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An improved ridge type estimator for logistic regression

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ABSTRACT

In this paper, an improved ridge type estimator is introduced to overcome the effect of multicollinearity in logistic regression. The proposed estimator is called a modified almost unbiased ridge logistic estimator. It is obtained by combining the ridge estimator and the almost unbiased ridge estimator. In order to asses the superiority of the proposed estimator over the existing estimators, theoretical comparisons based on the mean square error and the scalar mean square error criterion are presented. A Monte Carlo simulation study is carried out to compare the performance of the proposed estimator with the existing ones. Finally, a real data example is provided to support the findings.

Key words: Logistic Regression, Multicollinearity, ridge estimator, Modified almost unbiased ridge logistic estimator, Mean square error.

1. Introduction

The general form of logistic regression model is

$$y_i = \pi_i + \varepsilon_i, \ \ i = 1, ..., n, \tag{1}$$

where ε_i are independent with mean zero and variance $\pi_i(1-\pi_i)$ and π_i is the expected value of the response y_i when the ith value of dependent variable follows the Bernoulli distribution with parameter π_i as

$$\pi_i = \frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)}, \tag{2}$$

where x_i is the ith row of X, which is an $n \times p$ data matrix with p explanatory variables and β is a $p \times 1$ vector of coefficients. The Maximum likelihood method is the most common estimation technique to estimate the parameter vector β, and the maximum likelihood estimator (MLE) of β based on the sample model (1) can be obtained as follows:

$$\hat{\beta}_{MLE} = (X'\hat{W}X)^{-1}X'\hat{W}z, \tag{3}$$

where z is the column vector with ith element equals $logit(\hat{\pi}_i) + \frac{y_i - \hat{\pi}_i}{\hat{\pi}_i(1-\hat{\pi}_i)}$ and $\hat{W} = diag[\hat{\pi}_i(1-\hat{\pi}_i)]$, which is asymptotically unbiased estimate of β. The asymptotic covariance matrix of $\hat{\beta}_{MLE}$ is

$$Cov(\hat{\beta}_{MLE}) = (X'\hat{W}X)^{-1}. \tag{4}$$

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The asymptotic MSE and SMSE of $\hat{\beta}_{MLE}$ are

$$\begin{aligned} MSE[\hat{\beta}_{MLE}] &= Cov[\hat{\beta}_{MLE}] + B[\hat{\beta}_{MLE}]B'[\hat{\beta}_{MLE}] \\ &= \{X'\hat{W}X\}^{-1} \\ &= C^{-1} \end{aligned} \tag{5}$$

and

$$SMSE[\hat{\beta}_{MLE}] = tr[C^{-1}] \tag{6}$$

Since C is a positive definite matrix there exists an orthogonal matrix P such that $P'CP = \Delta = diag(\lambda_1, \lambda_2, ..., \lambda_p)$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p > 0$ are the ordered eigen values of C. Then,

$$SMSE[\hat{\beta}_{MLE}] = \sum_{j=1}^{p} \frac{1}{\lambda_j}.$$

The logistic regression model becomes unstable when there exists strong dependence among explanatory variables. This situation is referred to as multicollinearity. When the multicollinearity presents among explanatory variables, the estimation of the model parameters becomes inaccurate because of the need to invert near-singular information matrix $X'\hat{W}X$. As a result, the estimates have large variances and large confidence intervals, which produces inefficient estimates.

To overcome the problem of multicollinearity in the logistic regression, many estimators have been proposed in the literature alternative to the MLE. The most popular estimator to deal with this problem is called the Logistic Ridge Estimator (LRE), and was first proposed by Schaefer et al. (1984). Later, Aguilera et al. (2006) introduced the Principal Component Logistic Estimator (PCLE), Nja et al. (2013) proposed the Modified Logistic Ridge Regression Estimator (MLRE), Mansson et al. (2012) introduced the Liu-Estimator in logistic regression, Inan and Erdogan (2013) proposed Liu-type estimator, Xinfeng (2015) proposed the Almost Unbiased Liu Logistic Estimator (AULLE), Wu and Asar (2016) proposed the Almost Unbiased Ridge Logistic Estimator (AURLE), Varathan and Wijekoon (2019) proposed the Modified Almost Unbiased Liu Logistic Estimator (MAULLE), Jadhav (2020) proposed the Linearized ridge logistic estimator (LRLE), and the Modified ridge type logistic estimator was proposed by Lukman et al. (2020).

