STATISTICS IN TRANSITION new series, June 2022
Vol. 23 No. 2, pp. 49–68, DOI 10.2478/stattrans-2022-0016
Received – 20.07.2020; accepted – 01.03.2021

A Bayesian estimation of the Gini index and the Bonferroni index for the Dagum distribution with the application of different priors

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ABSTRACT

Bayesian estimators and highest posterior density credible intervals are obtained for two popular inequality measures, viz. the Gini index and the Bonferroni index in the case of the Dagum distribution. The study considers informative and non-informative priors, i.e. the Mukherjee-Islam prior and the extension of Jeffrey's prior, respectively, under the presumption of the Linear Exponential (LINEX) loss function. A Monte Carlo simulation study is carried out in order to obtain the relative efficiency of both the Gini and Bonferroni indices while taking into consideration different priors and loss functions. The estimated loss proves lower when using the Mukherjee-Islam prior in comparison to the extension of Jeffrey's prior and the LINEX loss function outperforms the squared error loss function (SELF) in terms of the estimated loss. Highest posterior density credible intervals are also obtained for both these measures. The study used real-life data sets for illustration purposes.

Key words: Inequality measures, Bayes estimator, credible interval, LINEX loss function.

1. Introduction

The Dagum distribution (also called the inverse Burr distribution; Dagum called it a generalized Logistic-Burr distribution (Kleiber and Kotz, 2003) is a well-known distribution popularly used to model income distribution. Camilo Dagum proposed the Dagum distribution in 1970, which is a skewed and heavy tailed distribution and is appropriate to model the distribution of financial, income as well as wealth distribution. The Dagum distribution was developed as an alternative to the Pareto distribution and lognormal distribution and it performs better than other two/three

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parameters income/wealth distribution models when applied to empirical data (Chotikapanich and Griffiths, 2006). One of the special cases of the Dagum distribution appeared for the first time in Burr (1942) as the third example (Burr III) of solutions of the Burr distribution system. The three parameter Dagum Type I distribution evolved from Dagum's experimentation with a shifted log-logistic distribution (Chotikapanich, 2008).

The probability density function of the Dagum distribution is given as

$$f(x;a,b,p) = \begin{cases} \frac{ap}{x}\left(\frac{\left(\frac{x}{b}\right)^{ap}}{\left(\left(\frac{x}{b}\right)^{a}+1\right)^{p+1}}\right), & x > 0;\; a,b,p > 0, \\ 0, & othwerwise. \end{cases} \tag{1}$$

The plot of probability density function of the Dagum distribution for various $p = 3.5, 4.5, 7.8$ with $a = 2.5, b = 1.5$ is shown in Figure 1.

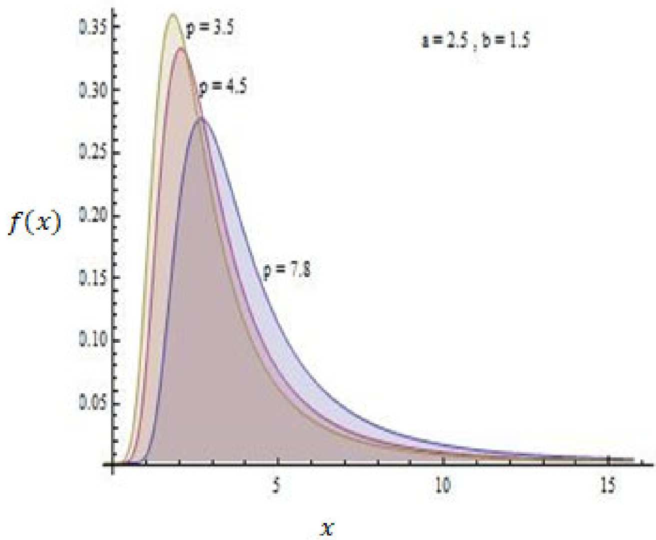

Figure 1. Probability density function of the Dagum distribution

The cumulative distribution function of the Dagum distribution is given by

$$F(x;a,b,p) = \begin{cases} \left(1+\left(\frac{x}{b}\right)^{-a}\right)^{-p}, & x > 0; \quad a,b,p > 0, \\ 0, & otherwise, \end{cases} \tag{2}$$

where a and p are shape parameters and b is a scale parameter. For $p = 1$, the Dagum distribution is also referred to as log-logistic distribution (Dagum, 1975).

Inequality is a vital characteristic of non-negative distribution. It is used to analyze data in socio-economic sciences, in the context of income distribution. In the context of income inequality, the Gini index (Gini, 1912; Foster et al., 1984) is generally defined as

$$G = 1 - 2\int_0^1 L(p)dp\,, \quad 0 \le p \le 1, \tag{3}$$

where $L(p) = (1/\mu)\int_0^p F^{-1}(t)dt$ is the equation of the Lorenz curve and $\mu = \int_0^1 F^{-1}(t)dt$ is the mean of the distribution.

The Bonferroni index is defined as

$$B = 1 - \int_0^1 B(p)dp, \qquad 0 \le p \le 1, \tag{4}$$

where $B(p) = \left(\frac{1}{p\mu}\right)\int_0^p F^{-1}(t)dt$ is the equation of the Bonferroni curve.

The curve was introduced by Bonferroni (1930) and has been analysed and studied by various authors: see for instance De Vergottini (1940), Tarsitano (1990), Giorgi & Crescenzi (2001) and Zenga (2013).

In the case of the Dagum distribution, the Gini index (G) and the Bonferroni index (B) are given by

$$G = \frac{\Gamma p\,\Gamma(2p+\frac{1}{a})}{\Gamma 2p\,\Gamma(p+\frac{1}{a})} - 1, \qquad a, p > 0, \tag{5}$$

where $\Gamma(.)$ is the Gamma function,

and $$B = p\left[\varphi\left(p + \frac{1}{a}\right) + \varphi(p)\right], \quad a, p > 0, \tag{6}$$

where $\varphi(x) = \frac{d}{dx} ln\sqrt{x} = \frac{\Gamma(x)'}{\Gamma(x)}$, is the Digamma function.

Note that both values are independent of the scale-parameter b.

