} \leqslant c\,\|\operatorname{rot} e\|_{L_2(\Omega)}. \tag{10.14}$$

Therefore (10.4), (10.7) and (10.14) imply (10.6). ■

LEMMA 10.4. *Let* $e \in H^1(\Omega)$ *be such that* $\operatorname{div} e = 0$, $e_\tau|_{\partial\Omega} = 0$. *Then*

$$\int_\Omega |\nabla e|^2 + \int_{\partial\Omega} e_n^2 \operatorname{div}\bar{n} = \int_\Omega (\operatorname{rot} e)^2. \tag{10.15}$$

Proof. Let $e \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then

$$\operatorname{div}(e \times \operatorname{rot} e - \nabla e\cdot e) = |\operatorname{rot} e|^2 - e\cdot\operatorname{rot}\operatorname{rot} e - \Delta e\cdot e - |\nabla e|^2.$$

Using $\operatorname{rot}\operatorname{rot} e = -\Delta e + \nabla \operatorname{div} e$, $\operatorname{div} e = 0$ and integrating the above identity over Ω one has

$$\int_{\partial\Omega} (e \times \operatorname{rot} e\cdot\bar{n} - \bar{n}\cdot\nabla e\cdot e) = \int_\Omega (|\operatorname{rot} e|^2 - |\nabla e|^2). \tag{10.16}$$

If now

$$e_\tau|_{\partial\Omega} = 0, \qquad (\bar{n}\cdot\nabla e_n + e_n \operatorname{div}\bar{n})|_{\partial\Omega} = 0, \tag{10.17}$$

then (10.16) implies (10.15). Now, for any $e \in H^1(\Omega)$ we choose a sequence $e^m \in C^2(\Omega) \cap C^1(\bar{\Omega})$ converging to $e \in H^1(\Omega)$ in $H^1(\Omega)$ such that $\operatorname{div} e^m = 0$ and (10.17), (10.15) are satisfied. Passing to the limit we see that the first condition of (10.17) and (10.15) are satisfied for an arbitrary $e \in H^1(\Omega)$ such that $\operatorname{div} e = 0$. This concludes the proof. ■

LEMMA 10.5. *Let e be a weak solution of* (3.9) *and* $\omega \in L_2(\Omega)$. *Then* $e \in H^1(\Omega)$ *and*

$$\|e\|_{H^1(\Omega)} \leqslant c\|\omega\|_{L_2(\Omega)}. \tag{10.18}$$

Proof. Putting $\eta = e$ into (10.1) and using (10.15) one gets

$$\|\mathrm{rot}\, e\|^2_{L_2(\Omega)} = \int_\Omega \omega\cdot e.$$

Hence Lemma 10.3 implies

$$\|\mathrm{rot}\, e\|_{L_2(\Omega)} \leqslant c\|\omega\|_{L_2(\Omega)}. \tag{10.19}$$

Now from (10.15) and (10.19) we have

$$\|\nabla e\|^2_{L_2(\Omega)} \leqslant c\|\mathrm{rot}\, e\|^2_{L_2(\Omega)} + \varepsilon\|\nabla e\|^2_{L_2(\Omega)} + c(\varepsilon)\|e\|^2_{L_2(\Omega)}.$$

Then for sufficiently small ε, by (10.6),

$$\|\nabla e\|_{L_2(\Omega)} \leqslant c\|\mathrm{rot}\, e\|_{L_2(\Omega)}. \tag{10.20}$$

Hence from (10.6), (10.19), and (10.20) we obtain (10.18). ∎

THEOREM 10.1. *There exists a unique solution of the problem* (10.1) *such that* $e \in H^1(\Omega)$, $e_\tau|_{\partial\Omega} = 0$ *and the estimate* (10.18) *is satisfied.*

Proof. By well-known methods [8], [9] we write (10.1) in the operator form

$$(e, \eta)_{H^1(\Omega)} + (A(e), \eta)_{H^1(\Omega)} = (\tilde{\omega}, \eta)_{H^1(\Omega)}, \quad \forall\, \eta \in H^1(\Omega),\ \eta_\tau|_{\partial\Omega} = 0, \tag{10.21}$$

where $(\,,)_{H^1(\Omega)}$ is the scalar product in $H^1(\Omega)$,

$$(A(e), \eta)_{H^1(\Omega)} = \int_{\partial\Omega} \mathrm{div}\, \bar{n} e_n \eta_n - \int_\Omega e\cdot\eta, \quad (\tilde{\omega}, \eta)_{H^1(\Omega)} = (\omega, \eta)_{L_2(\Omega)}. \tag{10.22}$$

From (10.22) it follows that A is a compact operator. Moreover (10.18) implies the uniqueness of generalized solutions (10.1). Then by the Fredholm theorem there exists a unique solution $e \in H^1(\Omega)$, $e_\tau|_{\partial\Omega} = 0$ of (10.21) such that (10.18) is satisfied. ∎

Let us assume that ω_i, $i = 1, 2$, belongs to either $W^l_{p,\mu}(\Omega)$ or $C^l_s(\Omega)$. Our aim is to show that the generalized solution of (3.9) belongs to $W^{l+2}_{p,\mu}(\Omega)$ or $C^{l+2}_s(\Omega)$, respectively. We only consider neighbourhoods of edges because the regularity problem for elliptic equations in the interior and near the smooth parts of the boundary is well known.

By using a partition of unity it is sufficient to consider a neighbourhood of a fixed point $\xi \in M$, where M is an edge of Ω. Let S_1, S_2 be two boundary surfaces intersecting along M and let $\vartheta_0 = \vartheta(\xi)$ be the angle they make at ξ. Then we denote by $\mathcal{D}_{\vartheta_0}$ the dihedral angle with edge M_0 tangent to M at ξ and with sides $\Gamma_i = T_\xi S_i$ which are the tangent spaces to S_i, $i = 1, 2$, at ξ. We assume that $\mathscr{D}_{\vartheta_0} \cap \Omega \neq \emptyset$.

Let $\eta(x)$ be a smooth function of compact support which belongs to a partition of unity and let there exist neighbourhoods ω_ξ, Ω_ξ of ξ such that $\omega_\xi \Subset \Omega_\xi$ and $\omega_\xi \cap \partial\Omega \neq \emptyset$. Moreover, we assume that $\operatorname{supp}\eta = \Omega_\xi$ and $\eta(x) = 1$ for $x \in \omega_\xi$, so in particular $\eta(x)|_{x\in\omega_\xi\cap\partial\Omega} = 1$.