In this research a new estimator is proposed by combining AURLE and LRE. Further, we compare the performance of the proposed MAURLE estimator with the existing MLE, LRE and AURLE estimators in the mean square error sense.

The organization of the paper is as follows. The construction of the proposed estimator is given in Section 2. In Section 3, the asymptotic properties of the estimators are given. In Section 4, the conditions for superiority of the proposed MAURLE estimator over the existing MLE, LRE, and AURLE estimators are derived with respect to mean square error (MSE) criterion. In Section 5, the conditions for superiority of the proposed MAURLE estimator over the existing MLE, LRE, and AURLE estimators are derived with respect to scalar mean square error (SMSE) criterion. The detail Monte Carlo simulation study is given to investigate the performance of the proposed estimator with some existing estimators in Section 6. A real data application is discussed in Section 7. Finally, some conclusive remarks are given in Section 8.

2. Construction of the proposed estimator

The new estimator is constructed by considering the Logistic ridge estimator (LRE) (Schaefer et al., 1984) and the Almost Unbiased Ridge Logistic Estimator (AURLE) (Wu and Asar, 2016). Note that LRE and AURLE are defined as

$$\begin{aligned} \hat{\beta}_{LRE} &= (X'\hat{W}X+kI)^{-1}X'\hat{W}X\hat{\beta}_{MLE} \\ &= (C+kI)^{-1}C\hat{\beta}_{MLE} \\ &= Z_k\hat{\beta}_{MLE} \end{aligned} \tag{7}$$

where $Z_k = (C+kI)^{-1}C$ and k is the ridge parameter, $k \geq 0$.

$$\hat{\beta}_{AURLE} = W_k\hat{\beta}_{MLE} \tag{8}$$

where $W_k = I - k^2(C+kI)^{-2}$, $k \geq 0$.

By substituting $\hat{\beta}_{LRE}$ in place of $\hat{\beta}_{MLE}$ in the estimator AURLE in (2.2), we propose a new estimator which is named the Modified almost unbiased ridge logistic estimator (MAURLE) and defined as

$$\begin{aligned} \hat{\beta}_{MAURLE} &= W_k\hat{\beta}_{LRE} \\ &= W_kZ_k\hat{\beta}_{MLE} \\ &= F_k\hat{\beta}_{MLE} \end{aligned} \tag{9}$$

where

$$\begin{aligned} F_k &= W_kZ_k \\ &= [I-k^2(C+kI)^{-2}][(C+kI)^{-1}C] \end{aligned} \tag{10}$$

$$\begin{aligned} SMSE(\hat{\beta}_{MAURLE}) &= \sum_{j=1}^{p} \frac{\lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &+ \sum_{j=1}^{p} \frac{(k^3+3k^2\lambda j+k\lambda j^2)^2}{(\lambda_j+k)^6}\alpha_j^2 \end{aligned} \tag{11}$$

3. Asymptotic properties of the proposed estimator

The mean vector, dispersion matrix and the bias vector of $\hat{\beta}_{MAURLE}$ are

$$\begin{aligned} E[\hat{\beta}_{MAURLE}] &= E[F_k\hat{\beta}_{MLE}] \\ &= F_k\beta, \end{aligned} \tag{12}$$

$$\begin{aligned} D(\hat{\beta}_{MAURLE}) &= Cov(\hat{\beta}_{MAURLE}) \\ &= Cov(F_k\hat{\beta}_{MLE}) \\ &= F_kC^{-1}F_k', \end{aligned} \tag{13}$$

and

$$\begin{aligned} Bias(\hat{\beta}_{MAURLE}) &= E[\hat{\beta}_{MAURLE}] - \beta \\ &= [F_k - I]\beta \\ &= \delta_{MAURLE} \end{aligned} \tag{14}$$

