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Metric generalizations of Banach algebras

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PRELIMINARIES*

§ 0. Introduction

Together with generalizations of Banach spaces one may consider corresponding generalizations of Banach algebras. This paper is devoted mainly to two metric generalizations of these algebras: locally bounded complete algebras and completely metrizable locally convex algebras. The class of Banach algebras is precisely intersection of these two classes as it follows from the theorem of Kolmogoroff stating that a topological linear space is a normed space if and only if it is both locally bounded and locally convex.

The paper covers material of lectures given by the author at Yale University in second term of 1963/64, it contains the earlier results of the author on above topics as well as some new ones. For sake of completeness there are given also some results of other authors: § 10 and in part § 11 are adopted from the paper of Michael [15], also some single theorems and examples are taken from the papers of Arens, Banach, Mitiagin, Rolewicz and Williamson.

Chapter I is devoted to the theory of locally bounded complete algebras. The essential results of this chapter were published by the author in papers [28]-[31]. Theorem 2.3 (due to A. Pełczyński) states that a complete locally bounded algebra is a p-normed algebra, i.e. its topology may be given by means of a p-norm, $0 < p \leqslant 1$, satisfying $\|xy\| \leqslant \|x\| \|y\|$, $\|tx\| = |t|^p \|x\|$, where x, y are elements of the algebra in question, t a scalar. As it may be seen through this chapter, all basic facts true for Banach algebras are also true for p-normed algebras. So the theory of wider class of p-normed algebras is the same as the theory of its subclass of Banach algebras. This follows that in the theory of Banach algebras essential is the fact that they are locally bounded spaces and the fact that they are locally convex is meaningless.

In the short chapter II are discussed some basic properties of complete metric algebras (F-algebras) to which is devoted § 7, and some topics on topological division algebras (§ 8). Among new results of this

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chapter we mention theorem 7.3, generalizing our earlier result on p-normed algebras ([28], theorem 2) on the possibility of completion of a metric algebra, and the concept of generalized topological divisors of zero together with theorem stating that a topological division algebra over real scalars either possesses such divisors or is homeomorphically isomorphic with either of three trivial division algebras (reals, complexes or quaternions). The last statement is an excuse for replacing this chapter after the first one, since in its proof we apply some results on p-normed algebras. The chapter contains also Arens' theorem on joint continuity of multiplication in F-algebras, and theorem of Banach stating that the operation of taking inverse in an F-algebra is continuous if and only if the set of all invertible elements of the algebra in question is a G_δ-set. The proof presented here is a combination of the original proof of Banach [5] and that of the author (lemma 7.5) published in [32], with a refinement made by Gleason when reviewing [32] (MR 23A (1962) # A3198). Proposition 8.5 is the generalization of a theorem of Arens [2].

The greatest portion of material and most of the new results are contained in chapter III devoted to completely metrizable locally convex algebras (B_0-algebras). In § 9 we give some basic preliminaries due to the author (cf. [28] and [32]). Next two sections (§ 10 and § 11) are devoted to multiplicatively-convex (m-convex) B_0-algebras. Theory of m-convex locally convex was created by Arens [3] and Michael [15]. The results of § 10 and partially of § 11 (proposition 11.2, theorem 11.8, and proposition 11.12) are adopted from [15]. Theorem 11.4 seems to be new. New is also concept of extension property (definition 11.10) and theorem 11.14 stating that m-convex B_0-algebra has extension property if and only if it is not a Q-algebra. Example 11.7 is due to Rolewicz [23]. In § 12 we discuss three examples of non-m-convex B_0-algebras. They are due to Arens [1] (example 12.1), to the author [28] (example 12.2), and to Williamson [26] (example 12.3). In § 13 there is given a characterization of commutative m-convex B_0-algebras as these algebras in which operate all entire functions of one variable. Theorem 13.8, giving this characterization, was published in the joint paper of Mitiagin, Rolewicz, and the author [17]. Other results of this section, as well as of remaining sections are new. Among them we mention the concept of extended spectrum (definition 13.1, proposition 13.2, theorem 13.5), and some applications of the characterization of m-convexity, as theorem 13.17 stating that any commutative B_0-algebra which is a Q-algebra must be necessarily multiplicatively-convex, and theorem 13.18 stating that any commutative radical algebra of type B_0 must be m-convex. In § 14 to a B_0-algebra A is attached an algebra $\mathscr{E}(A)$ of all entire functions operating in A and given some elementary properties of $\mathscr{E}(A)$. $\mathscr{E}(A)$ is called *trivial* if it consists only of polynomials. There is given following characterization of commutative B_0-algebras having non-trivial $\mathscr{E}(A)$: $\mathscr{E}(A)$ is non-trivial if and only if in A there exists an equivalent system of pseudonorms $\|x\|_i$ satisfying

$$\|x_1 x_2 \dots x_n\|_i \leqslant C_{n,i} \|x_1\|_{i+1} \|x_2\|_{i+1} \dots \|x_n\|_{i+1}$$

for arbitrary $x_1, \dots, x_n \in A$, where $C_{n,i}$ are positive constants. There is also given a suitable characterization of algebras with trivial $\mathscr{E}(A)$. In § 15 by means of entire operations there are given solutions of two problems stated in [17]: it is proved that entire functions are continuous mappings, and that superposition of two entire functions operating in a B_0-algebra again operates in the same algebra. The method used here is similar to that of the book of Hille and Philips [9], and was suggested to the author by S. Mazur, who also noticed that the use of more general concept of entire operations simplifies the problem. The proof of theorem 15.3 is the reproduction in a slightly simplified form of the proof communicated to the author by Professor S. Mazur. In § 16 are posed some unsolved problems (some problems were stated also through the paper), and proved some facts concerning these problems.

§ 1. Definitions and notation

1.1. DEFINITION. A *topological algebra* is a topological linear space equipped with an associative separately continuous multiplication xy satisfying $(\lambda x)(\mu y) = \lambda\mu xy$, where λ, μ are scalars and x, y — elements of an algebra. As the field of scalars we assume the field of complex numbers, only in a few places we shall consider algebras over field of real numbers.

1.2. DEFINITION. If q is any class of topological linear spaces, then by *q-algebra* we shall mean a topological algebra, which underlying linear topological space belongs to the class q. Applying a standard technique of taking direct products, we may imbed any topological algebra into an algebra with unit. *So, through this paper, we shall assume that any considered algebra possesses the unit element denoted by e.* Since usually considered classes of topological linear spaces possess the property that the direct product of a space with the complex plane again belongs to the class in question, in the most interesting cases algebras obtained from q-algebras by the process of adjoining unit are also q-algebras.

Taking as q the class of Banach spaces we get as q-algebras the Banach algebras, if q is the class of locally bounded or the class of locally convex spaces, we get locally bounded algebras or locally convex algebras, etc.

CHAPTER I

LOCALLY BOUNDED ALGEBRAS

§ 2. Basic facts and examples

2.1. DEFINITION. Let X be a topological linear space. A subset $Y \subset X$ is said to be *bounded* if for any neighbourhood U of zero element in X there is a scalar λ such that $\lambda Y \subset U$ or, what is equivalent, if for any sequence (x_n) of elements of Y, and any sequence of scalars (λ_n), with $\lim \lambda_n = 0$ the sequence of elements $(\lambda_n x_n)$ tends to zero in X.

2.2. DEFINITION. A topological linear space X is called *locally bounded* if there exists a bounded neighbourhood U of zero. In this case the family $\left\{\frac{1}{n} U\right\}$, $n = 1, 2, \ldots$, forms a basis of neighbourhoods of zero in X, and consequently X is metrizable. It is known (see [21] or [10], p. 162-166) that a topological linear space is locally bounded if and only if its topology may be given by means of a p-homogeneous norm $\|x\|$, $0 < p \leqslant 1$, i.e. functional satisfying

(2.2.1) $\quad \|x\| \geqslant 0, \quad$ and $\quad \|x\| = 0 \quad$ if and only if $\quad x = 0$.

(2.2.2) $$\|x+y\| \leqslant \|x\| + \|y\|.$$

(2.2.3) $$\|\lambda x\| = |\lambda|^p \|x\|, \quad 0 < p \leqslant 1 \ (p \text{ is fixed}).$$

So a complete locally bounded space is a special case of an F-space (cf. definition 7.1) in which the topology is given by means of a norm satisfying (2.2.1), (2.2.2), and such that $\|\lambda x\|$ is a function continuous in two variables λ and x.

The following theorem characterizes the complete locally bounded algebras.

2.3. THEOREM. *Let A be an F-algebra; then the following conditions are equivalent:*

(2.3.1) *There is an A in equivalent metric $\varrho(x, y)$ giving its topology and satisfying*

$$\varrho(xy, 0) \leqslant \varrho(x, 0)\varrho(y, 0).$$

(2.3.2) *A is locally bounded.*

(2.3.3) *The topology in A may be given by the means of a p-homogeneous $(0 < p \leqslant 1)$ norm satisfying*

$$\|xy\| \leqslant \|x\|\,\|y\| \quad \textit{and} \quad \|e\| = 1.$$

Proof. (2.3.1) $\to$ (2.3.2). It is to be shown that under assumption (2.3.1) there is in A a bounded open set. So we shall show that the unit ball

$$K = \{x : \varrho(x, 0) < 1\}$$

is a bounded subset of A. Indeed, let (λ_n) be a sequence of scalars tending to zero, and $x_n \in K$. We have to show that $\lambda_n x_n$ tends to zero in A but it follows from the inequality

$$\varrho(\lambda_n x_n, 0) = \varrho(x_n e \cdot x_n, 0) \leqslant \varrho(\lambda_n e, 0)\varrho(x_n, 0) \leqslant \varrho(x_n e, 0),$$

and from the fact that $\lambda_n e$ tends to zero in A. So we have proved (2.3.2).

Now suppose that A is locally bounded, so there exists an equivalent p-homogeneous norm $\|x\|'$. We put now

$$\|x\| = \sup_{y \neq 0} \frac{\|xy\|'}{\|y\|'}. \tag{2.3.4}$$

As it may be easily verified it is a submultiplicative and p-homogeneous norm for which $\|e\| = 1$. The proof that $\|x\|$ is equivalent with $\|x\|'$ is the same as in the Banach algebra case. We have $\|x\| \geqslant \|x\|'\|e\|'$, so it is sufficient to show that A is complete in the norm $\|x\|$ (cf. [4], p. 41). We may interpret A as a subalgebra of the algebra of all linear operators $A \to A$, interpreting any $x \in A$ as operator L_x: $L_x y = xy$. The norm $\|x\|$ is operator norm of L_x. Let $L_{x_n} \to L$ in $\|\ \|$. It is to be shown that L is of the form L_x. But for any $x, y \in A$ we have $L(xy) = \lim L_{x_n}(xy) = \lim L_{x_n}(x)\cdot y = L(x)\cdot y$. Setting $x = e$ we have $L(y) = L(e)\cdot y$ and L is of desired form. It shows that A is complete in $\|\ \|$, and $\|\ \|$ is equivalent with $\|\ \|'$. So (2.3.2) implies (2.3.3). That (2.3.3) $\to$ (2.3.1), it is evident. Thus the theorem is proved.

2.4. Remark. The general assumption that considered algebras possess unit elements is here essential. Indeed, taking any completely metrizable non-locally bounded space X, and setting $x \cdot y = 0$ for any $x, y \in X$ we get an F-algebra satisfying (2.3.1) but not satisfying (2.3.2).

2.5. Corollary. *Every complete locally bounded algebra is algebraically and topologically isomorphic with a closed subalgebra of the algebra of all linear bounded operators of a locally bounded space into itself.*

As an equivalent term with a "complete locally bounded algebra" we shall use the term "p-normed algebra".

We shall now give some examples of p-normed algebras.

2.6. Example. The algebra l_p, $0 < p \leqslant 1$, of all two-sided sequences $x = (x_n)_{n=-\infty}^{+\infty}$ of complex numbers satisfying

$$\|x\| = \sum_n |x_n|^p < \infty,$$

with the multiplication defined as convolution.

2.7. Example. The algebra A_p, $0 < p \leqslant 1$, of all holomorphic functions in the unit disc

$$x(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n$$

such that

$$\|x\| = \sum_n |x_n|^p < \infty,$$

with the pointwise multiplication.

2.8. Example. The algebra $B(X)$ of all bounded linear operators of a locally bounded space X into itself.

§ 3. Commutative p-normed algebras, spectral norm and p-normed field

In this section we assume A to be a commutative p-normed algebra over complex scalars equipped with the norm satisfying (2.3.3).

3.1. Definition. Let

$$K_s = \{x \in A : \lim \|x^n\| = 0\};$$

the *spectral norm* $\|x\|_s$ of A is defined as a p-homogeneous norm having K_s as the unit ball. Namely we put

$$\|x\|_s = (\sup\{|\lambda| : \lambda x \in K_s\})^{-p}.$$

To simplify our considerations we shall prove the following

3.2. Lemma. *$x \in K_s$ if and only if there exists an integer n_0 such that $\|x^{n_0}\| < 1$.*

Proof. In fact, if $x \in K_s$, then $\|x^n\| \to 0$ and such an integer exists. If $\|x^{n_0}\| < 1$, then $\|x^{kn_0}\| \leqslant \|x^{n_0}\|^k \to 0$, and setting $M = \max_{l<n_0} \|x^l\|$ we have for any integer $m = kn_0 + l$, $l < n_0$,

$$\|x^m\| = \|x^{kn_0+l}\| \leqslant M \|x^{kn_0}\| \to 0, \qquad \text{q.e.d.}$$

We shall prove the essential

3.3. THEOREM. *The spectral norm* $\|x\|_s$ *has the following properties.*

(3.3.1) $$\|x\|_s < 1 \text{ if and only if } x \in K_s.$$

(3.3.2) $$\|\lambda x\|_s = |\lambda|^p \|x\|_s.$$

(3.3.3) $$\|x+y\|_s \leqslant \|x\|_s + \|y\|_s.$$

(3.3.4) $$\|xy\|_s \leqslant \|x\|_s \|y\|_s.$$

(3.3.5) $$\|x^n\|_s = \|x\|_s^n.$$

(3.3.6) $$\|x\|_s \leqslant \|x\|.$$

(3.3.7) $$\text{If } x^{-1} \in A, \text{ then } \|x\|_s > 0.$$

(3.3.8) $$\|x\|_s = \lim \sqrt[n]{\|x^n\|}.$$

Proof. Ad (3.3.1). It follows immediately from definition (3.1), and from the fact that K_s is open; (3.3.2) is also obvious.

Ad (3.3.3). In virtue of (3.3.2), $\|x\|_s$ is p-homogeneous, so it is sufficient to show that $\|x\|_s + \|y\|_s < 1$ implies $\|x+y\|_s < 1$. Let $\|x\|_s = \alpha$, $\|y\|_s = \beta$; it is $\alpha+\beta < 1$. We choose $\bar{\alpha}$ and $\bar{\beta}$ in such a way that $\bar{\alpha} > \alpha$, $\bar{\beta} > \beta$, and $\bar{\alpha}+\bar{\beta} < 1$, and define $\bar{x}$ and $\bar{y}$ by the relations

$$x = \bar{\alpha}^{1/p}\bar{x}, \qquad y = \bar{\beta}^{1/p}\bar{y},$$

so $\|\bar{x}\|_s < 1$ and $\|\bar{y}\|_s < 1$. In presence of lemma 3.2 it is sufficient to find an integer N such that

(3.3.9) $$\|(x+y)^N\| < 1.$$

For this purpose observe that

(3.3.10) $$\|(x+y)^n\| = \|(\bar{\alpha}^{1/p}\bar{x}+\bar{\beta}^{1/p}\bar{y})^n\| \leqslant \sum_{k=0}^{n} \binom{n}{k}^p \alpha^{n-k}\beta^k \|\bar{x}^{n-k}\bar{y}^k\|.$$

Since from (3.3.1) it follows that $\|\bar{x}\|^n \to 0$ and $\|\bar{y}^n\| \to 0$, we have $a = \max_n \|\bar{x}^n\| < \infty$ and $b = \max_n \|\bar{y}^n\| < \infty$, and we can choose m_0 and n_0 such that

$$\|\bar{x}^n\| < \frac{1}{b} \quad \text{for} \quad n > m_0,$$

$$\|\bar{y}^n\| < \frac{1}{a} \quad \text{for} \quad n > n_0.$$

Thus for $N = m_0 + n_0$ we have

(3.3.11) $$\|\bar{x}^{N-k}\bar{y}^k\| < 1 \quad \text{for} \quad k = 0, 1, \ldots, N$$

Consequently, since $\binom{N}{k}^p \leqslant \binom{N}{k}$ for $0 < p \leqslant 1$, we get from (3.3.10) that

$$\|(x+y)^N\| \leqslant \sum \binom{N}{k} \alpha^{N-k}\beta^k = (\alpha+\beta)^N < 1,$$

and (3.3.9) holds.

Ad (3.3.4). In view of p-homogeneity of $\|x\|_s$ it is sufficient to show that $\|x\|_s < 1$ and $\|y\|_s < 1$ imply $\|xy\|_s < 1$, but it follows from the fact that $x^n \to 0$ and $y^n \to 0$ imply $(xy)^n \to 0$.

Ad (3.3.5). Again in view of p-homogeneity of $\|x\|_s$ it is sufficient to show that $\|x^n\|_s < 1$ is equivalent with $\|x\|_s < 1$, but both these relations, by lemma 3.2, are equivalent with $\|x^N\| < 1$ for some N.

(3.3.6) follows immediately from the definition; (3.3.7) follows from (3.3.4) and from the fact that $\|e\|_s = 1$.

Ad (3.3.8). The existence of the limit $\lim \sqrt[n]{\|x^n\|}$ follows from the general fact that for the sequence of reals a_n satisfying $a_{n+k} \leqslant a_n a_k$ it always exists $\lim \sqrt[n]{a_n}$. So put $\|x\|^* = \lim \sqrt[n]{\|x^n\|}$; it is a p-homogeneous functional defined on A. If $\|x\|^* < 1$, then $\|x^n\| < 1$ for large n, and so, by (3.3.1) and lemma 3.2, $\|x\|_s < 1$. If $\|x\|^* > 1$, then $\|x^n\| > 1$ for large n, and so, by lemma 3.2 and (3.3.1), $\|x\|_s > 1$, this follows $\|x\|^* = \|x\|_s$, q.e.d.

Now we pass to the proof of a theorem on p-normed fields. The proof here presented is a direct proof. Later on (in § 4) we shall give another proof reducing the problem to the normed algebra case.

3.4. LEMMA. *The set V of all invertible elements in A is an open set.*

3.5. LEMMA. *The operation $x \to x^{-1}$ is continuous on V.*

The proofs are the same as in the Banach algebra case.

3.6. LEMMA. *If A is a p-normed field, and A_0 is a closed subalgebra of A, then A_0 is also a subfield of A.*

Proof. Let $x \in A_0$, $x \neq 0$. We have to show that $x^{-1} \in A_0$. Observe that if $x_n \to x_0 \neq 0$, and if $x_n^{-1} \in A_0$, then by lemma 3.5 and completeness of A_0 it is $x_0^{-1} \in A_0$. If $x = \alpha e$, $\alpha \neq 0$, then $x^{-1} = \alpha^{-1} e \in A_0$, so we may assume that x is not of the form λe, and, for any complex λ, $(\lambda x - e)^{-1}$ exists in A. We put

$$\Lambda = \{\lambda : (\lambda x - e)^{-1} \in A_0\};$$

so, by our observation, Λ is a closed subset of complex plane. On the other hand, by lemma 3.4 it is an open set. Since $0 \in \Lambda$, it is non void and hence it is whole complex plane. We thus have

$$\left(x - \frac{1}{n} e\right)^{-1} \in A_0,$$

and again by our observation $x^{-1} \in A_0$, q.e.d.

3.7. LEMMA. *If A is a p-normed field, then for every complex $\lambda \neq 0$, every $\varepsilon > 0$, and every $x \in A$, x not of the form λe, there exists a polynomial $W(\lambda)$ with complex coefficients such that*

$$\|\lambda e - [e + xW(x)]^{-1}\| < \varepsilon. \tag{3.7.1}$$

Proof. Let $x \neq \lambda e$. Consider the closed subalgebra $A(x)$ of A generated by x, i.e. the closure in A of the algebra of all polynomials with complex coefficients of x. By lemma 3.6 it is a subfield of A, so $x^{-1} \in A(x)$. Therefore, there exists a sequence of polynomials $W_n(x)$ tending to $(\lambda^{-1}-1)x^{-1}$; hence $xW_n(x)+e \to \lambda^{-1}e$ and, by lemma 3.5, $(xW_n(x)+e)^{-1} \to \lambda e$, so (3.7.1) holds for large n, q.e.d.

3.8. THEOREM. *A p-normed field A is isomorphic and homeomorphic with the complex number field.*

Proof. Let $x \in A$. It is to be shown that for certain complex λ we have $x = \lambda e$. Suppose then that $x \neq \lambda e$ for every λ and try to get a contradiction. We put

$$f(\lambda) = \|(x^{-1}+\lambda e)^{-1}\|_s;$$

this is a continuous function of complex argument λ. Moreover,

$$\lim_{|\lambda| \to \infty} f(\lambda) = \lim |\lambda|^{-p} \|(\lambda^{-1}x^{-1}+e)^{-1}\|_s = 0,$$

so there exists a λ_0 such that $f(\lambda_0) \geqslant f(\lambda)$ for each λ. We put

$$y = f(\lambda_0)^{-1/p}(x^{-1}+\lambda_0 e)^{-1},$$

so we have $\|y\|_s = 1$, and

$$\|(y^{-1}+\lambda e)^{-1}\| \leqslant 1 \tag{3.8.1}$$

for every complex λ. We are going to show that there exists an n_0 such that $\|y^{n_0}\| < 1$, what by lemma 3.2 and (3.3.5) would be the contradiction completing the proof.

Let $V_n(\lambda)$ be an arbitrary polynomial of the form

$$V_n(\lambda) = \lambda^n + a_1\lambda^{n-1} + \quad + \lambda_n$$

where a_i are complex coefficients. We have

$$V_n(y^{-1}) = (y^{-1}-\beta_1 e) \dots (y^{-1}-\beta_n e),$$

where β_i are roots of V_n, so by (3.8.1) we have

$$\|[V_n(y^{-1})]^{-1}\|_s \leqslant \|(y^{-1}-\beta_1 e)^{-1}\|_s \dots \|(y^{-1}-\beta_n e)^{-1}\|_s \leqslant 1. \tag{3.8.2}$$

On the other hand,

$$[V_n(y^{-1})]^{-1} = y^n[e+yW_n(y)]^{-1},$$

where

$$W_n(y) = a_1 + a_2 y + \dots + a_n y^{n-1}. \tag{3.8.3}$$

For a given $\varepsilon > 0$ and given complex λ we can choose a V_n such that for $W_n(y)$ given by (3.8.3) holds (3.7.1) with $x = y$.

