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Estimation of $P(X \leq Y)$ for discrete distributions with non-identical support

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ABSTRACT

The Uniformly Minimum Variance Unbiased (UMVU) and the Maximum Likelihood (ML) estimations of $R = P(X \leq Y)$ and the associated variance are considered for independent discrete random variables X and Y. Assuming a discrete uniform distribution for X and the distribution of Y as a member of the discrete one parameter exponential family of distributions, theoretical expressions of such quantities are derived. Similar expressions are obtained when X and Y interchange their roles and both variables are from the discrete uniform distribution. A simulation study is carried out to compare the estimators numerically. A real application based on demand-supply system data is provided.

Key words: stress-strength model, uniformly minimum variance unbiased, maximum likelihood.

1. Introduction

In the stress-strength reliability literature, the quantity $R = P(X \leq Y)$, where X defines stress and Y defines strength, is the well-known reliability function, although a lot of development was carried out in the last few decades to explore inferential aspects concerning R, under the assumption of continuous X and Y. However, X and Y may be both discrete random variables and inference on R is required in many situations. For example, in demand analysis, if the number of demanded items is considered as the stress random variable X and the corresponding number of items supplied is regarded as strength random variable Y, then X and Y are both discrete and R represents the sensitivity of demand-supply system. Another example considered is the working of regular life gadgets, like scanners and Xerox machines, where measurable resistible voltage shocks are applied to the bulbs of the machines in a time interval. Then, the number of applied shocks may define stress (X) and the number of shocks the machine can withstand may define strength (Y) and consequently R measures the reliability of the system, where stress and strength random variables are both discrete. A comprehensive account of the details of stress-strength reliability can be found in the book-length coverage of Kotz et al. (2003).

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However, most of the authors considered identical distributions to represent stress and strength random variables. For example, Maiti (1995) took Geometric distribution to calculate UMVU and ML estimators of R, Ivshin and Lumelskii (1995) and Sathe and Dixit (2001) considered Negative Binomial distribution to represent both stress and strength random variables. Further, Balyaev and Lumelskii (1995) and Barbiero (2003) assumed Poisson distributions for both stress and strength random variables. On the contrary, Obradovic et al. (2015) regarded two different distributions for stress and strength random variables in a recent work. Specifically, Geometric distribution is used to model stress and Poisson distribution is assumed for the strength random variable.

In all these works, supports of the stress and strength random variables are assumed to be identical and the distributions are members of One Parameter Exponential Family (OPEF) of distributions. Strength and/ or stress random variable may be uniform, that is, supports may depend on the unknown parameters.

Assuming continuous uniform distributions, Ivshin (1996) and Ali et al. (2005) explored different inferential properties of R. But as far as our knowledge goes, no discrete counterpart of such work is developed. A motivating example may be the following. Suppose someone forgets his computer password. Now, if X denotes that he inputs the right password in second draw, then X follows Discrete Uniform distribution. Again, if Y represents the number of attempts the computer allows, then Y follows Poisson random variable. So R is the probability that he can open the computer successfully and hence Discrete Uniform-Poisson model is more appropriate. Consequently, in this work, we derive theoretical expressions of UMVU estimator of R and UMVU estimator of the associated variance assuming different discrete distributions for stress and strength random variables. In particular, assuming Y to be a member of the discrete OPEF and X as discrete uniform, we derive the UMVU estimator and then do the same when both X and Y are discrete uniform with different supports. UMVU estimation of R and the derivation of the UMVU estimators of the variances of the UMVU estimators are provided in Section 2. Section 3 gives the derivation of ML estimators of R for earlier mentioned combinations. In Section 4, we provide simplified expressions of R, associated three dimensional plots and also UMVU and ML estimators of R for specific members of OPEF. We compare efficiency of UMVU and ML estimators of R numerically for various combinations in Section 5. A real application based on demand-supply system data is discussed in Section 6. Finally, Section 7 concludes with a discussion of the related issues.

2. UMVU estimation of R

Derivation of the UMVU estimator of R depends on the nature of the distributions of stress (X) and strength (Y) random variables. Consequently, we start with the derivation for regular family of discrete distributions and then extend the methodology to cover distributions with parameter dependent supports. But, if either of $\mathscr{S}_{\mathscr{X}}$ (i.e. support of X) and $\mathscr{S}_{\mathscr{Y}}$ (i.e. support of Y) involves an unknown parameter, we need to develop afresh. Although a number of discrete distributions are available in the literature, we consider the Discrete Uniform distribution to model stress and discrete OPEF to represent strength and derive UMVU estimator of R. Similar expressions are also obtained for the combination (OPEF, Discrete Uniform). Further, considering (Discrete Uniform, Discrete Uniform) combination for (X,Y), we develop UMVU estimation of R.

2.1. UMVU estimation of R for regular family of discrete distributions

Suppose stress and strength are independent random variables having distributions in the regular family of discrete distributions with the supports (identical and/or non-identical) $\mathscr{S}_{\mathscr{X}}$ and $\mathscr{S}_{\mathscr{Y}}$, respectively. Naturally, $\mathscr{S}_{\mathscr{X}}$ and $\mathscr{S}_{\mathscr{Y}}$ are independent of the parameters and $\mathscr{S}_{\mathscr{X}} \bigcap \mathscr{S}_{\mathscr{Y}}$ is non-empty. Further, assume that single but different parameters are involved in the distributions of X and Y and complete sufficient statistics T_X and T_Y exist for the family of distributions of X and Y, respectively. Since we can write

$$R = P(X \leq Y) = \sum_{j \in \mathscr{S}_{\mathscr{Y}}} P(X \leq j)P(Y = j),$$

Blackwellisation (Rao, 1973) ensures that $\phi_j(T_Y) = P(Y_1 = j/T_Y)$ is the UMVU estimator of $P(Y = j)$ and $\phi_j(T_X) = P(X_1 \leq j/T_X)$ is that of $P(X \leq j)$ for every fixed $j \in \mathscr{S}_{\mathscr{Y}}$. Then, due to independence of the distributions of X and Y and the assumption of parameter independent of supports $\mathscr{S}_{\mathscr{X}}$ and $\mathscr{S}_{\mathscr{Y}}$ give the UMVU estimator of R as

$$\widehat{R}_{UMVUE} = \sum_{j \in \mathscr{S}_{\mathscr{Y}}} \phi_j(T_Y)\phi_j(T_X).$$

The available UMVU estimators of R (Kotz et al., 2003) can all be derived from the above expression.