Consequently, the mean square error and scalar mean square error can be obtained as,

$$\begin{aligned} MSE(\hat{\beta}_{MAURLE}) &= D(\hat{\beta}_{MAURLE}) + Bias(\hat{\beta}_{MAURLE})Bias(\hat{\beta}_{MAURLE})' \\ &= F_kC^{-1}F_k' + (F_k - I)\beta\beta'(F_k - I)' \end{aligned} \tag{15}$$

where

$$\begin{aligned} F_k &= W_kZ_k \\ &= [I - k^2(C+kI)^{-2}][(C+kI)^{-1}C] \\ &= (C+kI)^{-2}C(C+2kI)(C+kI)^{-1}C \\ &> 0 \text{ is a positive definite matrix.} \end{aligned} \tag{16}$$

and

$$\begin{aligned} SMSE(\hat{\beta}_{MAURLE}) &= \sum_{j=1}^{p} \frac{\lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &\quad + \sum_{j=1}^{p} \frac{(k^3+3k^2\lambda j + k\lambda j^2)^2}{(\lambda_j+k)^6}\alpha_j^2 \end{aligned} \tag{17}$$

where α_j is the jth element of $P'\beta$, P is an orthogonal matrix such that $P'CP = \Delta = diag(\lambda_1, \lambda_2, ..., \lambda_p)$, where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_p > 0$ are the ordered eigen values of C.

4. Mean square error comparison of estimators

To check the performance of the proposed MAURLE estimator with the existing MLE, LRE, and AURLE estimators, we compare the corresponding mean square errors of the estimators.

Note that, following Schaefer et al. (1984) and Wu and Asar (2016), the mean square errors of LRE and AURLE are respectively given by
$MSE[\hat{\beta}_{LRE}] = Z_k C^{-1} Z_k' + \delta_{LRE}\delta_{LRE}'$; where $\delta_{LRE} = (Z_k - I)\beta$
$MSE[\hat{\beta}_{AURLE}] = W_k C^{-1} W_k' + \delta_{AURLE}\delta_{AURLE}'$; where $\delta_{AURLE} = (W_k - I)\beta$

(I). **MAURLE Versus MLE**

$$\begin{aligned} MSE(\hat{\beta}_{MLE}) - MSE(\hat{\beta}_{MAURLE}) &= \{D(\hat{\beta}_{MLE}) + B(\hat{\beta}_{MLE})B'(\hat{\beta}_{MLE})\} \\ &\quad -\{D(\hat{\beta}_{MAURLE}) + B(\hat{\beta}_{MAURLE})B'(\hat{\beta}_{MAURLE})\} \\ &= C^{-1} - \{F_k C^{-1} F_k' + \delta_{MAURLE}\delta_{MAURLE}'\} \\ &= U_1 - V_1 \end{aligned} \tag{18}$$

where $U_1 = C^{-1}$ and $V_1 = F_k C^{-1} F_k' + \delta_{MAURLE}\delta_{MAURLE}'$. One can obviously say that $F_k C^{-1} F_k'$ and U_1 are positive definite matrices and $\delta_{MAURLE}\delta_{MAURLE}'$ is non-negative definite matrix. Further by Lemma 1 (see Appendix), it is clear that V_1 is a positive definite matrix. By Lemma 2 (see Appendix), if $\lambda_{\max}(V_1 U_1^{-1}) < 1$, then $U_1 - V_1$ is a positive definite matrix, where $\lambda_{\max}(V_1 U_1^{-1})$ is the largest eigen value of $V_1 U_1^{-1}$. Based on the above arguments, the following theorem can stated.
Theorem 1: The MAURLE estimator is superior to MLE if and only if $\lambda_{\max}(V_1 U_1^{-1}) < 1$.