Setting
$$z = \lambda e - [e - yW_n(y)]^{-1},$$
we have
$$\|z\|_s \leqslant \|z\| < \varepsilon.$$

It follows from (3.8.2) and (3.8.3) that
$$\|y^n(\lambda e - z)\|_s \leqslant 1,$$
consequently
$$1 \geqslant \|\lambda y^n - zy^n\|_s \geqslant |\lambda|^p \|y^n\|_s - \|z\|_s \|y^n\|_s \geqslant (|\lambda|^p - \varepsilon)\|y^n\|.$$
Hence
$$\|y^n\|_s < \frac{1}{|\lambda|^p - \varepsilon},$$
and by suitable choice of $|\lambda|$ and ε we can make $\|y^n\|_s < 1$ which is the announced contradiction, q.e.d.

In a similar way as in [2], proof of theorem 2, we may obtain the following generalization:

3.9. Theorem. *Let A be a p-normed division algebra over real numbers field. Then A is topologically isomorphic either with real numbers field or with complex numbers field, or with division algebra of real quaternions.*

In the next section we shall generalize this theorem onto non-complete p-normed division algebras.

§ 4. Commutative p-normed algebras (continued)

We shall list some applications of theorem 3.8 that are facts well known in the case of Banach algebras, and the proofs are actually the same.

Let A be a commutative complex p-normed algebra; the following propositions are easy corollaries to theorem 3.8:

4.1. Proposition. *Every ideal of A is contained in a maximal ideal. Every maximal ideal is closed and of codimension 1, so there is a 1-1 correspondence between multiplicative linear functionals and maximal ideals given by*
$$M = \{x \in A : f_M(x) = 0\},$$
$f_M(x) = \lambda$, if $x = m + \lambda e$, $m \in M$, and this decomposition follows from the fact that $A = M \oplus \{\lambda e\}$. Consequently each multiplicative and linear functional is continuous.

4.2. Proposition. *In A there exists at least one (non-zero) multiplicative and linear functional.*

4.3. Proposition. *If $\mathfrak{M}$ is the compact space of all maximal ideals of A (topologized exactly in the same way as in [7]), then there is a continuous homomorphism of A into algebra $C(\mathfrak{M})$ given by*

$$\Phi : x \to f_M(x) \overset{\text{def}}{=} x(M), \tag{4.3.1}$$

moreover

$$\sup_M |x(M)|^p \leqslant \|x\|_s \leqslant \|x\|, \tag{4.3.2}$$

the kernel of Φ is the radical of A.

(We shall see later that the left-hand inequality of (4.3.2) is actually an equality, so the radical of A is given by $\{x : \|x\|_s = 0\}$.)

4.4. Proposition. *An element $x \in A$ is invertible if and only if $x(M) \neq 0$, $M \in \mathfrak{M}$ or equivalently if and only if $f(x) \neq 0$ for each multiplicative linear functional f (for each $f \in \mathfrak{M}$ as we shall write for sake of brevity).*

Similarly as in the Banach algebra case we shall apply proposition 4.4 to trigonometrical series and prove the following generalization of Wiener's theorem:

4.5. Theorem. *Let $x(t)$ be a complex function of real variable t, $0 \leqslant t < 2\pi$, equal to its Fourier expansion*

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}; \tag{4.5.1}$$

then from the fact that $\sum |x_n|^p < \infty$ for a fixed p, $0 < p \leqslant 1$, and $x(t) \neq 0$, $0 \leqslant t < 2\pi$, it follows that

$$\frac{1}{x(t)} = \sum_n y_n e^{int},$$

where

$$\sum_n |y_n|^p < \infty.$$

Proof. Consider the algebra l_p (example 2.6). It may be interpreted as the algebra of functions of the form (4.5.1) with pointwise multiplication. We shall show that every multiplicative and linear functional in l_p is of the form

$$F(x) = F_{t_0}(x) = x(t_0), \qquad 0 \leqslant t_0 < 2\pi. \tag{4.5.2}$$

Put $z(t) = e^{it}$, it is an invertible element of l_p. Let F be a multiplicative linear functional in A; it is $|F(z)| = 1$. If $|F(z)| > 1$, then for suitable λ, with $|\lambda|<1$, it would be $|F(\lambda z)|=1$, which, by proposition 4.1, is impossible since $(\lambda z)^n=\lambda^n e^{int}$ tends to zero in A. Consequently $|F(z)|\leqslant 1$. Applying the same arguments to $z^{-1}=e^{-it}$, we get $|F(z^{-1})|\leqslant 1$ or $|F(z)|\geqslant 1$, which gives $|F(z)|=1$ and $F(z)=e^{it_0}$ $=z(t_0)$. Thus formula (4.5.2) holds for z, and consequently for any trigonometric polynomial

$$p(t)=\sum_{n=-K}^{s} p_n e^{int};$$

this follows that it holds for any $x\in A$ since polynomials of this form are dense in A. So the conclusion follows now from proposition 4.4.

We are able now to give the following generalization of theorem 3.8:

4.6. THEOREM. *Let A be a locally bounded space (not necessarily complete) over complexes, which is an algebra with separately continuous multiplication. Suppose that for any $x\neq 0$ there exists an inverse x^{-1}. Then A is topologically isomorphic with complex numbers field.*

Proof. It is to be shown that for any $x\in A$ it is $x=\lambda e$. Consider the smallest division subalgebra of A containing x. It is a subfield, and we denote it by $A(x)$. Let $\|x\|'$ be a p-homogeneous norm giving the topology in $A(x)$. The norm $\|x\|$ given by formula (2.3.4) is a submultiplicative norm in $A(x)$ (it must not be equivalent with $\|x\|'$). Let $\tilde{A}$ be the completion of $A(x)$ in the norm $\|\ \|$. This is a commutative p-normed algebra, containing algebraically $A(x)$. Consequently, by proposition 4.2, there exists at least one non-zero multiplicative linear functional f. This functional is non-zero on $A(x)$ since $e\in A(x)$, and it gives an isomorphism of $A(x)$ with complex numbers field $x=\lambda e$, where $\lambda=f(x)$.

4.7. Remark. Using the technique of [2], we may get under the assumptions of theorem 4.6 (with reals instead of complex coefficients) the conclusion of theorem 3.9.

4.8. Remark. In proving theorems 3.8 or 4.6 we could not use the standard tools of Riemann integral or of linear functionals, because in non-locally convex spaces there always exist continuous functions defined on the compact interval $[0,1]$ which are not Riemann-integrable, and there are also situations (as in $L_p(0,1)$) in which there are no continuous linear functionals. So we are not sure of the existence of suitable integrals. However, as the author was informed by S. Rolewicz the integrals in question do exist and the theory may also be based upon their existence (cf. [34]).

We shall now pass to the characterization of radicals in commutative p-normed algebras. We recall that in the commutative case the radical $\operatorname{rad} A$ is the intersection of all maximal ideals, or $\{x\in A$, that for any $y\in A$ there exists $(e+xy)^{-1}\in A\}$.

4.9. Theorem. *Let A be a p-normed commutative algebra; then*

$$\operatorname{rad} A = \{x \in A : \|x\|_s = 0\} \tag{4.9.1}$$

and

$$\|x\|_s = \sup_{f \in \mathfrak{M}} |f(x)|^p. \tag{4.9.2}$$

Proof. As stated in proposition 4.3, a radical is the kernel of embedding (4.3.1); therefore,

$$\operatorname{rad} A = \{x \in A : \sup_{f \in \mathfrak{M}} |f(x)| = 0\},$$

so to prove theorem 4.9 it suffices to prove formula (4.9.2). The proof will be based upon the following

4.10. Lemma. *The closure $\bar{K}_s$ of unit sphere K_s of spectral norm $\|x\|_s$* (see definition 3.1) *is a convex subset of A.*

Proof. To prove the convexity of a closed set $\bar{K}_s$ it is sufficient to prove that from $x, y \in \bar{K}_s$ it follows $\frac{1}{2}(x+y) \in \bar{K}_s$ or, what is the same, to prove that $\|x\|_s \leqslant 1$, $\|y\|_s \leqslant 1$ imply $\|\frac{1}{2}(x+y)\|_s \leqslant 1$. It may easily be verified that the last implication is equivalent with the following: $\|x\|_s < 1$, $\|y\|_s < 1$ imply $\|\frac{1}{2}(x+y)\|_s \leqslant 1$, so we shall prove it. We have

$$\|\tfrac{1}{2}(x+y)\|_s = \frac{1}{2^p} \lim \sqrt[n]{\|(x+y)^n\|} \leqslant \frac{1}{2^p} \lim \sqrt[n]{\sum_k \binom{n}{k}^p \|x^k y^{n-k}\|}.$$

Now by the same arguments as in proof of (3.3.3), formula (3.3.11), we can show that $\|x^k y^{n-k}\| < 1$ for large n and $k = 1, 2, \ldots, n$. It follows that

$$\|\tfrac{1}{2}(x+y)_s\| \leqslant \frac{1}{2^p} \lim_n \sqrt[2n]{\sum_k \binom{2n}{k}^p}$$
$$\leqslant \frac{1}{2^p} \lim \sqrt[2n]{(2n+1)\binom{2n}{n}^p} = \frac{1}{2^p} \lim \sqrt[2n]{\binom{2n}{n}^p} = 1.$$

As a corollary we get the following lemma:

4.11. Lemma. *The functional*

$$\|x\|^* = \|x\|_s^{1/p} \tag{4.11.1}$$

is a homogeneous continuous submultiplicative pseudonorm in A.

Now we pass to the proof of theorem 4.9.

By proposition 4.3, formula (4.3.2), we have

$$\sup_{f \in \mathfrak{M}} |f(x)|^p \leqslant \|x\|_s \quad \text{or} \quad \sup_{f \in \mathfrak{M}} |f(x)| \leqslant \|x\|^*$$

The set $I = \{x \in A : \|x\|^* = 0\}$ is a closed ideal of A, and every functional $f \in \mathfrak{M}$ is zero on I, so it is constant on the cosets of $A^* = A/I$, and we may define $\hat{f}$ on A^* by means of the formula $\hat{f}(X) = f(x)$, where $f \in \mathfrak{M}$, $x \in X \in A^*$ Moreover $\hat{f}$ is continuous with respect to $\|X\|^* = \inf_{x \in X} \|x\|^* = \|x\|^*$ for any $x \in X$, since $\|x\|^*$ is constant on cosets. It is also evident that

$$\sup_{\hat{f}} |\hat{f}(X)| = \sup_{f} |f(x)|$$

for any $X \in A^*$ and $x \in X$. But A^* equipped with the norm $\| \ \|^*$ is a normed algebra, so $\sup_{\hat{f}} |\hat{f}(X)| = \lim \sqrt[n]{\|X^n\|^*}$, and we have

$$\sup_{f} |f(x)| = \lim \sqrt[n]{\|X^n\|^*} = \lim \sqrt[n]{\inf_{x \in X} \|x^n\|^*} = \inf_{x \in X} \|x\|^* = \|x\|^*$$

which is equivalent with (4.9.2), q.e.d.

As a corollary we get another proof of theorem 3.8.

4.12. COROLLARY. *Any commutative p-normed division algebra is trivial (i.e. it is either real or complex numbers field).*

In fact, by (3.3.7), $\|x\|_s$ is a p-homogeneous norm there, so it is a normed field with the norm $\|x\|^*$ given by (4.11.1), and the conclusion follows from the theory of Banach algebras.

§ 5. Analytic functions in p-normed algebras

In this section we shall apply theorem 4.9 to the construction of analytic functions in a p-normed algebra.

5.1. DEFINITION. The *spectrum* of an element x of a topological algebra A is the subset of complex plane $\sigma(x) = \{\lambda : (x - \lambda e) \text{ is not invertible in } A\}$.

Since $\sigma(x)$ with respect to A is the same as $\sigma(x)$ with respect to maximal commutative subalgebra of A containing x, we shall limit ourselves in this section to commutative case.

Let A be a commutative p-normed algebra. By proposition 4.4 we have

$$\sigma(x) = \{f(x) : f \in \mathfrak{M}\},$$

where $\mathfrak{M}$ is the set of all multiplicative linear functionals (or maximal ideals; cf. proposition 4.1) of A, so $\sigma(x)$ is a compact subset of complex plane. In the case where A is a commutative Banach algebra there is known that if $\Phi(\lambda)$ is a holomorphic function defined in an open set $\mathcal{U}$ containing $\sigma(x)$, then there exists in A an element y such that

$$f(y) = \Phi(f(x)) \quad \text{for every} \quad f \in \mathfrak{M}. \tag{5.1.1}$$

For an abbreviation, we shall write also $y = \Phi(x)$, though y here is not given uniquely, and relation (5.1.1) holds also if we replace y by $y+r$, where $r \in \operatorname{rad} A$. We shall prove the same theorem in the case where A is a commutative p-normed algebra. We shall give a step-by-step construction of such a y, and the construction is based upon the following lemmas.

5.2. LEMMA. *Let $\Phi(\lambda)$ be a holomorphic function defined in an open subset $\mathcal{U}$ of complex plane, and let x be an element of a commutative p-normed algebra A such that*

$$\sigma(x) \subset K(\lambda_0, r) \subset \mathcal{U},$$

where $K(\lambda_0, r)$ is a disc with center λ_0 and radius r. Then there is a $y \in A$ such that (5.1.1) *holds.*

Proof. It may easily be verified that $\sigma(x-\lambda_0 e) \subset K(0, r)$. By formula (4.9.2),

$$\sup\{|\lambda|^p : \lambda \in \sigma(x-\lambda_0 e)\} = \lim \sqrt[n]{\|(x-\lambda_0 e)^n\|},$$

so for large n and suitable ϱ, $\max|\sigma(x)| < \varrho < r$,

$$\|(x-\lambda_0 e)^n\| < \varrho^{pn}.$$

But in $K(\lambda_0, r)$

$$\Phi(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda-\lambda_0)^n,$$

and $r \leqslant (\limsup \sqrt[n]{|a_n|})^{-1}$. It follows that the series $\sum a_n(x-\lambda_0 e)^n$ is absolutely convergent in A. In fact,

$$\sum \|a_n(x-\lambda_0 e)^n\| \leqslant \sum |a_n|^p \|(x-\lambda_0 e)^n\| \leqslant \sum |a_n|^p \varrho^{pn} \leqslant \sum \left(\frac{\varrho}{r}\right)^{pn} < \infty.$$

The estimations are true for large n but it does not change the desired convergence. Setting $y = \sum a_n(x-\lambda_0 e)^n$, we have

$$f(y) = f\Big(\sum a_n(x-\lambda_0 e)^n\Big) = \sum a_n\big(f(x)-\lambda_0\big)^n = \Phi\big(f(x)\big)$$

for any $f \in \mathfrak{M}$; hence (5.1.1) holds, q.e.d.

5.3. LEMMA. *If Φ is a holomorphic function defined on a simply connected open bounded subset $\mathcal{U}$ of complex plane, and if $\sigma(x) \subset \mathcal{U}$, then there is a $y \in A$ such that* (5.1.1) *holds.*

Proof. Let φ be a 1-1 conformal mapping of $\mathcal{U}$ onto $K(0, 1)$. Put $r = \max\{|\varphi(\sigma(x))|\}$. We have $r < 1$, so we put $\varepsilon = \frac{1}{5}(1-r)$, and define $\mathcal{U}_k = \varphi^{-1}\big(K(0, 1-\varepsilon k)\big)$ and $\Gamma_k = \varphi^{-1}\big(S(0, 1-\varepsilon k)\big)$, where $S(\lambda_0, r) = \{\lambda : |\lambda-\lambda_0| = r\}$, $k = 1, 2, \ldots, 5$. We have $\sigma(x) \subset \mathcal{U}_4$. Now by the theorem of Runge we can choose such a polynomial $p(\lambda)$ that $|p(\lambda)-\varphi(\lambda)| < \varepsilon$ for $\lambda \epsilon \mathscr{U}_1$. If $\lambda_0 \epsilon K(0, 1-2\varepsilon)$, then the equation $\varphi(\lambda) = \lambda_0$ has exactly one solution in $\mathscr{U}$; moreover we have

$$\min_{\lambda \epsilon \Gamma_1} |\varphi(\lambda)-\lambda_0| > \varepsilon$$

(because $|\varphi(\lambda)| = 1-\varepsilon$ on Γ_1 and $|\lambda_0| < 1-2\varepsilon$). So, by the theorem of Rouché, equation $p(\lambda) = \lambda_0$ has in $\mathscr{U}_1$ exactly one solution. We thus have

$$p\big(\sigma(x)\big) \subset K(0, 1-3\varepsilon) \subset p(\mathscr{U}_2),$$

and p is a 1-1 conformal mapping of $\mathscr{U}_2$ onto $p(\mathscr{U}_2)$. Now it may easily be verified that

$$p\big(\sigma(x)\big) = \sigma\big(p(x)\big),$$

and so we have

$$\sigma\big(p(x)\big) \subset K(0, 1-3\varepsilon) \subset p(\mathscr{U}_2),$$

and the function $\psi(\lambda) = \Phi\big(p^{-1}(\lambda)\big)$ is a holomorphic function defined on $p(\mathscr{U}_2)$. Thus, by lemma 5.2, we can define an element y such that $f(y) = \psi\Big(f\big(p(x)\big)\Big)$ for any $f \epsilon \mathfrak{M}$, but $\psi\Big(f\big(p(x)\big)\Big) = \psi\Big(p\big(f(x)\big)\Big) = \Phi\big(f(x)\big)$, and (5.1.1) holds, q.e.d.

5.4. LEMMA. *If $x \epsilon A$, $\sigma(x) \subset \mathscr{U}$, where $\mathscr{U}$ is a simply connected open subset of complex plane, having complement with non-void interior, then for any Φ holomorphic on $\mathscr{U}$ there exists a $y \epsilon A$ such that* (5.1.1) *holds.*

Proof. Let λ_0 be an interior point of complement of $\mathscr{U}$. The function

$$\psi(\lambda) = \Phi\left(\frac{1}{\lambda} + \lambda_0\right)$$

is a holomorphic function defined on the bounded, simply connected open set $\mathscr{V} = (\mathscr{U}-\lambda_0)^{-1}$. We have also $\sigma\big((x-\lambda_0 e)^{-1}\big) \subset \mathscr{V}$, and so by lemma 5.3 there is a $y \epsilon A$ that $f(y) = \psi\big(f[(x-\lambda_0 e)^{-1}]\big)$ for each $f \epsilon \mathfrak{M}$. But $\psi\big(f[(x-\lambda_0 e)^{-1}]\big) = \psi\Big(\big(f(x)-\lambda_0\big)^{-1}\Big) = \Phi\big(f(x)\big)$, and (5.1.1) holds, q.e.d.

5.5. LEMMA. *The conclusion of lemma* 5.4 *is also true in the case where $\mathscr{U}$ is an open, bounded and connected set, provided its complement consists of finite number of components each of them having interior points.*

Proof. Using the Cauchy integral formula we can write every holomorphic function Φ defined on $\mathscr{U}$ in the form

$$\Phi(\lambda) = \Phi_1(\lambda) + \quad + \Phi_n(\lambda),$$

where $\Phi_i(\lambda)$ is a holomorphic function defined on a simply connected set $\mathscr{U}_i$ being complement of a component of complement of $\mathscr{U}$, $\mathscr{U} = \bigcap \mathscr{U}_i$. Thus, by lemma 5.4, we can define $y_i \epsilon A$ such that (5.1.1) holds for x, y_i and Φ_i, and the desired element is $y = y_1 + \quad + y_n$, q.e.d.

5.6. LEMMA. *Let $\sigma(x) \subset \mathscr{U}_1 \cup \mathscr{U}_2$, where $\mathscr{U}_i$, $i = 1, 2$, is a connected, bounded, open set, and $\mathscr{U}_1 \cap \mathscr{U}_2 = \emptyset$. Assume also $\sigma(x) \cap \mathscr{U}_i \neq \emptyset$ $(i = 1, 2)$. Then there exists such a $z \in A$ that $f(z) = 0$ for any $f \in \mathfrak{M}$ such that $f(x) \in \mathscr{U}_1$, and $f(z) = 1$ for any $f \in \mathfrak{M}$ such that $f(x) \in \mathscr{U}_2$.*

Proof. Let α and β be two distinct complex numbers which lie in this component of the complement of $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$ which contains the boundary points of $\mathscr{U}_1$ and $\mathscr{U}_2$ (two possible situations are schematically given in Fig. 1).

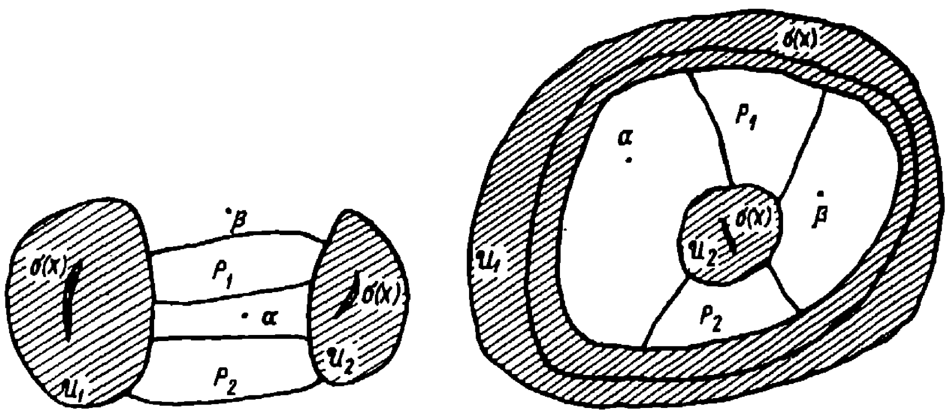

Fig. 1

Now we take two disjoint open sets P_1 and P_2 in such a way that $\alpha, \beta \notin S = \mathscr{U}_1 \cup P_1 \cup \mathscr{U}_2 \cup P_2$, S is connected, and α and β are not in the same component of complementation of S. We now take any branch $\varphi_1(\lambda)$ of

$$\log\left(\frac{1}{\alpha-\beta} - \frac{1}{\lambda-\beta}\right)$$

defined on $\mathscr{U}_1 \cup P_1 \cup \mathscr{U}_2$, put $\varphi_2(\lambda) = \varphi_1(\lambda)$ for $\lambda \in \mathscr{U}_1$ and extend it analytically onto P_2 and $\mathscr{U}_2$. We have

$$\varphi_1(\lambda) - \varphi_2(\lambda) = \begin{cases} 0 & \text{for} \quad \lambda \in \mathscr{U}_1 \\ \varepsilon 2\pi i & \text{for} \quad \lambda \in \mathscr{U}_2 \end{cases},$$

where $\varepsilon = -1$, or 1 The functions φ_1 and φ_2 are holomorphic on the connected open sets $\mathscr{U}_1 \cup P_1 \cup \mathscr{U}_2$ and $\mathscr{U}_1 \cup P_2 \cup \mathscr{U}_2$ containing $\sigma(x)$; thus, by lemma 5.5, there are elements x_1 and x_2 such that $f(x_1) = \varphi_1(f(x))$, and $f(x_2) = \varphi_2(f(x))$, $f \in \mathfrak{M}$. It is clear that the desired element z may be given by $z = (x_1 - x_2)/\varepsilon 2\pi i$, q.e.d.