A huge literature exists on the estimation of the Gini index and inequality measures using classical approach, i.e. parametric and non-parametric (Moothathu, 1985; Sen, 1988; Dixon, 1987; Bansal et al., 2011). But in the case of Bayesian set up, a lot of work still needs attention (Sathar et al., 2009; Bhattacharya and Chaturvedi, 1999) particularly in the context of income inequality. In the case of Pareto distribution Bayesian estimators of the Gini index (Kaur et al., 2015) are obtained using different priors under LINEX loss function. Some work regarding Bayesian estimation of the shape parameter p of the Dagum distribution is available under different loss functions using informative and non-informative priors (Naqash et al., 2017) while Layla et al. (2020) discussed the Bayesian estimation of the survival function using Gamma as informative and Jeffrey as non-informative prior, but the income inequality field still awaits the attention of researchers. In the present paper, Bayesian estimators for two famous inequality indices, viz. the Gini index and the Bonferroni index will be obtained for the Dagum distribution along with their credible intervals. These inequality indices are not only used in the economic set up but have applications in other fields such as survival analysis, reliability and bio-statistics.

When the Bayesian approach is used, the selection of a suitable prior distribution plays a major role. Basically, priors can be divided into informative (an informative prior depends on elicitation of prior distribution based on pre-existing scientific knowledge in the area of investigation), non-informative (a non-informative prior is usually improper, $i.e.$ it does not have a proper density function but the resulting posterior distribution is a proper density function), and conjugate prior (if the posterior distribution $p(\theta|x)$ is from the same family of probability distributions as the prior probability distribution $p(\theta)$) (Kass and Wasserman, 1996; Berger, 2006). In the Bayesian estimation, the benchmark for quality (good) estimator for the parameters of interest is the selection of the proper loss function. A squared error loss function is the simplest loss function among all the loss functions. It is also known as a quadratic loss function, defined as

$$L(\theta) = (\hat{\theta} - \theta)^2, \tag{7}$$

where $\hat{\theta}$ is the estimator of θ.

The squared error loss function (SELF) is symmetrical and shows equal importance to losses due to overestimation and underestimation of equal magnitude. One disadvantage of using the squared error loss function is that it penalizes overestimation or underestimation. Overestimation of a parameter can lead to more severe or less severe consequences than underestimation, or vice versa. In the case of income inequality under-estimation is more serious as compared to overestimation (Kaur et al., 2015). For this reason, the use of an asymmetrical loss function, which can provide greater importance to overestimation or underestimation, can be considered for the estimation of the parameters. Many asymmetrical loss functions are available in the statistical literature and one such Linear exponential loss function (LINEX) has been proposed by Varian (1975) as

$$L\left(\hat{\theta} - \theta\right) = e^{b(\hat{\theta}-\theta)} - b(\hat{\theta} - \theta) - 1, \ b \neq 0. \tag{8}$$

The posterior expectation of the LINEX loss function is

$$E(L\left(\hat{\theta} - \theta\right)) = e^{b\hat{\theta}}E(e^{-b\theta}) - b\left(\hat{\theta} - E(\theta)\right) - 1,$$

where $E(.)$ denotes posterior expectation with respect to the posterior density of θ.

By a result of Zellner (1986) the Bayes estimator of θ denoted by $\hat{\theta}$ under the LINEX loss function is the value which minimizes posterior expectation and is given by

$$\hat{\theta} = -\frac{1}{b}\ln\left[E\left(e^{-b\theta}\right)\right], \tag{9}$$

provided the expectation $E(e^{-b\theta})$ exists and is finite.

The LINEX loss function is approximately equal to the squared error loss function for the small values of b.

In this paper, the LINEX loss function is used for obtaining Bayesian estimators for two popular inequality indices, i.e. the Gini index and the Bonferroni index in the case of the Dagum distribution using Mukherjee-Islam prior (informative prior) and the extension of Jeffrey's prior (non-informative prior). The plan of the paper is as follows. In Section 2, prior and posterior distributions are discussed in the case of the Dagum distribution. In Section 3, Bayesian estimators are obtained for the Gini index and the Bonferroni index for the Dagum distribution under the assumption of the LINEX loss function. In Section 4, using simulation, relative efficiency of Bayesian estimates is obtained for both the Gini and Bonferroni index taking into consideration different priors and two loss functions, LINEX and SELF. In Section 5, the credible intervals are defined and highest posterior density credible intervals are carried out for both the Gini index and the Bonferroni index. Two real life examples to illustrate the method of Bayesian setup are given in Section 6.

2. Prior and posterior distribution

2.1. Case 1: Shape parameter p is unkown and a, b are known

Let $X = (x_1, x_2, \ldots, x_n)$ be a random sample from the Dagum distribution with shape parameters p and a and scale parameter b, i.e. $X \sim D(a, b, p)$, then the likelihood function for the Dagum distribution as a function of p (keeping a and b fixed) is given by

$$L = \left(\frac{ap}{b^{ap}}\right)^n \prod_{i=1}^{n} x_i{}^{ap-1} \prod_{i=1}^{n} \left(1 + \left(\frac{x_i}{b}\right)^a\right)^{-(p+1)} \tag{10}$$

$$L(a, b, p) \propto p^n \prod_{i=1}^{n} \left(\frac{x_i}{b^{ap}}\right) \prod_{i=1}^{n} \left[1 + \left(\frac{x_i}{b}\right)^a\right]^{-p}$$

$$= p^n e^{p \sum_{i=1}^{n} \ln\left(\frac{x_i}{b}\right)^a} e^{-p \sum_{i=1}^{n} \ln\left[1 + \left(\frac{x_i}{b}\right)^a\right]}$$

$$\Rightarrow \ L(p|x) \propto \ p^n e^{-p \sum_{i=1}^{n} \ln\left[1 + \left(\frac{x_i}{b}\right)^{-a}\right]}$$

$$= p^n e^{-pT}$$

where
$$T = \sum_{i=1}^{n} \ln\left[1 + \left(\frac{x_i}{b}\right)^{-a}\right]. \tag{11}$$

Posterior distribution under Mukherjee-Islam prior

Mukherjee Islam (1983) is a well-known probability distribution used by many researchers to model a failure distribution for the purpose of reliability and Bayesian analysis.

Assume that p has a Mukherjee-Islam prior with hyper parameters $(\alpha, \sigma) > 0$, defined by

$$\pi(p) = \alpha\sigma^{-\alpha}p^{\alpha-1} \;;\; p > 0, \alpha > 0, \sigma > 0. \tag{12}$$

Then, the posterior distribution of p under Mukherjee-Islam prior is given by

$$\pi_M(p|x) = \frac{L(p) * \pi(p)}{\int_0^\infty L(p) * \pi(p)\, dp}$$

$$\pi_M(p|x) \propto p^{n+\alpha-1}e^{-pT}$$

$$= hp^{n+\alpha-1}e^{-pT}$$

where h is the normalized constant given by

$$h = \int_0^\infty p^{n+\alpha-1}\, e^{-pT}\, dp$$

$$= \Gamma(n+\alpha)/T^{n+\alpha}\, .$$

Thus, the posterior distribution of p is given by

$$\pi_M(p|x) = \frac{T^{n+\alpha}}{\Gamma(n+\alpha)} p^{n+\alpha-1}e^{-pT}, \tag{13}$$

which is gamma density with parameters $T\ and\ \beta_1 = n + \alpha$.