2.2. UMVU estimation of R for (Discrete Uniform, OPEF) combination

Suppose X has a Discrete Uniform distribution over $\{1, 2, ..., N\}$ with probability mass function (PMF)

$$\begin{aligned} P(X = x) &= \frac{1}{N} \qquad \text{if } x = 1, ..., N \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and Y has OPEF with PMF

$$\begin{aligned} P(Y = y) &= c(\theta)h(y)exp(q(\theta)t(y)) \qquad \text{if } y = 0, 1, 2, ... \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

where, $c(\theta) = \left[\sum_{y=0}^{\infty} h(y)exp(q(\theta)t(y))\right]^{-1}$. Then, under the independence of X and Y, we have

$$R = P(X \leq Y) = E\left[\frac{Y}{N}\right].$$

However, if the roles of X and Y are interchanged, we get the following expression:

$$R = P(X \leq Y) \quad = \quad 1 - E\left[\frac{X-1}{N}\right].$$

In order to facilitate UMVU estimation of R, we assume that $(X_1, X_2, ..., X_{n_1})$ and $(Y_1, Y_2, ..., Y_{n_2})$ are independent samples from the distributions of X and Y, respectively. Then, it is well known that complete sufficient statistics for N and θ exist (Lehmann and Casella, 1998) and are respectively, $T_X = X_{(n_1)} = \max(X_1, X_2, ..., X_{n_1})$ and $T_Y = \sum_{i=1}^{n_2} t(Y_i)$ with respective PMFs

$$\begin{aligned} P(X_{(n_1)} = t_x) &= \frac{t_x^{n_1} - (t_x - 1)^{n_1}}{N^{n_1}} \quad \text{if } t_x = 1, 2, ..., N \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and

$$\begin{aligned} P(T_Y = t_y) &= [c(\theta)]^{n_2} h_0(t_y) exp(q(\theta) t_y) \quad \text{if } t_y = 0, 1, 2, ... \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $h_0(t_y)$ is the sum of $\prod_{j=1}^{n_2} h(y_j)$ over all $(y_1, y_2, ..., y_{n_2})$ for which $\sum_{j=1}^{n_2} t(y_j) = t_y$ (Ferguson, 1967).

Since the indicator function $I[X_1 \leq Y_1]$ is unbiased for R, the Rao-Blackwell theorem coupled with Lehman-Scheffe theorem (Lehmann and Casella, 1998) expresses the UMVU estimator of R as

$$\begin{aligned} \widehat{R}_{UMVUE} &= E(I[X_1 \leq Y_1] | X_{(n_1)} = t_x, T_Y = t_y) \\ &= P(X_1 \leq Y_1 | X_{(n_1)} = t_x, T_Y = t_y) \\ &= \frac{P(X_1 \leq Y_1, X_{(n_1)} = t_x, T_Y = t_y)}{P(X_{(n_1)} = t_x, T_Y = t_y)} \\ &= \frac{\sum_{y=1}^{\min(t_x, t_y)} P(X_1 \leq y, X_{(n_1)} = t_x, T_Y = t_y, Y_1 = y)}{P(X_{(n_1)} = t_x) P(T_Y = t_y)} \\ &= \frac{\sum_{y=1}^{\min(t_x, t_y)} P(X_1 \leq y, X_{(n_1)} = t_x) P(Y_1 = y, \sum_{i=1}^{n_2} t(Y_i) = t_y)}{P(X_{(n_1)} = t_x) P(T_Y = t_y)} \\ &= \frac{\sum_{y=1}^{\min(t_x, t_y)} h(y) h_0(t_y - t(y)) \sum_{x=1}^{y} P(X_1 = x | X_{(n_1)} = t_x)}{h_0(t_y)}. \end{aligned}$$

Further, using the fact that

$$\begin{aligned} P(X_1 = x | X_{(n_1)} = t_x) &= \frac{t_x^{n_1 - 1} - (t_x - 1)^{n_1 - 1}}{t_x^{n_1} - (t_x - 1)^{n_1}} \quad \text{if } x = 1, 2, ..., (t_x - 1) \\ &= \frac{t_x^{n_1 - 1}}{t_x^{n_1} - (t_x - 1)^{n_1}} \quad \text{if } x = t_x, \end{aligned}$$

$\widehat{R}_{UMVUE}$ can be simplified as

$$\begin{aligned}\widehat{R}_{UMVUE} &= \frac{1}{[T_X^{n_1} - (T_X - 1)^{n_1}]h_0(T_Y)} \times \\ &\sum_{y=1}^{\min(T_X,T_Y)} h(y)h_0(T_Y - t(y)) \times \\ &\sum_{x=1}^{y} \left\{ T_X^{n_1-1} - (T_X - 1)^{n_1-1} I[x \neq T_X] + T_X^{n_1-1} I[x = T_X] \right\}.\end{aligned}$$

However, for (OPEF, Discrete Uniform) combination), in a similar way, we derive the UMVU estimator of R as

$$\begin{aligned}\widehat{R}_{UMVUE} &= \frac{1}{[T_Y^{n_2} - (T_Y - 1)^{n_2}]h_0(T_X)} \times \\ &\sum_{y=1}^{\min(T_X,T_Y)} \left\{ (T_Y^{n_2-1} - (T_Y - 1)^{n_2-1}) I[y \neq T_Y] + T_Y^{n_2-1} I[y = T_Y] \right\} \times \\ &\sum_{x=0}^{y} h(x)h_0(T_X - t(x)).\end{aligned}$$

2.3. UMVU estimation of R for (Discrete Uniform, Discrete Uniform) combination

Now, assume that the distributions of both X and Y are Discrete Uniform with respective parameters N_1 and N_2. Then, the expression of R takes the form:

$$\begin{aligned}R = P(X \leq Y) &= \frac{2N_2 - N_1 + 1}{2N_2} \qquad \text{if } N_1 < N_2 \\ &= \frac{N_2 + 1}{2N_1} \qquad \text{if } N_1 \geq N_2.\end{aligned}$$