5.7. THEOREM. *Let A be a commutative p-normed algebra. Let $\mathscr{U}$ be an open subset of complex plane containing the spectrum $\sigma(x)$ of an $x \in A$.*

Then for any function Φ, holomorphic in $\mathscr{U}$ there exists an element $y \in A$ such that (5.1.1) *holds. If A is semisimple, then such a y is unique.*

Proof. Since $\sigma(x)$ is compact, it may be covered by a finite number of discs $K(\lambda, r_\lambda)$ contained in $\mathscr{U}$. Thus we can assume that $\mathscr{U}$ is itself a union of a finite number of discs, so $\mathscr{U} = \mathscr{U}_1 \cup \quad \cup \mathscr{U}_n$, where $\mathscr{U}_i$ is a connected bounded set having complement with a finite number of components each of them having interior points (the last we may obtain by changing a little radii of discs what is possible without removing them out of $\mathscr{U}$, or failing to cover $\sigma(x)$), and $\mathscr{U}_i \cap \mathscr{U}_j = \emptyset$, $i \neq j$, and $\sigma(x) \cap \mathscr{U}_i \neq \emptyset$. Now by lemma 5.6 and easy induction we may construct elements $e_1, \ldots, e_n \in A$ in such a way that

$$e_1 + e_2 + \ldots + e_n = e,$$

and

$$f(e_i) = \begin{cases} 1 & \text{if} \quad f(x) \in \mathscr{U}_i, \\ 0 & \text{if} \quad f(x) \notin \mathscr{U}_i, \end{cases} \qquad f \in \mathfrak{M}.$$

We now put $x_i = (x - a_i e) e_i + a_i e$, where a_i is an arbitrary complex number in $\mathscr{U}_i \cap \sigma(x)$. We have $\sigma(x_i) \subset \mathscr{U}_i$. By lemma 5.5 we can construct an element

$$y_i = \Phi(x_i) - \Phi(a_i)(e - e_i), \qquad i = 1, 2, \ldots, n,$$

and put $y = y_1 + \quad + y_n$. Let $f \in \mathfrak{M}$; thus there is a k such that $f(x) \in \mathscr{U}_k$. We have

$$f(y_i) = \Phi\big(f(x_i)\big) - \Phi(a_i)\big(1 - f(e_i)\big),$$

so if $i \neq k$ we have $f(x_i) = \big(f(x) - a_i\big) f(e_i) + a_i = a_i$, and $f(y_i) = \Phi(a_i) - \Phi(a_i) = 0$; if $i = k$ then $f(x_k) = f(x) - a_i + a_i = f(x)$, and $f(y_k) = \Phi\big(f(x)\big)$, so in any case $f(y) = \Phi\big(f(x)\big)$. The statement that y is unique if A is semisimple is obvious, so theorem is proved.

In the same way as theorem 4.5 we get as corollary the following generalization of classical theorem of Levy on trigonometrical series.

5.8. Theorem. *Let $x(t)$ be a complex function of real variable $0 \leqslant t < 2\pi$ equal to its Fourier expansion*

$$x(t) = \sum_{-\infty}^{\infty} x_n e^{int},$$

where $\sum |x_n|^p < \infty$ for a fixed p, $0 < p \leqslant 1$. Then if Φ is a holomorphic function defined in a neighbourhood of the set $\{x(t) \quad 0 \leqslant t < 2\pi\}$, then

$$\Phi\big(x(t)\big) = \sum y_n e^{int},$$

where $\sum |y_n|^p < \infty$.

§ 6. Final remarks

As we have seen, many facts true for Banach algebras are true for locally bounded complete algebras, though some proofs must be replaced by new ones. As author was informed by S. Rolewicz there is also true an analogue of theorem 5.7 in which instead of spectrum and analytic function of one complex variable there is taken joint spectrum and holomorphic function of several complex variables (the result is obtained by the use of suitable integrals; cf. remark 4.8). By the use of similar method as in the proof of theorem 4.9 it may be shown that a commutative p-normed algebra may be decomposed into a direct sum of ideals $A = I_1 \oplus I_2$ if and only if $\mathfrak{M}$ is disconnected. In this case we have $\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2$, where $\mathfrak{M}_i$, $i = 1, 2$, is open and closed in $\mathfrak{M}$, and there are two idempotents, e_1 and e_2, such that

$$e_1(M) = \begin{cases} 1 & \text{for} \quad M \in \mathfrak{M}_1, \\ 0 & \text{for} \quad M \in \mathfrak{M}_2, \end{cases} \qquad e_2(M) = \begin{cases} 0 & \text{for} \quad M \in \mathfrak{M}_1, \\ 1 & \text{for} \quad M \in \mathfrak{M}_2, \end{cases}$$

and $I_1 = e_1 A$, $I_2 = e_2 A$ (for the detailed proof, see [31]).

In a similar way as in Banach algebra theory it may be shown that in any (non-trivial) p-normed algebra there are topological divisors of zero. It may be shown also that for any two commutative semisimple p-normed algebras algebraic isomorphism implies the topological one. Thus there should exist a characterization of a commutative semisimple algebra in order to be isomorphic with a p-normed algebra. Let A be a commutative complex algebra with unit, $\mathfrak{M}$ the set of its maximal ideals. It is clear that the necessary condition in order A be p-normable in a semisimple way is that any element of $\mathfrak{M}$ should be of codimension one, so elements of $\mathfrak{M}$ may be identified, as in p-normed case, with multiplicative linear functionals. It must be, moreover,

$$\|x\|_s = \max_{f \in \mathfrak{M}} |f(x)|^p < \infty$$

for each $x \in A$ (here p is a number satisfying $0 < p \leqslant 1$), and it must be actually a norm, not a semi-norm.

6.1. Definition. A submultiplicative p-homogeneous norm $\| \ \|$ defined on A is called *semisimple* if the completion $[A, \| \ \|]$ of A in the norm $\| \ \|$ is semisimple. Let $S_p(A)$ denote the class of all semisimple p-homogeneous norms defined on A. If A satisfies necessary conditions mentioned above, $S_p(A)$ is non-void since $\| \ \|_s \in S_p(A)$.

6.2. Definition. An element $| \ |_1 \in S_p(A)$ is called to be *non-greater* than $| \ |_2$, in symbols $| \ |_1 \prec | \ |_2$, if $| \ |_1$ is continuous with respect to $| \ |_2$ or, what is the same, if there is a constant C such that $|x|_1 \leqslant C|x|_2$ for every $x \in A$. Two norms $|\ |_1$ and $|\ |_2$ are *equivalent* if $|\ |_1 \prec |\ |_2$ and $|\ |_1 \succ |\ |_2$. The set of $\sim$ equivalence classes $\tilde{S}_p(A)$ obtained from $S_p(A)$ is a partially ordered set which order $\prec$.

If A itself is a p-normed algebra with norm $\|\ \|$, then the $\sim$ class of $\|\ \|$ is a maximal element of $\tilde{S}_p(A)$. In fact, let $|\ | \in S_p(A)$, and let ψ be a natural homomorphism of A into $[A, |\ |]$. We shall show that the graph of ψ is closed. Let $x_n \to x_0$ in A, and $\psi(x_n) \to y_0$ in $[A, |\ |]$. For any $f \in \mathfrak{M}([A, |\ |])$, the functional $h(x) = f(\psi(x))$ belongs to $\mathfrak{M}(A)$; hence it is continuous in $\|\ \|$. It follows that

$$f(\psi(x_0)) = h(x_0) = \lim h(x_n) = \lim f(\psi(x_n)) = f(y_0),$$

and so $y_0 = \psi(x_0)$ since $[A, |\ |]$ is semisimple. It follows that ψ is continuous which means that $\lim \|x_n\| = 0$ implies $\lim |x_n| = 0$, thus $|\ | \prec \|\ \|$ and class of $\|\ \|$ is a maximal element of $\tilde{S}_p(A)$. So we pose the following

6.3. CONJECTURE. *Let A be a commutative complex algebra with unit. Then A is isomorphic with a p-normed algebra if and only if*

(6.3.1) *Every maximal ideal of A has codimension one, and $\|x\|_s = \max_{f \in \mathfrak{M}} |f(x)|$ is a finite norm in A.*

(6.3.2) *The family $\tilde{S}_p(A)$ has a maximal element $\|\ \|$.*

We conjecture also that under these conditions A is complete in maximal norm $\|\ \|$ and $\|x\|_s = \lim \sqrt[n]{\|x^n\|}$.

CHAPTER II

F-ALGEBRAS AND TOPOLOGICAL ALGEBRAS

We give now some general facts on F-algebras, and topological algebras. We shall be concerned here with basic problems connected with continuity of multiplication and of inversion in F-algebras, and with topological division algebras.

§ 7. F-algebras

7.1. DEFINITION. An *F-space* is a completely metrizable topological linear space (cf. [4], [6]). If X is an F-space, then its complete metric may be given by the means of an F-norm, i.e. functional $\|x\|$ satisfying:

(7.1.1) $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$.

(7.1.2) $\|x+y\| \leqslant \|x\|+\|y\|$.

(7.1.3) $\|\lambda x\| = \|x\|$ for $|\lambda| = 1$.

(7.1.4) $\lim |\lambda_n| = 0$ implies $\lim \|\lambda_n x\| = 0$ for each $x \in X$.

(7.1.5) $\lim \|x_n\| = 0$ implies $\lim \|\lambda x_n\| = 0$ for any scalar λ.

(7.1.6) The distance of two points $x, y \in X$ is given by $\|x-y\|$, and X is complete in this metric.

Of course, the topology in X may be also given by other (translation invariant, or not) metric functions. However we shall always consider metric functions in which X is a complete space.

7.2. THEOREM (Arens [2]). *Let A be an F-algebra; then the multiplication in A is jointly continuous.*

Proof. Let $\|x\|$ be an F-norm of A, and let $\bar{K}(x_0, r) = \{x \in A : \|x_0 - x\| \leqslant r\}$. Put $U_n = K(0, 1/n)$; thus (U_n) forms a basis of neighbourhoods of zero in A. Let U be any fixed neighbourhood of zero in A and let $A_n = \{x \in A : xU_n \subset U\}$. By separate continuity of multiplication in A we have $A = \bigcup A_n$. Since each A_n is closed, there is an n_0 such that interior of A_{n_0} is non-void, and so there is an x_0 and r_0 such that $K(x_0, r_0) \subset A_{n_0}$. Let $x \in K(0, r_0)$; we have $x = x' - x_0$, where $x' \in K(x_0, r_0)$, so $xU_{n_0} = x'U_{n_0} - x_0U_{n_0} \subset U+U$. Now let V be an arbitrary neighbourhood of zero in A. We can pick out a U in such a way that $U+U \subset V$; so $K(0, r_0)\cdot U_{n_0} \subset V$, and $[K(0, \min(r_0, 1/n_0))]^2 \subset V$ which means that multiplication in A is jointly continuous, q.e.d.

We now prove a theorem on the possibility of completion of a metric algebra.

7.3. Theorem. *Let A be a metric algebra; then A may be topologically embedded in an F-algebra (as dense subalgebra) if and only if the multiplication in A is jointly continuous.*

Proof. If A is subalgebra of an F-algebra, then by theorem 7.2 multiplication in A is jointly continuous (it is obvious that any subalgebra of an algebra with jointly continuous multiplication has the same property). Suppose now that multiplication in A is jointly continuous, i.e. for any neighbourhood of zero U there is another neighbourhood V such that $V^2 \subset U$. Consider a fixed metric in A. We shall show that if $(x_n), (y_n)$ are Cauchy sequences in A, then so is (x_ny_n). Therefore, let U be any neighbourhood of zero in A; it is to be shown that there exists such an N, that for $k, l > N$ we have $x_ky_k - x_ly_l \in U$.

We have

$$(7.3.1) \quad x_ky_k - x_ly_l = x_ky_k - x_ky_l + x_ky_l - x_ly_l = x_k(y_k - y_l) + (x_k - x_l)y_l$$
$$= (x_k - x_0)(y_k - y_l) + x_0(y_k - y_l) + (x_k - y_l)(y_l - y_0) + (x_k - y_l)y_0.$$

x_0, y_0 are elements of A which shall be defined later on. We find such a neighbourhood V of zero that $V+V+V+V \subset U$, and such a W that $W^2 \subset V$ Since $(x_n), (y_n)$ are Cauchy sequences, there exist x_0, y_0 and K such that for $k > K$ we have $x_k - x_0 \in W$ and $y_k - y_0 \in W$; there is also such an M that for $k, l > M$ we have $x_k - x_l \in W$ and $y_k - y_l \in W$ Setting $N = \max(K, M)$, we have, in view of (7.3.1), $x_ky_k - x_ly_l \in U$ for $k, l > N$, so (x_ky_k) is also a Cauchy sequence. If $\tilde{A}$ is the completion of A, then for $x, y \in \tilde{A}$ we have $x = \lim x_n$, $y = \lim y_n$, $x_n, y_n \in A$, and $x\cdot y$ we define as $\lim x_ny_n$; this is obviously a continuous associative multiplication in $\tilde{A}$, and $\tilde{A}$ is an F-algebra, so the natural mapping of A into $\tilde{A}$ is the desired embedding, q.e.d.

We now pass to the proof of a theorem on continuity of inverse in F-algebras.

7.4. Theorem (Banach [5]). *Let A be an F-algebra; then the inversion $x \to x^{-1}$ is continuous on the set V of all invertible elements of A if and only if V is a G_δ-set.*

Proof. The condition is necessary. In fact, the set H on which the oscillation of x^{-1} is zero is a G_δ-set. We have $V \subset H$. On the other hand, if $x \in H$, then $x = \lim x_n$, $x_n \in V$, and x_n^{-1} is a convergent sequence in A.

If $y = \lim x_n^{-1}$, then, by theorem 7.2, $e = \lim x_n \cdot x_n^{-1} = xy$, so $x \in V$ and $V = H$, so V is a G_δ-set. Now let V be a G_δ-set. Then there exists a metric ϱ on V, equivalent on V to the metric given on A by its F-norm $\| \ \|$ and such that V is complete in the metric ϱ; thus V is a complete space in ϱ, it is a group with multiplication, and multiplication $x \cdot y$ is jointly continuous in (V, ϱ); thus our conclusion follows from the following

7.5. LEMMA (Żelazko [32]). *Let G be a group, and at the same time a complete metric space with metric ϱ. Designate group operation as multiplication $x \cdot y$, and assume that the multiplication is separately continuous, i.e. $\lim x_n = x$ imply $\lim x_n y = xy$ and $\lim y x_n = yx$. Then the inversion in G is also continuous, i.e. $\lim x_n^{-1} = x^{-1}$.*

Proof. It is clearly sufficient to prove that if x_n is any sequence of elements of the group G, which tends to the unit e, then there exists a subsequence $\bar{x}_n$ such that $\bar{x}_n^{-1} \to e$. The subsequence $\bar{x}_n$ will be defined by an induction. Suppose that elements $\bar{x}_1, \ldots, \bar{x}_n$ are chosen from the sequence (x_n) in such a way that

$$\varrho(p_k, p_{k+1}) < \frac{1}{2^{k+1}}, \tag{7.5.1}$$

$$\varrho(q_{s,k}, q_{s,k+1}) < \frac{1}{2^{k+1}} \tag{7.5.2}$$

for $k = 1, 2, \ldots, n-1$, $s = 1, 2, \ldots, n, n+1$, where

$$p_k = \bar{x}_1, \ldots, \bar{x}_k, \qquad q_{s,k} = \begin{cases} p_k \bar{x}_s^{-1} & \text{for} \quad k \geqslant s, \\ p_k & \text{for} \quad k < s. \end{cases}$$

The possibility of choosing of suitable $\bar{x}_{n+1}$ for which also hold relations (7.5.1) and (7.5.2) follows from the fact that there is only finite number of these relations to be satisfied and that $x_n \to e$. It follows from (7.5.1) and (7.5.2) that the limits $p = \lim\limits_n p_n$ and $q_s = \lim\limits_n q_{s,m}$ exist and that

$$\varrho(p, p_k) \leqslant \frac{1}{2^k}, \qquad \varrho(q_s, q_{s,k}) \leqslant \frac{1}{2^k}.$$

We also have

$$\varrho(p, q_s) \leqslant \varrho(p, q_{s,s}) + \varrho(q_{s,s}, q_s) \stackrel{\prime}{=} \varrho(p, p_{s-1}) + \varrho(q_s, q_{s,s}) \leqslant \frac{1}{2^{s-1}} + \frac{1}{2^s} < \frac{1}{2^{s-2}},$$

so it follows that $p = \lim\limits_s q_s$. On the other hand, it is $\bar{x}_s^{-1} = p^{-1} q_s$, so there exists the limit $\lim \bar{x}_s^{-1} = p^{-1} \lim q_s = e$, q.e.d.

7.6. Corollary. *If A is a division algebra of type F, then the inversion is continuous in A.*

Or, giving the following

7.7. Definition. A *Q-algebra* is a topological algebra in which the set of invertible elements is open,

we have also

7.8. Corollary. *In any Q-algebra of type F the inversion is continuous on the set of all invertible elements.*

The following question is open:

7.9. Problem. Is every division algebra of type F isomorphic and homeomorphic either with real or complex field, or with the division algebra of real quaternions?

Without the assumption of completeness the answer is in negative, even in the case where the inverse is continuous. The counter-example is the algebra of all rational functions of one real variable with metrizable topology given by asymptotic convergence. However, there are some informations about topological division algebras, which we give in the next section.

§ 8. Topological division algebras

In this section we shall consider topological algebras for which the multiplication is jointly continuous, i.e. for any neighbourhood of zero V there exists another such a neighbourhood satisfying $W^2 \subset V$

8.1. Definition. A topological linear space X is called *to possess short lines* if for any neighbourhood of zero V there exists an $x \neq 0$ such that $\lambda x \in V$ for every scalar λ.

8.2. Definition. A topological algebra is called *to possess small ideals* if for every neighbourhood V of zero there exists a non-zero ideal contained in V.

8.3. Proposition. *A topological algebra possesses small ideals if and only if it possesses short lines as a space.*

Proof. If A possesses small ideals, then it obviously possesses short lines. Suppose that A possesses short lines, and let U be an arbitrary neighbourhood of zero in A. Let $V^2 \subset U$ and $\lambda x \in V$ for each λ, where $x \neq 0$. We shall show that the ideal $I = xA$ is contained in U. In fact, let $y \in A$. We can choose such a $\lambda \neq 0$ that $\lambda y \in V$ But $\frac{1}{\lambda} x \in V$, and so

$$xy = \frac{1}{\lambda} x \cdot \lambda y \in V^2 \subset U,$$

q. e. d.

8.4. Corollary. *If A is a division algebra, then A as a space possesses no short lines.*

8.5. Proposition. *Let A be a topological division algebra over complexes, and let the inversion be continuous on the set of non-zero elements of A. If there is in A at least one non-zero continuous linear functional, then A is topologically isomorphic with complex numbers field.*

Proof. Let f be a continuous (complex) linear functional in A. The set $\{x \in A : f(x) \neq 0\}$ is open in A, so it is sufficient to show any element of this set is of the form $x = \lambda e$, λ— a complex scalar. Suppose then that there is a $y \in A$, not of the form λe, such that $f(y) \neq 0$. Consider the function $\varphi(\lambda) = f[(y^{-1} - \lambda e)^{-1}]$; it is a complex function defined on the whole complex plane. Moreover, there exists

$$\lim_{\lambda=0} \frac{\varphi(\lambda+\lambda_0) - \varphi(\lambda_0)}{\lambda} = f\big[\lim_{\lambda=0} (y^{-1} - (\lambda+\lambda_0)e)^{-1}(y^{-1} - \lambda_0 e)^{-1}\big]$$
$$= f\big((y^{-1} - \lambda_0 e)^{-2}\big),$$

so that $\varphi(\lambda)$ is an entire function. Moreover, it is

$$\lim_{|\lambda|\to\infty} \varphi(\lambda) = \lim_{|\lambda|=\infty} \lambda^{-1} f\left(\lim\left(\frac{y^{-1}}{\lambda} - e\right)^{-1}\right) = 0 \cdot f(-e) = 0;$$

this follows $\varphi(\lambda) \equiv 0$, which is impossible since $\varphi(0) = f(y) \neq 0$, q.e.d.

8.6. Remark. By use of the technique mentioned in remark 4.7, proposition 8.5 may be proved under assumption that A is a division algebra over real numbers. In this case the conclusion would be that A is isomorphic with either one of three standard trivial division algebras.

8.7. Corollary. *If A is a non-trivial F-division algebra, then convex envelope of any neighbourhood of zero in A is the whole of A.*

Now we introduce a new concept of generalized topological divisors of zero.

8.8. Definition. Two subsets X and Y of a topological algebra A are called *generalized topological divisors of zero in A*, if $0 \notin \bar{X} \cup \bar{Y}$, but $0 \in \overline{X \cdot Y}$ ($\bar{X}$ denotes the closure of X).