(II). **MAURLE Versus LRE**

$$\begin{aligned} MSE(\hat{\beta}_{LRE}) - MSE(\hat{\beta}_{MAURLE}) &= \{D(\hat{\beta}_{LRE}) + B(\hat{\beta}_{LRE})B'(\hat{\beta}_{LRE})\} \\ &\quad -\{D(\hat{\beta}_{MAURLE}) + B(\hat{\beta}_{MAURLE})B'(\hat{\beta}_{MAURLE})\} \\ &= \{Z_k C^{-1} Z_k' + \delta_{LRE}\delta_{LRE}'\} \\ &\quad -\{F_k C^{-1} F_k' + \delta_{MAURLE}\delta_{MAURLE}'\} \\ &= U_2 - V_2 \end{aligned} \tag{19}$$

where $U_2 = Z_k C^{-1} Z_k' + \delta_{LRE}\delta_{LRE}'$ and $V_2 = F_k C^{-1} F_k' + \delta_{MAURLE}\delta_{MAURLE}'$. One can easily say that $F_k C^{-1} F_k'$ and $Z_k C^{-1} Z_k'$ are positive definite matrices and $\delta_{LRE}\delta_{LRE}'$ and $\delta_{MAURLE}\delta_{MAURLE}'$ are non-negative definite matrices. Further by Lemma 1, it is clear that U_2 and V_2 are positive definite matrices. By Lemma 2, if $\lambda_{\max}(V_2 U_2^{-1}) < 1$, then $U_2 - V_2$ is a positive definite matrix, where $\lambda_{\max}(V_2 U_2^{-1})$ is the largest eigen value of $V_2 U_2^{-1}$. Based on the above results, the following theorem can be stated.
Theorem 2: The MAURLE estimator is superior to LRE if and only if $\lambda_{\max}(V_2 U_2^{-1}) < 1$.

(III). **MAURLE Versus AURLE**

$$\begin{aligned} MSE(\hat{\beta}_{AURLE}) - MSE(\hat{\beta}_{MAURLE}) &= \{D(\hat{\beta}_{AURLE}) + B(\hat{\beta}_{AURLE})B'(\hat{\beta}_{AURLE})\} \\ &\quad -\{D(\hat{\beta}_{MAURLE}) + B(\hat{\beta}_{MAURLE})B'(\hat{\beta}_{MAURLE})\} \\ &= \{W_kC^{-1}W_k' + \delta_{AURLE}\delta_{AURLE}'\} \\ &\quad -\{F_kC^{-1}F_k' + \delta_{MAURLE}\delta_{MAURLE}'\} \\ &= U_3 - V_3 \end{aligned} \tag{20}$$

where $U_3 = W_kC^{-1}W_k' + \delta_{AURLE}\delta_{AURLE}'$ and $V_3 = F_kC^{-1}F_k' + \delta_{MAURLE}\delta_{MAURLE}'$. One can easily say that $F_kC^{-1}F_k'$ and $W_kC^{-1}W_k'$ are positive definite matrices and $\delta_{AURLE}\delta_{AURLE}'$ and $\delta_{MAURLE}\delta_{MAURLE}'$ are non-negative definite matrices. Further by Lemma 1, it is clear that U_3 and V_3 are positive definite matrices. By Lemma 2, if $\lambda_{\max}(V_3U_3^{-1}) < 1$, then $U_3 - V_3$ is a positive definite matrix, where $\lambda_{\max}(V_3U_3^{-1})$ is the largest eigen value of $V_3U_3^{-1}$. Based on the above results, the foolowing theorem can be stated.
Theorem 3: The MAURLE estimator is superior to AURLE if and only if $\lambda_{\max}(V_3U_3^{-1}) < 1$.

5. Scalar mean square error comparison

In this section, we compare the scalar mean square error of the proposed MAURLE estimator with the existing MLE, LRE, and AURLE estimators. According to Schaefer et al. (1984) and Wu and Asar (2016), the mean square errors of LRE and AURLE are respectively given by:
$SMSE[\hat{\beta}_{LRE}] = \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j+k)^2} + \sum_{j=1}^{p} \frac{k^2\alpha_j^2}{(\lambda_j+k)^2}$
$SMSE[\hat{\beta}_{AURLE}] = \sum_{j=1}^{p} \frac{\lambda_j(\lambda_j+2k)^2}{(\lambda_j+k)^4} + \sum_{j=1}^{p} \frac{k^4\alpha_j^2}{(\lambda_j+k)^4}$