Introducing the notation $\tilde{e}(x) = e(x)\eta(x)$ we replace the problem (3.9) by

$$(10.23)\qquad \begin{aligned} -\Delta\tilde{e} &= -2\nabla e\nabla\eta - e\Delta\eta + \tilde{\omega} \quad \text{in } \Omega_\xi,\\ \tilde{e}_\tau|_{\partial\Omega_\xi} &= 0,\\ (\operatorname{div}\tilde{e} - e\cdot\nabla\eta)|_{\partial\Omega_\xi} &= 0. \end{aligned}$$

Let $\boldsymbol{R}^3$ be the Euclidean space endowed with the usual metric. By a dot we shall denote the scalar product in $\boldsymbol{R}^3$. Now we introduce the transformation $\Phi\colon \Omega_\xi \to \mathscr{D}_{\vartheta_0} \subset \boldsymbol{R}^3$, $\Omega_\xi \subset \boldsymbol{R}^3$ such that $\Phi\colon \Omega_\xi \cap M \to M_0$ and

$$\Phi'(\xi) = \frac{\partial y}{\partial x}(\xi) = I,$$

where $\boldsymbol{R}^3 \supset \Omega_\xi \ni x \to \Phi(x) = y \in \mathscr{D}_{\vartheta_0} \subset \boldsymbol{R}^3$ $(0 = \Phi(\xi))$ and I is the identity transformation. Moreover, we write

$$f'(y) = f(x)|_{x=\Phi^{-1}(y)}, \qquad \hat{f}(y) = f'(y)\eta'(y),$$

$$\hat{\nabla}_i = \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k} \equiv \frac{\partial y_k}{\partial x_i}\nabla'_k, \qquad \hat{\Delta} = \hat{\nabla}_i\hat{\nabla}_i, \qquad \Delta' = \nabla'_i\nabla'_i,$$

here and in the sequal the summation convention over repeated indices is used. We assume that

$$\Phi_*\colon T(S_i \cap \Omega_\xi) \to \Gamma_i,\ i = 1, 2, \quad \text{and} \quad \Phi_*\colon T(M \cap \Omega_\xi) \to M_0.$$

Let $\bar{\tau} \in T(S_i \cap \Omega_\xi)$. Hence

$$\Phi_*\tau = \hat{\tau} \in \Gamma_i, \qquad \hat{\tau}^k(y) = \tau^i(x)|_{x=\Phi^{-1}(y)}\frac{\partial y_k}{\partial x_i},$$

because $\partial/\partial x_i$ and $\partial/\partial y_i$, $i = 1, 2, 3$, are connected with the canonical bases in $\boldsymbol{R}^3$ and $\Phi_*\boldsymbol{R}^3 \equiv \boldsymbol{R}^3$, respectively, and

$$\Phi_*\frac{\partial}{\partial x_i} = \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j}.$$

Therefore instead of (10.23) we have

$$(10.24)\qquad \begin{aligned} -\Delta'\hat{e} &= -(\Delta' - \hat{\Delta})\hat{e} - 2\hat{\nabla}e'\hat{\nabla}\eta' - e'\hat{\Delta}\eta' + \hat{\omega},\\ \hat{e}\cdot\bar{\tau}'_{si}|_{\Gamma_i} &= (\hat{e}\cdot\bar{\tau}'_{si} - \hat{e}\cdot\hat{\tau}_{si})|_{\Gamma_i}, \quad i = 1, 2,\ s = 1, 2,\\ \operatorname{div}'\hat{e}|_{\Gamma_i} &= (\operatorname{div}'\hat{e} - \widehat{\operatorname{div}}\hat{e} + e'\hat{\nabla}\eta')|_{\Gamma_i}, \quad i = 1, 2, \end{aligned}$$

where $\bar{\tau}'_{si}$, $s = 1, 2$, are tangent vectors to Γ_i, $i = 1, 2$, $\hat{\tau}_{si} = \Phi_* \bar{\tau}_{si}$ and $\bar{\tau}_{si} \in T(S_i \cap \Omega_\xi)$, $i = 1, 2$, $s = 1, 2$. In the above considerations we have used the fact that the transformation Φ transforms a vector e as a set scalars.

To express the problem (10.24) in such a form that the previous results of this paper could be used we have to define the transformation $y = \Phi(x)$ explicitly. Let $S_i \cap \Omega_\xi$ be described by equations $x_2 = f_i(x_1, x_3)$, $i = 1, 2$. Then $M = \{x \in \boldsymbol{R}^3\colon\ x_2 = f_1(x_1, x_3) = f_2(x_1, x_3)\}$. Now we define the transformation Φ by

$$y_2 = x_2 - f_1(x_1, x_3) - y_2^0,$$
$$y_1 \sin\vartheta_0 - y_2 \cos\vartheta_0 = x_2 - f_2(x_1, x_3) - y_1^0 \cos\vartheta_0 - y_2^0 \sin\vartheta_0,$$
$$y_3 = \varphi(x_1, x_2, x_3),$$

where the constants $y_i^0 = y_i^0(\xi)$, $i = 1, 2$, and the functions φ are such that $0 = \Phi(\xi)$. This definition ensures that $\Gamma_i = \Phi(S_i \cap \Omega_\xi)$, $i = 1, 2$, are described by $y_2 = 0$ and $y_1 \sin\vartheta_0 - y_2 \cos\vartheta_0 = 0$, respectively. Moreover, the edge M_0 is defined by $y_1 = y_2 = 0$.

Let $\bar{\eta}_i$, $i = 1, 2, 3$, be the orthonormal system of vectors connected with the axes y_i, $i = 1, 2, 3$, and let $\hat{e}_i = \hat{e}\cdot\bar{\eta}_i$, $i = 1, 2, 3$. Then the problem (10.24) can be replaced by the following system of two problems which are connected with each other by their right-hand sides:

$$\text{(10.25)}\qquad \begin{aligned} -\Delta'\hat{e}_3 &= -(\Delta' - \hat{\Delta})\hat{e}_3 - 2\hat{\nabla} e'_3 \hat{\nabla}\eta' - e'_3 \hat{\Delta}\eta' + \hat{\omega}_3, \\ \hat{e}_3|_{\Gamma_i} &= (\hat{e}_3 - \hat{e}\cdot\hat{\tau}_{1i})|_{\Gamma_i}, \qquad i = 1, 2, \end{aligned}$$