If one of the sets X, Y consists of single point, it is an ordinary topological divisor of zero; if both consist of single points, there are divisors of zero. We shall prove the following theorem:

8.9. Theorem. *If A is a topological division algebra over real numbers field, then either A possesses generalized topological divisors of zero, or A is isomorphic and homeomorphic either with real numbers field, or complex numbers field or with the division algebra of real quaternions.*

Proof. Let W be a neighbourhood of zero in A such that $e \notin W$, and let V be another such a neighbourhood satisfying $V \cup V^2 \subset W$ We put $U = (A \dot{-} V)^{-1} \cup \{0\}$, where $A \dot{-} V$ denotes the complementation of V in A. We claim that $0 \in \operatorname{int} U$. If not, then $0 \in \overline{A \dot{-} U}$, so $A \dot{-} U$ intersects with V and there is an $x \in V \cap (A \dot{-} U)$. This follows $x \neq 0$, and $x \notin (A \dot{-} V)^{-1}$, and so $x^{-1} \notin (A \dot{-} V)$ and $x^{-1} \in V$ But we have also $x \in V$, so $e = x \cdot x^{-1} \in V^2 \subset W$, and we get a contradiction. Therefore U is a neighbourhood of zero in A. If U was a bounded set in A, then A would be a locally bounded division algebra, and so, by Remark 4.7, it would be one of either three standard division algebras. To prove our theorem it remains to show that if $(A \dot{-} V)^{-1}$ is an unbounded set, then there are generalized topological divisors of zero in A. But in this case there exists a neighbourhood Q of zero in A such that for every scalar $\lambda \neq 0$ there exists an $x_\lambda \in (A \dot{-} V)^{-1}$ such that $\lambda x_\lambda \notin Q$. We have $x_\lambda^{-1} \in (A \dot{-} V)$, $\lambda x_\lambda \in (A \dot{-} Q)$. Obviously $0 \notin \overline{(A \dot{-} V) \cup (A \dot{-} Q)}$, but $\lambda e = \lambda x_\lambda x_\lambda^{-1} \in (A \dot{-} V) \cdot (A \dot{-} Q)$ for every $\lambda \neq 0$, so $0 \in \overline{(A \dot{-} V)(A \dot{-} Q)}$, q.e.d.

Theorem 8.9 is obtained under very general assumptions, but probably there is true also a more general fact (which may be considered as a generalization of the fact that every Banach algebra either possesses topological divisors of zero, or is one of three division algebras), and we pose the following

8.10. CONJECTURE. *Let A be a topological algebra with jointly continuous multiplication. Then either A has generalized topological divisors of zero, or A is homeomorphically isomorphic either with real or complex numbers field, or with the division algebra of real quaternions.*

CHAPTER III

B_0-ALGEBRAS

§ 9. Basic facts

Now we pass to another generalization of Banach algebras, namely to locally convex, completely metrizable topological algebras.

9.1. DEFINITION. A *B_0-space* is a locally convex F-space (cf. definition 7.1). The topology in a B_0-space X may be given by means of a countable family $(|x|_i)$ of homogeneous pseudonorms. The sequence (x_n) of elements of X tends to x_0 if and only if

$$\lim_n |x_n - x_0|_i = 0 \qquad \text{for} \qquad i = 1, 2, \ldots;$$

this convergence is equivalent with convergence given by distance

$$\varrho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x-y|_i}{1+|x-y|_i},$$

which gives an F-norm $\|x\| = \varrho(x, 0)$. Replacing the system $(|\ |_i)$ by $\|x\|_n = \max(|x|_1, \ldots, |x|_n)$ we get an equivalent system (i.e. giving the same topology) satisfying

(9.1.1) $$|x|_1 \leqslant |x|_2 \leqslant \ldots,$$

so we shall assume that considered pseudonorms satisfy this relation, unless otherwise stated. A pseudonorm $\|x\|$ is continuous in X if and only if there is a constant C and an integer n such that $\|x\| \leqslant C|x|_n$ for every $x \in X$. Any two systems of pseudonorms satisfying (9.1.1) give the same topology if and only if any pseudonorm of one system is continuous with respect to the other one. In particular any subsequence of system satisfying (9.1.1) gives the same topology as the system itself (for properties of B_0-spaces see e.g. [13] and [14]).

9.2. THEOREM. *Let A be a B_0-algebra; then the system of pseudonorms* (9.1.1), *giving the topology of A, may be chosen in such a way that*

(9.2.1) $$|xy|_i \leqslant |x|_{i+1}|y|_{i+1}, \qquad i = 1, 2,$$

Proof. By theorem 7.2 the multiplication in A is jointly continuous. It follows that for every continuous pseudonorm $\|x\|$ there are a constant C and an integer n_0 such that $\|xy\| \leqslant C|x|_{n_0}|y|_{n_0}$, otherwise there would exist sequences (x_n), (y_n) of elements of A such that $\|x_n y_n\| \geqslant n^2|x_n|_n|y_n|_n$, and it would be $\lim \xi_n = \lim \eta_n = 0$, where

$$\xi_n = \frac{1}{n}\frac{x_n}{|x_n|_n}, \qquad \eta_n = \frac{1}{n}\frac{y_n}{|y_n|_n},$$

while $\|\xi_n\eta_n\| \geqslant 1$, what is impossible. Having this, we put $\|x\|_1 = |x|_1$, we pick such an $i > 1$ that $\|xy\|_1 \leqslant C|x|_i|y|_i$, and we put $\|x\|_2 = \sqrt{C}|x|_i$; then for $\|x\|_2$ we get a $j > i$ such that $\|xy\|_2 \leqslant C|x|_j|y|_j$ and so on. We can always increase constants C to make the new system satisfying (9.1.1). It clearly satisfies (9.1.2) and is equivalent with the old one, q.e.d.

9.3. Remark. We do not know whether it is possible to have an equivalent system of pseudonorms satisfying (9.1.1), (9.2.1), and such that $|e|_i = 1$ for $i = 1, 2, \ldots$ We can do it in the case when in A there exists at least one closed maximal ideal and A is over real or complex scalars. To prove this we formulate

9.4. Theorem. *Any B_0-division algebra (over real numbers) is isomorphic and homeomorphic with one of three standard division algebras.*

This theorem is an immediate consequence of remark 8.6.

Suppose now that M is a closed maximal ideal in A, so its codimension is finite (1, 2 or 4 in case of real scalars, 1 in case of complex scalars), and so A is a direct sum, $A = M \oplus D$, where D is a division algebra with modulus $|\xi\eta| = |\xi||\eta|$, for $\xi, \eta \in D$, and $|e| = 1$. We may assume also that $|\xi x|_i = |\xi||x|_i = |x\xi|_i$ for $\xi \in D$, $x \in A$ (we may replace, if necessary, $|x|_i$ by $\sup_{|\eta|=|\xi|=1} |\xi x\eta|_i$). Having decomposition $x = m + \xi$, where $x \in A$, $m \in M$, $\xi \in D$ we put now $\|x\|_i = |m|_i + |\xi|$. So if $x_i = m_i + \xi_i$, $i = 1, 2$, we have $\|x_1x_2\|_i = \|m_1m_2 + \xi_1m_2 + m_1\xi_2 + \xi_1\xi_2\|_i = |m_1m_2 + \xi_1m_2 + m_1\xi_2|_i + |\xi_1||\xi_2| \leqslant |m_1m_2|_i + |\xi_1||m_2|_i + |\xi_2||m_1|_i + |\xi_1||\xi_2| \leqslant |m_1|_{i+1}|m_2|_{i+1} + |\xi_1||m_2|_{i+1} + |\xi_2||m_1|_{i+1} + |\xi_1||\xi_2| = \|x_1\|_{i+1}\|x_2\|_{i+1}$. Obviously, $\|e\|_i = 1$, and the system $\|x\|_i$ is equivalent with the old one. Unfortunately, there are B_0-algebras without closed maximal ideals, so the question is still open.

9.5. Remark. Theorem 9.4 is false without assumption of completeness (see example 12.3).

9.6. Remark. In many cases one may pick out a system (9.1.1) in such a way that

$$|xy|_i \leqslant |x|_i|y|_i \tag{9.6.1}$$

holds. The next two sections will be devoted to this case.

§ 10. Multiplicatively convex B_0-algebras ([1])

10.1. Definition. A B_0-algebra A is called *multiplicatively-convex* or *m-convex*, if there exists an equivalent system of pseudonorms (9.1.1) satisfying (9.6.1) (in [15] Michael calls these algebras the F-algebras).

10.2. Definition. A subset X of a topological algebra is called *idempotent* if $X^2 \subset X$. A topological algebra is called *locally idempotent* if there is a basis of neighbourhoods of zero consisting of idempotent sets.

10.3. Lemma. *Convex envelope of an idempotent set is again an idempotent set.*

Proof. Let U be an idempotent, and $V = \operatorname{conv} U$. It is to be shown that $V^2 \subset V$ Let $x, y \in V$, so

$$x = \sum_{n=1}^{M} a_n x_n, \quad y = \sum_{n=1}^{N} \beta_n y_n,$$

and $\sum a_n = \sum \beta_n = 1$, where $0 \leqslant a_i,\ \beta_i \leqslant 1$. We have

$$xy = \sum_{k=1}^{M} \sum_{n=1}^{N} a_k \beta_n x_k y_n.$$

But $x_k y_n \in U$, $\sum\limits_{k,n} a_k \beta_n = 1$, and $0 \leqslant a_k \beta_n \leqslant 1$, so $xy \in V$, q.e.d.

The following proposition gives a characterization of multiplicatively convex B_0-algebras.

10.4. Proposition. *A B_0-algebra is m-convex if and only if it is locally idempotent.*

Proof. Let $K_i(r) = \{x \in A : |x|_i < r\}$. If A is multiplicatively convex then $\{K_i(r)\}$, $0 < r \leqslant 1$, $i = 1, 2, \ldots$, forms a basis of idempotent neighbourhoods of A. If (U_i) is a basis of A consisting of idempotent sets, then setting $V_i = \bigcup\limits_{|\lambda| \leqslant 1} \lambda U$ we get a basis consisting of balanced idempotent set, and if we put $W_i = \operatorname{conv} V_i$ then we get, by lemma 10.3 and local convexity of A, a basis consisting of balanced convex idempotent sets. So if $\|x\|_i$ is a pseudonorm such that $\{x : \|x\|_i \leqslant 1\} = \bar{V}$ — closure of V, then the system $(\|x\|_i)$ gives the same topology as basis (U_i), and satisfies (9.6.1). To have the system satisfying moreover (9.1.1) we may put $|x|_k = \max\limits_{i \leqslant k} \{\|x\|_i\}$, q.e.d.

10.5. Example. Any closed subalgebra of an enumerable cartesian product of Banach algebras, with coordinatewise multiplication, is an m-convex B_0-algebra. This is in a sense universal example as we shall see in the following

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([1]) Presented here results are adopted from [15].

10.6. Proposition. *Any m-convex B_0-algebra is a closed subalgebra of cartesian product of Banach algebras.*

Proof. Let $N_i = \{x : |x|_i = 0\}$, where the system $(|x|_i)$ satisfies (9.6.1); it is a closed ideal in A. Let $A_i = A/N_i$, and let $\bar{A}_i$ be completion of A_i in the norm $|x|_i$, so $\bar{A}_i$ is a Banach algebra. If $\pi_i(x)$ is the natural mapping of A into $\bar{A}_i$ we get a natural isomorphism of A into cartesian product $\prod_{i=1}^{\infty} \bar{A}_i$, given by $x \to (\pi_i(x))$. It is a topological isomorphism, since topology of A is identical with cartesian product topology of its image. The image is closed, since A is complete, q.e.d.

10.7. Examples. The algebras $C(-\infty, \infty)$ of all complex valued continuous functions of real variable λ, $-\infty < \lambda < \infty$, and E of all entire functions of one complex variable λ, with pointwise multiplication and pseudonorms $|x|_i = \max_{|\lambda| \leqslant i} |x(\lambda)|$ are m-convex B_0-algebras. For $C(-\infty, \infty)$, $\bar{A}_i = C[-i, i]$; for E, $\bar{A}_i$ is Banach algebra of all functions continuous on the disc $|\lambda| \leqslant i$, and holomorphic in its interior.

10.8. Theorem. *A B_0-algebra A is m-convex if and only if there exists a system of pseudonorms giving its topology such that the multiplication is separately continuous with respect to each pseudonorm of this system, i.e. $|y_n|_i \underset{n}{\to} 0$ implies $|x_n y_n|_i \underset{n}{\to} 0$ for $i = 1, 2, \ldots$, $x \in A$.*

Proof. Suppose that $V_i = \{x \in A : xK_i(|e|_i) \subset K_i(|e|_i)\}$, where $K_i(r) = \{x : |x|_i < r\}$. Since for any $x \in A$ there is a $\lambda_{x,i}$ such that $|xy|_i \leqslant \lambda_{x,i}|y|_i$, it follows that V_i are symmetric and absorbing closed sets. Each V_i is also convex and idempotent, so it is a neighbourhood of zero. Since $V_i \subset K_i(|e|_i)$, it follows that the system of pseudonorms given by sets V_i is equivalent with $(|x|_i)$ and satisfy (9.6.1); thus A is multiplicatively convex, q.e.d.

10.9. Remark. By the above theorem one cannot expect any mixed situation of type $|xy|_i \leqslant |x|_{i+1}|y|_i$; this condition would imply existence of an equivalent system satisfying (9.6.1).

We recall that if X_n is a sequence of topological linear spaces, and π_{ij}, $i \leqslant j$, are linear mappings $\pi_{ij} : X_j \to X_i$ such that π_{ii} is an identity mapping and $\pi_{ij}\pi_{jk} = \pi_{ik}$ for $i \leqslant j \leqslant k$, then the inverse, or projective limit of sequence X_n with mappings π_{ij} is a subspace of Cartesian product $\prod_{n=1}^{\infty} X$ given by $X = \{x = (x_n) \in \prod X_n : \pi_{ij}(x_j) = x_i\}$. If X_n are Banach spaces, then inverse limit X is a B_0-space (i.e. it is complete). Now let A be an m-convex B_0-algebra, $\bar{A}_i$ — B-algebras as in proposition 10.6, and $\pi_i(x)$ a natural mapping of A into $\bar{A}_i$ (cf. 10.6). Define $\pi_{ij} : A_j \to A_i$ by $\pi_{ij}(\pi_j(x)) = \pi\ (x)$, where $A_i = A/N_i$ equipped with norm $|\ \ |_i$ (cf. 10.6). Since π_{ij} are continuous algebra homomorphisms, they may be extended by continuity to mappings $\pi_{ij}: \bar{A}_j \to \bar{A}_i$. With this notation we shall prove the following

10.10. THEOREM. *If A is an m-convex B_0-algebra, then A is homeomorphically isomorphic with projective limit of algebras $\bar{A}_i$ with mappings π_{ij}.*

Proof. Let $\tilde{A}$ be a projective limit of $(\bar{A}_i)$ with π_{ij}. This is a closed subalgebra of $\prod \bar{A}_i$ and clearly A interpreted as subalgebra of $\prod \bar{A}_i$ is contained in $\tilde{A}$ (cf. 10.6), so it is sufficient to show that A is dense in $\tilde{A}$, since A is also closed. But it follows immediately from the fact that $\pi_i(A) = A_i$, and A_i is dense in $\bar{A}_i$ for $i = 1, 2, \ldots$, q.e.d.

10.11. COROLLARY. *If for an element $(x_i) \in \prod \bar{A}_i$ it is $\pi_{ij}(x_j) = x_i$, then there exist an $x \in A$ such that $\pi_i(x) = x_i$.*

From this corollary we deduce the following

10.12. THEOREM. *If $x \in A$, and $\pi_i(x)$ is invertible in $\bar{A}_i$, then x is invertible in A.*

Proof. If $[\pi_i(x)]^{-1} \in \bar{A}_i$ for $i = 1, 2, \ldots$, then $\pi_i(e) = \pi_{ij}(\pi_j(e)) = \pi_{ij}([\pi_j(x)]^{-1}\pi_j(x)) = \pi_{ij}([\pi_j(x)]^{-1})\pi_{ij}(\pi_j(x)) = \pi_{ij}([\pi_j(x)]^{-1})\cdot\pi_i(x)$, and since $\pi_i(e)$ is unit in $\bar{A}_i$, it follows that $\pi_{ij}([\pi_j(x)]^{-1}) = [\pi_i(x)]^{-1}$. It follows, by corollary 10.11, that there is a $y \in A$ that $\pi_i(y) = [\pi_i(x)]^{-1}$. We have $\pi_i(xy) = \pi_i(x)\cdot\pi_i(y) = \pi_i(e)$, so $xy = e$ and $y = x^{-1}$, q.e.d.

Let $\mathfrak{M}$ denote the set of all non-identically zero continuous multiplicative linear functionals of A. We have the following

10.13. THEOREM (Wiener property of m-convex B_0-algebras). *Let A be a commutative m-convex B_0-algebra. Then $x \in A$ is invertible in A if and only if $f(x) \neq 0$ for each $f \in \mathfrak{M}$.*

Proof. If x is invertible, then clearly $f(x) \neq 0$ for each $f \in \mathfrak{M}$. If $\bar{x}$ is not invertible, then by theorem 10.12 there is an i_0 such that $\pi_{i_0}(\bar{x})$ is not invertible in $\bar{A}_{i_0}$. But from the same property of Banach algebras it follows that there is a multiplicative linear functional F defined on $\bar{A}_{i_0}$ such that $F(\pi_{i_0}(\bar{x})) = 0$. Setting $f(x) = F(\pi_{i_0}(x))$, we have $f \in \mathfrak{M}$ and $f(\bar{x}) = 0$, q.e.d.

In the sequel of this section we shall consider only commutative algebras.

10.14. PROPOSITION. *If A is a commutative m-convex B_0-algebra, then the intersection of all maximal ideals coincides with the intersection of all closed maximal ideals.*

Proof. It is to be shown that if $x \in \bigcap_{M \in \mathfrak{M}} M$, then x belongs to the intersection of all maximal ideals. We have $f(e+xy) = 1$ for each $f \in \mathfrak{M}$, $y \in A$, so, by theorem 10.14, $(e+xy)^{-1} \in A$ for any $y \in A$, but this means that x is an intersection of all maximal ideals of A, q.e.d.

Usually in m-convex B_0-algebras there are maximal ideals which are non-closed (and consequently dense in A).

10.15. EXAMPLE. Let $A = C(-\infty, \infty)$, as in 10.7, and put $I = \{x \in A : x(t) = 0 \text{ for } t > T_x\}$. I is obviously a dense ideal in A, so every maximal ideal containing I is also dense.

For any maximal ideal in a complex B_0-algebra it is true that either its codimension is one, or infinite (Frobenius theorem), and by theorem 9.4 any maximal ideal of infinite codimension must be dense. It is still open problem whether every maximal ideal of codimension one is always closed, equivalent to the question, whether every multiplicative and linear functional is continuous.

10.16. Remark. Theorem 9.4 may be extended also on topological algebras which are dense in m-convex B_0-algebras. In fact, any such a division algebra equipped with any $|\ |_i$ may be considered as a normed division algebra. In non-m-convex case it remains not to be true (example 12.3).

Generally speaking the set $V = \{x \in A : x^{-1} \in A\}$ is not open in an m-convex B_0-algebra, e.g. in $C(-\infty, \infty)$ there is a sequence non-invertible functions tending to the unit. However, it is always a G_δ-set as it follows from theorem 7.4 and from the following

10.17. PROPOSITION. *In an m-convex B_0-algebra the operation of taking inverse $x \to x^{-1}$ is continuous.*

Proof. Let V be the set of all invertible elements of A, $x_n, x_0 \in V$, and $\lim x_n = x_0$. It follows that $\lim\limits_n \pi_i(x_n) = \pi_i(x_0)$ for each i, and so $\lim\limits_n \pi_i(x_n^{-1}) = \pi_i(x_0^{-1})$ which means that $\lim x_n^{-1} = x_0^{-1}$, q.e.d.

§ 11. Spectra and power series in commutative m-convex B_0-algebras (1)

In this section we assume A to be a commutative, complex, m-convex B_0-algebra.

11.1. DEFINITION. Let $x \in A$. The *spectrum* $\sigma_A(x)$ of x is the subset of complex plane consisting of all λ's such that $x - \lambda e$ is not invertible in A.

11.2. PROPOSITION. *Put*

$$\sigma_1(x) = \{f(x) : f \in \mathfrak{M}\},$$

where $\mathfrak{M}$ is the set of all continuous multiplicative linear functionals of A, and

$$\sigma_2(x) = \bigcup_{i=1}^{\infty} \sigma_{\tilde{A}_i}(\pi_i(x)),$$

then $\sigma_A(x) = \sigma_1(x) = \sigma_2(x)$.

(1) Part of presented here results is adopted from [15], cf. Introduction.

Proof. Clearly $\mathfrak{M} = \bigcup_i \mathfrak{M}_i$, where $\mathfrak{M}_i$ is the set of all multiplicative linear functionals continuous with respect to $|x|_i$. $\mathfrak{M}_i$ may be identified with maximal ideal space of $\bar{A}_i$. Since $\sigma_{\bar{A}_i}(\pi_i(x)) = \{f(x) : f \in \mathfrak{M}_i\}$, it follows that $\sigma_1(x) = \sigma_2(x)$. If $\lambda \in \sigma_1(x)$, then there is an $f \in \mathfrak{M}$ with $f(x) = \lambda$, so $f(x-\lambda e) = 0$, $x-\lambda e$ is not invertible in A, and $\lambda \in \sigma_A(x)$. If $\lambda \notin \sigma_1(x)$, then $f(x-\lambda e) \neq 0$ for any $f \in \mathfrak{M}$, and so, by theorem 10.13, $x-\lambda e$ is invertible in A, and $\lambda \notin \sigma_A(x)$. Thus $\sigma_1(x) = \sigma_A(x)$, q.e.d.

11.3. The spectral radius $\varrho_A(x)$ is defined as $\sup\{|\lambda| : \lambda \in \sigma_A(x)\}$. We have the following

11.4. THEOREM. *Let*

(11.4.1) $r_1(x) = \sup_{|\,|}\limsup \sqrt[n]{|x^n|}$, *where* $\sup_{|\,|}$ *designates supremum over all continuous pseudonorms of* A;

(11.4.2) $r_2(x) = \sup_{f \in A^*}\limsup \sqrt[n]{|f(x^n)|}$, *where* A^* *designates the conjugate space of space* A;

(11.4.3) $r_3(x) = \sup_{f \in \mathfrak{M}} |f(x)|$;

(11.4.4) $r_4(x) = \inf\{R : (x-\lambda e)^{-1}$ *exists for each* λ, $|\lambda| > R\}$;

(11.4.5) $r_5(x) = \inf\{R :$ *there exists a power series with complex coefficients* $\sum a_n\lambda^n$, *having radius of convergence* R, *such that* $\sum a_n x^n$ *converges in* $A\}$;

(11.4.6) $r_6(x) = \inf\{R :$ *for each power series with complex coefficients* $\sum a_n\lambda^n$, *having radius of convergence* R, $\sum a_n x^n$ *converges in* $A\}$;

then $r_1(x) = r_2(x) = r_3(x) = r_4(x) = r_5(x) = r_6(x) = \varrho_A(x)$.