(I). **MAURLE Versus MLE**

$$\begin{aligned} & SMSE(\hat{\beta}_{MLE}) - SMSE(\hat{\beta}_{MAURLE}) \\ &= \sum_{j=1}^{p} \frac{1}{\lambda_j} - [\sum_{j=1}^{p} \frac{\lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &\quad + \sum_{j=1}^{p} \frac{(k^3+3k^2\lambda j + k\lambda j^2)^2}{(\lambda_j+k)^6}\alpha_j^2] \\ &= \sum_{j=1}^{p} \frac{(\lambda_j+k)^6 - (\lambda j^4\lambda_j+2k)^2}{\lambda_j(\lambda_j+k)^6} \\ &\quad - \sum_{j=1}^{p} \frac{(k^3+3k^2\lambda j + k\lambda j^2)^2\alpha_j^2}{(\lambda_j+k)^6} \\ &= \Delta_1^* \end{aligned} \tag{21}$$

Based on the above comparison, it can be noted that MAURLE is superior to MLE in the SMSE sense if and only if $\Delta_1^* > 0$.

(II). **MAURLE Versus LRE**

$$\begin{aligned} SMSE(\hat{\beta}_{LRE}) - SMSE(\hat{\beta}_{MAURLE}) &= \sum_{j=1}^{p} \frac{\lambda_j}{(\lambda_j+k)^2} + \sum_{j=1}^{p} \frac{k^2\alpha_j^2}{(\lambda_j+k)^2} - \sum_{j=1}^{p} \frac{\lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &\quad - \sum_{j=1}^{p} \frac{(k^3+3k^2\lambda j+k\lambda j^2)^2}{(\lambda_j+k)^6}\alpha_j^2 \\ &= \sum_{j=1}^{p} \frac{\lambda_j(\lambda_j+k)^4 - \lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &\quad + \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j+k)^6} k^2(\lambda_j+k)^4 \\ &\quad - \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j+k)^6} (k^2+3k^2\lambda j+k\lambda j^2)^2 \\ &= \Delta_2^* \end{aligned} \tag{22}$$

From the above comparison, it can be concluded that MAURLE is superior to LRE in the SMSE sense if and only if $\Delta_2^* > 0$.

(III). **MAURLE Versus AURLE**

$$\begin{aligned} SMSE(\hat{\beta}_{AURLE}) - SMSE(\hat{\beta}_{MAURLE}) &= \sum_{j=1}^{p} \frac{\lambda_j(\lambda_j+2k)^2}{(\lambda_j+k)^4} + \sum_{j=1}^{p} \frac{k^4\alpha_j^2}{(\lambda_j+k)^4} \\ &\quad - \sum_{j=1}^{p} \frac{\lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &\quad - \sum_{j=1}^{p} \frac{(k^3+3k^2\lambda j+k\lambda j^2)^2}{(\lambda_j+k)^6}\alpha_j^2 \\ &= \sum_{j=1}^{p} \frac{\lambda_j(\lambda_j+2k)^2(\lambda_j+k)^2 - \lambda_j^3(\lambda_j+2k)^2}{(\lambda_j+k)^6} \\ &\quad + \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j+k)^6} k^2(\lambda_j+k)^4 \\ &\quad - \sum_{j=1}^{p} \frac{\alpha_j^2}{(\lambda_j+k)^6} (k^2+3k^2\lambda j+k\lambda j^2)^2 \\ &= \Delta_3^* \end{aligned} \tag{23}$$

Based on the above comparison, it can be said that MAURLE is superior to AURLE in the SMSE sense if and only if $\Delta_3^* > 0$.