and

$$\text{(10.26)}\qquad \begin{aligned} -\Delta'\hat{e}_i &= -(\Delta' - \hat{\Delta})\hat{e}_i - 2\hat{\nabla} e'_i \hat{\nabla}\eta' - e'_i \hat{\Delta}\eta' + \hat{\omega}_i, \qquad i = 1, 2, \\ \hat{e}_1|_{\Gamma_1} &= (\hat{e}_1 - \hat{e}\cdot\hat{\tau}_{21})|_{\Gamma_1}, \\ (\hat{e}_1 \cos\vartheta_0 + \hat{e}_2 \sin\vartheta_0)|_{\Gamma_2} &= [(\hat{e}_1 \cos\vartheta_0 + \hat{e}_2 \sin\vartheta_0) - \hat{e}\cdot\hat{\tau}_{22}]|_{\Gamma_2}, \\ \frac{\partial \hat{e}_2}{\partial y_2}\bigg|_{\Gamma_1} &= \left(\frac{\partial \hat{e}_2}{\partial y_2} - \widehat{\operatorname{div}}\,\hat{e} + e'\cdot\hat{\nabla}\eta'\right)\bigg|_{\Gamma_1}, \\ \bar{n}\cdot\nabla'\hat{e}_n|_{\Gamma_2} &= (\bar{n}\cdot\nabla'\hat{e}_n - \widehat{\operatorname{div}}\,\hat{e} + e'\cdot\hat{\nabla}\eta')|_{\Gamma_2} \end{aligned}$$

where we have used the fact that $\bar{\eta}_1, \bar{\eta}_3 \in T\Gamma_1$, $\bar{\eta}_2 \perp T\Gamma_1$, $\bar{\eta}_3$, $\bar{\tau} = \cos\vartheta_0 \bar{\eta}_1 + \sin\vartheta_0 \bar{\eta}_2 \in T\Gamma_2$ and $\bar{n} = \sin\vartheta_0 \bar{\eta}_1 - \cos\vartheta_0 \bar{\eta}_2 \perp T\Gamma_2$, together with the equalities

$$\operatorname{div}'\hat{e}|_{\Gamma_1} = \sum_{i=1}^{3} \frac{\partial \hat{e}_i}{\partial y_i}\bigg|_{\Gamma_1}, \qquad \operatorname{div}'\hat{e}|_{\Gamma_2} = \left(n\cdot\nabla'\hat{e}_n + \bar{\tau}\cdot\nabla'\hat{e}_\tau + \frac{\partial \hat{e}_3}{\partial y_3}\right)\bigg|_{\Gamma_2}$$

and

$$\bar{\tau}_{1i}\cdot\bar{\tau}_0|_{M\cap\Omega_\xi} \neq 0, \qquad \tau_{2i}\cdot\bar{n}_0|_{M\cap\Omega_\xi} \neq 0,$$

where $\bar{\tau}_0 \in T(M \cap \Omega_\xi)$, $\bar{n}_0 \perp T(M \cap \Omega_\xi)$, $i = 1, 2$.

Now we extend the functions $\hat{e}_i$, $i = 1, 2, 3$, by zero on $\mathscr{D}_{\vartheta_0}\backslash\Phi\Omega_\xi$. We denote the extensions by the same letters. Therefore we can regard the problem (10.25) as the Dirichlet problem in $\mathscr{D}_{\vartheta_0}$, which is investigated in [11], [12], and consider only the problem (10.26), which is in fact the problem (3.13).

To prove the existence of solutions of (10.26) we have to know that the condition (5.4) is satisfied, which in this case has the following form

$$(10.27)\quad \Big[\frac{\partial}{\partial y_1}(\hat{e}_1-\hat{e}\cdot\hat{\tau}_{21})-\Big(\cos\vartheta_0\frac{\partial}{\partial y_1}+\sin\vartheta_0\frac{\partial}{\partial y_2}\Big)(\hat{e}_1\cos\vartheta_0+\hat{e}_2\sin\vartheta_0-\hat{e}\cdot\hat{\tau}_{22})$$
$$+\frac{\partial\hat{e}_2}{\partial y_2}-\Big(\sin\vartheta_0\frac{\partial}{\partial y_1}-\cos\vartheta_0\frac{\partial}{\partial y_2}(\hat{e}_1\sin\vartheta_0-\hat{e}_2\cos\vartheta_0)\Big)\Big]\Big|_{y'=0}=0,$$

where $y' = (y_1, y_2)$. From (10.27) after some calculations one has

$$(10.28)\quad \Big[\frac{\partial}{\partial y_1}\hat{e}\cdot\hat{\tau}_{21}-\Big(\cos\vartheta_0\frac{\partial}{\partial y_1}+\cos\vartheta_0\frac{\partial}{\partial y_2}\Big)\hat{e}\cdot\hat{\tau}_{22}\Big]\Big|_{y'=0}=0.$$

Now we show that (10.28) is satisfied. First, taking inverse images in the transformation Φ gives, in place of (10.28),

$$(10.29)\quad \Big[\Phi_*^{-1}\Big(\frac{\partial}{\partial y_1}\Big)e\cdot\bar{\tau}_{21}-\Phi_*^{-1}\Big(\cos\vartheta_0\frac{\partial}{\partial y_1}+\sin\vartheta_0\frac{\partial}{\partial y_2}\Big)e\cdot\bar{\tau}_{22}\Big]\Big|_{x\in M\cap\Omega_\xi}=0.$$

By the properties of the transformation Φ_* (Φ_*: $T(S_i\cap\Omega_\xi)\to\Gamma_i$, $i = 1, 2$) and $(10.23)_2$ we see that (10.29) is satisfied.

Finally, we turn to the discussion of domains with edges which are intersection of two surfaces making angle π/m, $m\in N$, $m \geqslant 2$. We keep all the previous notation with the additional assumption that $\vartheta_0 = \vartheta(\xi) = \pi/m$ for every $\xi\in M$. In this case we can define Φ more explicitly. Let $\tilde{\gamma}_i(x)\in T_x(S_i\cap\Omega_\xi)$, $x\in S_i\cap\Omega_\xi$, $i = 1, 2$, $\tilde{\gamma}_0\in T_x(M\cap\Omega_\xi)$, $x\in M\cap\Omega_\xi$ be such that $\tilde{\gamma}_i(x)\cdot\tilde{\gamma}_0(x) = 0$ for $x\in M\cap\Omega_\xi$, $i = 1, 2$. Then we determine Φ by the condition that $\gamma_i(y) = \Phi_*\tilde{\gamma}_i(x)\in\Gamma_i$, $x\in S_i\cap\Omega_\xi$, $i = 1, 2$, $\gamma_0(y) = \Phi_*\tilde{\gamma}_0(x)\in TM_0$, $x\in M\cap\Omega_\xi$, $y = \Phi(x)$ and γ_i, $i = 1, 2$, are the same as at the beginning of Section 2. The vectors $\gamma_i(y)$, γ_0, $i = 1, 2$, are constants and $\gamma_i\cdot\gamma_0 = 0$, $i = 1, 2$. Let $\tilde{d}_{\vartheta_0}(x)$ be generated by the vectors $\tilde{\gamma}_1(x)$, $\tilde{\gamma}_2(x)$, $x\in M\cap\Omega_\xi$ and $d_{\vartheta_0} = d_{\vartheta_0}(y)$ by the vectors γ_1, γ_2, as in Section 2. Therefore the transformation $\Phi_*(x)$: $\tilde{d}_{\vartheta_0}(x)\to d_{\vartheta_0}(y)$, $x\in M\cap\Omega_\xi$, $y = \Phi(x)$ is a superposition of a translation and a rotation only (i.e. a rigid motion as a transformation of $\boldsymbol{R}^3$). Considering the problems (10.25), (10.26) in $\mathscr{D}_\vartheta$ for $\vartheta = \pi/m$ requires solving them first in the angle d_{ϑ_0} needs in general some compatibility conditions at $y\in M_0$. The fact that Φ is a rigid motion implies that the necessary compatibility conditions in $d_{\vartheta_0}(y)$ follow from those in $\tilde{d}_{\vartheta_0}(x)$, $x\in M\cap\Omega_\xi$, the latter appear in the problem (10.23) or (3.9). In general it is difficult to show this equivalence so we shall prove it in a particular case.