Proof. Clearly, $r_1(x) \geqslant r_2(x) \geqslant r_3(x) = \varrho_A(x) = r_4(x)$. At first we shall prove $r_1(x) = r_4(x)$. Let $R > r_4(x)$; then $\pi_i[(x-\lambda e)^{-1}]$ exists in each $\bar{A}_i$ for each λ with $|\lambda| \geqslant R$, or there exists $\pi_i[(\xi x - e)^{-1}]$ for $|\xi| < 1/R$. It follows that in $\bar{A}_i$ the series $\sum \xi^n \pi_i(x)^n$ is convergent and consequently $\lim \sqrt[n]{|\xi^n x^n|_i} < 1$, or $\lim \sqrt[n]{|x^n|_i} < R$. If $|x|$ is any continuous pseudonorm in A, then there is an i and a constant C such that $|x| \leqslant C|x|_i$ for each $x \in A$. It follows that

$$\limsup \sqrt[n]{|x^n|} \leqslant \lim \sqrt[n]{C|x^n|_i} < R,$$

and so $r_1(x) < R$. Since it holds for every $R > r_4(x)$, it follows that $r_4(x) \geqslant r_1(x)$, and so $r_1(x) = r_2(x) = r_3(x) = r_4(x) = \varrho_A(x)$. Clearly $r_5(x) \leqslant r_6(x)$. If $R > r_5(x)$, then there is a sequence of integers k_n such that $\sum_n (x/R)^{k_n}$ is convergent in A. So $\sum [f(x/n)]^n$ is convergent for any $f \in \mathfrak{M}$, and $|f(x/R)| < 1$, or $|f(x)| < R$. This follows that $r_3(x) \leqslant R$, and

$r_5(x) \geqslant r_3(x)$. It remains to be shown that $r_6(x) \leqslant \varrho(x) = r_1(x)$. So let $R > r_1(x)$. We have for any continuous pseudonorm $|x|$

$$\limsup \sqrt[n]{\left|\left(\frac{x}{R}\right)^n\right|} \leqslant \frac{r_1(x)}{R} < 1,$$

so the series $\sum_{n=1}^{\infty} (x/R)^n$ is convergent in A. It follows that for any power series $\sum a_n \lambda^n$ with radius of convergence $\geqslant R$, the series $\sum a_n x^n$ is convergent in A, so $r_6(x) \leqslant R$. It follows that $r_6(x) \leqslant r_1(x)$ which proves the theorem, q.e.d.

11.5. Definition. The *radical* of A, $\mathrm{rad}A$, is defined as the intersection of all maximal ideals of A. So, by proposition 10.14, we have the following

11.6. COROLLARY. *If A is a commutative m-convex B_0-algebra, then its radical is characterized by*

$$\mathrm{rad}A = \{x : \varrho_A(x) = 0\} = \{x : r_i(x) = 0\}, \quad i = 1, 2, \ldots, 6.$$

An algebra is called *semisimple* if its radical consists only of zero. Obviously A is semisimple if every $\bar{A}_i$ is semisimple. The converse statement, however, remains not to be true as is shown by the following

11.7. EXAMPLE (Rolewicz [23]). It shall be given a construction of a semisimple, commutative m-convex B_0-algebra, such that for any system of submultiplicative pseudo-pseudonorms giving its topology there exists a non-semisimple $\bar{A}_i$. Let A_0 be a semisimple Banach algebra with norm $\|x\|$, and let $\|x\|_0$ be a submultiplicative continuous norm in A_0 such that the completion $[A_0, \|x\|_0]$ of A_0 in $\|x\|_0$ is not semisimple, and $\mathrm{rad}[A_0, \|x\|_0] \cap A_0 \neq \emptyset$. Such algebras exist, e.g. $A_0 = l_1$ with

$$\|x\| = \sum_0^{\infty} |\xi_n|, \quad \|x\|_0 = \sum_0^{\infty} \frac{1}{n^n} |\xi_n|$$

and with convolution multiplication. Define now A as an algebra of sequences $x = (x_n)_{n=0}^{\infty}$, $x_n \in A_0$, such that

$$(11.7.1) \qquad |x|_i = \sup\{\|x_1\|, \|x_2\|, \ldots \|x_i\|, \|x_{i+1}\|_0, \|x_{i+2}\|_0 \ldots\} < \infty.$$

This is obviously an m-convex B_0-algebra with pseudonorms (11.7.1) and with coordinatewise multiplication. Moreover, A is semisimple. Let $\|x\|_i$ be any increasing system of submultiplicative pseudonorms, equivalent with system (11.7.1). We shall show that, for large i, $\bar{A}_i$ with respect to the new system are not semisimple. In fact, there are constants C_1, C_2, integers i and m such that

$$|x|_1 \leqslant C_1 \|x\|_i \leqslant C_2 |x|_m, \quad x \in A.$$

Since $|x|_1$ is a norm, it follows that $\bar{A}_i$ is a completion of A with respect to $\|x\|_i$. By our assumptions there is in A_0 an element x_0 such that $\lim\sqrt[n]{\|x_0^n\|} = 0$ (if $A_0 = l$, we may take as x_0 any $\neq 0$ element of the form $(0, \xi_1, \xi_2, \ldots)$). Put $\bar{x} = (0, \ldots 0, x_0, x_0 \ldots)$; it is an element of A. We have $|\bar{x}|_m = \|x_0\|_0$, so if we treat $\bar{x}$ as an element of $\bar{A}_i$ we have

$$\|\bar{x}^n\|_i \leqslant \frac{C_2}{C_1}|\bar{x}^n|_m = \frac{C_2}{C_1}\|x_0^n\|_0,$$

so $\lim\sqrt[n]{\|\bar{x}^n\|_i} = 0$, $\bar{x} \neq 0$, thus $\bar{A}_i$ is not semisimple. Suppose now that $|||x|||_i$ is an arbitrary system of submultiplicative pseudonorms in A, equivalent to (11.7.1). We shall show that in this case there also exists an integer i such that $\bar{A}_i$ is not semisimple. Put

$$\|x\|_n = \max_{i \leqslant n}(|||x|||_i).$$

Suppose that all $\bar{A}_i$ are semisimple; then

$$\lim\sqrt[n]{\|x^n\|_i} = \lim\sqrt[n]{\max_{k \leqslant i}|||x^n|||_k} = \max_{k \leqslant i}\lim\sqrt[n]{|||x^n|||_k},$$

so $\lim\sqrt[n]{\|x^n\|_i} = 0$ implies $|||x|||_k = 0$ for $k \leqslant i$, and $\|x\|_i = 0$, which would mean that $\bar{A}_i$ for the increasing family $\|x\|_i$ are all semisimple, which, as was shown before, is impossible.

In a similar way as for Banach algebras one may construct analytic functions in m-convex B_0-algebras.

11.8. THEOREM. *If $x \in A$, $\sigma_A(x)$ is contained with its closure in an open subset U of complex plane, and if Φ is holomorphic in U, then there exists a $y \in A$ such that for every $f \in \mathfrak{M}$ it is*

(11.8.1) $$f(y) = \Phi(f(x)).$$

Proof. By proposition 12.1, $\sigma_A(x) = \bigcup_i \sigma_{\bar{A}_i}(\pi_i(x))$, so $\sigma_{\bar{A}_i}(\pi_i(x)) \subset U$. If Γ_i is any rectifiable Jordan curve contained in U, and surrounding $\sigma_{\bar{A}_i}(\pi_i(x))$, then

$$y_i = \frac{1}{2\pi_i}\int_{\Gamma_i}\Phi(\xi)(\xi e - \pi_i(x))^{-1}d\xi$$

is a well-defined element of $\bar{A}_i$, independent of Γ_i. Moreover

$$\pi_{ij}(y_j) = \frac{1}{2\pi_i}\int_{\Gamma_j}\Phi(\xi)\left(\xi e - \pi_{ij}(\pi_j(x))\right)^{-1}d\xi$$

$$= \frac{1}{2\pi_i}\int_{\Gamma_j}\Phi(\xi)(\xi e - \pi_i(x))^{-1}d\xi = y_i,$$

so by corollary 10.11 there exists a $y \in A$ such that $\pi_i(y) = y_i$. It is evident that y satisfies (11.8.1), q.e.d.

11.9. Remark. Usually $\sigma_A(x)$ fails to be compact, it may be even the whole complex plane. Anyway, if Φ is an entire function, there is always the well-defined element $y = \Phi(x)$ for all $x \in A$. It is $y = \sum a_n x^n$, where $\Phi(\lambda) = \sum a_n \lambda^n$. Let us observe that, in certain cases, from the fact that Φ is defined in an open subset of an algebra it follows that Φ is an entire function. E.g. if Φ is a function defined in an open subset of $C(-\infty, \infty)$, then Φ must be an entire function, since every open set in $C(-\infty, \infty)$ contains elements with arbitrary large spectra. We shall speak rather about power series than about analytic functions, so we give the following

11.10. Definition. A metric algebra A is called *to have extension property* if from the fact that a series $\sum a_n x^n$ with complex coefficients is convergent for every x from an open subset of A it follows that this series must be convergent for every $x \in A$. We shall show that any m-convex B_0-algebra possesses the extension property if and only if it is not a Q-algebra (see definition 7.7). Before proving this we shall give some facts about m-convex B_0-algebras which are Q-algebras. At first we give an example of such an algebra.

11.11. Example. $C^\infty(0,1)$ consists of all complex functions defined on the closed segment $[0,1]$, which are continuous together with all their derivatives. The multiplication is defined pointwise, and the pseudonorms are given by

$$|x|_i = \max_{k\leqslant i} 2^k \max_t \left| \frac{d^k}{dt^k} x(t) \right| .$$

It is

$$|xy|_i = \max_{k\leqslant i} 2^k \max_t |(xy)^{(k)}| = \max_{k\leqslant i} 2^k \max_t \left| \sum_i \binom{k}{l} x^{(l)} y^{(k-l)} \right|$$

$$\leqslant \max_{k\leqslant i} 2^k \max_t \sum_i \binom{k}{l} |x^{(l)}|\,|y^{(k-l)}| \leqslant \max_{k\leqslant i} 2^k \sum_i \binom{k}{l} \frac{|x|_i}{2^k} \frac{|y|_i}{2^k} = |x|_i |y|_i.$$

So $C^\infty(0,1)$ is m-convex. It is obviously a Q-algebra.

It is easy to see that A is a Q-algebra if and only if the set $V = \{x \in A : x^{-1} \in A\}$ has non-void interior, and that every maximal ideal in a Q-algebra must be closed.

11.12. Proposition. *Let A be an m-convex B_0-algebra. Put*

$$B = \{x \in A : \varrho(x) \leqslant 1\}, \quad i = 1, 2, \ldots,$$

$$B_\infty = \{x \in A : \varrho(x) < \infty\},$$

where $\varrho(x)$ *is spectral radius of* x; *then the following assertions are equivalent:*

(11.12.1) *A is a Q-algebra,*

(11.12.2) *B is a neighbourhood of zero in A.*

Proof. If A is a Q-algebra, then there is such a symmetric convex neighbourhood U of zero that the set $\{e+x : x \in U\}$ consists of invertible elements. We shall show that $U \subset B$. If not, there is an $x \in U$ with $\varrho(x) > 1$, so there is a λ, $|\lambda| > 1$, that $x+\lambda e$ is not invertible, and $e+x/\lambda$ is also non-invertible in A, but $|1/\lambda| < 1$, so $x/\lambda \in U$, and $e+x/\lambda$ must be invertible, thus we have proved (11.12.1) $\to$ (11.12.2). If $0 \in \operatorname{int} B_1$, then setting $U = \frac{1}{2}B_1$ we have $\sum x^n$ convergent for each $x \in U$. The sum of this series is $(e-x)^{-1}$, so each element of $e-U$ has an inverse, and A is a Q-algebra, q.e.d.

11.13. Corollary. *If A is a Q-algebra, then $A = B_\infty$.*

As another corollary we get the following

11.14. Theorem. *An m-convex B_0-algebra has the extension property if and only if it is not a Q-algebra.*

Proof. If A is a Q-algebra, then $U = \frac{1}{2}B_1$ (cf. proposition 11.12) is an open set having the property that for each element $x \in U$ the series $\sum x^n$ is convergent. This series is clearly divergent for $x = e$ so A has not extension property. Suppose now that A is not a Q-algebra and that the series $\sum a_n x^n$ is convergent for any x belonging to an open subset $U \subset A$. We claim that $\limsup \sqrt[n]{|a_n|} = 0$. If not, then $\limsup \sqrt[n]{|a_n|} = 1/r > 0$, and so, by theorem 11.4, $r_5(x) = \varrho(x) \leqslant r$ for each $x \in U$. By the same theorem we have $r_6(x) \leqslant r$, and so $\sum (x/2r)^n$ converges for each $x \in U$. The sum is clearly $(e-x/2r)^{-1}$ and there is an open set

$$V = e - \frac{1}{2r} U$$

consisting of invertible elements, so A is a Q algebra which is a contradiction. Therefore we have $\lim \sqrt[n]{|a_n|} = 0$, and hence $\sum a_n x^n$ is convergent for each $x \in A$ and A has extension property, q.e.d.

Later on we shall show that any commutative B_0-algebra, which is a Q-algebra must be automatically multiplicatively convex.

It would be interesting to extend theory of multiplicatively convex algebras onto non-locally convex algebras. There may be many approaches to the definition itself (see e.g. [28]); it seems to be interesting to see whether every locally idempotent topological algebra must be a subalgebra of cartesian product of p-normed algebras.

Now we pass to non-m-convex B_0-algebras.

§ 12. Examples of non-m-convex B_0-algebras

For some time it was not clear whether there exist non-m-convex B_0-algebras. The first example of such an algebra was constructed by Arens [1]. It is the following algebra called L^ω.

12.1. Example (Arens [1]). L^ω consists of functions summable with every power $p \geqslant 1$ on the interval $[0, 1]$, with pointwise multiplication and norms

$$|x|_k = \Big(\int_0^1 |x(t)|^k dt\Big)^{1/k}, \qquad k = 1, 2, \ldots;$$

the continuity of multiplication follows from the inequality

$$\Big(\int_0^1 |x(t)y(t)|^n dt\Big)^{1/n} \leqslant \Big(\int_0^1 |x(t)|^{2n} dt\Big)^{1/2n} \Big(\int_0^1 |y(t)|^{2n} dt\Big)^{1/2n}$$

It is clearly a B_0-space, so it is a commutative B_0-algebra. For L^ω fail the following facts true for (commutative) m-convex B_0-algebras:

(12.1.1) *In A there is at least one multiplicative-linear functional.*

(12.1.2) *The operation $x \to x^{-1}$ is continuous on $V = \{x \in A : x^{-1} \in A\}$.*

(12.1.3) *In A there is defined every entire function φ, i.e. for every entire $\varphi(\lambda) = \sum a_n \lambda^n$, the series $\sum a_n x^n$ converges for each $x \in A$.*

So showing failure of these facts we shall show that L^ω is not m-convex.

Proof of (12.1.1). Suppose to the contrary that in L^ω there exists a non-zero multiplicative and linear functional f. So $f(e) = 1$, $e = e(t) \equiv 1$ is the unit of L^ω Therefore f considered as a multiplicative linear functional on $C(0,1) \subset L^\omega$ would be of the form $f(x) = x(t_0)$, $0 \leqslant t_0 \leqslant 1$, for each $x \in C(0,1)$. It is easy to construct a function $\tilde{x}(t)$ such that $\tilde{x}(t)$ is continuous on $[0,1] \dot{-} \{t_0\}$, $\tilde{x}(t) > C > 0$, $\lim_{t=t_0} \tilde{x}(t) = \infty$, and $\tilde{x} \in L^\omega$

Clearly $\tilde{x}$ is invertible in L^ω, and its inverse $\tilde{y}$ is a continuous function, moreover $\tilde{y}(t_0) = 0$. It follows $1 = f(e) = f(\tilde{x}\tilde{y}) = f(\tilde{x})f(\tilde{y}) = f(\tilde{x})\tilde{y}(t_0) = 0$, and this is a contradiction. So (12.1.1) is proved. It also follows that any maximal ideal in L^ω has infinite codimension, and it is dense. It may be proved that any ideal of L^ω which is not dense is properly contained in a closed ideal $\neq L^\omega$, and if I is any closed ideal of L^ω, then $\inf \alpha(f) > 0$, where $\alpha(f)$ is the Lebesgue measure of the set $\{t \in [0,1] : f(t) = 0\}$ (see [27]).

Proof of (12.1.2). It is easy to construct a sequence of positive reals $a_n \to \infty$ such that sequence

$$x_n(t) = a_n \chi_{\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right]}(t) + 1$$

diverges in L^ω Here $\chi_A(t)$ denotes the characteristic function of a set A, i.e.

$$\chi_A(t) = \begin{cases} 1 & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \notin A. \end{cases}$$

On the other hand, there exist x_n^{-1}, and $x_n^{-1} \to e$.

Proof of (12.1.3). We shall show that only entire functions defined in L^ω are polynomials. Suppose to the contrary that $\varphi(\lambda) = \sum a_n \lambda^n$ is a transcendental entire function defined in L^ω. This follows that $\sum a_n t^n x^n$ converges for every $x \in A$, $t > 0$, so for any i

$$\lim_n |a_n x^n|_i t^n = 0 \quad \text{and} \quad \lim \sqrt[n]{|a_n|\,|x^n|_i} = 0.$$

In particular, $\lim \sqrt[n]{|a_n|\,|x^n|_1} = 0$. But

$$|x^n|_1 = \int_0^1 |x(t)|^n dt = |x|_n^n.$$

This follows $\lim \sqrt[n]{|a_n|}\,|x|_n = 0$ for every $x \in A$, so if we put

$$\|x\| = \sup_n \sqrt[n]{|a_n|}\,|x|_n$$

we would obtain a homogeneous norm giving in L^ω the same topology or system $(|x|_i)$, since this system is an increasing family of pseudonorms, and $|a_n| \neq 0$ for infinite many n. It follows that L^ω is a Banach space, which e.g. by (12.1.1) is impossible.

12.2. Example (Żelazko [28]). An algebra l^{1+} consists of all complex sequences $x = (x_n)_0^\infty$ summable with any power $p > 1$, with convolution multiplication. So $l^{1+} = \bigcap_{p>1} l^p$, and as pseudonorms we take p-norms of l_{p_n} where p_n is a sequence of reals decreasing to 1. From the fact that if $x \in l_p$, $y \in l_q$, then $x * y \in l_r$, and

$$\|x * y\|_r \leqslant \|x\|_p \|y\|_q \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r},$$

where $\|x\|_p = \left(\sum |x_n|^p\right)^{1/p}$, it follows that if we put $p_n = 1 + 1/t_n$, $t_1 = 1$, $t_{k+1} = 2t_k + 1$, and $|x|_n = \|x\|_{p_n}$, we get $\|x * y\|_n \leqslant \|x\|_{n+1} \|y\|_{n+1}$, so l^{1+} is a B_0-algebra. We shall show that in l^{1+} there exists a total family of multiplicative linear functionals, but there fails "Wiener property" (true by 10.13 for m-convex algebras), so l^{1+} is a non-m-convex B_0-algebra. So we are going to show that in l^{1+} there is an element z such that $f(z) \neq 0$ for each multiplicative and linear functional of l^{1+}, but z^{-1}

does not exist in l^{1+}. To this end we shall show that any (non-zero) multiplicative and linear functional in l^{1+} is of the form

(12.2.1) $$F(x) = F_\lambda(x) = \sum_0^\infty x_n \lambda^n,$$

where λ is a fixed complex number with $|\lambda| < 1$. In this aim we shall show that if

$$x(\lambda) = \sum_{n=0}^\infty x_n \lambda^n$$

is a holomorphic function defined in the unit disc, $\sum |x_n|^p < \infty$ for $p > 1$, and $x(\lambda_0) = 0$, where $|\lambda_0| < 1$, then

(12.2.2) $$x(\lambda) = (\lambda - \lambda_0) y(\lambda),$$

where $\sum |y_n|^p < \infty$. In fact, let $x(\lambda) = (\lambda - \lambda_0) y(\lambda)$, so $y(\lambda)$ is a holomorphic function defined in unit disc. Let $y(\lambda) = \sum y_n \lambda^n$. We have $x_n = y_{n-1} - \lambda_0 y_n$, so

$$\infty > \Big(\sum_0^\infty |x_n|^p\Big)^{1/p} \geqslant \Big(\sum_1^N |y_{n-1} - \lambda_0 y_n|^p\Big)^{1/p}$$
$$\geqslant \Big|\Big(\sum_0^N |y_{n-1}|^p\Big)^{1/p} - |\lambda_0| \Big(\sum_1^N |y_n|^p\Big)^{1/p}$$

for $N = 1, 2, \ldots$ Therefore

(12.2.3) $$|S_{n-1} - |\lambda_0| S_n| \leqslant \Big(\sum_{n=1}^\infty |x_n|^p\Big)^{1/p} + |y_0| < \infty,$$

where

$$S_n = \Big(\sum_{k=0}^n |y_k|^p\Big)^{1/p}, \quad n = 1, 2, \ldots$$

It is to be shown that $\lim_n S_n < \infty$. Suppose, to the contrary, that $\lim S_n = \infty$. By (12.2.3)

(12.2.4) $$\lim \frac{S_n}{S_{n+1}} = |\lambda_0| < 1.$$

Consequently, it is also $\lim |y_n| = \infty$, otherwise for certain subsequence y_{n_k} it would be

$$|\lambda_0| = \lim \frac{S_{n+1}}{S_n} = \lim \frac{S_{n_k-1} + |y_{n_k}|}{S_{n_k-1}} = 1,$$

which does not hold. Since $x_{n+1} = y_n - \lambda_0 y_{n+1}$ tends to zero, it follows that y_n / y_{n+1} tends to λ_0, and radius of convergence of $\sum y_n \lambda^n$ would be

$|\lambda_0| < 1$ which contradicts the fact that $y(\lambda)$ is holomorphic in the unit disc; thus $\lim S_n < \infty$ and $y \in l^p$. Let now $z = (z_n) \in l^{1+}$ be defined by $z_1 = 1$, $z_i = 0$ for $i \neq 1$, and let F be an arbitrary multiplicative and linear functional in l^{1+}. Put $\lambda_0 = F(z)$. We shall show that $|\lambda_0| < 1$. To this aim interpret any $x \in l^{1+}$ as holomorphic function in the unit disc $x(\lambda) = \sum x_n \lambda^n$. Multiplication in l^{1+} is then pointwise multiplication of these functions. So $z(\lambda) \equiv \lambda$.