It is well known that for such distributions, complete sufficient statistics exist and are given by $T_X = X_{(n_1)} = \max(X_1, X_2, ..., X_{n_1})$ and $T_Y = Y_{(n_2)} = \max(Y_1, Y_2, ..., Y_{n_2})$, respectively. Thus the UMVU estimator of R takes the form

$$\begin{aligned}
\widehat{R}_{UMVUE} &= E(I[X_1 \leq Y_1]|X_{(n_1)} = t_x, Y_{(n_2)} = t_y) \\
&= P(X_1 \leq Y_1|X_{(n_1)} = t_x, Y_{(n_2)} = t_y) \\
&= \frac{P(X_1 \leq Y_1, X_{(n_1)} = t_x, Y_{(n_2)} = t_y)}{P(X_{(n_1)} = t_x, Y_{(n_2)} = t_y)} \\
&= \frac{\sum_{y=1}^{\min(t_x,t_y)} P(X_1 \leq y, X_{(n_1)} = t_x, Y_{(n_2)} = t_y, Y_1 = y)}{P(X_{(n_1)} = t_x)P(Y_{(n_2)} = t_y)} \\
&= \frac{\sum_{y=1}^{\min(t_x,t_y)} P(X_1 \leq y, X_{(n_1)} = t_x)P(Y_1 = y, Y_{(n_2)} = t_y)}{P(X_{(n_1)} = t_x)P(Y_{(n_2)} = t_y)} \\
&= \sum_{y=1}^{\min(t_x,t_y)} \sum_{x=1}^{y} P(X_1 = x|X_{(n_1)} = t_x)P(Y_1 = y|Y_{(n_2)} = t_y).
\end{aligned}$$

Now, using the expressions of the conditional PMF's $P(X_1 = x|X_{(n_1)} = t_x)$ and $P(Y_1 = y|Y_{(n_2)} = t_y)$, we derive the simplified expression

$$\begin{aligned}
\widehat{R}_{UMVUE} = & \frac{1}{\left\{T_X^{n_1} - (T_X - 1)^{n_1}\right\}\left\{T_Y^{n_2} - (T_Y - 1)^{n_2}\right\}} \times \\
& \sum_{y=1}^{\min(T_X,T_Y)} \left\{(T_Y^{n_2-1} - (T_Y - 1)^{n_2-1})I[y \neq T_Y] + T_Y^{n_2-1}I[y = T_Y]\right\} \times \\
& \sum_{x=1}^{y} \left\{(T_X^{n_1-1} - (T_X - 1)^{n_1-1})I[x \neq T_X] + T_X^{n_1-1}I[x = T_X]\right\}.
\end{aligned}$$

2.4. UMVU estimation of $Var(\widehat{R}_{UMVUE})$

For the UMVU estimation of $Var(\widehat{R}_{UMVUE})$, we consider the representation

$$\begin{aligned}
Var(\widehat{R}_{UMVUE}) &= E([\widehat{R}_{UMVUE}]^2) - E^2(\widehat{R}_{UMVUE}) \\
&= E([\widehat{R}_{UMVUE}]^2) - R^2
\end{aligned}$$

Therefore, if we can derive the UMVU estimator $\widehat{Q}_{UMVUE}$ of $Q = R^2$, then we can write

$$Var(\widehat{R}_{UMVUE}) = E([\widehat{R}_{UMVUE}]^2) - E(\widehat{Q}_{UMVUE}),$$

and hence obtain the UMVU estimator of $Var(\widehat{R}_{UMVUE})$ as $\widehat{Var}(\widehat{R}_{UMVUE}) = [\widehat{R}_{UMVUE}]^2 - \widehat{Q}_{UMVUE}$. Consequently, we move our attention to deriving $\widehat{Q}_{UMVUE}$.

For the relevant derivation, first of all, we note that the events $(X_1 \leq Y_1)$ and $(X_2 \leq Y_2)$ are independent and so are the corresponding indicator functions $I[X_1 \leq Y_1]$ and $I[X_2 \leq Y_2]$.

Then, naturally $I[X_1 \leq Y_1, X_2 \leq Y_2] = I[X_1 \leq Y_1]I[X_2 \leq Y_2]$ is unbiased for $Q = R^2$ and corresponding to the (Discrete Uniform, One Parameter Exponential family) combination, we derive

$$\begin{aligned}
\widehat{Q}_{UMVUE} &= E(I[X_1 \leq Y_1, X_2 \leq Y_2] | T_X = t_x, T_Y = t_y) \\
&= P(X_1 \leq Y_1, X_2 \leq Y_2 | T_X = t_x, T_Y = t_y) \\
&= \frac{P(X_1 \leq Y_1, X_2 \leq Y_2, T_X = t_x, T_Y = t_y)}{P(T_X = t_x, T_Y = t_y)} \\
&= \frac{\sum_{y_1=1}^{\min(t_x,t_y)} \sum_{y_2=1}^{y_1} P(X_1 \leq y_1, X_2 \leq y_2, T_X = t_x, T_Y = t_y, Y_1 = y_1, Y_2 = y_2)}{P(T_X = t_x)P(T_Y = t_y)} \\
&= \frac{\sum_{y_1=1}^{\min(t_x,t_y)} \sum_{y_2=1}^{y_1} P(X_1 \leq y_1, X_2 \leq y_2, T_X = t_x)P(T_Y = t_y, Y_1 = y_1, Y_2 = y_2)}{P(T_X = t_x)P(T_Y = t_y)}.
\end{aligned}$$

Since, for $x_2 \leq x_1$,

$$\begin{aligned}
P(X_1 = x_1, X_2 = x_2 / X_{(n_1)} = t_x) &= \frac{t_x^{n_1-2} - (t_x - 1)^{n_1-2}}{t_x^{n_1} - (t_x - 1)^{n_1}} \qquad \text{if } x_1 = 1, 2, ..., (t_x - 1) \\
&= \frac{t_x^{n_1-2}}{t_x^{n_1} - (t_x - 1)^{n_1}} \qquad \text{if } x_1 = t_x,
\end{aligned}$$

we get the simplified expression of $\widehat{Q}_{UMVUE}$ for the (Discrete Uniform, OPEF) combination as

$$\begin{aligned}
\widehat{Q}_{UMVUE} &= \frac{1}{h_0(T_Y)[T_X^{n_1} - (T_X - 1)^{n_1}]} \times \\
&\sum_{y_1=1}^{\min(T_X,T_Y)} \sum_{y_2=1}^{y_1} h(y_1)h(y_2)h_0(T_Y - t(y_1) - t(y_2)) \times \\
&\sum_{x_1=1}^{y_1} \sum_{x_2=1}^{y_2} \left\{ T_X^{n_1-2} - (T_X - 1)^{n_1-2} I[x_1 \neq T_X] + T_X^{n_1-2} I[x_1 = T_X] \right\} I[x_2 \leq x_1].
\end{aligned}$$