Posterior distribution under extension of Jeffreys' prior

Jeffreys' prior is a particular case of the extension of Jefferys' prior proposed by Kutubi and Ibrahim (2009). The extension of Jeffreys' prior is defined as

$$\pi(p) \propto [I(p)]^m\, ; m > 0,$$

where $[I(p)]$ is the Fisher Information given by

$$[I(p)] = -E\left[\frac{\partial^2 l}{\partial p^2}\right] = \frac{n}{p^2},$$

where l is the log-likelihood function. For $m = 0.5$, it reduces to Jeffreys' prior. Thus, the extension of Jeffreys' prior is given by

$$\pi(p) \propto \frac{1}{p^{2m}}, m > 0. \tag{14}$$

The posterior distribution is defined by

$$\pi_{EJ}(p|x) \propto p^{n-2m}e^{-pT} = Kp^{n-2m}e^{-pT},$$

where k is the normalized constant given by

$$K = \int_0^\infty p^{n-2m}\, e^{-pT}\, dp = \frac{\Gamma(n-2m+1)}{T^{n-2m+1}}.$$

Thus, the posterior distribution of p/x is given by

$$\pi_{EJ}(p|x) = \frac{T^{n-2m+1}}{\Gamma(n-2m+1)} p^{n-2m}e^{-pT}, \tag{15}$$

which is a gamma density with parameters $T and\ \beta_2 = n - 2m + 1$.

2.2. Case 2: Shape parameter a is unkown and p, b are known

Let $X = (x_1, x_2, \ldots, x_n)$ be a random sample from $D(a, b, p)$ Dagum distribution. Then, the likelihood function of the scale parameter a (keeping p and b fixed) is given by

$$L = \left(\frac{ap}{b^{ap}}\right)^n \prod_{i=1}^n x_i^{ap-1} \prod_{i=1}^n \left(1 + \left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}.$$

Posterior distribution under Mukherjee-Islam prior

Assume that a has a Mukherjee-Islam prior with hyper parameters $(\alpha, \sigma) > 0$ defined by

$$g(a) = \alpha\sigma^{-\alpha} a^{\alpha-1} \;\; ; \alpha > 0, \sigma > 0. \tag{16}$$

The posterior distribution of a is

$$\pi_M(a|x) = \frac{(a^{n+\alpha-1})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}}{\int_0^\infty (a^{n+\alpha-1})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)} da} \tag{17}$$

Posterior distribution under extension of Jeffreys' prior

The extension of Jeffreys' prior is given by

$$g(a) \propto \frac{1}{a^{2m}}, m > 0. \tag{18}$$

The posterior distribution of a is

$$\pi_{EJ}(a|x) = \frac{(a^{n-2m})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}}{\int_0^\infty (a^{n-2m})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)} da} \tag{19}$$

3. Bayesian estimation under Linear Exponential (LINEX) loss function using different priors

3.1. Case 1: Shape parameter p is unkown and a, b are known

Bayesian estimators using Mukherjee-Islam prior

Using the posterior distribution given in (13) the Bayesian estimator $\widehat{G_{ML}}$ of the Gini index G using Mukherjee-Islam prior is

$$\widehat{G_{ML_1}} = \frac{-1}{b}\log\, E[e^{-bG}]$$

$$= \frac{-1}{b}\log\left[\int_0^\infty e^{-b\left(\frac{\Gamma p\, \Gamma\left(2p+\frac{1}{a}\right)}{\Gamma 2p\, \Gamma\left(p+\frac{1}{a}\right)}-1\right)} \frac{T^{n+\alpha}}{\Gamma(n+\alpha)} p^{n+\alpha-1} e^{-pT} dp\right]$$

$$= \frac{-1}{b}\log\left[\frac{T^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^{\infty} e^{-\left(b\frac{\Gamma p\,\Gamma\left(2p+\frac{1}{a}\right)}{\Gamma 2p\,\Gamma\left(p+\frac{1}{a}\right)}-b-pT\right)} p^{n+\alpha-1}dp\right]. \tag{20}$$

The Bayesian estimator $\widehat{B_{ML}}$ of the Bonferroni index B using Mukherjee-Islam prior is

$$\widehat{B_{ML_1}} = \frac{-1}{b}\log\ E[e^{-bB}]$$
$$= \frac{-1}{b}\log\left[\int_0^{\infty} e^{-bp\left[\varphi\left(p+\frac{1}{a}\right)+\varphi(p)\right]}\frac{T^{n+\alpha}}{\Gamma(n+\alpha)}p^{n+\alpha-1}e^{-pT}dp\right]$$
$$= \frac{-1}{b}\log\left[\frac{T^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^{\infty} e^{-bp\left[\varphi\left(p+\frac{1}{a}\right)+\varphi(p)\right]-pT}\,p^{n+\alpha-1}dp\right]. \tag{21}$$

Bayesian estimators using extension of Jeffreys' Prior

Using the posterior distribution given in (15) the Bayesian estimator $\widehat{G_{EL}}$ of the Gini index G using the extension of Jeffreys' prior is

$$\widehat{G_{EL_1}} = \frac{-1}{b}\log\ E[e^{-bG}]$$
$$= \frac{-1}{b}\log\left[\int_0^{\infty} e^{-b\left(\frac{\Gamma p\,\Gamma\left(2p+\frac{1}{a}\right)}{\Gamma 2p\,\Gamma\left(p+\frac{1}{a}\right)}-1\right)}\frac{T^{n-2m+1}}{\Gamma(n-2m+1)}p^{n-2m}e^{-pT}dp\right]$$
$$= \frac{-1}{b}\log\left[\frac{T^{n-2m+1}}{\Gamma(n-2m+1)}\int_0^{\infty} e^{-\left(b\frac{\Gamma p\,\Gamma\left(2p+\frac{1}{a}\right)}{\Gamma 2p\,\Gamma\left(p+\frac{1}{a}\right)}-b-pT\right)} p^{n-2m}dp\right]. \tag{22}$$

The Bayesian estimator $\widehat{B_{EL}}$ of the Bonferroni index B using the extension of Jeffreys' prior is

$$\widehat{B_{EL_1}} = \frac{-1}{b}\log\ E[e^{-bB}]$$
$$= \frac{-1}{b}\log\left[\int_0^{\infty} e^{-bp\left[\varphi\left(p+\frac{1}{a}\right)+\varphi(p)\right]}\frac{T^{n-2m+1}}{\Gamma(n-2m+1)}p^{n-2m}e^{-pT}dp\right]$$
$$= \frac{-1}{b}\log\left[\frac{T^{n-2m+1}}{\Gamma(n-2m+1)}\int_0^{\infty} e^{-bp\left[\varphi\left(p+\frac{1}{a}\right)+\varphi(p)\right]-pT}\,p^{n-2m}dp\right]. \tag{23}$$