It cannot be $|\lambda_0| > 1$, since $z - \lambda_0 e$ is an invertible element of l^{1+} To show that $|\lambda_0| \neq 1$, it is sufficient to show that $|\lambda_0| \neq 1$, since rotations $x(\lambda) \to x(e^{it_0}\lambda)$ are automorphism of l^{1+} into itself. To show that $\lambda_0 \neq 1$ consider the function

$$v(\lambda) = \frac{1}{\lambda}\ln(1-\lambda).$$

It may easily be verified that $v(\lambda) \in l^{1+}$. On the other hand,

$$\frac{1}{v(\lambda)} = \Sigma p_n \lambda^n, \quad \text{where} \quad p_n = o\left(\frac{1}{n[\ln n]^2}\right)$$

(see [35], p. 93, example 8.4), and so $v^{-1} \in l^{1+}$ We have not only this but also $v^{-1} \in l^1$. If we had $F(z) = 1$, then F restricted to $l^1 \subset l^{1+}$ would be of the form $F(u) = u(1)$. So it would be $F(v^{-1}) = v^{-1}(1) = 0$ which contradicts the fact that v^{-1} is invertible in l^{1+} So $F(z) = \lambda_0$, and $|\lambda_0| < 1$. Now let $x \in l^{1+}$; by (12.2.2) we may write $x - x(\lambda_0)e = (z - \lambda_0 e)y$, so that $F\big(x - x(\lambda_0)e\big) = \big(F(z) - \lambda_0\big)F(y) = 0$, and $F(x) = x(\lambda_0)$ which proves formula (12.2.1). To prove that l^{1+} has not the Wiener property we observe that $F(e-z) \neq 0$ for each multiplicative and linear functional in l^{1+}, but $e - z$ is not invertible in l^{1+}, since its convolution inverse would be $\{1, 1, \ldots\}$.

12.3. EXAMPLE (Williamson [26]). W is an example of a non-m-convex B_0-algebra such that there is a dense subalgebra isomorphic with the division algebra of all rational functions of one variable. Let

$$a(n, r) = \begin{cases} (1-r)^{n(1-r)} & \text{for} \quad r \leqslant -1, \\ 1 & \text{for} \quad r = 0, \qquad n = 1, 2, \ldots, \\ (1+r)^{-(1+r)/n} & \text{for} \quad r \geqslant 1, \end{cases} \tag{12.3.1}$$

It may be verified that $a(n, r+s) \leqslant a(4n, r)a(4n, s)$. The algebra W consists of all formal power series

$$x(\lambda) = \sum_{n=-\infty}^{\infty} x_n \lambda^n$$

such that

$$|x|_n = \sum_k a(n, k)|x_k| < \infty, \qquad n = 1, 2, \ldots,$$

with "pointwise multiplication" (convolution multiplication of coefficients). W is clearly a B_0-space, moreover,

$$|xy|_n = \sum_k a(n,k)\Big|\sum_l x_{k-l}y_l\Big| \leqslant \sum_{k,l} a(n,k)|x_{k-l}|\,|y_l|$$
$$\leqslant \sum_{k,l} a(4n, k-l)\,a(4n, l)\,|x_{k-l}|\,|y_l| = |x|_{4n}|y|_{4n},$$

so W is a B_0-algebra. To prove that all rational functions belong to W it is sufficient to show that $x_a(\lambda) = (1-a\lambda)^{-1}$, and $1/\lambda$ belongs to W; but this follows from the fact that $(1-a\lambda)^{-1} = \sum a^n\lambda^n$, and $\sum_{k=0}^{\infty} a(n,k)a^k$ converges for each n. In W there are also no multiplicative linear (non-trivial) functionals, since every such a functional would give an isomorphism between the field of all rational functions and field of complexes.

For other examples, see [17] and [24].

§ 13. Extended spectrum; theorem on entire functions and its applications to Q-algebras and radicals

In the algebra W (example 12.3) there is a dense subalgebra which is a division algebra; therefore there is a dense subset consisting of elements with void spectrum. However, if we take e.g. $x(\lambda)\in W$, $x(\lambda)\equiv\lambda$, then the function

(13.0.1) $$R(\mu, x): (\mu, x) \to (x-\mu e)^{-1}$$

is continuous (with fixed $x = x(\lambda)\equiv\lambda$) at each point $\mu\neq 0$, but is discontinuous at the point 0. Taking $x(\lambda) = \lambda^{-1}$, we infer that $R(\mu, x)$ is continuous for each complex μ, but the function $\mu\to R(\mu x, 1)$ is discontinuous for $\mu = 0$. So we give the following

13.1. Definition. Let A be a topological algebra. Let $x\in A$. The *extended spectrum* $\Sigma_A(x)$ of x is a subset of extended complex plane (Riemann sphere) $C\cup(\infty)$ and it is defined as

(13.1.1) $\Sigma_A(x) = \sigma_A(x)\cup\{\lambda_0 : R(\lambda, x)$ is discontinuous at $\lambda=\lambda_0\}\cup$ $\{\infty$, if and only if $R(1, \lambda x)$ is discontinuous for $\lambda = 0\}$.

13.2. Proposition. *Let A be a B_0-algebra; then the extended spectrum $\Sigma_A(x)$ is never void, moreover, either x or x^{-1} possesses spectrum containing finite complex numbers, and $\Sigma_A(x) = \{\infty\}$ if and only if $\Sigma_A(x^{-1}) = \{0\}$.*

Proof. If x is non-invertible in A, then $\Sigma_A(x)$ is non-void since $\sigma_A(x)$ is non-void. Suppose then that x is invertible in A and that there is no finite $\lambda \in \Sigma_A(x)$. Then setting for any continuous linear functional f defined on A

$$\varphi_f(\lambda) = f\big(R(\lambda, x)\big)$$

we get an entire function of λ. In fact, this function is defined for each complex λ and moreover

$$\lim_{\lambda=0} \frac{\varphi_f(\lambda+\lambda_0)-\varphi(\lambda_0)}{\lambda} = \lim_{\lambda=0} f\left[\frac{1}{[x-(\lambda+\lambda_0)e][x-\lambda_0 e]}\right] = f\big([R(\lambda_0, x)]^2\big),$$

so it is an entire function of λ. We have also

$$\varphi_f(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\varphi_f^{(k)}(0)\lambda^k = \sum_{k=1}^{\infty} f(x^{-k})\lambda^{k-1}.$$

It follows that $\lambda^k x^{-k}$ is for each complex λ weakly convergent to zero, therefore it is a bounded sequence and $t^k\lambda^k x^{-k}$ converges for $|t| < 1$ to zero in A, i.e. $\lambda^k x^{-k}$ converges to zero for each λ. We shall show that $\infty \in \Sigma_A(x)$, and $0 \in \Sigma_A(x^{-1})$. In fact, if $\infty \notin \Sigma_A(x)$, then $R(1, \lambda x)$ would be continuous at $\lambda = 0$, this would follow that

$$\lim_{|\lambda|=\infty} \varphi_f(\lambda) = \lim_{|\lambda|=\infty} f\big(R(\lambda, x)\big) = \lim_{|\lambda|=\infty} \frac{1}{\lambda} f\left(R\left(1, \frac{1}{\lambda}x\right)\right) = \lim_{t=0} t f\big(R(1, tx)\big) = 0\cdot f(-e) = 0.$$

So, since φ_f is an entire function, it would follow that $\varphi_f = 0$ for each f, or $R(\lambda, x) = 0$ for each λ, which is impossible. So $\Sigma_A(x) = \{\infty\}$. It is clear that no λ different from 0 or ∞ can belong to $\Sigma_A(x^{-1})$. If $0 \notin \Sigma_A(x^{-1})$, then it would follow that $\lim x^n = 0$, and since $\lim x^{-n} = 0$ it would be $e = \lim x^n x^{-n} = 0$, which is impossible. So theorem is proved, q.e.d.

13.3. Corollary. *If $\infty^{-1} = 0$, $0^{-1} = \infty$, then we have $\Sigma_A(x^{-1}) = [\Sigma_A(x)]^{-1}$.*

13.4. Definition. The *extended spectral radius* $R_A(x)$ of an element $x \in A$ is defined as

$$R_A(x) = \sup|\Sigma_A(x) \div \{\infty\}| \quad \text{for} \quad \Sigma_A(x) \neq \{\infty\}.$$

If $\Sigma_A(x) = \{\infty\}$, we put $R_A(x) = 0$. $R_A(x)$ may be defined also as $R_A(x) = \inf\{R : R(\lambda, x)$ is continuous for $|\lambda| > R\}$.

We shall prove now a theorem generalizing a part of theorem 11.4.

13.5. Theorem. *Let A be a B_0-algebra, and let $r_1(x)-r_6(x)$ be defined as in theorem* 11.4; *then* $r_1(x) = r_2(x) = r_6(x) = R_A(x)$.

Proof. It is $r_1(x) \geqslant r_2(x)$. Let $t > r_2(x)$. We have $\lim\sup\sqrt[n]{|f(x^n)|} < t$ for each $f \in A^*$, so $(x/t)^n$ tends weakly to zero, and $(x/t')^n = (x/t)^n(t/t')^n$ tends to zero in A for $t' > t$. Therefore, since t is arbitrary $> r_2(t)$, it follows that $(x/t)^n$ tends to zero for each $t > r_2(x)$, so series $\sum(x/t)^n$ converges in A, and it follows that any series $\sum a_n x^n$ converges in A if its radius of convergence is greater than $r_2(x)$. So $r_2(x) \geqslant r_6(x)$. But again, if $|\lambda| > r_6(x)$, then $\sum(x/\lambda)^n$ converges in A. Its sum equals $(e-x/\lambda)^{-1}$, so there exists $(\lambda e-x)^{-1}$. Let $|\lambda|, |\lambda_0| > R > r_6(x)$. It is

$$\|(\lambda e-x)^{-1}-(\lambda_0 e-x)^{-1}\| = \|(\lambda-\lambda_0)(\lambda e-x)^{-1}(\lambda_0 e-x)^{-1}\|$$

$$\leqslant |\lambda-\lambda_0| \sum\frac{1}{|\lambda|}\left\|\left(\frac{x}{\lambda}\right)^n\right\| \sum\frac{1}{|\lambda_0|}\left\|\left(\frac{x}{\lambda_0}\right)^n\right\| \leqslant |\lambda-\lambda_0| \frac{1}{R^2}\left(\sum\left\|\left(\frac{x}{R}\right)^n\right\|\right)^2,$$

where $\|\ \|$ is an arbitrary continuous pseudonorm in A. It follows that $R(\lambda, x)$ is continuous for $|\lambda| > r_6(x)$, and therefore $r_6(x) \geqslant R_A(x)$. It remains to show that $R_A(x) \geqslant r_1(x)$. Let $|\lambda| > R_A(x)$; so $(e-x/\lambda)^{-1}$ is a continuous function of λ, or $(e-\lambda x)^{-1}$ is a continuous function of λ, for $|\lambda| < 1/R_A(x)$. Applying to this function any continuous $f \in A^*$ we have that $\varphi_f(\lambda) = f[(e-\lambda x)^{-1}]$ is a holomorphic function of λ, $|\lambda| < 1/R_A(x)$, and $\varphi_f(\lambda) = \sum f(x^n)\lambda^n$; from this follows that $\lambda^n x^n$ tends weakly to zero for any $|\lambda| < 1/R(x)$, so that it tends also strongly, and for any continuous pseudonorm $\|x\|$ in A it is $\lim\|\lambda^n x^n\| = 0$, or $\lim\sup\sqrt[n]{\|\lambda^n x^n\|} \leqslant 1$, or $\lim\sup\sqrt[n]{\|x^n\|} \leqslant 1/|\lambda|$. This follows

$$r_1(x) = \sup_{\|\ \|}\lim\sup\sqrt[n]{\|x^n\|} \leqslant \frac{1}{|\lambda|}$$

for each λ with $1/|\lambda| > R(x)$. This follows $r_1(x) \leqslant R_A(x)$ which proves the theorem, q.e.d.

13.6. Remark. Since even in commutative case $\mathfrak{M}$ may be a void set (as e.g. for L^ω), we have $r_3(x) = 0$, but $\varrho_A(x)$ may be arbitrary large, so it may be $r_3(x) < \varrho_A(x) = r_4(x)$. Since for $x(t) = 1/(1-t)$, $x \in W$, we have $\sigma_W(x) = \emptyset$, and $\Sigma_W(x) = \{1\}$; this follows that it is possible to have $\varrho(x) = r_4(x) < R_A(x)$. The problem, whether $r_5(x) = R_A(x)$ is still open (we know only that $r_5(x) \leqslant R_A(x)$).

13.7. Remark. In the case where A is an m-convex B_0-algebra it is $\Sigma_A(x) = \sigma_A(x)$.

Now we pass to the proof of a theorem characterizing m-convexity of commutative B_0-algebras by means of entire functions. We have seen that in any m-convex B_0-algebra there are defined all entire functions, and that in certain non-m-convex algebras (e.g. L^ω, example 12.1) there were defined no transcendental entire functions. We have the following

13.8. THEOREM. *A commutative B_0-algebra A is m-convex if and only if there is defined every entire function, i.e. for each entire function $\varphi(\lambda) = \sum a_n\lambda^n$, and for each $x \in A$, the series $\sum a_n x^n$ is convergent in A.*

We shall prove this theorem by means of some lemmas. At the beginning we give the following

13.9. DEFINITION. Let A be a topological algebra, and $X \subset A$. We define *m-convex hull* $H(X)$ of X as

$$H(X) = \operatorname{conv} \bigcup_{n=1}^{\infty} X^n;$$

it is the smallest convex idempotent subset of A containing X, so that X is convex and idempotent if and only if $X = H(X)$.

13.10. LEMMA. *Let A be a B_0-algebra, and let there exists a matrix $C_{i,n}$ of positive reals, $i, n = 1, 2, \ldots$, such that*

$$|x_1 \ldots x_n|_i \leqslant C_{i,n}|x_1|_{i+1} \ldots |x_n|_{i+1}, \quad x_k \in A, \tag{13.10.1}$$

and

$$\sup_n \sqrt[n]{C_{i,n}} = p_i < \infty; \tag{13.10.2}$$

then A is a multiplicatively convex algebra.

Proof. Put $K_i(r) = \{x \in A : |x|_i < r\}$; by (13.10.1) and (13.10.2) we have

$$|x_1 \ldots x_n|_i \leqslant p_i^n |x_1|_{i+1} \quad |x_n|_{i+1};$$

this follows

$$K_{i+1}\left(\frac{1}{p_i}\right) \subset U_i = H\left(K_{i+1}\left(\frac{1}{p_i}\right)\right) \subset K_i(1). \tag{13.10.3}$$

Since U_i is a convex, idempotent and balanced neighbourhood of zero, it forms unit ball of a submultiplicative pseudonorm $\|x\|_i$, satisfying in view of (13.10.3)

$$|x|_i \leqslant \|x\|_i \leqslant p_i\|x\|_{i+1},$$

so system $(\|x\|_i)$ is equivalent to system $(|x|_i)$ and consists of submultiplicative pseudonorms, and so A is m-convex, q.e.d.

13.11. LEMMA. *If A is a commutative B_0-algebra, and there is a matrix $C_{i,n}$ of positive reals such that*

$$|x^n|_i \leqslant C_{i,n}|x|_{i+1}^n, \tag{13.11.1}$$

and (13.10.2) *holds, then A is m-convex.*

Proof. Consider the generalization of formula $xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$, namely

$$x_1 \quad x_n = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k W_k^n(x_1 \ldots x_n), \tag{13.11.2}$$

where

$$W_k^n = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} (x_{i_1} + \quad + x_{i_k})^n$$

We have

(13.11.3) $$|x_1 \ldots x_n|_i = |x_1|_{i+1} \quad |x_n|_{i+1} \left| \frac{x_1}{|x_1|_{i+1}} \cdots \frac{x_n}{|x_n|_{i+1}} \right|_i$$

$$\leqslant |x_1|_{i+1} \quad |x_n|_{i+1} \frac{1}{n!} \sum_{k=1}^{\infty} |W_k^n(\bar{x}_1 \ldots \bar{x}_n)|_i,$$

where $\bar{x}_k = x_k/|x_k|_{i+1}$. We have also the following estimation:

$$|W_k^n(\bar{x}_1 \ldots \bar{x}_n)|_i \leqslant \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} |(\bar{x}_{i_1} + \quad + \bar{x}_{i_k})^n|_i$$

$$\leqslant \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} C_{i,n} |(\bar{x}_{i_1} + \quad + \bar{x}_{i_k})|_{i+1}^n \leqslant C_{i,n} \binom{n}{k} k^n \leqslant C_{i,n} \binom{n}{k} n^n.$$

From (13.11.3) it follows

$$|x_1 \ldots x_n|_i \leqslant C_{i,n} (2n)^n \frac{1}{n!}.$$

Our conclusion follows from lemma 13.10, and from the fact that $\lim \sqrt[n]{(2n)^n/n!} = 2e$, q.e.d.

As a corollary we get also the following

13.12. Proposition. *Let A be a commutative B_0-algebra, and U a convex subset of A. Put $V = \operatorname{conv} U \cup (-U)$, and*

$$Q^n_{\| \|, P} = \sqrt[n]{\sup_{x_i \in P} \|x_1 \ldots x_n\|}, \qquad \tilde{Q}^n_{\| \|, P} = \sqrt[n]{\sup_{x \in P} \|x^n\|};$$

then

(13.12.1) $$\tilde{Q}^n_{\| \|, U} \leqslant \tilde{Q}^n_{\| \|, V} \leqslant Q^n_{\| \|, V} = Q^n_{\| \|, U} \leqslant M \tilde{Q}^n_{\| \|, U},$$

where

$$M = \max_n \frac{2n}{\sqrt[n]{n!}} < \infty.$$

Proof. It is clear that $\tilde{Q}^n_{\| \|, U} \leqslant \tilde{Q}^n_{\| \|, V} \leqslant Q^n_{\| \|, V}$. It remains to prove that $Q^n_{\| \|, V} = Q^n_{\| \|, U}$ and $Q^n_{\| \|, U} \leqslant M \tilde{Q}^n_{\| \|, U}$. To prove the first equality let us observe that

$$\sup_{x_i \in U} \|x_1 \ldots x_n\| = \sup_{x_i \in U \cup (-U)} \|x_1 \ldots x_n\|,$$

so it remains to show that if W is any set, then

$$\sup_{x_i \in W} \|x_1 \ldots x_n\| = \sup_{x_i \in \operatorname{conv} W} \|x_1 \ldots x_n\|.$$

Let $x_1, \ldots, x_n \epsilon \operatorname{conv} W$; so

$$x_i = \sum_{m=1}^{M_i} a_{i,m}\bar{x}_{i,m}, \quad \bar{x}_{i,m} \epsilon W, \quad 0 \leqslant a_{i,m} \leqslant 1, \quad \text{and} \quad \sum_{m=1}^{M_i} a_{i,m} = 1.$$

So we have

$$x_1 \ldots x_n = \sum_{m_1=1}^{M_1} \quad \sum_{m_n=1}^{M_n} a_{1,m_1} \ldots a_{n,m_n}\bar{x}_{1,m_1} \ldots \bar{x}_{n,m_n}, \quad \text{and} \quad \sum_m a_{1,m_1} \ldots a_{n,m_n} = 1,$$

the terms being between 0 and 1. So

$$\|x_1 \ldots x_n\| \leqslant \sum_m a_{1,m_1} \quad a_{n,m_n}\|\bar{x}_{1,m_1} \ldots \bar{x}_{n,m_n}\|$$
$$\leqslant \sum_m a_{1,m_1} \ldots a_{n,m_n} \sup_W \|x_1 \ldots x_n\| = \sup_{x_i \epsilon W} \|x_1 \ldots x_n\|.$$

So

$$\sup_{x_i \epsilon \operatorname{conv} W} \|x_1 \ldots x_n\| \leqslant \sup_{x_i \epsilon W} \|x_1 \ldots x_n\|,$$

but having $W \subset \operatorname{conv} W$ we obtain the equality. To prove $Q^n_{\| \|, U} \leqslant M\tilde{Q}^n_{\| \|, U}$ let us observe that in view of (13.11.2)

$$\|x_1 \ldots x_n\| \leqslant \frac{1}{n!} \sum_{k=1}^{n} \|W^n_k(x_1 \ldots x_n)\|,$$

and since U is convex

$$\sup_{x_i \epsilon U} \|W^n_k(x_1 \ldots x_n)\| \leqslant \sum_{i_p < i_{p+1}} \sup_{x_i \epsilon U} \|(x_{i_1} + \quad + x_{i_k})^n\| \leqslant \binom{n}{k} n^n \sup_{x \epsilon U} \|x^n\|,$$

so

$$\text{(13.12.2)} \qquad \sup_{x_i \epsilon U} \|x_1 \ldots x_n\| \leqslant \frac{(2n)^n}{n!} \sup_{x \epsilon U} \|x^n\| \leqslant M^n \sup_{x \epsilon U} \|x^n\|,$$

and $Q^n_{\| \|, U} \leqslant M\tilde{Q}^n_{\| \|, U}$, q.e.d.

13.13. LEMMA. *If A is a commutative B_0-algebra, and*

$$p_i(x) = \sup_n \sqrt[k]{|x^n|_i} < \infty$$

for each $x \epsilon A$, $i = 1, 2, \ldots$, then A is multiplicatively convex.

Proof. We have

$$p_i(x) = \lim_n \max_{k \leqslant n} \sqrt[n]{|x^k|_i},$$

so $p_i(x)$ is a function of the first Baire class, which is defined on complete metric space. This follows that there is a point $x_0 \epsilon A$ such that $p_i(x)$ is continuous for $x = x_0$, and so there is a constant C and neighbourhood U of x_0 such that $p_i(x) < C$ for $x \in U$. Let $V = U - x_0$; thus V is a neighbourhood of zero in A. For any $z \in V$ we have $z = x - x_0$, $x \in U$. This follows

$$|z^n|_{i-1} \leqslant \sum_k \binom{n}{k} |x^{n-k} x_0^k|_{i-1} \leqslant \sum_k \binom{n}{k} |x^{n-k}|_i |x_0^k|_i \leqslant (2C)^n.$$

We can now find $j(i) > i$, $r(i)$, such that

$$K_{j(i)}(r(i)) \subset V, \quad \text{where} \quad K_a(r) = \{x : |x|_a \leqslant r\}.$$

For any

$$x \in A \frac{x}{|x|_{j(i)}} r(i) \in V,$$

this implies

$$\left| \left(\frac{x}{|x|_{j(i)}} r(i) \right)^n \right|_{i-1} \leqslant (2C)^n,$$

or

$$|x^n|_{i-1} \leqslant \left(\frac{2C}{r(i)} \right)^n |x|^n_{j(i)}. \tag{13.13.1}$$

So if we put $|x|_1' = |x|_1$, and having by induction $|x|_k' = |x|_{n_k}$, we define $|x|_{k+1}' = |x|_{j(n_k)}$. In view of (13.13.1) the new system satisfies (13.11.1), so by lemma 13.11 algebra A is m-convex, q.e.d.