In a similar way, we derive the UMVU estimator of Q for (OPEF, Discrete Uniform) combination as

$$\begin{aligned}
\widehat{Q}_{UMVUE} &= \frac{1}{h_0(T_X)[T_Y^{n_2} - (T_Y - 1)^{n_2}]} \times \\
&\sum_{y_1=1}^{\min(T_X,T_Y)} \sum_{y_2=1}^{y_1} \left\{ (T_Y^{n_2-2} - (T_Y - 1)^{n_2-2}) I[y_1 \neq T_Y] + T_Y^{n_2-2} I[y_1 = T_Y] \right\} I[y_2 \leq y_1] \\
&\times \sum_{x_1=0}^{y_1} \sum_{x_2=0}^{y_2} h(x_1)h(x_2)h_0(T_X - t(x_1) - t(x_2)).
\end{aligned}$$

Finally, replacing T_Y by $Y_{(n_2)}$ and proceeding in a similar manner, we get the UMVU estimator of Q for (Discrete Uniform, Discrete Uniform) combination as

$$
\begin{aligned}
\widehat{Q}_{UMVUE} &= \frac{1}{[T_X^{n_1} - (T_X - 1)^{n_1}][T_Y^{n_2} - (T_Y - 1)^{n_2}]} \times \\
& \sum_{y_1=1}^{\min(T_X,T_Y)} \sum_{y_2=1}^{y_1} \left\{ T_Y^{n_1-2} - (T_Y - 1)^{n_1-2} I[y_1 \neq T_Y] + T_Y^{n_1-2} I[y_1 = T_Y] \right\} I[y_2 \leq y_1] \\
& \times \sum_{x_1=1}^{y_1} \sum_{x_2=1}^{y_2} \left\{ T_X^{n_1-2} - (T_X - 1)^{n_1-2} I[x_1 \neq T_X] + T_X^{n_1-2} I[x_1 = T_X] \right\} I[x_2 \leq x_1].
\end{aligned}
$$

3. Maximum likelihood (ML) estimation of R

Suppose $(X_1, X_2, \ldots\ldots, X_{n_1})$ and $(Y_1, Y_2, \ldots\ldots, Y_{n_2})$ are independent random samples from the distribution of random variables X and Y respectively. It is well known that ML estimator $\widehat{N}_{MLE}$ of N is $X_{(n_1)}$ and ML estimator $\widehat{\theta}_{MLE}$ of θ is obtained by solving

$$
\frac{n_2 c^{(1)}(\theta)}{c(\theta)} + q^{(1)}(\theta) \sum_{i=1}^{n_2} t(y_i) = 0,
$$

where superscript indicate the first order derivative.

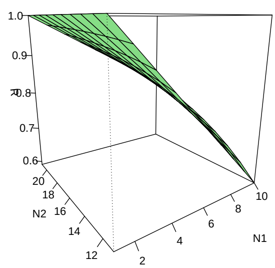
Then, by virtue of the invariance property of ML estimator we can obtain $\widehat{R}_{MLE}$ by substituting values of θ and N by $\widehat{\theta}_{MLE}$ and $\widehat{N}_{MLE}$ in the corresponding expression of R. Similarly, ML estimator of R can be obtained for OPEF - Discrete Uniform combination. $\widehat{R}_{MLE}$ for Discrete Uniform - Discrete Uniform combination can be written as

$$
\begin{aligned}
\widehat{R}_{MLE} &= \frac{2Y_{(n_2)} - X_{(n_1)} + 1}{2Y_{(n_2)}} \qquad \text{if } X_{(n_1)} < Y_{(n_2)} \\
&= \frac{Y_{(n_2)} + 1}{2X_{(n_1)}} \qquad \text{if } X_{(n_1)} \geq Y_{(n_2)}.
\end{aligned}
$$

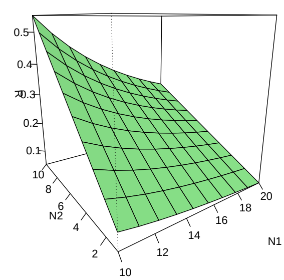
4. Expressions of R and $\widehat{R}_{UMVUE}$

Theoretical expressions of R and its UMVU estimators are derived for discrete OPEF and Discrete Uniform (DU) distributions in the previous section. Now, we provide such expressions in the simplified form for different members of OPEF. In particular, apart from the well-known Binomial, Poisson, Negative Binomial (Neg Bin), Log Series distributions, we consider One Parameter Discrete Lindley (OPDL) distribution of Hussain et al. (2016). OPDL is also a member of the discrete OPEF having the PMF, $P(X = x) = (1-\phi)^2(1+x)\phi^x$, where $x = 0, 1, 2, \ldots.,$ $0 < \phi < 1$ and complete sufficient statistic $T_X = \sum_{i=1}^{n} X_i$.

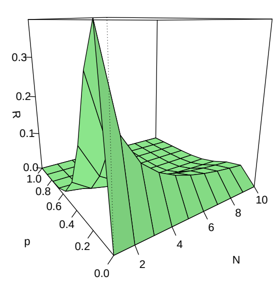
Further, for brevity, we provide three dimensional plots (Figure 1-4) of R for different members of OPEF along with the concerned expressions of R. We also provide simplified expressions of UMVU and ML estimators of R for different combinations of stress and strength distributions in Tables 1-2.



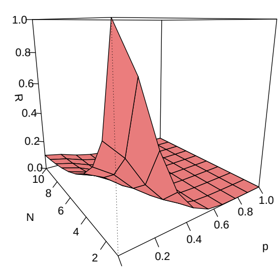
(a) Discrete Uniform (N_1) - Discrete Uniform (N_2) Model, $R = \frac{2N_2 - N_1 + 1}{2N_2} I[N_1 < N_2]$



(b) Discrete Uniform (N_1) - Discrete Uniform (N_2) Model, $R = 1 - \frac{N_2 + 1}{2N_1} I[N_1 \geq N_2]$