Let $\vartheta_0 = \pi/2$. First we find the compatibility conditions for the problem (3.9). For simplicity we assume that $M \cap \Omega'_\xi$ is a straight line, $S_i \cap \Omega'_\xi$ are planes, $i = 1, 2$, and $\Omega'_\xi \subset \Omega_\xi$ is a small neighbourhood of ξ. Ω'_ξ can be arbitrarily small because we are interested in the behaviour of solutions of (3.9) at ξ only. We also assume that $\bar{\gamma}_i(\xi)$ is the x_i-axis, $i = 1, 2$, and $\bar{\gamma}_0(\xi)$ is the x_3-axis. Then locally in these coordinates the problem (3.9) has the form

$$
\begin{gathered}
-\Delta e = \omega, \\
e_1|_{S_1} = 0, \quad e_2|_{S_2} = 0, \quad e_3|_{S_1 \cup S_2} = 0, \\
\left.\frac{\partial e_2}{\partial x_2}\right|_{S_1} = 0, \quad \left.\frac{\partial e_1}{\partial x_1}\right|_{S_2} = 0
\end{gathered} \tag{10.30}
$$

in Ω'_ξ. Considering solutions of (10.30) with third derivatives continuous one has the following compatibility conditions. From $(10.30)_{1,2}$ one has

$$
\left.\frac{\partial^2}{\partial x_2^2}\frac{\partial}{\partial x_1} e_1\right|_{x'=0} = 0, \quad \left.\frac{\partial^3}{\partial x_1^3} e_1\right|_{x'=0} = 0, \quad \left.\frac{\partial^2}{\partial x_3^2}\frac{\partial}{\partial x_1} e_1\right|_{x'=0} = 0,
$$

$$
-\left.\frac{\partial}{\partial x_1}\Delta e_1\right|_{x'=0} = \left.\frac{\partial}{\partial x_1}\omega_1\right|_{x'=0} = 0,
$$

and similarly

$$
\left.\frac{\partial}{\partial x_1}\omega_1\right|_{x'=0} = 0, \quad \left.\frac{\partial}{\partial x_2}\omega_2\right|_{x'=0} = 0, \quad \omega_i|_{x'=0} = 0, \; i = 1, 2. \tag{10.31}
$$

Consider the compatibility conditions for the problem (10.25), (10.26) where in the latter we assume that $\vartheta_0 = \pi/2$ and $\bar{n}\cdot\nabla'\hat{e}_n = \partial\hat{e}_1/\partial y_1$. From $(10.25)_2$ and (10.26) one has

$$
\begin{aligned}
&\frac{\partial}{\partial y_1}\left[\frac{\partial^2}{\partial y_1^2}(\hat{e}_1 - \bar{\tau}_{21}\cdot\hat{e}) + \frac{\partial^2}{\partial y_2^2}(\hat{e}_2 - \hat{\tau}_{22}\cdot\hat{e}) + \frac{\partial^2}{\partial y_3^2}(\hat{e}_3 - \hat{\tau}_{11}\hat{e})\right]\Bigg|_{y'=0} \\
&\qquad = \frac{\partial}{\partial y_1}[(\Delta' - \hat{\Delta})\hat{e}_1 + 2\hat{\nabla}e'_1\cdot\hat{\nabla}\eta' + e'_1\hat{\Delta}\eta' - \hat{\omega}_1]\Big|_{y'=0}
\end{aligned}
$$

$$
\frac{\partial}{\partial y_1}\left[\frac{\partial^2}{\partial y_1^2}\hat{\tau}_{21}\cdot\hat{e} + \frac{\partial^2}{\partial y_2^2}\hat{\tau}_{22}\cdot\hat{e} + \frac{\partial^2}{\partial y_3^2}\hat{\tau}_{11}\cdot\hat{e}\right]\Bigg|_{y'=0} = \frac{\partial}{\partial y_1}[(\hat{\Delta}e'_1 + \omega'_1)]\Big|_{y'=0}.
$$

Going back to the x-coordinates by means of the transformation Φ we obtain

$$
\frac{\partial}{\partial x_1}\left[\frac{\partial^2}{\partial x_1^2}e_1 + \frac{\partial^2}{\partial x_2^2}e_2 + \frac{\partial^2}{\partial x_3^2}e_3\right]\Bigg|_{x'=0} = \frac{\partial}{\partial x_1}[(\Delta e + \omega)\eta]\Big|_{x'=0},
$$

which is satisfied under the conditions (10.31). Similarly, the other compatibility conditions for the problem (10.25), (10.26) are also satisfied.

Now we are in a position to show that the weak solution of the problem (3.9) belongs either to $W^{l+2}_{p,\mu}(\Omega)$ or to $C^{l+2}_s(\Omega)$ in a neighbourhood of any edge for ω sufficiently regular. Assume that $\eta(x) = \zeta_\lambda((x-\xi)/\lambda)$ and $\Omega_\lambda(\xi) = \Omega \cap K_\lambda(\xi)$, where $\lambda < d/2$ and d is described in Theorem 2.7.

We only consider the case of weighted Sobolev spaces; the case of weighted Hölder spaces is similar.