The proof of theorem 13.8 follows from the above lemma and from the following one:

13.14. Lemma. *If A is a B_0-algebra, and in A there is defined every entire function, then*

$$p_i(x) = \sup_n \sqrt[n]{|x^n|} < \infty$$

for $i = 1, 2, \ldots$ and each $x \in A$.

Proof. If not, then for certain i_0, x_0, and sequence of integers k_n it is $\sqrt[k_n]{|x_0^{k_n}|_{i_0}} > n$, and $|x_0^{k_0}|_{i_0}/n^{k_n} > 1$, so that the function

$$\varphi(\lambda) = \sum \frac{\lambda^{k_n}}{n^{k_n}}$$

is an entire function for which $\sum x_0^{k_n}/n^{k_n}$ diverges, q.e.d.

The following question is open:

13.15. Problem. Is the conclusion of theorem 13.8 true for non-commutative B_0-algebras, or, what is equivalent, is it true that if every commutative subalgebra of A is m-convex, then the B_0-algebra A is m-convex?

13.16. Remarks. It may be proved that for every entire function φ there is a non-m-convex B_0-algebra for which the function φ is defined. The properties of non-possessing entire functions and non-possessing multiplicative linear functionals are independent for commutative B_0-algebras: there are algebras with functionals and without entire functions, and without functionals and with transcendental entire functions (cf. [17]). There are even, for any entire φ, commutative B_0-algebras having as dense subsets division algebras (as e.g. W, example 12.3), for which φ is defined (see [24]).

We give now some applications of theorem 13.8.

13.17. THEOREM. *Let A be a commutative B_0-algebra which is a Q-algebra* (see definition 7.7); *then A is multiplicatively convex.*

Proof. By corollary 7.8, the inversion is continuous in A, so that for every $x \in A$ it is $\Sigma_A(x) = \sigma_A(x)$. Let U be a neighbourhood of zero in A such that $e+U$ consists of invertible elements. In a similar way as in proposition 11.12 one may prove that $\varrho(x) < 1$ for each $x \in U$. Since $\varrho(x) = R(x)$, it follows from theorem 13.5 that $R_A(x) = r_6(x) < 1$, so for each $x \in U$ the series $\sum x^n$ converges. But since U absorbs, it follows that for every entire function $\varphi(\lambda) = \sum a_n \lambda^n$ the series $\sum a_n x^n$ converges for each $x \in A$, and so, by theorem 13.8, A is m-convex, q.e.d.

As a corollary we get the following

13.18. THEOREM. *Let A be a commutative B_0-algebra, and $A_0 = \operatorname{rad} A$ its radical; then A_0 is an m-convex algebra, provided it is closed.*

Proof. Since A_0 is closed, $A_1 = A_0 \oplus \{\lambda e\}$ is a closed subalgebra of A. It may easily be verified that $A_0 = \operatorname{rad} A_1$, so every element $x \in A_1 \dot{-} A_0$ is invertible in A_1, and A_1 is a Q-algebra (here $X \dot{-} Y$ denotes the complementation of Y in X), so by the previous theorem it is m-convex, q.e.d.

However the following question is open:

13.19. PROBLEM. Is the radical of a B_0-algebra necessarily closed?

Let us remark that the positive answer to the problem 13.15 would imply proof of theorems 13.17 and 13.18 also in non-commutative case.

§ 14. Elementary properties of entire functions and characterization of commutative B_0-algebras with and without entire functions

14.1. DEFINITION. $\mathscr{E}(A)$ will denote the family of all entire functions defined in a B_0-algebra A. We shall also write $(a_n) \in \mathscr{E}(A)$ instead of $\varphi \in \mathscr{E}(A)$, where $\varphi(\lambda) = \sum a_n \lambda^n$.

14.2. LEMMA. $(a_n) \in \mathscr{E}(A)$ *if an only if* $\lim a_n x^n = 0$ *for each* $x \in A$.

Proof. If $\varphi \in \mathscr{E}(A)$, then obviously $\lim a_n x^n = 0$. Conversely, let $\| \ \|$ be a continuous pseudonorm in A. We have $\lim \|a_n 2^n x^n\| = 0$ for each $x \in A$, so $\sum \|a_n x^n\|$ converges for each $\| \ \|$, and this follows that $\sum a_n x^n$ converges in A, and $\varphi \in \mathscr{E}(A)$, q.e.d.

14.3. Proposition. *$\mathscr{E}(A)$ is an algebra under pointwise addition and multiplication (or convolution multiplication of coefficients).*

Proof. Let $(a_n), (b_n) \in \mathscr{E}(A)$. Obviously $(\lambda a_n + \mu b_n) \in \mathscr{E}(A)$. Let

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k .$$

We have

$$|c_n x^n|_i = \Big|\sum_{k=0}^{n} a_{n-k} x^{n-k} b_k x^k\Big|_i \leqslant \sum_{k=0}^{n} |a_{n-k} x^{n-k}|_{i+1} \, |b_k x^k|_{i+1} .$$

Since $|a_n x^n|_{i+1}$, and $|b_n x^n|_{i+1}$ are summable, then so is $|c_n x^n|_i$, $i = 1, 2, \ldots$, and so, by lemma 14.2, $(c_n) \in \mathscr{E}(A)$, q.e.d.

It shall be convenient to introduce certain relations in the set E of all entire functions of one variable. Therefore, we give the following

14.4. Definition. Let $(a_n) \in E$, and put $p_n = \sqrt[n]{|a_n|}$. We shall write $\varphi' \prec \varphi''$ if for corresponding sequences p_n' and p_n'' it holds

$$0 < \liminf \frac{p_n''}{p_n'} \leqslant \limsup \frac{p_n''}{p_n'} = \infty .$$

(Here we assume $a/0 = \infty$, for $a \neq 0$, and $0/0 = 1$.)

We shall write $\varphi' \sim \varphi''$ if $\varphi' \prec \varphi''$ and $\varphi'' \prec \varphi'$ It is obviously an equivalence relation. Equivalence class containing φ will be denoted by $\{\varphi\}$. $\{\varphi'\} \prec \{\varphi''\}$ means the same relation for each pair of elements.

14.5. Proposition. *Let A be a B_0-algebra. If $\varphi \in \mathscr{E}(A)$, and if $\psi \prec \varphi$, then $\psi \in \mathscr{E}(A)$.*

Proof. Let $\varphi = (a_n)$, $\psi = (b_n)$. There is a constant C such that $\sqrt[n]{|b_n|}/\sqrt[n]{|a_n|} < C$; if $|a_n| \neq 0$, and if $|a_n| = 0$, and n is sufficiently large, then $b_n = 0$. It follows that $|b_n| < M^n |a_n|$ for large n, so $\sum b_n x^n$ converges in A for every x, since $\sum a_n M^n x^n$ converges, q.e.d.

14.6. Corollary. *If $\varphi \in \mathscr{E}(A)$, then $\{\varphi\} \subset \mathscr{E}(A)$, and $\mathscr{E}(A) = \bigcup_{\varphi \in \mathscr{E}(A)} \{\varphi\}$.*

14.7. Definition. $\mathscr{E}^*(A) = \{\{\varphi\} : \varphi \in \mathscr{E}(A)\}$. Obviously $\mathscr{E}(A) = \bigcup \mathscr{E}^*(A)$, and $\{\psi\} \prec \{\varphi\} \in \mathscr{E}^*(A)$ implies $\{\psi\} \in \mathscr{E}^*(A)$. We shall call $\mathscr{E}^*(A)$ (or $\mathscr{E}(A)$) *trivial*, if it consists of single element $\{0\}$ (or $\mathscr{E}(A)$ is the set of all polynomials). A B_0-algebra will be called an *algebra with entire functions*, if $\mathscr{E}^*(A)$ is non-trivial, otherwise it would be called an *algebra without entire functions*.

We shall characterize now commutative B_0-algebras with entire functions.

14.8. PROPOSITION. *Let A be a commutative B_0-algebra; if for any continuous pseudonorm $\| \|$ there exists an open subset $U \subset A$ such that for infinitely many n*

$$\xi_n = \sup_{x \in U} \|x^n\| < \infty, \tag{14.8.1}$$

then $\mathscr{E}(A)$ is non-trivial.

Proof. Choose an $\varepsilon > 0$, and open $V \subset A$ in such a way that

$$\{x + \lambda e : x \in V, |\lambda| < \varepsilon\} \subset U.$$

Let $\mathscr{N}$ denote the set of integers for which $\xi_n < \infty$, where ξ_n are given by (14.8.1). So $\mathscr{N}$ is infinite by our assumption. For a given $n \in \mathscr{N}$ choose reals a_p, $p = 0, 1, \ldots, n$, in such a way that $|a_p| < \varepsilon$, and $a_p \neq a_q$, $p \neq q$. We have

$$(x + a_p e)^n = \sum_{l=0}^{n} \binom{n}{l} a_p^{n-l} x^l, \quad p = 0, 1, \ldots, n.$$

This is a system of simultaneous linear equations with matrix $m_{p,l} = \binom{n}{l} a_p^{n-l}$; thus $\det m_{p,l} \neq 0$ (it is Vandermonde's determinant multiplied by binomial coefficients). So there is a matrix β_p^k, $p, k = 0, \ldots, n$, such that

$$x^k = \sum_{p=0}^{n} \beta_p^k (x + a_p e)^n, \quad k = 1, 2, \ldots, n.$$

This follows

$$\sup_{x \in V} \|x^k\| \leqslant \sum_{p=0}^{n} |\beta_p^k| \sup_{x \in V} \|(x + a_p e)^n\| \leqslant \sum_{p=0}^{n} |\beta_p^k| \sup_{x \in U} \|x^n\| = \xi_n \sum_{p=0}^{n} |\beta_p^k|,$$

whence

$$\eta_n = \sup_{x \in V} \|x^n\| < \infty \quad \text{for} \quad n = 1, 2, \ldots$$

Now by proposition 13.12, formula (13.12.1), we have

$$\sup_{x \in W} \|x^n\| \leqslant M^n \eta_n < \infty,$$

where $W = \operatorname{conv} V \cup (-V)$ is a neighbourhood of zero in A. So for a neighbourhood of zero W it is

$$\gamma_n = \sup_{x \in W} \|x^n\| < \infty, \quad n = 1, 2, \ldots \tag{14.8.2}$$

Taking now as $\| \ \|$ the norms $| \ |_i$, and denoting the corresponding numbers, and neighbourhoods, as given in (13.8.2) by γ_n^i, W_i and setting

$$a_n = \frac{1}{n!} \cdot \frac{1}{\max\limits_{i \leqslant n} \gamma_n^i},$$

we get a transcendental entire function defined in A. In fact, let $x \in A$. For given i we choose positive $t_i \neq 0$ in such a way that $x \cdot t_i \in W_i$. So

$$|a_n x^n|_i = |a_n| t_i^{-n} |(x t_i)^n|_i \leqslant |a_n| t_i^{-n} \cdot \gamma_n^i = \frac{t_i^{-n}}{n!} \cdot \frac{\gamma_n^i}{\max\limits_{l \leqslant n} \gamma_n^l} \leqslant \frac{t_i^{-n}}{n!}$$

for $n \geqslant i$, and so $\lim a_n x^n = 0$, whence, by lemma 14.2, $(a_n) \in \mathscr{E}(A)$, q.e.d.

From this proposition we get the following characterization of B_0-algebras with entire functions:

14.9. Theorem. *Let A be a commutative B_0-algebra. Then $\mathscr{E}^*(A)$ is non-trivial if and only if there exists a matrix $C_{i,n}$ of positive numbers $i, n = 1, 2, \ldots$, such that there exists an equivalent system of pseudonorms satisfying*

(14.9.1) $$|x_1 \ldots x_n|_i \leqslant C_{i,n} |x_1|_{i+1}, \ldots |x_n|_{i+1}$$

for each $x_i \in A$, $i, n = 1, 2, \ldots$

Proof. If such a system exists, then obviously

$$\left(\frac{1}{n! \max\limits_{i \leqslant n} C_{i,n}}\right) \in \mathscr{E}(A),$$

so $\mathscr{E}^*(A)$ is non-trivial. Suppose now that $\mathscr{E}^*(A)$ is non-trivial. So there is an $(a_n) \in \mathscr{E}(A)$ with infinitely many $a_n \neq 0$. Put

$$A_n^i = \{x \in A : |a_k x^k|_i \leqslant 1, \text{ for } k \geqslant n\}.$$

Every A_n^i is closed in A, and for fixed i we have

$$A = \bigcup_{n=1}^{\infty} A_n^i,$$

so there is an $n(i)$ such that $U_i = \operatorname{int} A_{n(i)}^i \neq \emptyset$. We have

$$\sup_{x \in U_i} |a_k x^k| \leqslant 1$$

for each $k \geqslant n(i)$. Let $\mathscr{N}_i$ denote infinite set of integers $\geqslant n(i)$ such that $a_n \neq 0$ for $n \in \mathscr{N}_i$. So we have

$$\xi_n^i = \sup_{x \in U_i} |x^n| \leqslant \frac{1}{|a_n|} < \infty \quad \text{for} \quad n \in \mathscr{N}_i.$$

Thus, by proposition 14.8, formula (14.8.2), there are neighbourhoods of zero W_i and numbers γ_n^i such that

$$\sup_{x \in W_i} |x^n|_i = \gamma_n^i < \infty, \quad i, n = 1, 2, 3, \ldots$$

We may assume that W_i is convex and symmetric, so in view of proposition 13.12 we have

$$\sup_{x_j \in W_i} |x_1 \ldots x_n|_i \leqslant M^n \gamma_n^i .$$

We can choose now such $j(i) > i$, $r(i) > 0$ that $\bar{K}_{j(i)}(r(i)) \subset W_i$, where $\bar{K}_a(r) = \{x \in A : |x|_a \leqslant r\}$. We have then

$$r(i) \frac{x}{|x|_{j(i)}} \in W_i$$

for each $x \in A$, so

$$\left| r(i)^n \frac{x_1}{|x_1|_{j(i)}} \cdots \frac{x_n}{|x_n|_{j(i)}} \right|_i \leqslant M^n \gamma_n^i ,$$

or

$$|x_1 \ldots x_n|_i \leqslant M^n \cdot \frac{1}{r(i)^n} \gamma_n^i |x_1|_{j(i)} \cdots |x_n|_{j(i)} .$$

If we introduce now a new system of pseudonorms setting $|x|_1' = |x|_1$, and if $|x|_k' = |x|_{n_k}$, then $|x'|_{k+1} = |x|_{j(n_k)}$, and then we get an equivalent system satisfying (14.9.1) with

$$C_{i,n} = \frac{M^n}{r(i)^n} \gamma_n^i, \quad \text{q.e.d.}$$

14.10. Corollary. *If $\mathscr{E}^*(A)$ is non-trivial, then there exists an element $(a_k) \in \mathscr{E}(A)$ with every $a_k \neq 0$.*

The following theorem characterizes commutative B_0-algebras without entire functions.

14.11. Theorem. *Let A be a commutative B_0-algebra. Then $\mathscr{E}^*(A)$ is trivial if and only if there exists a continuous pseudonorm $\|x\|$ such that for each open subset $U \subset A$ there is an integer $N(U)$, such that*

$$\sup_{x \in U} \|x^n\| = \infty \quad \text{for} \quad n \geqslant N(U). \tag{14.11.1}$$

Proof. If $\mathscr{E}^*(A)$ is trivial, then by Proposition 14.8, (14.11.1) holds. If $\mathscr{E}^*(A)$ is non-trivial, then $\|x\| \leqslant C|x|_i$ for certain i, and by (14.9.1)

$$\sup_{|x|_{i+1} \leqslant 1} \|x^n\| \leqslant C_{i,n} < \infty, \quad \text{q.e.d.}$$

§ 15. Entire operations in B_0-spaces and their applications to entire functions

15.1. Definitions. Let X, Y be topological linear spaces. A function $y = F(x_1, \ldots, x_n) : X^n \to Y$ is called a *symmetric n-linear form* if it is linear and continuous in each variable separately, and if it is a symmetric function of the variables. A *homogeneous polynomial of degree n* is a function $y = p_n(x) : X \to Y$ of the form $p_n(x) = F(\underbrace{x, \ldots, x}_{n \text{ times}})$, where F is a symmetric n-linear form.

It may easily be verified that if X, Y are B_0-spaces with systems of pseudonorms respectively $(|\ |_i)$ and $(\|\ \|_i)$, and if $P_n(x)$ is a homogeneous polynomial of degree n acting from X to Y, then for any i there is an integer $j(i)$ and a constant C_i such that

$$\|P_n(x)\|_i \leqslant C_i |x|_{j(i)}^n.$$

If $P_n(x)$ is a homogeneous polynomial of degree n, then $P_n(\lambda x) = \lambda^n P_n(x)$ and $P_n(x+\lambda y) = \sum_{k=0}^{n} a_k(x, y)\lambda^k$ where $a_k(x, y)$ are independent of λ. If $P(x)$ is a continuous mapping acting from X to Y and satisfying the above relation, then it is a homogeneous polynomial of degree n, that is there exists a unique symmetric n-linear form $F(x_1, \ldots, x_n)$, called the *polar form* of P such that $P(x) = F(x \ldots x)$. If P is a homogeneous polynomial, F its polar form, then

$$P(x+y) = \sum_{k=0}^{n} \binom{n}{k} F(\underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{n-k});$$

it is also $P(\lambda x + \mu y)$ strongly differentiable, i.e. there exists the limit

$$\lim_{\lambda=0} \frac{P[(\lambda+\lambda_0)x+\mu y]-P[\lambda_0 x+\mu_y]}{\lambda}\, dt = \frac{d}{d\lambda} P(\lambda x+\mu y)\Big|_{\lambda=\lambda_0}$$

and the same with respect to μ. $P(\lambda x + \mu y) = \varphi(\lambda, \mu)$ is infinitely many derivable (with fixed x, y), and

$$Q(x) = \frac{d^k}{d\lambda^k} P(\lambda x + y)\Big|_{\lambda=0}$$

is a homogeneous polynomial of degree k, provided P is a homogeneous polynomial of any degree. The sum of two homogeneous polynomials of a fixed degree n is again such a polynomial. If $P_n^{(k)}(x)$ is a sequence of polynomials of fixed degree n, $k = 1, 2, \ldots$, and for each $x \in X$ there exists in Y the limit

$$Q_n(x) = \lim_{k=\infty} P_n^{(k)}(x),$$

then $Q_n(x)$ is a homogeneous polynomial of degree n, provided X, Y are spaces of type F (see Mazur and Orlicz [12]).

An entire operation is a mapping $x \to A(x)$ of the form

$$A(x) = \sum_{n=0}^{\infty} P_n(x)$$

where $P_n(x)$ are homogeneous polynomials of degree n, and the series is convergent in Y for each $x \in X$.

We are going to prove that any entire operation acting in B_0-spaces is a continuous mapping.

15.2. LEMMA. *Let X, Y be B_0-spaces, and $y = A(x)$ an entire operation acting from X to Y; then $A(x)$ is a continuous mapping at the point $x = 0$.*

Proof. Let us observe that formula (13.11.2) is also true for homogeneous polynomials and corresponding polar forms (the proof of (13.11.2) not depends upon associativity of multiplication), more exactly, if $P_n(x) = F(\underbrace{x, \ldots, x}_{n \text{ times}})$, then

$$F(x_1, \ldots, x_n) = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k W_k^n(x_1, \ldots, x_n),$$

where

$$W_k^n(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} P_n(x_{i_1} + \quad + x_{i_k}),$$

so in the same way as in Proposition 13.12 we get

$$\text{(15.2.1)} \qquad \sup_W \|P_n(x)\| \leqslant M^n \sup_V \|P_n(x)\|,$$

where V is an arbitrary open and convex subset of X, W is a neighbourhood of zero in X given by $W = \operatorname{conv}[V \cup (-V)]$, and $\| \ \|$ is an arbitrary continuous pseudonorm in Y. Since $\sum P_n(2x) = \sum 2^n p_n(x)$ is convergent in Y, it follows that for any continuous pseudonorm $\| \ \|$ we have $\lim 2^n \|P_n(x)\| = 0$, which implies $\varphi_{\| \|}(x) = \sum \|P_n(x)\| < \infty$ for each $x \in X$ and each continuous pseudonorm $\| \ \|$ in Y. Thus, with fixed $\| \ \|$, $\varphi_{\| \|}(x)$ is a function of the first Baire class in X, so there is an open set $V \subset X$ such that $\varphi_{\| \|}(x) < C$ for every $v \in V$ Hence $\sum \|P_n(x)\| < C$, which implies $\|P_n(x)\| < C$ on V. It follows, in view of (15.2.1), that

$$\sup_{x \in W} \|P_n(x)\| \leqslant CM^n$$

If

$$U = \frac{1}{2M} W,$$

U is a neighbourhood of zero in X, and

$$\sup_{x\in U}\|P_n(x)\| \leqslant \frac{C}{2^n},$$

so for given ε we can choose an integral N such that

$$\sum_{k=N+1}^{\infty}\sup_{x\in U}\|P_k(x)\| < \frac{\varepsilon}{2}.$$

We can find now a neighbourhood $U_0 \subset U$ of zero such that for $x \in U_0$ we have

$$\Big\|\sum_{k=0}^{N}P_k(x) - P_0(0)\Big\| < \frac{\varepsilon}{2}.$$

Thus for any $x\in U_0$ we have

$$\|A(x)-A(0)\| = \Big\|\sum_{k=0}^{N}P_k(x)-P_0(0)+\sum_{k=N+1}^{\infty}P_k(x)\Big\|$$
$$\leqslant \Big\|\sum_{k=0}^{N}P_k(x)-P_0(0)\Big\| + \Big\|\sum_{k=N+1}^{\infty}P_k(x)\Big\| < \varepsilon.$$

Thus for every continuous pseudonorm $\|\ \|$ in Y we have

$$\lim_{x=0}\|A(x)-A(0)\| = 0,$$

so $A(x)$ is a mapping continuous at the point $x = 0$, q.e.d.

To prove continuity of $A(x)$ in an arbitrary point we shall use Taylor expansions.