3.2. Case 2: Shape parameter a is unkown and p, b are known

Bayesian estimators using Mukherjee-Islam prior

Using the posterior distribution given in (17) the Bayes estimator $\widehat{G_{ML}}$ of the Gini index G using Mukherjee-Islam prior is

$$\widehat{G_{ML_2}} = \frac{-1}{b}\log\ E[e^{-bG}]$$
$$= \frac{-1}{b}\log\left[\int_0^{\infty} e^{-b\left(\frac{\Gamma p\,\Gamma\left(2p+\frac{1}{a}\right)}{\Gamma 2p\,\Gamma\left(p+\frac{1}{a}\right)}-1\right)}\frac{(a^{n+\alpha-1})(\frac{1}{b^{ap}})^n\prod_{i=1}^n x_i^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}}{\int_0^{\infty}(a^{n+\alpha-1})(\frac{1}{b^{ap}})^n\prod_{i=1}^n x_i^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}da}\right]. \tag{24}$$

The Bayes estimator $\widehat{B_{ML}}$ of the Bonferroni index B using Mukherjee-Islam prior is

$$\widehat{B_{ML_2}} = \frac{-1}{b}\log\ E[e^{-bB}]$$
$$= \frac{-1}{b}\log\left[\int_0^\infty e^{-bp\left[\varphi\left(p+\frac{1}{a}\right)+\varphi(p)\right]} \frac{(a^{n+\alpha-1})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i{}^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}}{\int_0^\infty (a^{n+\alpha-1})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i{}^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)} da}\right]. \quad (25)$$

Bayesian estimators under extension of Jeffreys' prior

The Bayes estimator $\widehat{G_{EL}}$ of the Gini index G using the extension of Jeffreys' prior is

$$\widehat{G_{EL_2}} = \frac{-1}{b}\log\ E[e^{-bG}]$$
$$= \frac{-1}{b}\log\left[\int_0^\infty e^{-b\left(\frac{\Gamma p\,\Gamma\left(2p+\frac{1}{a}\right)}{\Gamma 2p\,\Gamma\left(p+\frac{1}{a}\right)}-1\right)} \frac{(a^{n-2m})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i{}^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}}{\int_0^\infty (a^{n-2m})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i{}^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)} da}\right]. \quad (26)$$

The Bayes estimator $\widehat{B_{EL}}$ of the Bonferroni index B using the extension of Jeffreys' prior is

$$\widehat{B_{EL_2}} = \frac{-1}{b}\log\ E[e^{-bB}]$$
$$= \frac{-1}{b}\log\left[\int_0^\infty e^{-bp\left[\varphi\left(p+\frac{1}{a}\right)+\varphi(p)\right]} \frac{(a^{n-2m})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i{}^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)}}{\int_0^\infty (a^{n-2m})(\frac{1}{b^{ap}})^n \prod_{i=1}^n x_i{}^{ap-1}\left(1+\left(\frac{x_i}{b}\right)^a\right)^{-(p+1)} da}\right] \quad (27)$$

Remark As all these expressions cannot be simplified further, the Bayesian estimators have been obtained using simulation techniques in R software.

4. Simulation study

In order to assess the statistical performance of these estimators for the Gini index and the Bonferroni index, a simulation study is conducted. The $VGAM$ package in R software is used to draw the sample from the Dagum distribution and using simulation the Bayes estimates and their corresponding losses are computed. Theprocess is replicated 10000 times and the average of the results has been presented in the tables below (Tables 1-4). The estimated losses are computed for both LINEX and the squared error loss function (SELF) using generated random samples from the Dagum distribution and by considering three sample sizes, (i) small sample size $n = 25$, (ii) moderate sample size $n = 50$, (iii) large sample size $n = 100$. The estimated losses are repeated for Mukherjee-Islam prior and the extension of Jeffreys' prior using different configuration of scale and shape parameters, viz. $b(known)$ =1.5, 1.3, 1.2, $a(known)$ =2.5, 1.75, 1.6, $p(unkown)$ =7.8, 4.5, 3.5 and $b(known)$ =1.2, 2.2, 3.2, $p(known)$ =1.4, 1.7, 1.9, $a(unkown)$ =5.8, 2.5, 1.5. The hyper parameters values are α = 2.2, 3.5 and m = 1, 1.7 are chosen using MLE values in R software.

Table 1. Bayesian Estimates under LINEX loss function and Estimated loss (in parenthesis) for Gini Index and Bonferroni Index under Mukherjee-Islam prior (when p is unknown, a and b are known)

n	b	a	p	α =2.2		α =3.5	
				$\hat{G}_{ML_1}$	$\hat{B}_{ML_1}$	$\hat{G}_{ML_1}$	$\hat{B}_{ML_1}$
25	1.5	2.5	7.8	0.14119 (0.03647, 0.03848)	0.17720 (0.05690,0.05817)	0.14053 (0.03672,0.03825)	0.17590 (0.05748,0.05950)
	1.3	1.75	4.5	0.141006 (0.03654, 0.03789)	0.25390 (0.08675,0.08836)	0.14177 (0.03626,0.03798)	0.25368 (0.08785,0.08837)
	1.2	1.6	3.5	0.14341 (0.03566, 0.036148)	0.27770 (0.06531,0.06661)	0.14179 (0.03625,0.03775)	0.27523 (0.06627,0.06819)
50	1.5	2.5	7.8	0.14139 (0.03640, 0.03715)	0.17759 (0.05686,0.05739)	0.14063 (0.03668,0.03799)	0.17684 (0.05706,0.05874)
	1.3	1.75	4.5	0.14121 (0.03647, 0.03718)	0.25448 (0.08648,0.08739)	0.14205 (0.03616,0.03701)	0.25394 (0.08673,0.08735)
	1.2	1.6	3.5	0.14404 (0.03543, 0.03608)	0.27925 (0.06472,0.06597)	0.14376 (0.03553,0.03691)	0.27860 (0.06497,0.06691)
100	1.5	2.5	7.8	0.14152 (0.03638, 0.03699)	0.17746 (0.05522,0.05669)	0.14088 (0.03660,0.03709)	0.17722 (0.05688,0.05797)
	1.3	1.75	4.5	0.14292 (0.03584,0.03615)	0.25450 (0.08531,0.08649)	0.14235 (0.03605,0.03700)	0.25447 (0.08548,0.08646)
	1.2	1.6	3.5	0.14587 (0.03481,0.03513)	0.27879 (0.06390,0.06459)	0.14519 (0.03501,0.03611)	0.27933 (0.06369,0.06479)