From $e \in H^1(\Omega)$ it follows that $e' \in H^1(\mathscr{D}_\vartheta)$, and instead of the problem (10.26) we consider the following problem

$$
\begin{aligned}
-\Delta' \hat{e}_i &= -(\Delta' - \hat{\Delta})\hat{e}_i + g_i, \quad i = 1, 2,\\
\hat{e}_1|_{\Gamma_1} &= (\hat{e}_1 - \hat{e}\cdot\hat{\tau}_{21})|_{\Gamma_1},\\
(\hat{e}_1 \cos\vartheta_0 + \hat{e}_2 \sin\vartheta_0)|_{\Gamma_2} &= [(\hat{e}_1\cos\vartheta_0 + \hat{e}_2 \sin\vartheta) - \hat{e}\cdot\hat{\tau}_{22}]|_{\Gamma_2},\\
\frac{\partial \hat{e}_2}{\partial y_2}\Big|_{\Gamma_1} &= \left(\frac{\partial \hat{e}_2}{\partial y_2} - \hat{\operatorname{div}}\,\hat{e}\right)\Big|_{\Gamma_1} + \varphi_1,\\
\bar{n}\cdot\nabla' \hat{e}_n|_{\Gamma_2} &= (\bar{n}\nabla'\hat{e}_n - \hat{\operatorname{div}}\,\hat{e})|_{\Gamma_2} + \varphi_2,
\end{aligned}
\tag{10.32}
$$

where $\bar{n}$ is the normal vector to Γ_2 and

$$
\begin{aligned}
g_i &= -2\hat{\nabla} e'_i \hat{\nabla}\eta' - e'_i \hat{\Delta}\eta' + \hat{\omega}_i, \quad i = 1, 2,\\
\varphi_i &= e'\hat{\nabla}\eta'|_{\Gamma_i}, \quad i = 1, 2.
\end{aligned}
\tag{10.33}
$$

We have $g_i \in L_{2,\mu}(\mathscr{D}_{\vartheta_0})$, $\varphi_i \in H^{1/2}_\mu(\mathscr{D}_{\vartheta_0})$, $\mu \geqslant 0$, $i = 1, 2$.

To prove the existence of solutions of the problem (10.32) we use the following method of successive approximations:

$$
\begin{aligned}
-\Delta' \hat{e}^m_i &= -(\Delta' - \hat{\Delta})\hat{e}^{m-1}_i + g_i, \quad i = 1, 2,\\
\hat{e}^m_1|_{\Gamma_1} &= (\hat{e}^{m-1}_1 - \hat{e}^{m-1}\cdot\hat{\tau}_{21})|_{\Gamma_1},\\
(\hat{e}^m_1 \cos\vartheta_0 + \hat{e}^m_2 \sin\vartheta_0)|_{\Gamma_2} &= [(\hat{e}^{m-1}_1\cos\vartheta_0 + \hat{e}^{m-1}_2 \sin\vartheta_0) - \hat{e}^{m-1}\cdot\bar{\tau}_{22}]|_{\Gamma_2},\\
\frac{\partial \hat{e}^m_2}{\partial y_2}\Big|_{\Gamma_2} &= \left(\frac{\partial \hat{e}^{m-1}_2}{\partial y_2} - \hat{\operatorname{div}}\,\hat{e}^{m-1}\right)\Big|_{\Gamma_1} + \varphi_1,\\
\bar{n}\cdot\nabla \hat{e}^m_n|_{\Gamma_2} &= (\bar{n}\cdot\nabla\hat{e}^{m-1}_n - \hat{\operatorname{div}}\,\hat{e}^{m-1})|_{\Gamma_2} + \varphi_2
\end{aligned}
\tag{10.34}
$$

and $\hat{e}^0_1 = 0$, $i = 1, 2$. In view of the properties of the transformation Φ, (10.33) and Theorem 8.3, from (10.34) we have

$$
\sum_{i=1}^{2} \|\hat{e}^m_i\|_{H^2_\mu(\mathscr{D}_{\vartheta_0})} \leqslant c\sum_{i=1}^{2} (\|g_i\|_{L_{2,\mu}(\mathscr{D}_{\vartheta_0})} + \|\varphi_i\|_{H^{1/2}_\mu(\Gamma_i)}) + c\lambda \sum_{i=1}^{3} \|\hat{e}^{m-1}_i\|_{H^2_\mu(\mathscr{D}_{\vartheta_0})}.
\tag{10.35}
$$

Since the norm of $\hat{e}^{m-1}_3$ occurs here, we have to consider the problems (10.25) and (10.26) together. For the convenience of the reader we recall the theorem about the existence and regularity of solutions to the Dirichlet problem in $\mathscr{D}_\vartheta \subset \boldsymbol{R}^n$,

$$\begin{aligned} \Delta u &= f \quad \text{in } \mathscr{D}_\vartheta, \\ u|_{\Gamma_i} &= \psi_i, \quad i = 1, 2. \end{aligned} \tag{10.36}$$

which follows directly from the results of [11], [12].

THEOREM 10.2. *Let* $f \in W^l_{p,\mu}(\mathscr{D}_\vartheta)$, $\psi_i \in W^{l+2-1/p}_{p,\mu}(\Gamma_i)$, $i = 1, 2$, $l \in \boldsymbol{Z}$, $p, \mu \in \boldsymbol{R}$, $p \geqslant 1$, $\mu > -2/p$ *and* $\pi/\vartheta > l+1-(\mu+2/p) > 0$. *Then there exists a unique solution of the problem* (10.36) *such that* $u \in W^{l+2}_{p,\mu}(\mathscr{D}_\vartheta)$ *and*

$$\|u\|_{W^{l+2}_{p,\mu}(\mathscr{D}_\vartheta)} \leqslant c\Big(\|f\|_{W^l_{p,\mu}(\mathscr{D}_\vartheta)} + \sum_{i=1}^{2} \|\psi_i\|_{W^{l+2-1/p}_{p,\mu}(\Gamma_i)}\Big). \tag{10.37}$$

Since $e \in H^1(\Omega)$ we can replace (10.25) by the problem

$$\begin{aligned} -\Delta' \hat{e}_3 &= -(\Delta' - \hat{\Delta})\hat{e}_3 + h, \\ \hat{e}_3|_{\Gamma_i} &= (\hat{e}_3 - \hat{e}\cdot\hat{\tau}_{1i})|_{\Gamma_i}, \quad i = 1, 2, \end{aligned} \tag{10.38}$$

where $h = -2\hat{\nabla} e'_3 \hat{\nabla}\eta' - e'_3 \hat{\Delta}\eta' + \hat{\omega}_3 \in L_{2,\mu}(\mathscr{D}_{\vartheta_0})$ which will be solved by the following method of successive approximations:

$$\begin{aligned} -\Delta' \hat{e}^m_3 &= -(\Delta' - \hat{\Delta})\hat{e}^{m-1}_3 + h, \\ \hat{e}^m_3|_{\Gamma_i} &= (\hat{e}^{m-1}_3 - \hat{e}^{m-1}\cdot\hat{\tau}_{1i})|_{\Gamma_i}, \quad i = 1, 2, \end{aligned} \tag{10.39}$$

where $\hat{e}^0_3 = 0$. Applying Theorem 10.2 to (10.39) we have $\hat{e}^m_3 \in H^2_\mu(\mathscr{D}_{\vartheta_0})$ and

$$\|\hat{e}^m_3\|_{H^3_\mu(\mathscr{D}_{\vartheta_0})} \leqslant c\|h\|_{L_{2,\mu}(\mathscr{D}_{\vartheta_0})} + c\lambda \sum_{i=1}^{3} \|\hat{e}^{m-1}_i\|_{H^2_\mu(\mathscr{D}_{\vartheta_0})}. \tag{10.40}$$