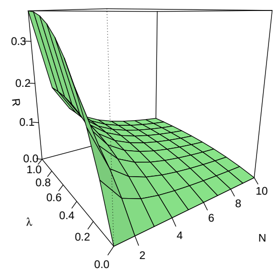


(c) Discrete Uniform (N) - Binomial (m, p) Model, $R = \frac{mp}{N}$

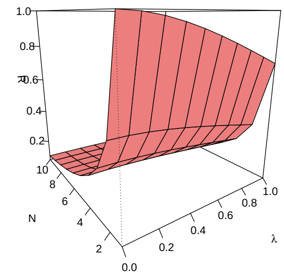


(d) Binomial (m, p) - Discrete Uniform (N) Model, $R = 1 - \frac{mp-1}{N}$

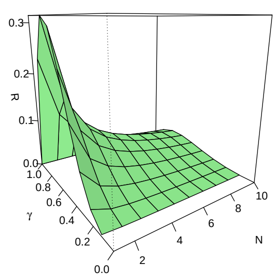
Figure 1: Plot of R for Discrete Uniform - Discrete Uniform and Discrete Uniform - Binomial, Binomial - Discrete Uniform models



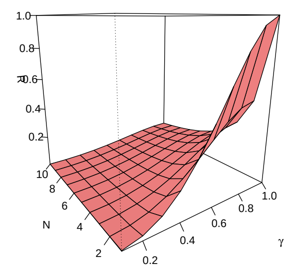
(a) Discrete Uniform (N) - Poisson (λ) Model, $R = \frac{\lambda}{N}$



(b) Poisson (λ) - Discrete Uniform (N) Model, $R = 1 - \frac{\lambda - 1}{N}$

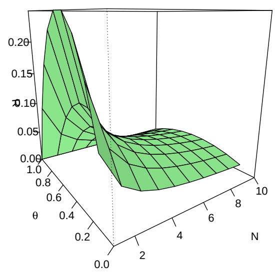


(c) Discrete Uniform (N) - Negative Binomial (r, γ) Model, $R = \frac{r(1-\gamma)}{\gamma N}$

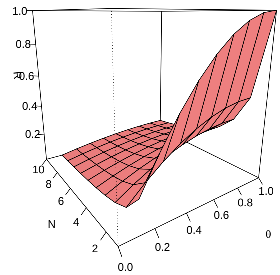


(d) Negative Binomial (r, γ) - Discrete Uniform (N) Model, $R = 1 - \frac{r-(r+1)\gamma}{\gamma N}$

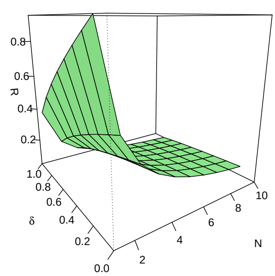
Figure 2: Plot of R for and Discrete Uniform - Poisson, Poisson - Discrete Uniform and Discrete Uniform - Negative Binomial, Negative Binomial - Discrete Uniform models



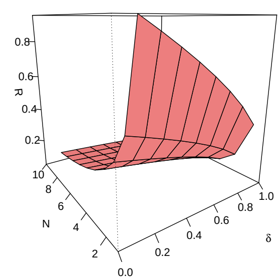
(a) Discrete Uniform (N) - Geometric (θ) Model, $R = \frac{1-\theta}{\theta} N$



(b) Geometric (θ) - Discrete Uniform (N) Model, $R = 1 - \frac{1-2\theta}{\theta N}$

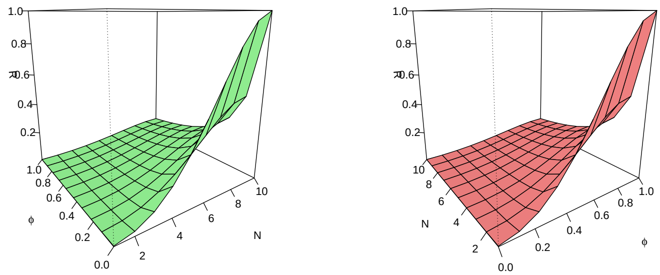


(c) Discrete Uniform (N) - Log series (δ) Model, $R = -\frac{\delta}{N(1-\delta)\log(1-\delta)}$



(d) Log series (δ) - Discrete Uniform (N) Model, $R = 1 + \frac{\lambda+(1-\lambda)\log(1-\delta)}{N(1-\delta)\log(1-\delta)}$

Figure 3: Plot of R for Discrete Uniform - Geometric, Geometric - Discrete Uniform and Discrete Uniform - Log series, Log series - Discrete Uniform models



(a) Discrete Uniform (N) - OPDL (ϕ) Model, $R = \frac{2\phi}{(1-\phi)N}$

(b) OPDL (ϕ) - Discrete Uniform (N) Model, $R = 1 - \frac{3\phi-1}{N(1-\phi)}$

Figure 4: Plot of R for Discrete Uniform - OPDL and OPDL - Discrete Uniform model

5. Simulation study

In this section, it is of our interest to compare the efficiency of the estimates $\widehat{R}_{UMVUE}$ and $\widehat{R}_{MLE}$. Although estimators of $Var(\widehat{R}_{UMVUE})$ have a closed form, neither $MSE(\widehat{R}_{MLE})$ nor its estimator is analytically tractable. Therefore for the purpose of comparison, we run a simulation study with specific choices of (n_1, n_2) and different choices of stress and strength distributions.

For each such choice, we estimate, $\widehat{R}_{UMVUE}$ and $\widehat{R}_{MLE}$ together with their MSE. Finally, we report the empirical relative efficiency (ERE), defined by

$$ERE = \frac{MSE(\widehat{R}_{MLE})}{Var(\widehat{R}_{UMVUE})}$$

for different choices of parameters and distributions. Naturally $\widehat{R}_{UMVUE}$ is better or worse than $\widehat{R}_{MLE}$ as efficiency exceeds or does not exceed unity. Figure of Tables 3-8 reveal the superiority of $\widehat{R}_{MLE}$ over $\widehat{R}_{UMVUE}$ for most of the assumed configuration.