Notation used: Estimated loss (under LINEX, under SELF)

Table 2. Bayesian Estimates under LINEX loss function and Estimated loss (in parenthesis) for Gini and Bonferroni index under the extension of Jeffreys' prior (when p is unknown, a and b are known)

n	b	a	p	m =1		m =1.7	
				$\hat{G}_{ML_1}$	$\hat{B}_{ML_1}$	$\hat{G}_{ML_1}$	$\hat{B}_{ML_1}$
25	1.5	2.5	7.8	0.14072 (0.036684,0.03801)	0.17732 (0.05741,0.05964)	0.14007 (0.03689,0.03816)	0.17682 (0.05707,0.05936)
	1.3	1.75	4.5	0.21439 (0.06362,0.06560)	0.25300 (0.08727,0.08902)	0.21493 (0.06339,0.06549)	0.25368 (0.08685,0.08894)
	1.2	1.6	3.5	0.24123 (0.06760,0.06996)	0.17982 (0.20488,0.22827)	0.23999 (0.06808,0.06971)	0.18100 (0.20398,0.22873)
50	1.5	2.5	7.8	0.14098 (0.03655,0.03715)	0.17735 (0.05683,0.05806)	0.14070 (0.03665,0.03793)	0.17737 (0.05682,0.05815)
	1.3	1.75	4.5	0.21519 (0.06366,0.06499)	0.25428 (0.08657,0.08748)	0.21505 (0.06335,0.06499)	0.25449 (0.08647,0.08764)
	1.2	1.6	3.5	0.24176 (0.06739,0.06861)	0.18305 (0.20241,0.21953)	0.24153 (0.06748,0.06879)	0.18294 (0.20149,0.22238)
100	1.5	2.5	7.8	0.14103 (0.03643,0.03709)	0.17806 (0.05583,0.05696)	0.14088 (0.03659,0.03704)	0.17727 (0.05586,0.05619)
	1.3	1.75	4.5	0.21478 (0.06346,0.064185)	0.25431 (0.08556,0.08602)	0.21471 (0.06320,0.06435)	0.25471 (0.08546,0.08675)
	1.2	1.6	3.5	0.24193 (0.06720,0.06818)	0.18424 (0.20006,0.21174)	0.24098 (0.06700,0.06861)	0.18328 (0.20106,0.21352)

Notation used: Estimated loss (under LINEX, under SELF)

Table 3. Bayesian Estimates under LINEX loss function and Estimated loss (in parenthesis) for Gini Index and Bonferroni Index under Mukherjee-Islam prior (when a is unknown, p and b are known)

n	b	p	a	α =2.2		α =3.5	
				$\hat{G}_{ML_2}$	$\hat{B}_{ML_2}$	$\hat{G}_{ML_2}$	$\hat{B}_{ML_2}$
25	1.2	1.4	5.8	0.41101	0.37029	0.41738	0.37936
				(0.00324,0.00349)	(0.00260,0.00277)	(0.00325,0.00344)	(0.00261,0.00285)
	2.2	1.7	2.5	0.48465	0.53917	0.49700	0.53748
				(0.00558,0.00582)	(0.00362,0.00382)	(0.00564,0.00595)	(0.00377,0.00390)
	3.2	1.9	1.5	0.50321	0.55357	0.50791	0.55153
				(0.00665,0.00683)	(0.00161,0.00187)	(0.00655,0.00689)	(0.00167,0.00185)
50	1.2	1.4	5.8	0.41186	0.37684	0.41071	0.37579
				(0.00324,0.00341)	(0.00251,0.00268)	(0.00322,0.00340)	(0.00261,0.00275)
	2.2	1.7	2.5	0.49196	0.53931	0.49994	0.53254
				(0.00557,0.00579)	(0.00361,0.00379)	(0.00556,0.00590)	(0.00362,0.00382)
	3.2	1.9	1.5	0.50789	0.55070	0.51258	0.55503
				(0.00654,0.00680)	(0.00160,0.00179)	(0.00654,0.00681)	(0.00161,0.00176)
100	1.2	1.4	5.8	0.41284	0.38057	0.41004	0.38513
				(0.00320,0.00333)	(0.00244,0.00255)	(0.00322,0.00331)	(0.00241,0.00263)
	2.2	1.7	2.5	0.49897	0.54934	0.49148	0.54864
				(0.00555,0.00569)	(0.00358,0.00362)	(0.00555,0.00586)	(0.00350,0.00371)
	3.2	1.9	1.5	0.51124	0.56258	0.51245	0.56856
				(0.00641,0.00679)	(0.00157,0.00165)	(0.00644,0.00677)	(0.00158,0.00160)

Notation used: Estimated loss (under LINEX, under SELF)

Table 4. Bayesian Estimates under LINEX loss function and Estimated loss (in parenthesis) for Gini and Bonferroni index under the extension of Jeffreys' prior (when a is unknown, p and b are known)

n	b	p	a	m =1		m =1.7	
				$\hat{G}_{ML_2}$	$\hat{B}_{ML_2}$	$\hat{G}_{ML_2}$	$\hat{B}_{ML_2}$
25	1.2	1.4	5.8	0.38574	0.35028	0.38974	0.35525
				(0.00347,0.00365)	(0.00275,0.00288)	(0.00351,0.00369)	(0.00268,0.00279)
	2.2	1.7	2.5	0.46251	0.50124	0.46925	0.502173
				(0.00562,0.00581)	(0.00367,0.00379)	(0.00573,0.00588)	(0.00368,0.00379)
	3.2	1.9	1.5	0.48196	0.45218	0.48202	0.45869
				(0.00667,0.00679)	(0.00179,0.00183)	(0.00660,0.00671)	(0.00172,0.00189)
50	1.2	1.4	5.8	0.38159	0.35585	0.38874	0.35968
				(0.00349,0.00350)	(0.00267,0.00271)	(0.00342,0.00359)	(0.00258,0.00266)
	2.2	1.7	2.5	0.46095	0.50143	0.46748	0.50147
				(0.00560,0.00579)	(0.00366,0.00370)	(0.00563,0.00571)	(0.00367,0.00372)
	3.2	1.9	1.5	0.47259	0.45748	0.47179	0.45321
				(0.00659,0.00661)	(0.00168,0.00176)	(0.00659,0.00669)	(0.00167,0.00170)
100	1.2	1.4	5.8	0.39561	0.35249	0.39095	0.35648
				(0.00335,0.00341)	(0.00259,0.00266)	(0.00335,0.00349)	(0.00249,0.00251)
	2.2	1.7	2.5	0.47259	0.50852	0.47125	0.50357
				(0.00558,0.00561)	(0.00359,0.00362)	(0.00559,0.00569)	(0.00359,0.00368)
	3.2	1.9	1.5	0.50014	0.45354	0.50934	0.45258
				(0.00649,0.00656)	(0.00158,0.00163)	(0.00649,0.00655)	(0.00158,0.00166)