6. Monte Carlo Simulation study

In this simulation study we compare the performance of proposed MAURLE estimator with the existing MLE, LRE, and AURLE estimators in the scalar mean square error criteria. The sample sizes *n*= 20, 50, and 100 are considered. Following McDonald and Galarneau (1975) and Kibria (2003), we generate the explanatory variables as follows:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i,p+1}, i = 1,2,...,n, \quad j = 1,2,...,p$$

where z_{ij} are pseudo- random numbers from standardized normal distribution and ρ^2 represents the correlation between any two explanatory variables. For the multicollinearity, different levels of ρ, such as ρ= 0.9, 0.95, 0.99 and 0.999 are used. Further, for the biasing parameter k, we consider some selected values in the range $0 < k < 1$. The simulation is repeated 1000 times by generating new pseudo- random numbers, and the simulated SMSE values of the estimators are obtained using the following equation.

$$SM\hat{S}E(\hat{\beta}) \quad = \quad \frac{1}{1000} \sum_{r=1}^{1000} (\hat{\beta}_r - \beta)'(\hat{\beta}_r - \beta)$$

where $\hat{\beta}_r$ is any estimator considered in the r^{th} simulation. The simulated scalar mean square errors of estimators are reported for different values of d, ρ, and n in Tables 1 - 3.

Table 1: The estimated MSE values for different k when $n = 20$

		$\rho = 0.9$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.999$
$k = 0.1$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	33.6541	45.7819	64.8760	37.0857
	AURLE	48.3103	74.0211	137.5215	108.2149
	MAURLE	**30.1161**	**37.8386**	**40.0568**	**10.4364**
$k = 0.2$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	25.0801	30.8425	33.9479	15.0765
	AURLE	39.5509	54.7585	77.9639	40.7710
	MAURLE	**21.1814**	**23.5606**	**18.3130**	**5.8744**
$k = 0.3$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	20.4920	23.7132	22.6380	9.7370
	AURLE	33.8980	44.0296	53.3815	23.4383
	MAURLE	**16.7268**	**17.3486**	**11.6077**	**5.0841**
$k = 0.4$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	17.6061	19.5145	16.9926	7.6181
	AURLE	29.8427	37.0186	40.0643	16.3491
	MAURLE	**14.1271**	**13.9827**	**8.7388**	**4.8320**
$k = 0.5$	MLE	66.8246	135.8973	698.06386	7023.9774
	LRE	15.6389	16.7747	13.7330	6.5873
	AURLE	26.7593	32.0282	31.8030	12.7009
	MAURLE	**12.4977**	**11.9727**	**7.3495**	**4.7941**
$k = 0.6$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	14.2318	14.8758	11.6889	6.0421
	AURLE	24.3250	28.2792	26.2346	10.5413
	MAURLE	**11.4422**	**10.7165**	**6.6648**	**4.8842**
$k = 0.7$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	13.1936	13.5077	10.3412	5.7533
	AURLE	22.3515	25.3560	22.2638	9.1379
	MAURLE	**10.7527**	**9.9205**	**6.3619**	**5.0612**
$k = 0.8$	MLE	66.8246	135.8973	698.06386	7023.9774
	LRE	12.4114	12.4961	9.4255	5.6153
	AURLE	20.7202	23.0144	19.3148	8.1649
	MAURLE	**10.3085**	**9.4233**	**6.2837**	**5.2989**
$k = 0.9$	MLE	66.8246	135.8973	698.0638	7023.9774
	LRE	11.8137	11.7350	8.7934	**5.5725**
	AURLE	19.3512	21.0998	17.0568	7.4588
	MAURLE	**10.0342**	**9.1285**	**6.3450**	5.5784