15.3. THEOREM. *If X, Y are B_0-spaces, and $A(x)$ is an entire operation acting from X to Y, then $A(x)$ is a continuous mapping.*

Proof. Let $f\in Y^*$. Consider function

$$\varphi_f(\lambda) = f[A(x+\lambda h)] = \sum_{n=0}^{\infty}f\big(P_n(x+\lambda h)\big), \tag{15.3.1}$$

where x, h are fixed elements of X, λ is a complex variable. Consider an auxiliary function

$$\varphi_f(\xi,\lambda) = f[A(\xi x+\lambda h)] = \sum_{n=0}^{\infty}f[P_n(\xi x+\lambda h)]$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}f[F_n(\underbrace{x\ldots x}_{k}\underbrace{h\ldots h}_{n-k})]\,\xi^k\lambda^{n-k},$$

where F_n is the polar form of P_n, $(\xi,\lambda)\in C^2$. Since the right-hand series converges for each $(\xi,\lambda)\in C^2$, (x,h)-fixed, it follows by a theorem of

Hartogs (cf. [9], p. 108) that the series converges uniformly on compact subsets of C^2. Setting $\xi = 1$, we get that the right-hand series of (15.3.1) converges almost uniformly (uniformly on compact subsets of C), so that $\varphi_f(\lambda)$ is an entire function of λ (with fixed x and h). We can therefore differentiate the right-hand series of 15.3.1 term by term, so

$$\frac{d^n}{d\lambda^n}\varphi_f(\lambda)\Big|_{\lambda=0} = \sum_{k=0}^{\infty}\frac{d^n}{d\lambda^n}f\big(P_k(x+\lambda h)\big)\Big|_{\lambda=0}$$

But

$$\frac{d^n}{d\lambda^n}f\big(P_k(x+\lambda h)\big)\Big|_{\lambda=0} = f\left(\frac{d^n}{d\lambda^n}P_k(x+\lambda h)\right)\Big|_{\lambda=0}.$$

Write

$$Q_n^{(k)}(h) = \frac{d^n}{d\lambda^n}P_k(x+\lambda h)\Big|_{\lambda=0};$$

it is a homogeneous polynomial in h of degree n. We have also

$$\sum_{k=0}^{\infty} f\big(Q_n^{(k)}(h)\big) < \infty$$

for any $f \in Y^*$. It follows that $Q_n^{(k)}(h)$ weakly tends to zero as $k \to \infty$, so with fixed h (and x) it is a bounded sequence in Y, so for each continuous pseudonorm $\| \ \|$ in Y there is a constant C such that

$$\|Q_n^{(k)}(h)\| \leqslant C, \quad k = 1, 2, \dots$$

Applying the same reasoning to $A(2x) = \sum 2^n P_n(x)$, we get $\|2^k Q_n^{(k)}(h)\| \leqslant C'$, so

$$\|Q_n^{(k)}(h)\| \leqslant \frac{C'}{2^k}, \quad k = 0, 1,$$

It follows that $\sum\limits_{k=0}^{\infty} Q_n^{(k)}(h)$ converges in Y, so its sum $Q_n(h)$ is a homogeneous polynomial of degree n (cf. [12]). We have

$$f\big(Q_n(h)\big) = \sum_{k=0}^{\infty} f\big(Q_n^{(k)}(h)\big) = \frac{d^n}{d\lambda^n}\varphi_f(\lambda)\Big|_{\lambda=0}$$

It follows that for any $f \in Y^*$

$$\varphi_f(\lambda) = \sum \frac{1}{n!} f\big(Q_n(h)\big)\lambda^n,$$

the series being convergent for each complex λ. So, as before, $Q_n(h)\lambda^n/n!$ is weakly convergent to zero, and it is bounded, thus it is strongly convergent to zero, and therefore summable. This follows

$$f\big(A(x+\lambda h)\big) = f\Big(\sum \frac{1}{n!} Q_n(h)\lambda^n\Big)$$

for each $f \in Y^*$, $h \in X$, so

$$A(x+h) = \sum \frac{1}{n!} Q_n(h),$$

and $B(h) = A(x+h)$ is an entire operation in h. By lemma 15.2, $B(h)$ is continuous at the point $h = 0$, but it proves that $A(x)$ is continuous at point x, so it is continuous in X, q.e.d.

15.4. COROLLARY. *Every entire function defined in a B_0-algebra A (commutative or not) is a continuous operation $A \to A$.*

(This is a positive answer to a question stated in [17].)

15.5. Remark. From 15.3 it also follows that any entire function with coefficients $a_n \in A$ is also continuous. Let us remark that it is possible for certain B_0-algebras to possess entire functions of this type, having at the same time $\mathscr{E}^*(A)$ trivial. So we give the following

15.6. EXAMPLE. Let A be the cartesian product of L^ω (example 12.1), and Banach algebra $C(0, 1)$. Clearly $\mathscr{E}^*(A)$ is trivial. On the other hand, if

$$a_n = \Big(0, \frac{1}{n!}\Big), \qquad 0 \in L^\omega,$$

$1/n!$ is a constant function from $C(0, 1)$, then clearly the series $\sum a_n x^n$ converges for each $x \in A$.

15.7. PROPOSITION. *Let $A(x) = \sum P_n(x)$ be an entire operation acting from X to Y, where X, Y are B_0-spaces. Then for each continuous pseudonorm $\| \;\|$ in Y and each $x_0 \in X$, there is a neighbourhood V of x_0 such that*

$$\sum_n \sup_V \|P_n(x)\| < \infty. \tag{15.7.1}$$

Proof. We know, by theorem 15.3, that $A(x)$ is a continuous mapping and that $\varphi_f(\lambda) = f\big(A(\lambda x)\big)$ is for any $f \in Y^*$ an entire function of λ. $\varphi_f(\lambda) = \sum f\big(P_n(x)\big)\lambda^n$, so, by the Cauchy formula

$$f\big(P_n(x)\big) = \frac{1}{n!}\varphi_f(\lambda)\Big|_{\lambda=0}^{(n)} = \frac{1}{2\pi i}\int_\Gamma \frac{f\big(A(\lambda x)\big)}{\lambda^{n+1}}\,d\lambda.$$

Since $A(\lambda x)$ is Riemann integrable on compact subsets of complex plane, it follows

$$P_n(x) = \frac{1}{2\pi i}\int_\Gamma \frac{A(\lambda x)}{\lambda^{n+1}}\,d\lambda,$$

where as Γ we take $\{\lambda : |\lambda| = 2\}$. So

$$(15.7.2) \qquad \|P_n(x)\| = \frac{1}{2\pi}\left\| \int_\Gamma \frac{A(\lambda x)}{\lambda^{n+1}}\, d\lambda \right\| \leqslant 2^{-n} \sup_{|\lambda|=2} \|A(\lambda x)\|.$$

For any complex λ_0, $A(\lambda_0 x)$ is continuous function of x, so there is a neighbourhood V_{λ_0} of x_0 such that

$$\sup_{x \in V_{\lambda_0}} \|A(\lambda_0 x)\| \leqslant M_{\lambda_0} < \infty.$$

This follows that

$$\sup_{x \in V_{\lambda_0}} \|A(\lambda x)\| < 2M_{\lambda_0}$$

holds for a neighbourhood of λ_0, so covering Γ by a finite number of such neighbourhoods corresponding, say, to $\lambda_0, \lambda_1, \ldots, \lambda_p$ we have

$$\sup_{x \in V} \sup_{\lambda \in \Gamma} \|A(\lambda x)\| \leqslant M,$$

where $V = V_{\lambda_0} \cap \quad \cap V_{\lambda_p}$ is a neighbourhood of the point x_0 and $M = \max(2\lambda_0, \ldots, 2\lambda_p)$. So, by (15.7.2),

$$\sup_V \|P_n(x)\| \leqslant 2^{-n} M,$$

which implies (15.7.1), q.e.d.

We pass now to the proof of a theorem on superposition of entire operations. Before we give the following

15.8. Definition. A mapping $\lambda \to A(\lambda)$ from complex plane to a metric space X is called (*strongly*) *differentiable* at $\lambda = \lambda_0$, if the limit

$$\lim_{\lambda=0} \frac{A(\lambda+\lambda_0) - A(\lambda_0)}{\lambda}$$

exists in X. If f is a continuous linear functional defined on X, then

$$f\left(\frac{d^n}{d\lambda^n} A(\lambda)\right) = \frac{d^n}{d\lambda^n} f\big(A(\lambda)\big),$$

provided $A(\lambda)$ is n times differentiable. The following lemma is obvious.

15.9. Lemma. *If X is a B_0-space, $a_n \in X$, and the series $A(\lambda) = \sum a_n \lambda^n$ converges in X in a neighbourhood of $\lambda = 0$, then*

$$\frac{d^n}{d\lambda^n} A(\lambda)\bigg|_{\lambda=0}$$

exists and equals $n!\,a_n$.

15.10. Lemma. *If X, Y are two metric spaces, $A(\lambda) = \sum a_n\lambda^n$ a differentiable function on X, and P a polynomial from X to Y, then $P(A(\lambda))$ is also differentiable.*

Proof. Let F be the polar form of P. We have

$$\begin{aligned}
&\lambda^{-1}[P(A(\lambda+\lambda_0))-P(A(\lambda_0))]\\
&= \lambda^{-1}[P([A(\lambda+\lambda_0)-A(\lambda_0)+A(\lambda_0)]-P(A(\lambda_0))]\\
&= \lambda^{-1}\sum_{k=0}^{n-1}\binom{n}{k}F(\underbrace{A(\lambda+\lambda_0)-A(\lambda_0),\ldots,A(\lambda+\lambda_0)-A(\lambda_0)}_{n-k\text{ times}},\underbrace{A(\lambda_0)\ldots A(\lambda_0)}_{k\text{ times}})\\
&= \sum_{k=0}^{n-1}\binom{n}{k}F\Big(\frac{A(\lambda+\lambda_0)-A(\lambda_0)}{\lambda},\underbrace{A(\lambda+\lambda_0)-A(\lambda_0),\ldots,A(\lambda+\lambda_0)-A(\lambda_0)}_{n-k-1\text{ times}},\\
&\qquad\underbrace{A(\lambda_0),\ldots,A(\lambda_0)}_{k\text{ times}}\Big) \to nF\Big(\frac{d}{d\lambda}A(\lambda),\underbrace{A(\lambda)\ldots A(\lambda)}_{n-1\text{ times}}\Big)\Big|_{\lambda=\lambda_0}
\end{aligned}$$

In this way it may be also established the existence of further derivatives of $P(A(\lambda))$, q.e.d.

15.11. Theorem. *Let X, Y, Z be B_0-spaces with pseudonorms respectively $(x)_n$, $|x|_n$ and $\|x\|_n$. Let A', A'' be entire operations: $y = A'(x)$: $X\to Y$, $z = A''(y): Y\to Z$. Then $z = A(x) = A''[A'(x)]$ is an entire operation acting from X to Z, and*

$$A(x) = \sum_{n=0}^{\infty} P_n(x),$$

where $P_n(x)$ are homogeneous polynomials satisfying

$$n!P_n(x) = \frac{d^n}{d\lambda^n}A(\lambda x)\Big|_{\lambda=0}. \tag{15.11.1}$$

Proof. Let

$$A'(x) = \sum_{n=0}^{\infty} P'_n(x), \qquad A''(y) = \sum_{n=0}^{\infty} P''_n(y),$$

where P'_n, P''_n are homogeneous polynomials of degree n. Let $f\in Z^*$ and put

$$\varphi_f^x(\lambda) = f(A(\lambda x)) = \sum_{n=0}^{\infty} f\big(P''_n(A'(\lambda x))\big).$$

First of all we shall show that $\varphi_f^x(\lambda)$ is an entire function of λ. Since

$$A'(\lambda x) = \sum_n P'_n(x)\lambda^n,$$

it follows that with fixed x it is a differentiable mapping, and, by lemma 15.10, $P_n''(A'(\lambda x))$ is also differentiable. Moreover, if we set

$$a(\lambda) = \sum_{l=0}^{n} P_l'(\lambda x), \qquad b(\lambda) = \sum_{l=n+1}^{\infty} P_l'(\lambda x)\lambda^{-n-1}$$

we get two differentiable mappings of C into Y. But

$$P_k''(A'(\lambda x)) = P_k''(a(\lambda)+\lambda^{n+1}b(\lambda))$$
$$= \sum_{l=0}^{k}\binom{k}{l} F_k''[\underbrace{a(\lambda)\dots a(\lambda)}_{k-l \text{ times}}, \underbrace{b(\lambda)\dots b(\lambda)}_{l \text{ times}}]\lambda^{(n+1)},$$

where F_k'' is polar form of P_k''. Since $F_k''[\dots a(\lambda)\dots b(\lambda)]$ is differentiable, it follows that

(15.11.2)
$$\frac{d^n}{d\lambda^n} P_k''[A'(\lambda x)]\Big|_{\lambda=0} = \frac{d^n}{d\lambda^n} F_k''[a(\lambda)\dots a(\lambda)]\Big|_{\lambda=0}$$
$$= \frac{d^n}{d\lambda^n} P_k''\Big[\sum_{l=0}^{n} P_l'(\lambda x)\Big]\Big|_{\lambda=0}$$

Take complex parameter η and consider the mapping $(\lambda, \eta) \to X$ given by

$$\sum_{l=0}^{n} P_l'(\lambda x)\eta^{n-l}.$$

The series

$$\sum_k P_k''\Big(\sum_{l=0}^{n} P_l'(\lambda x)\eta^{n-l}\Big)$$

is convergent in Z, and its k-th term is a homogeneous polynomial of degree kn in two variables (λ, η). So by theorem of Hartogs (cf. [9], p. 108) the series

$$\varphi(\lambda, \eta) = \sum_k f\Big[P_k''\Big(\sum_{l=0}^{n} P_\lambda'(\lambda x)^{n-l}\Big)\Big], \qquad f \in Z^*,$$

is convergent uniformly on compact subsets of C^2 and may be differentiated term by term, moreover $\varphi(\lambda, 1)$ is an entire function. So

(15.11.3)

$$\frac{d^n}{d\lambda^n}\varphi(\lambda, \eta)\Big|_{\lambda=0,\eta=1} = f\Big(\sum_k \frac{d^n}{d\lambda^n} P_k''(a(\lambda))\Big) = f\Big(\sum_k \frac{d^n}{d\lambda^n} P_k''(A'(\lambda x))\Big)\Big|_{\lambda=0},$$

which holds by (15.11.2). We have then

$$\frac{d^n}{d\lambda^n}\varphi(\lambda,\eta)\Big|_{\lambda=0,\eta=1} = \frac{d^n}{d\lambda^n}\varphi_f^x(\lambda)\Big|_{\lambda=0},$$

so $\varphi_f^x(\lambda) = \varphi(\lambda, 1)$ and it is an entire function. Moreover

$$\frac{d^n}{d\lambda^n}P_k''\big(A'(\lambda x)\big)\Big|_{\lambda=0} = \frac{d^n}{d\lambda^n}P_k''\Big(\sum_{l=0}^{n}P_l'(\lambda x)\Big),$$

so it is a homogeneous polynomial of degree n. We shall denote this polynomial by $Q_n^{(k)}(x)$, it acts from X to Z. So, by (15.11.3),

$$\frac{d^n}{d\lambda^n}\varphi_f^x(\lambda)\Big|_{\lambda=0} = \sum_{k=0}^{\infty}f\big(Q_n^{(k)}(x)\big).$$

By the same arguments as in the proof of theorem 15.3, we can see that the series

$$\sum_{k=0}^{\infty}Q_n^{(k)}(x)$$

converges in Z, and by a theorem of Mazur and Orlicz [12] its sum is a polynomial $Q_n(x)$ of degree n acting from X to Z. We have therefore

$$\varphi_f^x(\lambda) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n}{d\lambda^n}\varphi_f^x(\lambda)\Big|_{\lambda=0} = \sum_{n=0}^{\infty}\frac{1}{n!}f\big(Q_n(x)\big),$$

or

$$f\big(A(\lambda x)\big) = \sum f\Big(\frac{1}{n!}Q_n(x)\Big)\lambda^n,$$

and again, as before, going through weak convergence to zero of $2^nQ_n(x)/n!$ its boundedness, and therefore strong summability we get that the series

$$\sum\frac{1}{n!}Q_n(x)$$

converges in Z, so

$$f\big(A(\lambda x)\big) = f\Big[\sum\frac{1}{n!}Q_n(\lambda x)\Big]$$

for every $x\in X$, $f\in Z^*$, $\lambda\in C$, and

$$A(x) = \sum\frac{1}{n!}Q_n(x) = \sum P_n(x).$$

Thus $A(x)$ is an entire operation $(P_n(x) = Q_n(x)/n!)$. Moreover, since $A(\lambda x) = \sum P_n(\lambda x) = \sum P_n(x)\lambda^n$, it follows by lemma 15.9 that (15.11.1) holds, q.e.d.

As a corollary we get the following

15.12. THEOREM. *Let A be a B_0-algebra and $(a_n), (b_n) \in \mathcal{E}(A)$. If $\varphi_1 = \sum a_n\lambda^n$, $\varphi_2 = \sum b_n\lambda^n$, and $\psi = \varphi_1(\varphi_2(\lambda)) = \sum C_n\lambda^n$, then $(C_n) \in \mathcal{E}(A)$.*

Proof. $\varphi_1(\varphi_2(x))$ is an entire operation acting from A to A. It may be easily verified that

$$\frac{1}{n!}\frac{d^n}{d\lambda^n}\varphi_1(\varphi_2(\lambda x))\Big|_{\lambda=0} = C_n x^n$$

for each $x \in A$. So, by theorem 15.11, the series $\sum C_n x^n$ converges in A for each x, q.e.d.

This solves a problem stated in [17].

§ 16. Final remarks

Let A be a commutative B_0-algebra with non-trivial $\mathcal{E}^*(A)$. We may assume that its topology is given by a system of pseudonorms satisfying (14.9.1). Setting $Q^n_{a,V} = Q^n_{|\ |a,V}$, $Q^n_{a,\beta} = Q^n_{a,V}$, where $V = \{x \in A : |x|_\beta < 1\}$ ($Q^n_{|\ |,V}$ is defined as in proposition 13.12), we have $Q^n_{a,\beta} < \infty$ for $a < \beta$.

The following Lemma is evident:

16.1. LEMMA. *$Q^n_{a,\beta}, Q^n_{a,v}$ being defined as above, we have the following inequalities:*

(16.1.1) $$U \subset V \text{ implies } Q^n_{a,U} \leqslant Q^n_{a,V},$$

(16.1.2) $$Q^n_{a,\beta} \geqslant Q^n_{a,\beta+1},$$

(16.1.3) $$Q^n_{a,\beta} \leqslant Q^n_{a+1,\beta},$$

(16.1.4) $$Q^{2n}_{a,\beta} \leqslant Q^n_{a+1,\beta},$$

(16.1.5) $$Q^{2n}_{a,\beta+1} \leqslant [Q^n_{a,\beta}]^{1/2}.$$

We have the following

16.2. PROPOSITION. *If A is a commutative B_0-algebra, then for any $(a_n) \in \mathcal{E}(A)$, and any integer a there is a $\beta > a$ such that*

(16.2.1) $$\sup_n p_n Q^n_{a,\beta} < \infty,$$

where $p_n = \sqrt[n]{|a_n|}$.

Proof. By propositions 15.7 and 13.12 there is in A a neighbourhood V of zero such that

(16.2.2) $$\sum_n |a_n| \sup_{x_i \in V} |x_1 \ldots x_n|_a < \infty.$$

We can choose now a $\beta > \alpha$ such that $\{x \in A : |x|_\beta < r\} \subset V$ for a certain $r > 0$.

But from this it follows that

$$\limsup \sqrt[n]{a_n} \sup_{|x_i| < 1/r} \sqrt[n]{|x_i \dots x_n|_\alpha} < 1,$$

which implies (16.2.1), q.e.d.

We do not know the answer to the following

16.3. Problem. Let $(a_n) \in \mathscr{E}(A)$. Is it true that for any α there exists a β such that

(16.3.1) $$\lim_n p_n Q^n_{\alpha,\beta} = 0,$$

where $p_n = \sqrt[n]{|a_n|}$?

The following problem is also open.

16.4. Problem. Does the class of non-m-convex B_0-algebras possess the extension property (Definition 11.10)?

The answer to the following question would give a positive answer to the problems 16.3 and 16.4.

16.5. Problem. Let A be a non-m-convex B_0-algebra. Does there exist a system of pseudonorms $(|x|_\alpha)$ giving its topology and such that

(16.5.1) $$Q^n_{\alpha,\beta} \leqslant Q^{n+1}_{\alpha,\beta} \quad (\text{or } Q^n_{\alpha,\beta} \leqslant Q^{2n}_{\alpha,\beta})$$

for any $\alpha < \beta$?

Note that always for fixed α, β formula (16.5.1) is true for infinitely many n (if α is sufficiently large).

16.6. Let $(a_n) \in \mathscr{E}(A)$. By (16.2.2) we have

$$\lim_{n=\infty} |a_n| \sup_{x_i \in V} |x_1 \dots x_n|_\alpha = 0;$$

therefore setting

(16.6.1) $$|\varphi|_{\alpha,V} = \sup_n |a_n| \sup_{x_i \in V} |x_1 \dots x_n|_\alpha,$$

we have a Banach space $\mathscr{E}_{\alpha,V}$ of all complex sequences such that $|\varphi|_{\alpha,V} < \infty$. Moreover, by (16.1.1), we have $|\varphi|_{\alpha,V} < |\varphi|_{\alpha,U}$ for $V \subset U$, and therefore $\mathscr{E}_{\alpha,V} \supset \mathscr{E}_{\alpha,U}$, and the natural embedding is continuous. We may put now

$$\mathscr{E}_\alpha = \liminf_{\{U\}} \mathscr{E}_{\alpha,U},$$

where $\{U\}$ is a set of all neighbourhoods of zero in A directed by inclusion. This inductive limit is formed by all sequences a_n such that $\sum a_n |x^n|_\alpha$

is convergent in some neighbourhood of zero of A. We have $\mathscr{E}_a \supset \mathscr{E}_{a+1}$ and the natural embedding is continuous. Setting

$$\tilde{\mathscr{E}} = \operatorname*{lim\,proj}_{a} \mathscr{E}_a$$

we have $\tilde{\mathscr{E}} \supset \mathscr{E}(A)$. The problem is whether the two sets are equal.

There are also some questions connected with the space $\mathscr{E}(A)$ (which may be topologized in a way as in 16.6). Namely $\mathscr{E}(A)$ is somewhat analogous to the "approximative dimension" of topological linear spaces. $\mathscr{E}(A)$ is invariant under isomorphisms, moreover, if $A = H(B)$, where H is an algebra homomorphism of B onto A, then $\mathscr{E}(B) \subset \mathscr{E}(A)$; if B is a subalgebra of A, then $\mathscr{E}(A) \subset \mathscr{E}(B)$. It would be interesting to check what are common properties of B_0-algebras having the same $\mathscr{E}(A)$, and whether $\mathscr{E}(A) = \mathscr{E}(B)$ as sets implies that they have the same topology given by 16.6.

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