Table 1: UMVU and ML estimators of R for Discrete Uniform - Binomial, Discrete Uniform - Poisson, Discrete Uniform - Negative Binomial, Discrete Uniform - Geometric, Discrete Uniform - Log series, Discrete Uniform - OPDL models

$Model$	$\widehat{R}_{UMVUE}$	$\widehat{R}_{MLE}$
$DiscreteUniform(N)$ $Binomial(m,p)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\binom{m}{y}\binom{m(n_2-1)}{T_Y-y}\sum_{x=1}^{y}\left\{(T_X^{n_1-1}-(T_X-1)^{n_1-1})I[x\neq T_X]+T_X^{n_1-1}I[x=T_X]\right\}}{[T_X^{n_1}-(T_X-1)^{n_1}]\binom{mn_2}{T_Y}}$	$\frac{T_Y}{n_2T_X}$
$DiscreteUniform(N)$ $Poisson(\lambda)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\frac{(n_2-1)^{T_Y-y}}{y!(T_Y-y)!}\sum_{x=1}^{y}\left\{(T_X^{n_1-1}-(T_X-1)^{n_1-1})I[x\neq T_X]+T_X^{n_1-1}I[x=T_X]\right\}}{[T_X^{n_1}-(T_X-1)^{n_1}]\frac{n_2^{T_Y}}{T_Y!}}$	$\frac{T_Y}{n_2T_X}$
$DiscreteUniform(N)$ $NegativeBinomial(r,\gamma)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\binom{r+y-1}{y}\binom{(n_2-1)r+T_Y-y-1}{T_Y-y}\sum_{x=1}^{y}\left\{(T_X^{n_1-1}-(T_X-1)^{n_1-1})I[x\neq T_X]+T_X^{n_1-1}I[x=T_X]\right\}}{[T_X^{n_1}-(T_X-1)^{n_1}]\binom{n_2r+T_Y-1}{T_Y}}$	$\frac{n_2r^2}{T_YT_X}$
$DiscreteUniform(N)$ $Geometric(\theta)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\binom{n_2+T_Y-y-2}{T_Y-y}\sum_{x=1}^{y}\left\{(T_X^{n_1-1}-(T_X-1)^{n_1-1})I[x\neq T_X]+T_X^{n_1-1}I[x=T_X]\right\}}{[T_X^{n_1}-(T_X-1)^{n_1}]\binom{n_2+T_Y-1}{T_Y}}$	$\frac{n_2}{T_YT_X}$
$DiscreteUniform(N)$ $Logseries(\delta)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\frac{(n_2-1)!\lvert S(T_Y-y,n_2-1)\rvert}{y(T_Y-y)!}\sum_{x=1}^{y}\left\{(T_X^{n_1-1}-(T_X-1)^{n_1-1})I[x\neq T_X]\right\}}{[T_X^{n_1}-(T_X-1)^{n_1}]\frac{n_2!\lvert S(T_Y,n_2)\rvert}{T_Y!}}$	$-\frac{\widehat{\delta}_{MLE}}{T_X(1-\widehat{\delta}_{MLE})\log(1-\widehat{\delta}_{MLE})}$
$DiscreteUniform(N)$ $OPDL(\phi)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}(y+1)\binom{2n_2+T_Y-y-3}{T_Y-y}\sum_{x=1}^{y}\left\{(T_X^{n_1-1}-(T_X-1)^{n_1-1})I[x\neq T_X]+T_X^{n_1-1}I[x=T_X]\right\}}{[T_X^{n_1}-(T_X-1)^{n_1}]\binom{2n_2+T_Y-1}{T_Y}}$	$\frac{2log(\frac{2n_2+T_Y}{T_Y})}{(1-T_Ylog(\frac{2n_2+T_Y}{T_Y}))}$

Table 2: UMVU and ML estimators of R for Binomial - Discrete Uniform, Poisson - Discrete Uniform, Negative Binomial - Discrete Uniform, Geometric - Discrete Uniform, Log series - Discrete Uniform, OPDL - Discrete Uniform models

$Model$	$\widehat{R}_{UMVUE}$	$\widehat{R}_{MLE}$
$Binomial(m,p)$ $DiscreteUniform(N)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\left\{(T_Y^{n_2-1}-(T_Y-1)^{n_2-1})I[y\neq T_Y]+T_Y^{n_2-1}I[y=T_Y]\right\}\sum_{x=0}^{y}\binom{m}{x}\binom{m(n_1-1)}{T_X-x}}{[T_Y^{n_2}-(T_Y-1)^{n_2}]\binom{mn_1}{T_X}}$	$1-\frac{T_X-n_1}{T_Y n_1}$
$Poisson(\lambda)$ $DiscreteUniform(N)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\left\{(T_Y^{n_2-1}-(T_Y-1)^{n_2-1})I[y\neq T_Y]+T_Y^{n_2-1}I[y=T_Y]\right\}\sum_{x=0}^{y}\frac{(n_1-1)^{T_X-x}}{x!(T_X-x)!}}{[T_Y^{n_2}-(T_Y-1)^{n_2}]\frac{n_1^{T_X}}{T_X!}}$	$1-\frac{T_X-n_1}{T_Y n_1}$
$NegativeBinomial(r,\gamma)$ $DiscreteUniform(N)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\left\{(T_Y^{n_2-1}-(T_Y-1)^{n_2-1})I[y\neq T_Y]+T_Y^{n_2-1}I[y=T_Y]\right\}\sum_{x=0}^{y}\binom{r+x-1}{x}\binom{(n_1-1)r+T_X-x-1}{T_X-x}}{[T_Y^{n_2}-(T_Y-1)^{n_2}]\binom{n_1r+T_X-1}{T_X}}$	$1-\frac{n_1r^2-T_X}{T_Y T_X}$
$Geometric(\theta)$ $DiscreteUniform(N)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\left\{(T_Y^{n_2-1}-(T_Y-1)^{n_2-1})I[y\neq T_Y]+T_Y^{n_2-1}I[y=T_Y]\right\}\sum_{x=0}^{y}\binom{n_1+T_X-x-2}{T_X-x}}{[T_Y^{n_2}-(T_Y-1)^{n_2}]\binom{n_1+T_X-1}{T_X}}$	$1-\frac{n_1-T_X}{T_Y T_X}$
$Logseries(\delta)$ $DiscreteUniform(N)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\left\{(T_Y^{n_2-1}-(T_Y-1)^{n_2-1})I[y\neq T_Y]+T_Y^{n_2-1}I[y=T_Y]\right\}\sum_{x=1}^{y}\frac{(n_1-1)!\lvert S(T_X-x,n_1-1)\rvert}{x(T_X-x)!}}{[T_Y^{n_2}-(T_Y-1)^{n_2}]\frac{n_1!\lvert S(T_X,n_1)\rvert}{T_X!}}$	$1+\frac{\log(1-\widehat{\delta}_{MLE})}{N(1-\widehat{\delta}_{MLE})\log(1-\widehat{\delta}_{MLE})}$
$OPDL(\phi)$ $DiscreteUniform(N)$	$\frac{\sum_{y=1}^{\min(T_X,T_Y)}\left\{(T_Y^{n_2-1}-(T_Y-1)^{n_2-1})I[y\neq T_Y]+T_Y^{n_2-1}I[y=T_Y]\right\}\sum_{x=0}^{y}(x+1)\binom{2n_1+T_X-x-3}{T_X-x}}{[T_Y^{n_2}-(T_Y-1)^{n_2}]\binom{2n_1+T_X-1}{T_X}}$	$1-\frac{3log(\frac{2n_1+T_X}{T_X})-1}{T_Y(1-log(\frac{2n_1+T_X}{T_X}))}$

$S(u,k)$ is the Stirling number of the first kind. $\widehat{\delta}_{MLE}$ is a root of equation $(1-\delta)^{\frac{1-\delta}{\delta}}=e^{-\frac{1}{x}}$. The first (second) distribution in each cell of the first column represents Stress (Strength) distribution.