Notation used: Estimated loss (under LINEX, under SELF)

Comments: One can observe that

1) The estimated loss in each case decreases as sample size n increases for all the configurations of various parameters.
2) The estimated loss using the LINEX loss function is smaller as compared with the squared error loss function (SELF) for both Mukherjee-Islam prior and the extension of Jeffrey's prior.
3) The estimated loss is also lower using Mukherjee-Islam prior than the extension of Jeffrey's prior.

5. Credible interval

According to Eberly and Casella (2003), the $100(1-\gamma)\%$ equal tail credible interval for the exact posterior distribution can be defined as

$$P(\theta < L) = \int_{-\infty}^{L} \pi(\theta/x)d\theta = \frac{\gamma}{2}, \qquad P(\theta < U) = \int_{U}^{\infty} \pi(\theta/x)d\theta = \frac{\gamma}{2}$$

where $\pi(\theta/x)$ is the posterior distribution of θ and (L, U) are the lower and upper limits of the credible interval respectively for the specified value of γ level of significance.

Highest Posterior Density (HPD) Credible Intervals

Chen and Shao (1999) introduced the algorithm to find the HPD credible intervals. $100(1-\gamma)\%$ HPD credible interval is the $100(1-\gamma)\%$ credible interval with smallest width among all possible $100(1-\gamma)\%$ credible intervals. Once the posterior sample is generated for parameter $\theta_i \left(i = 1,2,\dots,(N-N_0)\right)$, then $\theta_{(1)} \le \theta_{(2)} \le \cdots \le \theta_{(N-N_0)}$ denote the ordered values of $\theta_1, \theta_2, \dots, \theta_{(N-N_0)}$. The $100(1-\gamma)\%$ HPD interval for θ is defined by $\left(\theta_{(j)}, \theta_{(j+[(1-\gamma)(N-N_0)])}\right)$, where j is chosen such that

$$\theta_{(j+[(1-\gamma)(N-N_0)])} - \theta_{(j)} = \min_{1\le j\le M}\left(\theta_{(j+[(1-\gamma)(N-N_0)])} - \theta_{(j)}\right), \qquad j = 1,2,\dots,(N-N_0),$$

where $[x]$ denotes to greatest integer less than or equal to x.

Table 5. 95% HPD Credible Intervals, width of the interval and Bayesian estimates (in 2nd row) for Gini Index under Mukherjee–Islam Prior

n	b	a	p	α =2.2	α =3.5
				(Credible interval) (width) (Bayes Estimate)	(Credible interval) (width) (Bayes Estimate)
25	1.5	2.5	7.8	(0.08764,1.159667)	(0.098866,1.418709)
				(1.072027) (0.141195)	(1.319843)(0.140532)
	1.3	1.75	4.5	(0.034647,1.065941)	(0.090902,1.285003)
				(1.031294) (0.1410067)	(1.194101)(0.141778)
	1.2	1.6	3.5	(0.086689,1.750493)	(0.029834,1.230873)
				(1.663804)(0.1434128)	(1.201039)(0.141793)

Table 5. 95% HPD Credible Intervals, width of the interval and Bayesian estimates (in 2nd row) for Gini Index under Mukherjee–Islam Prior (cont.)

n	b	a	p	α =2.2	α =3.5
50	1.5	2.5	7.8	(0.04379,1.016813) (0.973023)(0.1413926)	(0.079860,1.239483) (1.159623) (0.1406394)
	1.3	1.75	4.5	(0.010254,1.01777) (1.007516)(0.1412134)	(0.006467,1.069109) (1.062642)(0.1420578)
	1.2	1.6	3.5	(0.072973,1.210653) (1.13768)(0.1440414)	(0.094263,1.058927) (0.964664)(0.1437612)
100	1.5	2.5	7.8	(0.050407,1.00031) (0.949903) (0.1415200)	(0.046396,1.091680) (1.045284) (0.1408886)
	1.3	1.75	4.5	(0.259496,1.00071) (0.741214)(0.1429269)	(0.001569,1.031742) (1.030173) (0.1423524)
	1.2	1.6	3.5	(0.005600,1.000285) (0.994685) (0.1458737)	(0.047262,1.007538) (0.960276) (0.1451941)

Table 6. 95% Credible Intervals, width of the interval and Bayesian estimates for Bonferroni Index under Mukherjee–Islam Prior

n	b	a	p	α =2.2	α =3.5
				(Credible interval) (width) (Bayes Estimate)	(Credible interval) (width) (Bayes Estimate)
25	1.5	2.5	7.8	(0.155791,1.996669) (1.840878)(0.1772007)	(0.118866, 1.907854) (1.788988)(0.1759073)
	1.3	1.75	4.5	(0.134647,1.901148) (1.766501)(0.2539019)	(0.110902,1.903905) (1.793003)(0.2536816)
	1.2	1.6	3.5	(0.116897,1.537517) (1.42062)(0.2777098)	(0.129834,1.986688) (1.856854)(0.2752306)
50	1.5	2.5	7.8	(0.11965,1.743861) (1.624211)(0.1775938)	(0.109801,1.618079) (1.508278)(0.1768471)
	1.3	1.75	4.5	(0.103105,1.349774) (1.246669)(0.2544812)	(0.106467,1.322476) (1.216009)(0.2539437)
	1.2	1.6	3.5	(0.102973,1.182072) (1.079099)(0.2792508)	(0.194236,1.460026) (1.26579)(0.2786031)
100	1.5	2.5	7.8	(0.045070,1.109478) (1.064408)(0.1774652)	(0.054969,1.535899) (1.48093)(0.1772273)
	1.3	1.75	4.5	(0.059796,1.179001) (1.119205)(0.2545054)	(0.043519,1.017423) (0.973904)(0.2544711)
	1.2	1.6	3.5	(0.006575,1.000118) (0.993543)(0.2787902)	(0.004072,1.087538) (1.083466)(0.279333)