Therefore for sufficiently small λ we have the existence of solutions of the problem (10.32), (10.38) and the estimate

$$\sum_{i=1}^{3} \|\hat{e}_i\|_{H^2_\mu(\mathscr{D}_{\vartheta_0})} \leqslant c \sum_{i=1}^{2} \big(\|g_i\|_{L_{2,\mu}(\mathscr{D}_{\vartheta_0})} + \|\varphi_i\|_{H^{1/2}_\mu(\Gamma_i)}\big) + c\|h\|_{L_{2,\mu}(\mathscr{D}_{\vartheta_0})}. \tag{10.41}$$

Since a neighbourhood of an arbitrary edge M of Ω is covered by a finite number of $\Omega_{\lambda/2}(\xi^k)$, $\xi^k \in M$, $k = 1, \dots, N(M)$, we go back in (10.41) to $\Omega_\lambda(\xi^k)$ and then sum the results over all sets $\Omega_{\lambda/2}(\xi^k)$ (and also over neighbourhoods of interior points and points of the smooth part of the boundary). We get

$$\sum_{i=1}^{3} \|e_i\|_{H^2_\mu(\Omega)} \leqslant c \sum_{i=1}^{3} \|\omega_i\|_{L_{2,\mu}(\Omega)} + c(\lambda) \sum_{i=1}^{3} \|e_i\|_{H^1(\Omega)} \leqslant c \sum_{i=1}^{3} \|\omega_i\|_{L_{2,\mu}(\Omega)}, \tag{10.42}$$

where (10.18) has been used to obtain the last inequality. Hence $e_i \in H^2_\mu(\Omega)$, $i = 1, 2$, and our result is proved for $p = 2$ and $k = 0$.

To show higher regularity we use the embedding theorem [20]

$$\|D^\nu f\|_{L_{q,\mu}(\Omega)} \leqslant c\|f\|_{W^j_{p,\mu}(\Omega)}, \quad j-\nu-3\left(\frac{1}{p}-\frac{1}{q}\right) \geqslant 0. \tag{10.43}$$

Hence using (10.42) for $j = 2$, $p = 2$, $\nu = 1$, gives $q \leqslant 6$ and implies h, $g_i \in L_{q,\mu}(\mathscr{D}_{\vartheta_0})$, $\varphi_i \in W^{1-1/q}_{q,\mu}(\Gamma_i)$, $i = 1, 2$. Then repeating the above considerations gives $e_i \in W^2_{q,\mu}(\Omega)$, $i = 1, 2$, and

$$\sum_{i=1}^{3} \|e_i\|_{W^2_{q,\mu}(\Omega)} \leqslant c \sum_{i=1}^{3} \|\omega_i\|_{L_{q,\mu}(\Omega)}. \tag{10.44}$$

Now using (10.43) for $j = 2$, $p \leqslant 6$, $\nu = 1$ we obtain (10.44) for arbitrary q.

For $\omega_i \in W'_{p,\mu}(\Omega)$, $i = 1, 2$, the above results give h, $g_i \in W^1_{p,\mu}(\mathscr{D}_{\vartheta_0})$ and $\varphi_i \in W^{2-1/p}_{p,\mu}(\Gamma_i)$, $i = 1, 2$, so from (10.32), (10.38) after applying the method of successive approximations we get $\hat{e}_i \in W^3_{p,\mu}(\mathscr{D}_{\vartheta_0})$ hence $e_i \in W^3_{p,\mu}(\Omega)$, $i = 1, 2, 3$, and

$$\sum_{i=1}^{3} \|e_i\|_{W^3_{p,\mu}(\Omega)} \leqslant c \sum_{i=1}^{3} \|\omega_i\|_{W^1_{p,\mu}(\Omega)}. \tag{10.45}$$

In the case $\omega_i \in W^l_{p,\mu}(\Omega)$, $i = 1, 2, 3$, the above considerations must be done step by step so we get $e_i \in W^{l+2}_{p,\mu}(\Omega)$, $i = 1, 2, 3$, and

$$\|e\|_{W^{l+2}_{p,\mu}(\Omega)} \leqslant c\|\omega\|_{W^l_{p,\mu}(\Omega)}. \tag{10.46}$$

Suppose that

(a) $\Omega \subset \mathbf{R}^3$ is a bounded domain.

(b) $\partial\Omega = \bigcup_{\nu=1}^{r} S_\nu$, where S_ν are 2-dimensional manifolds of class $C^{\varrho+2}$.

(c) The edges do not intersect each other.

(d) If $\bar{S}_{\nu_1} \cap \bar{S}_{\nu_2} \neq \emptyset$ then $L = \bar{S}_{\nu_1} \cap \bar{S}_{\nu_2}$ is a 1-dimensional closed curve of class $C^{\varrho+2}$ which is an edge of Ω.

(f) Ω has q edges L_σ, $\sigma = 1, \ldots, q$.

(g) Ω has only edges which are intersection of two boundary surfaces S_ν, $S_{\nu'}$, at an angle ϑ either everywhere equal to π/m or everywhere different from π/m, $m \in \mathbf{N}$, $m \geqslant 2$,

By using the results of [13], [14] about the existence of solutions of the Neumann problem (3.2) the above considerations and Theorems 8.3, 9.3 imply

THEOREM 10.3. *Let the assumptions (a)–(g) ($\varrho = k$) be satisfied. Let $\omega \in W^k_{p,\mu}(\Omega)$, $b \in W^{k+1-1/p}_{p,\mu}(\partial\Omega)$ where b satisfies the compatibility condition* (1.4) *with $k \in \mathbf{Z}$, $p, \mu \in \mathbf{R}$, $p \geqslant 1$, $\mu > -2/p$. Assume that*

$$\Lambda(\vartheta(z)) > k+2-\left(\mu+\frac{2}{p}\right) > 0, \quad \text{for every } z \in L_\sigma, \sigma = 1, \ldots, q, \tag{10.47}$$

where $\Lambda(\vartheta)$ is described by (6.8). *For domains with edges with two-surface angle equal to π/m, $m \in \mathbf{N}$, $m \geqslant 2$, the condition* (10.47) *can be omitted but some* *compatibility conditions on the right-hand side functions of* (1.1)–(1.3) *at points of edges have to be imposed (for an example of the compatibility conditions, see* (10.31)). *The compatibility conditions contain derivatives up to order* $[k+2-(\mu+2/p)]$. *Then there exists a unique solution of the problem* (1.1)–(1.3) *such that* $v \in W^{k+1}_{p,\mu}(\Omega)$, *and*

$$\|v\|_{W^{k+1}_{p,\mu}(\Omega)} \leqslant c\,(\|\omega\|_{W^{k}_{p,\mu}(\Omega)} + \|b\|_{W^{k+1-1/p}_{p,\mu}(\partial\Omega)}). \tag{10.48}$$