Table 3: Poisson(λ) and DU(N)

		Poisson – DU		*DU – Poisson*	
(n_1, n_2)	(λ, N)	*R*	*ERE*	*R*	*ERE*
(15 , 30)	(0.3 , 15)	0.064	0.430	0.015	0.006
(15 , 30)	(0.5 , 15)	0.061	1.577	0.020	1.728
(15 , 30)	(0.8 , 15)	0.054	0.523	0.024	0.359
(15 , 30)	(0.3 , 30)	0.032	1.947	0.007	2.125
(15 , 30)	(0.5 , 30)	0.030	0.040	0.010	0.403
(15 , 30)	(0.8 , 30)	0.027	0.939	0.012	0.435
(30 , 30)	(0.3 , 15)	0.064	0.041	0.015	1.922
(30 , 30)	(0.5 , 15)	0.061	0.032	0.020	0.783
(30 , 30)	(0.8 , 15)	0.054	0.956	0.024	0.005
(30 , 30)	(0.3 , 30)	0.032	0.110	0.007	0.533
(30 , 30)	(0.5 , 30)	0.030	4.819	0.010	0.829
(30 , 30)	(0.8 , 30)	0.027	3.461	0.012	0.034
(45 , 30)	(0.3 , 15)	0.064	0.246	0.015	0.432
(45 , 30)	(0.5 , 15)	0.061	0.398	0.020	0.602
(45 , 30)	(0.8 , 15)	0.054	0.102	0.024	0.205
(45 , 30)	(0.3 , 30)	0.032	0.202	0.007	1.148
(45 , 30)	(0.5 , 30)	0.030	1.701	0.010	0.015
(45 , 30)	(0.8 , 30)	0.027	0.601	0.012	0.060

Table 4: Binomial($m = 8$, p) and DU(N)

		Binomial − DU		*DU − Binomial*	
(n_1 , n_2)	(p , N)	*R*	*ERE*	*R*	*ERE*
(15 , 30)	(0.3 , 15)	0.017	0.077	0.013	0.013
(15 , 30)	(0.5 , 15)	0.002	0.138	0.002	0.001
(15 , 30)	(0.8 , 15)	1.84×10^{-6}	0.415	5.46×10^{-6}	1.096
(15 , 30)	(0.3 , 30)	0.008	0.102	0.007	0.252
(15 , 30)	(0.5 , 30)	0.001	0.001	0.001	0.726
(15 , 30)	(0.8 , 30)	2.82×10^{-6}	0.387	2.73×10^{-6}	1.396
(30 , 30)	(0.3 , 15)	0.017	1.604	0.013	1.923
(30 , 30)	(0.5 , 15)	0.002	0.553	0.002	0.207
(30 , 30)	(0.8 , 15)	5.63×10^{-6}	0.859	5.46×10^{-6}	0.487
(30 , 30)	(0.3 , 30)	0.008	0.241	0.007	2.175
(30 , 30)	(0.5 , 30)	0.001	0.598	0.001	0.330
(30 , 30)	(0.8 , 30)	2.82×10^{-6}	0.214	2.73×10^{-6}	0.498
(45 , 30)	(0.3 , 15)	0.017	0.002	0.013	2.436
(45 , 30)	(0.5 , 15)	0.002	1.181	0.002	0.438
(45 , 30)	(0.8 , 15)	5.63×10^{-6}	0.642	5.46×10^{-6}	1.456
(45 , 30)	(0.3 , 30)	0.008	0.005	0.007	0.405
(45 , 30)	(0.5 , 30)	0.001	0.072	0.001	0.162
(45 , 30)	(0.8 , 30)	2.82×10^{-6}	0.266	2.73×10^{-6}	0.633

Table 5: Geometric(θ) and DU(N)

		$Geometric - DU$		$DU - Geometric$	
(n_1 , n_2)	(θ , N)	R	ERE	R	ERE
(15 , 30)	(0.3 , 15)	0.034	0.041	0.014	0.664
(15 , 30)	(0.5 , 15)	0.050	0.791	0.017	0.867
(15 , 30)	(0.8 , 15)	0.064	0.290	0.011	0.731
(15 , 30)	(0.3 , 30)	0.017	0.808	0.007	0.505
(15 , 30)	(0.5 , 30)	0.025	0.001	0.008	2.873
(15 , 30)	(0.8 , 30)	0.032	0.030	0.005	0.058
(30 , 30)	(0.3 , 15)	0.034	2.547	0.014	2.256
(30 , 30)	(0.5 , 15)	0.050	0.045	0.017	1.628
(30 , 30)	(0.8 , 15)	0.064	0.102	0.011	0.567
(30 , 30)	(0.3 , 30)	0.017	0.929	0.007	1.621
(30 , 30)	(0.5 , 30)	0.025	0.469	0.008	0.879
(30 , 30)	(0.8 , 30)	0.032	0.791	0.005	0.761
(45 , 30)	(0.3 , 15)	0.034	2.036	0.014	0.120
(45 , 30)	(0.5 , 15)	0.050	0.935	0.017	0.183
(45 , 30)	(0.8 , 15)	0.064	0.519	0.011	1.833
(45 , 30)	(0.3 , 30)	0.017	0.845	0.007	0.663
(45 , 30)	(0.5 , 30)	0.025	0.122	0.008	0.984
(45 , 30)	(0.8 , 30)	0.032	0.364	0.005	0.020

Table 6: Neg Bin($r = 3$, γ) and DU(N)