Table 7. 95% HPD Credible Intervals, width of the interval and Bayesian estimates for the Gini index under the extension of Jeffreys' Prior

n	**b**	**a**	**p**	**m** =1	**m** =1.7
				(Credible interval) (width) (Bayes Estimate)	(Credible interval) (width) (Bayes Estimate)
25	1.5	2.5	7.8	(0.102227,1.930575) (1.828348)(0.1407223)	(0.17144,1.901145) (1.729705)(0.1400771)
	1.3	1.75	4.5	(0.140647,1.674105) (1.533458)(0.2143965)	(0.16113,1.693517) (1.532387)(0.2149375)
	1.2	1.6	3.5	(0.166116,1.744702) (1.578586)(0.2412356)	(0.147876,1.842412) (1.694536)(0.239996)
50	1.5	2.5	7.8	(0.113269,1.459546) (1.346277)(0.1409815)	(0.154915,1.654938) (1.500023)(0.1407081)
	1.3	1.75	4.5	(0.10900,1.310285) (1.201285)(0.2151987)	(0.174812,1.450959) (1.276147)(0.2150537)
	1.2	1.6	3.5	(0.131304,1.105901) (0.974597)(0.2417637)	(0.135988,1.437119) (1.301131)(0.2415304)
100	1.5	2.5	7.8	(0.176421,1.111614) (0.935193)(0.1410336)	(0.13556,1.147504) (1.011944)(0.140881)
	1.3	1.75	4.5	(0.157634,1.133661) (0.976027)(0.2147873)	(0.139321,1.037455) (0.898134)(0.2147116)
	1.2	1.6	3.5	(0.103214,1.057451) (0.954237)(0.2419366)	(0.035624,1.098726) (1.063102)(0.240987)

Table 8. 95% HPD Credible Intervals, width of the interval and Bayesian estimates for Bonferroni Index under the extension of Jeffreys' Prior

n	b	a	p	m =1	m =1.7
				(Credible interval) (width) (Bayes Estimate)	(Credible interval) (width) (Bayes Estimate)
25	1.5	2.5	7.8	(0.115489,1.799312) (1.683823)(0.1773219)	(0.109144,1.86549) (1.756346)(0.1768241)
	1.3	1.75	4.5	(0.180611,1.841856) (1.661245)(0.2530068)	(0.150197,1.699720) (1.549523)(0.2536889)
	1.2	1.6	3.5	(0.13116,1.971017) (1.839857)(0.1798252)	(0.132148,1.360320) (1.228172)(0.1810069)
50	1.5	2.5	7.8	(0.113269,1.272277) (1.159008)(0.1773576)	(0.134027,1.590231) (1.456204)(0.1773717)
	1.3	1.75	4.5	(0.150900,1.479691) (1.328791)(0.2542884)	(0.168629,1.265633) (1.097004)(0.2544995)
	1.2	1.6	3.5	(0.113309,1.698651) (1.585342)(0.1830509)	(0.145988,1.243351) (1.097363)(0.1829449)
100	1.5	2.5	7.8	(0.169611,1.029145) (0.859534)(0.1780651)	(0.16900,1.080353) (0.911353)(0.1772798)
	1.3	1.75	4.5	(0.163091,1.154255) (0.991164)(0.2543103)	(0.122073,1.105629) (0.983556)(0.2547158)
	1.2	1.6	3.5	(0.136478,1.175373) (1.038895)(0.1842456)	(0.118012,1.006210) (0.888198)(0.1832878)

Comment: One can further infer that as sample sizes increases, the width of the credible interval decreases for 95% credible intervals for both Mukherjee-Islam prior and the extension of Jeffreys' prior. The width of HPD is smaller in the case of Mukherjee-Islam prior.

6. For illustration, two real data sets are taken up in this section

Example 1. A real data is considered for the illustration of the proposed study. This data (Daren et al. (2014)) set represents the degree of reading power (DRP) scores for a sample of 30 third grade students.

40, 26, 39, 14, 42, 18, 25, 43, 46, 27, 19, 47, 19, 26, 35, 34, 15, 44, 40, 38, 31, 46, 52, 25, 35, 35, 33, 29, 34, 41. By using easy fit software, it is seen that data fit well to the Dagum distribution and p-value for the Kolmogorov-Smirnov test is 0.87284 at 5% level of significance. The value of shape parameters and scale parameter $p = 0.14, a = 18.5,\ b = 45.7$ are obtained using easy fit software and the Bayes estimates are obtained along with HPD credible intervals for the Gini and Bonferroni Index using both Mukherjee and the extension of Jeffrey' Priors. The results have been presented in the table below (Table 9).

Table 9: Bayesian estimates along with 95% HPD Credible Intervals under LINEX loss function and Estimated loss under LINEX and SELF (in parenthesis) for Gini index and Bonferroni index under Mukherjee-Islam prior and the extension of Jeffrey's Prior

Priors		$\hat{G}_{ML}$	$\hat{B}_{ML}$
Mukherjee-Islam prior	$\alpha = 2.2$	0.11480 (0.04757,0.06558)	0.06611 (0.0026,0.00277)
	95% HPD Credible Intervals (width)	(0.07145,1.54261) (1.47115)	(0.00483,0.01642) (0.01159)
Extension of Jeffrey's prior	m =1	0.12146 (0.08620,0.1982)	0.06919 (0.07093,0.14253)
	95% HPD Credible Intervals (width)	(0.0531,1.8642) (1.8111)	(0.0015,0.9631) (0.9616)

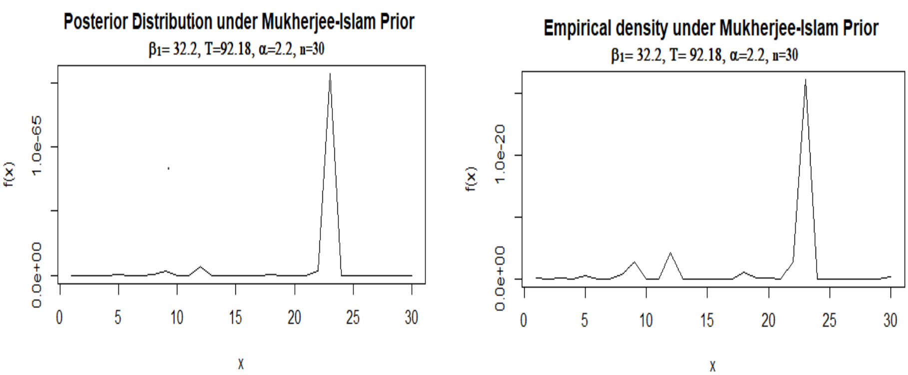

Figure 2. Comparison of Posterior density with Empirical density under Mukherjee-Islam Prior