THEOREM 10.4. *Let the assumptions* (a)–(g) ($\varrho = l$) *and the compatibility condition* (1.4) *be satisfied and let* $S_\nu \in C^{l+2}_{s+1}$, $\nu \leqslant r$, $\omega \in C^{l}_{s-2}(\Omega)$, $b \in C^{l+1}_{s-1}(\partial\Omega)$, *where* $l, s \in \boldsymbol{R}$ *and*

$$0 < s < \Lambda(\vartheta(z)) \tag{10.49}$$

for every $z \in L_\sigma$, $\sigma = 1,\ldots,q$, *here* $\Lambda(\vartheta(z))$ *is the same as in Theorem* 10.3.
For angles equal to π/m, $m \in \boldsymbol{N}$, $m \geqslant 2$, *see the remark in Theorem* 10.3 *with the difference that instead of* (10.47) *the condition* (10.49) *can be omitted. Moreover, the compatibility conditions contain derivatives up to order* $[l+1]$. *Then there exists a unique solution of the problem* (1.1)–(1.3) *such that* $v \in C^{l+1}_{s-1}(\Omega)$, *and*

$$|v|_{C^{l+1}_{s-1}(\Omega)} \leqslant c(|\omega|_{C^{l}_{s-2}(\Omega)} + |b|_{C^{l+1}_{s-1}(\partial\Omega)}). \tag{10.50}$$

Appendix. The distinguished case: $\mu + \dfrac{2}{p} \in \boldsymbol{Z}$

In this section we shall extend Theorem 4.3 to the case $k+2-(\mu+2/p) = 1$, where 1 is one of the eigenvalues obtained in Lemma 4.1. Moreover, Lemma 4.1 implies that the eigenfunctions corresponding to the eigenvalue 1 have the form

$$e_1 = \alpha x_2, \quad e_2 = -\alpha x_1, \tag{A.1}$$

where α is an arbitrary parameter. In addition, the functions (A.1) belong to $\ker D^2_x$.

To obtain the estimate (4.43) for $\mu+2/p \in \boldsymbol{Z}$ such that $1 = k+2-(\mu+2/p)$ we use interpolation. Assume that μ_j, p_j, $j = 1, 2$, are such that (4.40) is satisfied and

$$2 > k+2-\left(\mu_1+\frac{2}{p_1}\right) > 1 > k+2-\left(\mu_2+\frac{2}{p_2}\right) > 0. \tag{A.2}$$

Therefore Theorem 4.3 implies that there exists a unique solution of the problem (3.15) such that

(A.3) $$\sum_{i=1}^{2} \|e_i\|_{L^{k+2}_{p_j,\mu_j}(d_\vartheta)} \leqslant C \sum_{i=1}^{2} \|f_i\|_{L^{k}_{p_j,\mu_j}(d_\vartheta)}, \quad j = 1, 2.$$

Moreover, the operator corresponding to the problem (3.15) generate an isomorphism

(A.4) $$\mathscr{P}\colon L^{k+2}_{p_j,\mu_j}(d_\vartheta) \times L^{k+2}_{p_j,\mu_j}(d_\vartheta) \to L^{k}_{p_j,\mu_j}(d_\vartheta) \times L^{k}_{p_j,\mu_j}(d_\vartheta),$$

where $j = 1, 2$, and $\mathscr{P}(e_1, e_2) = (\Delta' e_1, \Delta' e_2)$, with

$$e_1|_{\gamma_1} = \frac{\partial e_2}{\partial n}\bigg|_{\gamma_1} = (e_1 \cos\vartheta + e_2 \sin\vartheta)|_{\gamma_2} = \left(\frac{\partial e_1}{\partial n}\sin\vartheta - \frac{\partial e_2}{\partial n}\cos\vartheta\right)\bigg|_{\gamma_2} = 0.$$

Let $\theta \in [0, 1]$ and

(A.5) $$\mu = \mu(\theta) = (1-\theta)\mu_1 + \theta\mu_2, \quad \frac{1}{p} = \frac{1}{p(\theta)} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Then (A.2) implies that there exists $\theta_0 \in (0, 1)$ such that $\mu_0 + 2/p_0$ is an integer and $k+2-(\mu_0+2/p_0) = 1$, where $\mu_0 = \mu(\theta_0)$, $p_0 = p(\theta_0)$. Introducing the function $v = \sum_{|\alpha|=k} D^\alpha u$ we have

(A.6) $$C_1 \|u\|_{L^k_{p,\mu}(d_\vartheta)} \leqslant \|v\|_{L_{p,\mu}(d_\vartheta)} \leqslant C_2 \|u\|_{L^k_{p,\mu}(d_\vartheta)},$$

where C_1, C_2 are constants.

Let us recall some notation and results from [18]. Let $A_i = L_{p_i,\mu_i}(d_\vartheta)$, $i = 1, 2$. From 1.3.1 in [18] we have the K-functional

$$K(t, v) = \inf_{v = v_1 + v_2} (\|v_1\|_{A_1} + t\|v_2\|_{A_2}), \quad v \in A_1 + A_2,$$

and we introduce the norm (see 1.3.2 in [18])

(A.7) $$\|v\|_{K,\theta,p} = \left(\int_0^\infty [t^{-\theta} K(t, v)]^p \frac{dt}{t}\right)^{1/p}, \quad 0 < \theta < 1,\ 1 \leqslant p < \infty.$$

From 1.4.2 in [18] we have the L-functional

$$L(t, v) = \inf_{v = v_1 + v_2} (\|v_1\|^{p_1}_{A_1} + t\|v_2\|^{p_2}_{A_2}),$$

$$1 \leqslant p_i < \infty,\ p = (1-\eta)p_1 + \eta p_2,\ \eta = \theta \frac{p}{p_2},$$

and

(A.8) $$\|v\|_{L,\eta,p} = \left(\int_0^\infty t^{-\eta} L(t, v) \frac{dt}{t}\right)^{1/p}.$$

Theorem 1.4.2 of [18] shows that the norms (A.7) and (A.8) are equivalent.

Therefore we introduce the following space

(A.9) $$(A_1, A_2)_{\theta,p} = \{v \in A_1 + A_2 \colon \|v\|_{K,\theta,p} < \infty\}.$$

Now we rewrite Theorem 1.18.5 from [18] in the form useful for us.

LEMMA A.1. *Let* $1 \leqslant p_1, p_2 < \infty$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Then

(A.10) $$\big(L_{p_1,\mu_1}(d_\vartheta), L_{p_2,\mu_2}(d_\vartheta)\big)_{\theta,p} = L_{p,\mu}(d_\vartheta),$$

where $\mu = (1-\theta)\mu_1 + \theta\mu_2$.