Table 2: The estimated MSE values for different k when $n = 50$

		$\rho = 0.9$	$\rho = 0.95$	$\rho = 0.99$	$\rho = 0.999$
$k = 0.1$	MLE	12.1526	22.3349	102.87942	1008.1701
	LRE	11.1597	18.8554	51.6995	59.7663
	AURLE	12.0889	21.8856	81.6874	165.7703
	MAURLE	**11.1069**	**18.5386**	**44.1989**	**19.2800**
$k = 0.2$	MLE	12.1526	22.3349	102.8794	1008.1701
	LRE	10.3536	16.4178	34.0311	22.8387
	AURLE	11.9366	21.0209	63.2961	68.7981
	MAURLE	**10.2011**	**15.6829**	**25.0449**	**5.3163**
$k = 0.3$	MLE	12.1526	22.3349	102.8794	1008.1701
	LRE	9.6877	14.5909	24.9083	12.7929
	AURLE	11.7342	20.0544	50.7573	38.4925
	MAURLE	**9.4312**	**13.5255**	**16.2785**	**3.1866**
$k = 0.4$	MLE	12.1526	22.3349	102.8794	1008.1701
	LRE	9.1311	13.1668	19.4025	8.6338
	AURLE	11.5041	19.0916	41.8789	25.0999
	MAURLE	**8.7818**	**11.8640**	**11.5592**	**2.6042**
$k = 0.5$	MLE	12.1526	22.3349	102.8794	1008.1701
	LRE	8.6620	12.0274	15.7713	6.5190
	AURLE	11.2596	18.1716	35.3344	17.9971
	MAURLE	**8.2368**	**10.5637**	**8.7654**	**2.3981**
$k = 0.6$	MLE	12.1526	22.3348	102.8794	1008.1701
	LRE	8.2642	11.0987	13.2338	5.3059
	AURLE	11.0095	17.3081	30.3489	13.7716
	MAURLE	**7.7814**	**9.5338**	**7.0081**	**2.3256**
$k = 0.7$	MLE	12.1526	22.3348	102.8794	1008.1701
	LRE	7.9257	10.3315	11.3873	4.5556
	AURLE	10.7591	16.5045	26.4491	11.0495
	MAURLE	**7.4028**	**8.7115**	**5.8587**	**2.3167**
$k = 0.8$	MLE	12.1526	22.3348	102.8794	1008.1701
	LRE	7.6369	9.6912	10.0031	4.0692
	AURLE	10.5121	15.7596	23.3318	9.1899
	MAURLE	**7.0903**	**8.0521**	**5.0896**	**2.3459**
$k = 0.9$	MLE	12.1526	22.3348	102.8794	1008.1701
	LRE	7.3902	9.1526	8.9421	3.7456
	AURLE	10.2707	15.0700	20.7951	7.8607
	MAURLE	**6.8348**	**7.5224**	**4.5706**	**2.4017**

Table 3: The estimated MSE values for different k when $n = 100$

		$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$	$\rho = 0.99$
$k = 0.1$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.8749	8.4967	30.7126	72.6411
	AURLE	5.0144	8.9688	38.7927	170.6357
	MAURLE	**4.8725**	**8.4807**	**29.6393**	**37.2474**
$k = 0.2$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.7454	8.0613	24.5283	31.8073
	AURLE	5.0072	8.9218	35.9068	88.1286
	MAURLE	**4.7370**	**8.0075**	**22.1917**	**10.0957**
$k = 0.3$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.6275	7.6724	20.2308	18.2460
	AURLE	4.9959	8.8527	32.8470	53.9915
	MAURLE	**4.6107**	**7.5696**	**17.0628**	**4.4179**
$k = 0.4$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.5201	7.3234	17.0880	12.0657
	AURLE	4.9811	8.7669	29.9663	36.6658
	MAURLE	**4.4936**	**7.1672**	**13.4521**	**2.6290**
$k = 0.5$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.4226	7.0095	14.7042	8.7289
	AURLE	4.9633	8.6686	27.3633	26.6724
	MAURLE	**4.3859**	**6.7990**	**10.8469**	**1.9207**
$k = 0.6$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.3342	6.7263	12.8461	6.7242
	AURLE	4.9427	8.5610	25.0478	20.3845
	MAURLE	**4.2874**	**6.4634**	**8.9250**	**1.5979**
$k = 0.7$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.2543	6.4704	11.3661	5.4283
	AURLE	4.9200	8.4465	22.9995	16.1707
	MAURLE	**4.1979**	**6.1586**	**7.4803**	**1.4386**
$k = 0.8$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.1821	6.2388	10.1670	4.5452
	AURLE	4.8952	8.3273	21.1888	13.2082
	MAURLE	**4.1172**	**5.8824**	**6.3771**	**1.3585**
$k = 0.9$	MLE	5.0169	8.9865	40.3501	392.0983
	LRE	4.1173	6.0288	9.1817	3.9194
	AURLE	4.8688	8.2047	19.5857	11.0458
	MAURLE	**4.0451**	**5.6329**	**5.5238**	**1.3212**