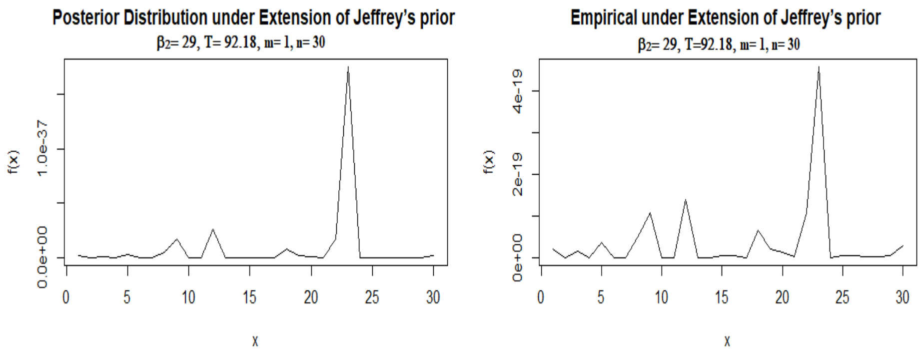

Figure 3. Comparison of Posterior density with Empirical density under Extension of Jeffreys's Prior

From the above findings of graph, we can see the posterior density and empirical density under Mukherjee-Islam prior and the extension of Jeffreys' prior are nearly the same.

Example 2. The data (Sanku et al. (2017)) set consists of 30 observations on breaking stress of carbon fibres (in Gba). The data are: 3.7, 2.74, 2.73, 3.11, 3.27, 2.87, 4.42, 2.41, 3.19, 3.28, 3.09, 1.87, 3.75, 2.43, 2.95, 2.96, 2.3, 2.67, 3.39, 2.81, 4.2, 3.31, 3.31, 2.85, 3.15, 2.35, 2.55, 2.81, 2.77, 2.17. By using easy fit software, it is seen that data fit well to the Dagum distribution and the p-value for the Kolmogorov-Smirnov test is 0.99668 at 5% level of significance. The values of shape parameters and scale parameter $p = 0.97, a = 9.7, b = 2.9$ are obtained using easy fit software and the Bayes estimates are obtained along with HPD credible intervals for the Gini and Bonferroni Index using both Mukherjee and Uniform Prior. The results have been presented in the table below (Table 10).

Table 10. Bayesian estimates along with 95% HPD Credible Intervals under LINEX loss function and Estimated loss under LINEX and SELF (in parenthesis) for Gini index and Bonferroni index.

Priors		$\widehat{\mathbf{G}}_{\mathbf{ML}}$	$\widehat{\mathbf{B}}_{\mathbf{ML}}$
Mukherjee-Islam prior	α = 1.2	0.02634	0.01152
		(0.03926,0.07219)	(0.0011,0.00161)
	95% HPD Credible Intervals (width)	(0.00926,1.12018) (1.11092)	(0.00916,0.01849) (0.00933)
Extension of Jeffrey's prior	m =1	0.18001	0.04629
		(0.04165,0.21013)	(0.06282,0.09932)
	95% HPD Credible Intervals (width)	(0.01406,1.50674) (1.49268)	(0.00132,1.73965) (1.73833)

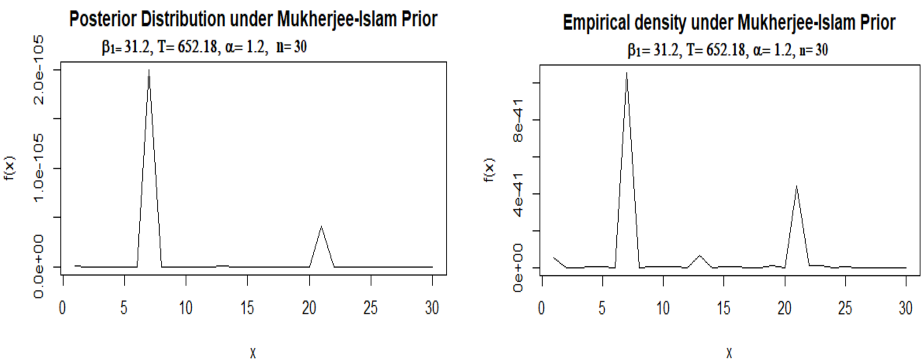

Figure 4. Comparison of Posterior density with Empirical density under Mukherjee-Islam Prior.

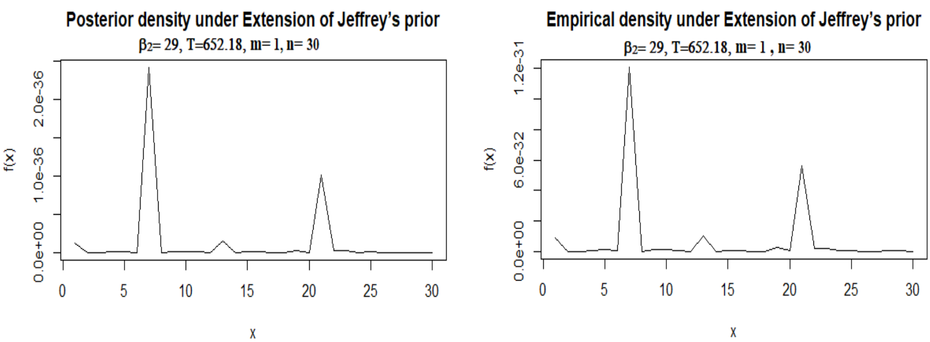

Figure 5. Comparison of Posterior density with Empirical density under Extension of Jeffrey's Prior.

From the above findings of the graph, we can see the posterior density and empirical density under Mukherjee-Islam prior and the extension of Jeffrey's prior are nearly the same.

As seen above, the findings from real life examples are in accordance with those of the simulation study. One can see that in the case of the real data set also Mukherjee-Islam prior results in smaller estimated loss in comparison with the extension of Jeffrey's prior. Even the width of HPD credible interval is smaller in the case of Mukherjee prior. The estimated loss is also smaller in the case of LINEX than SELF irrespective of the prior being used. The findings from the real life example are in accordance with those of the simulation study.

7. Conclusion

Bayes estimates of two inequality indices are obtained in the case of the Dagum distribution, an important income distribution. As seen from the simulation study it is observed that Mukherjee-Islam prior performs better than the extension of Jeffrey's prior in terms of having smaller estimated loss. It is also observed that the LINEX loss function results in smaller loss as compared to SELF for small, medium and large sample sizes irrespective of the choice of prior. One can further see that the expected loss decreases as the sample size increases. The real data set is also in conformity with above results.

Acknowledgement

The authors acknowledge with thanks the valuable suggestions/comments made by the anonymous reviewers, which has led to a considerable improvement of the manuscript.

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