Proof. The proof is the same as the proof of Theorem 1.18.5 from [18]. We repeat it in a simpler form

$$\|v\|^p_{(L_{p_1,\mu_1}(d_\vartheta), L_{p_2,\mu_2}(d_\vartheta))_{\theta,p}}$$

$$= \int_0^\infty t^{-\eta} \inf_{v_1+v_2=v} \int_{d_\vartheta} [|v_1|^{p_1} r^{p_1\mu_1} + t|v_2|^{p_2} r^{p_2\mu_2}] r\,dr\,d\varphi \frac{dt}{t}$$

$$= \int_{d_\vartheta} r\,dr\,d\varphi\, r^{p_1\mu_1} \int_0^\infty t^{-\eta} \inf_{v_1+v_2=v} [|v_1|^{p_1} + t r^{p_2\mu_2 - p_1\mu_1} |v_2|^{p_2}] \frac{dt}{t},$$

where $\eta = \theta p/p_2$.

By introducing the new variable $\tau = t r^{p_2\mu_2 - p_1\mu_1}$ and using the equality

$$(1-\eta)p_1\mu_1 + \eta p_2\mu_2 = (1-\theta)p\mu_1 + \theta p\mu_2 = p$$

which follows from

$$1-\eta = (1-\theta)\frac{p}{p_1}, \qquad \eta = \theta\frac{p}{p_2},$$

the above integral is equal to

$$\int_{d_\vartheta} r\,dr\,d\varphi\, r^{p\mu} \int_0^\infty \tau^{-\eta} \inf_{v_1+v_2=v} [|v_1|^{p_1} + \tau|v_2|^{p_2}] \frac{d\tau}{\tau},$$

which, by Theorem 1.4.2 of [18], is equivalent to

$$\int_{d_\vartheta} r\,dr\,d\varphi\, r^{p\mu} \int_0^\infty \tau^{-\theta p} \inf_{v_1+v_2=v} [|v_1| + \tau|v_2|]^p \frac{d\tau}{\tau}$$

$$= \int_{d_\vartheta} r\,dr\,d\varphi |v|^{p\mu} \int_0^\infty \tau^{-\theta p} \min(1, \tau)^p \frac{d\tau}{\tau} \sim \|v\|^p_{L_{p,\mu}(d_\vartheta)}.$$

This concludes the proof. ∎

Using (A.6), (A.9), (A.10) and Theorem 4 in 1.18.7 of [18] from (A.3) and (A.4) we have

(A.11) $$\mathscr{P}\colon L^{k+2}_{p,\mu}(d_\vartheta)\times L^{k+2}_{p,\mu}(d_\vartheta)\to L^{k}_{p,\mu}(d_\vartheta)\times L^{k}_{p,\mu}(d_\vartheta)$$

that the operator $\mathscr{P}$ is isomorphism and

(A.12) $$\sum_{i=1}^{2}\|e_i\|_{L^{k+2}_{p,\mu}(d_\vartheta)}\leqslant C\sum_{i=1}^{2}\|f_i\|_{L^{k}_{p,\mu}(d_\vartheta)}.$$

Therefore we have proved

THEOREM A.1. *Let $1\leqslant p<\infty$, $\mu\geqslant 0$, let k be natural such that*

(A.13) $$k+2-\left(\mu+\frac{2}{p}\right)=1,$$

and let $f_i\in L^{k}_{p,\mu}(d_\vartheta)$, $i=1,2$. Then there exist a unique solution of the problem (3.15) *such that $e_i\in L^{k+2}_{p,\mu}(d_\vartheta)$ and* (A.12) *is valid.*

References

[1] E. B. Byhovsky, *Solvability of mixed problem for the Maxwell equations for ideal conductive boundary*, Vest. Len. Univ., Ser. Mat. Mekh. Astr. 13 (1957), 50–66 (in Russian).

[2] R. R. Coifman, C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 101(3) (1971), 241–250.

[3] D. M. Eidus, *About existence of normal derivative of solution of the Dirichlet problem*, Vest. Len. Univ. 13 (1956), 147–150 (in Russian).

[4] D. K. Faddeyev, V. A. Solonnikov, *et al.*, *Selected Topics of Analysis and Algebra*, Isd. Len. Univ., Leningrad 1981 (in Russian).

[5] N. E. Kochin, *Vectorial Calculus and Introduction to Tensor Calculus*, Moscow 1951 (in Russian).

[6] V. A. Kondratiev, *About smoothness of solutions of the Dirichlet problem for elliptic equations of second order in domains with sectionally smooth boundary*, Diff. Uravn. 6 (1970), 1831–1843 (in Russian).

[7] –, *Boundary value problems for elliptic equations in domains with conical and angular points*, Trudy Mosk. Mat. Obshch. 16(1967), 209–292 (in Russian).

[8] O. A. Ladyzhenskaya, *Boundary Value Problems for Mathematical Physics*, Moscow 1973 (in Russian).

[9] O. A. Ladyzhenskaya, N. N. Uraltzeva, *Linear and Quasilinear Equations of Elliptic Type*, Moscow 1973 (in Russian).

[10] V. G. Maz'ya, B. A. Plamenevsky, *L_p estimates for elliptic boundary value problems in domains with edges*, Trudy Mosk. Mat. Obshch. 37(1978), 49–93 (in Russian).

[11] –, –, *Estimates in L_p and in Hölder spaces and the Miranda–Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary*, Math. Nachr. 81(1978), 25–82 (in Russian).

[12] –, –, *About coefficients in asymptotic of solutions of elliptic boundary value problems in domains with conical points*, Math. Nachr. 76(1977), 29–60 (in Russian).

[13] V. A. Solonnikov, W. M. Zajączkowski, *About the Neumann problem for elliptic equations of second order in domains with edges on boundary*, Zap. Nauch. Sem. LOMI 127(1983), 7–48 (in Russian).

[14] V. A. Solonnikov, *Estimates for solutions of the Neumann problem for elliptic equations of second order in domains with edges on boundary*, Preprint LOMI, P-4-83, 1983 (in Russian).

[15] –, *Overdetermined elliptic boundary value problems*, Zap. Nauch. Sem. LOMI 21(1971), 112–158 (in Russian).

[16] –, *Solvability of three-dimensional problem with a free boundary for the stationary system of Navier–Stokes equations*, Zap. Nauch. Sem. LOMI 84(1979), 252–285 (in Russian).

[17] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.

[18] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Berlin 1978.
[19] P. Urbański, *Dirichlet type boundary value problems in magnetostatics*, to appear.
[20] W. M. Zajączkowski, *On theorem of embedding for weighted Sobolev spaces*, Bull. Pol. Acad. Sc., Ser. Math. 32(3–4) (1985), 115–121.

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