From the results of Tables 1 - 3 it can be observed that, the proposed MAURLE estimator outperforms the MLE, LRE, and AURLE estimators in the scalar mean square error sense for almost all the values of biasing parameter k in the range $0 < k < 1$ and for all sample sizes $n = 20$, 50, and 100, except the case of k=0.9, ρ=0.999, and n=20. The LRE gives the second best performance compared to MLE, and AURLE for all the values of k, ρ, and n considered in this study. It is further noted that, comparatively MLE gives the worst performance by giving large values of SMSE.

7. A real data application

To illustrate the performance of the proposed MAURLE estimator with the existing MLE, LRE and AURLE estimators, in this research, we consider a real data application, which is obtained from the Statistics Sweden website (http://www.scb.se/). This example was used in Mansson et al. (2012), Asar and Genç (2016), and Wu and Asar (2016) to illustrate results of their papers. The data describes the information of 100 municipalities of Sweden. The following variables are considered in this study.
$x1$: Population,
$x2$: Number unemployed people,
$x3$: Number of newly constructed buildings,
$x4$: Number of bankrupt firms,
y: Net population change and is defined as

$$y = \begin{cases} 1 & ; \quad \text{if there is an increase in the population;} \\ 0 & ; \quad o/w. \end{cases}$$

Note that the Variance Inflation Factor (VIF) values for the above data are 488.17, 344.26, 44.99, and 50.71. VIF measure how much the variance of the estimated regression coefficients are inflated as compared to when the predictor variables are not linearly related. According to the literature, multicollinearity is high if VIF > 10. Hence, a clear high multicollinearity exists in this data set. Further, the condition number, which is used as a measure of the degree of multicollinearity, is obtained as 188. This indicates the sign of severe multicollinearity in this data set.
The SMSE values of MLE, LRE, AURLE, and MAURLE for some selected values of biasing parameter k in the range $0 < k < 1$ are given in the Table 4. Results reveal that the proposed MAURLE estimator outperforms the MLE, LRE, and AURLE estimators in the SMSE sense, with respect to all values of k in the range $0 < k < 1$, and LRE performs well compared to MLE and AURLE.

Table 4: The SMSE values (in 10^{-4}) of estimators for the real world application data

	MLE	LRE	AURLE	MAURLE
$k = 0.1$	7.596226	7.595180	7.759226	**7.595180**
$k = 0.2$	7.596226	7.594137	7.759225	**7.541370**
$k = 0.3$	7.596226	7.593097	7.759225	**7.593096**
$k = 0.4$	7.596226	7.592059	7.759225	**7.592058**
$k = 0.5$	7.596226	7.591024	7.759224	**7.591022**
$k = 0.6$	7.596226	7.589991	7.759223	**7.589988**
$k = 0.7$	7.596226	7.588961	7.759222	**7.588957**
$k = 0.8$	7.596226	7.587934	7.759221	**7.587929**
$k = 0.9$	7.596226	7.586909	7.759220	**7.586903**

8. Concluding Remarks

In this paper, an improved estimator called Modified almost unbiased ridge logistic estimator (MAURLE) is proposed for logistic regression model when the multicollinearity problem exists. The superiority conditions for the proposed estimator with the existing MLE, LRE, and AURLE estimators are derived with respect to MSE and SMSE criterions. Further, from the real data application and the Monte Carlo simulation study we notice that the proposed estimator performs well compared to MLE, LRE, and AURLE when the multicollinearity among the explanatory variables is high.

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Appendix

Lemma 1: (Rao and Toutenburg, 1995) Let $A : n \times n$ and $B : n \times n$ such that A is positive definite and B is non-negative definite. Then $(A+B)$ is positive definite.
Lemma 2: (Rao et al., 2008) Let the two $n \times n$ matrices M be positive definite, N be non-negative definite, then $M-N$ is positive definite if and only if $\lambda_{\max}(NM^{-1}) < 1$.