		$NegBin - DU$		$DU - NegBin$	
(n_1 , n_2)	(γ , N)	R	ERE	R	ERE
(15 , 30)	(0.3 , 15)	0.006	0.014	0.004	0.085
(15 , 30)	(0.5 , 15)	0.021	0.154	0.012	4.031
(15 , 30)	(0.8 , 15)	0.055	4.173	0.020	0.118
(15 , 30)	(0.3 , 30)	0.003	0.015	0.002	1.790
(15 , 30)	(0.5 , 30)	0.010	2.942	0.006	2.391
(15 , 30)	(0.8 , 30)	0.027	0.447	0.010	4.302
(30 , 30)	(0.3 , 15)	0.006	0.630	0.004	1.851
(30 , 30)	(0.5 , 15)	0.021	1.315	0.012	0.397
(30 , 30)	(0.8 , 15)	0.055	0.828	0.020	0.209
(30 , 30)	(0.3 , 30)	0.003	0.014	0.002	1.082
(30 , 30)	(0.5 , 30)	0.010	0.003	0.006	0.031
(30 , 30)	(0.8 , 30)	0.027	1.847	0.010	0.307
(45 , 30)	(0.3 , 15)	0.006	0.543	0.004	0.443
(45 , 30)	(0.5 , 15)	0.021	1.050	0.012	3.684
(45 , 30)	(0.8 , 15)	0.055	0.313	0.020	0.076
(45 , 30)	(0.3 , 30)	0.003	1.123	0.002	1.337
(45 , 30)	(0.5 , 30)	0.010	0.006	0.006	2.508
(45 , 30)	(0.8 , 30)	0.027	2.156	0.010	0.698

Table 7: OPDL(ϕ) and DU(N)

		OPDL – DU		*DU – OPDL*	
(n_1, n_2)	(ϕ, N)	*R*	*ERE*	*R*	*ERE*
(15 , 30)	(0.3 , 15)	0.013	0.004	0.007	0.003
(15 , 30)	(0.5 , 15)	0.026	0.001	0.013	0.020
(15 , 30)	(0.8 , 15)	0.041	0.089	0.017	0.031
(15 , 30)	(0.3 , 30)	0.006	0.021	0.004	0.003
(15 , 30)	(0.5 , 30)	0.013	0.033	0.006	0.041
(15 , 30)	(0.8 , 30)	0.021	0.019	0.008	0.005
(30 , 30)	(0.3 , 15)	0.013	0.026	0.007	0.017
(30 , 30)	(0.5 , 15)	0.026	0.059	0.013	0.005
(30 , 30)	(0.8 , 15)	0.041	0.832	0.017	0.045
(30 , 30)	(0.3 , 30)	0.006	0.004	0.004	0.025
(30 , 30)	(0.5 , 30)	0.013	0.009	0.006	0.065
(30 , 30)	(0.8 , 30)	0.021	0.063	0.008	0.013
(45 , 30)	(0.3 , 15)	0.013	0.001	0.007	0.003
(45 , 30)	(0.5 , 15)	0.026	0.003	0.013	0.202
(45 , 30)	(0.8 , 15)	0.041	0.886	0.017	0.090
(45 , 30)	(0.3 , 30)	0.006	0.003	0.004	0.098
(45 , 30)	(0.5 , 30)	0.013	0.012	0.006	0.103
(45 , 30)	(0.8 , 30)	0.021	0.003	0.008	0.006

Table 8: Log Series(δ) and DU (N)

		$LogSeries-DU$		$DU-LogSeries$	
$(n_1\ ,n_2)$	$(\delta\ ,N)$	R	ERE	R	ERE
(15 , 30)	(0.3 , 15)	0.080	0.115	0.056	0.023
(15 , 30)	(0.5 , 15)	0.096	0.014	0.048	0.535
(15 , 30)	(0.8 , 15)	0.165	0.192	0.033	0.143
(15 , 30)	(0.3 , 30)	0.040	0.910	0.028	0.683
(15 , 30)	(0.5 , 30)	0.048	0.004	0.024	3.074
(15 , 30)	(0.8 , 30)	0.082	0.238	0.016	0.014
(30 , 30)	(0.3 , 15)	0.080	0.666	0.056	1.627
(30 , 30)	(0.5 , 15)	0.096	0.060	0.048	0.226
(30 , 30)	(0.8 , 15)	0.165	0.073	0.033	1.028
(30 , 30)	(0.3 , 30)	0.040	0.009	0.028	0.150
(30 , 30)	(0.5 , 30)	0.048	0.021	0.024	3.418
(30 , 30)	(0.8 , 30)	0.082	1.262	0.016	1.915
(45 , 30)	(0.3 , 15)	0.080	0.004	0.056	1.156
(45 , 30)	(0.5 , 15)	0.096	0.068	0.048	0.050
(45 , 30)	(0.8 , 15)	0.165	0.323	0.033	0.235
(45 , 30)	(0.3 , 30)	0.040	0.031	0.028	1.785
(45 , 30)	(0.5 , 30)	0.048	0.041	0.024	3.425
(45 , 30)	(0.8 , 30)	0.082	0.386	0.016	0.050

6. A real application

A uniform distribution may be used to model demand-supply system data [Hadley and Whitin (1963), Wanke (2008)]. Here, we use a demand-supply system data (Naikan et al. 2014) of spare parts from an auto ancillary unit in India, reported in Table 9. We fit Discrete Uniform with $\hat{N}_1 = 56$ for demand (X) and Discrete Uniform with $\hat{N}_2 = 45$ for supply (Y). Now, by using expression of UMVU and ML estimators of R, we obtain $\widehat{R}_{MLE} = 0.409$ and $\widehat{R}_{UMVUE} = 0.411$. Therefore, the expressions derived theoretically are well applicable in real problems to estimate reliability (R) of demand-supply system data.

Table 9: Demand and supply system data for spare parts

Week	Demand	Supply	Week	Demand	Supply	Week	Demand	Supply
1	25	30	12	50	43	23	27	31
2	38	37	13	27	31	24	36	36
3	2	16	14	19	21	25	25	30
4	28	32	15	27	31	26	30	33
5	23	29	16	18	27	27	27	31
6	7	21	17	18	27	28	17	26
8	23	29	18	34	35	29	22	29
9	56	45	19	34	35	30	12	24
10	48	41	20	34	35			
11	6	21	21	26	31			

7. Concluding Remarks

We have discussed so far the UMVU and ML estimation of $P(X \leq Y)$ considering a discrete uniform distribution to represent stress and/or strength. However, an assumption of equal (but unknown) probability for stress and/or strength is less practical. Consequently, we intend further development with a general class of distributions to model stress and/or strength, allowing non-identical and parameter dependent supports.

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