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On the theory of order statistics of the flexible Lomax distribution

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Abstract

This paper studies the flexible Lomax distribution's order statistics with graphical and numerical findings. Along with the quantitative measurements, some plots are furnished, including those for the skewness and kurtosis measures. We will dwell on the numerous results that relate to statistics of moments of order. We consider the single and product moment of order statistics from the new distribution. Further, we establish some recurrence relation for single moments of order statistics. We have sought to apply the derived relations to empirically evaluate the moments of smallest (largest) order statistics to establish well-known moments and related measures. For order statistics of a flexible Lomax distribution, exact analytical expressions of entropy, residual entropy, and past latent entropy are determined.

Keywords: *Weibull distribution, weighted family, moments, simulation, maximum likelihood estimation*

1. Introduction

Order statistics have been used in a wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distribution, goodness of fit-tests, quality control, and analysis of censored samples. One of the most essential techniques in non-parametric statistics and inference is order statistics. An order statistic tree, for instance, is a variation of the binary search tree used in data structuring. Order statistics are applied in life testing to speed up component testing. The use of recurrence relations for the moments of order statistics is quite well-known in the statistical literature (see e.g. [2, 8]). For the improved form of these results, Samuel and Thomes [15], Arnold et al. [2], and Ali and Khan [1] reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Until recently, Dar and Abdullah [4]

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studied the sampling distribution of order statistics of the two parametric Lomax distributions and derived the exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of Lomax distribution. Asadi et al. [3] explored Rényi information and entropy, also referred to as dynamic information divergence, concerning the order of entropy. By establishing a residual entropy of order statistics premised on quantiles, Sunoj et al. [17] investigated the ordered observations for used products. Mustafa et al. [12] derived the order statistics of inverse Pareto distribution, however, the numerical application of these expressions seems a far-fetched notion. Keeping this in view, we tried to fill this void by providing the practical utility of these recurrence relations. The fact that many of these relations and identities express higher-order moments in terms of lower-order moments facilitates the ability to evaluate higher-order moments. Furthermore, these relations and identities offer some streamlined verifications to determine whether the computation of moments of order statistics was accurate.

1.1. Definitions

A random variable X with a range of values $(0,\ \infty)$ is said to have the flexible Lomax distribution (FLD), if its probability density function (pdf) is given by [9]

$$f(x) = \frac{\alpha\lambda}{\theta} x^{\lambda-1}\left(1+\left(\frac{x}{\theta}\right)^{\lambda}\right)^{-\alpha-1}, \quad x \geq 0,\ \alpha, \lambda, \theta \geq 0 \tag{1}$$

Here α is the shape parameter. The hazard rate function of FLD can model failure rates of both monotonic and non-monotonic nature.

$$F(x) = 1-\left(1+\left(\frac{x}{\theta}\right)^{\lambda}\right)^{-\alpha} \tag{2}$$

$$\overline{F}(x) = \left(1+\left(\frac{x}{\theta}\right)^{\lambda}\right)^{-\alpha} \tag{3}$$

Noting that all FLDs possess the upside-down bathtub shape for their hazard rates, we consider a flexible Lomax distribution that can be effectively used to model the upside-down bathtub shape hazard rate data. The following functional relationship exists between the pdf and cdf of the FLD

$$f(x) = \frac{\alpha\lambda}{\theta} x^{\lambda-1}\left(1+\left(\frac{x}{\theta}\right)^{\lambda}\right)^{-1}(1-F(x)) \tag{4}$$

Table 1 provides the empirical findings for some basic quantities of the FLD distribution which includes the first four ordinary moments, variance, standard deviation, coefficient of variation (CoV) and the coefficients of skewness (CoS) and kurtosis (CoK). Plots of the Bowley skewness and Moors kurtosis are depicted in Figure 1. Fayomi et al. [7] employed another method, termed MacGillivray (MGs) skewness (due to [11]) to evaluate the skewness measure based on the quantile. These plots are displayed in Figure 2. These plots show the flexible behavior of FLD for selected parameter combinations.

To study the distributional behaviour of the set of observations, we can use minimum and maximum (min-max) plots of the order statistics. Min-max plot depends on extreme order statistics. It is introduced to capture all information about the distribution's tails and the whole data distribution.

Table 1. Moments and related measures of the FLD $(\alpha, \lambda, \theta)$ for selected parameters

Parameters $(\alpha,\ \lambda,\ \theta)$	$\mathbb{E}(X)$	$\mathbb{E}(X^2)$	$\mathbb{E}(X^3)$	$\mathbb{E}(X^4)$	$\mathbb{V}(X)$	$\mathbb{SD}(X)$	$\mathbb{C}oV$	$\mathbb{C}oS$	$\mathbb{C}oK$
(0.3, 1.6, 2.7)	23.593	986.803	60427.71	4339151	430.193	20.741	0.879	0.553	2.323
(1.3, 0.6, 1.5)	4.288	152.132	8915.759	627496	133.745	11.565	2.697	4.785	28.807
(1.9, 1.6, 1.5)	1.607	4.575	39.09	1075.887	1.991	1.411	0.878	5.443	120.065
(2.9, 1.6, 1.5)	1.079	1.601	4.141	22.867	0.437	0.661	0.613	1.702	21.747
(3.3, 2.1, 1.3)	0.554	0.435	0.461	0.669	0.128	0.358	0.647	1.713	10.024
(3.5, 2.1, 1.3)	0.533	0.4	0.4	0.533	0.116	0.34	0.637	1.614	8.982
(3.9, 2.2, 1.3)	0.499	0.345	0.312	0.363	0.096	0.31	0.622	1.463	7.606

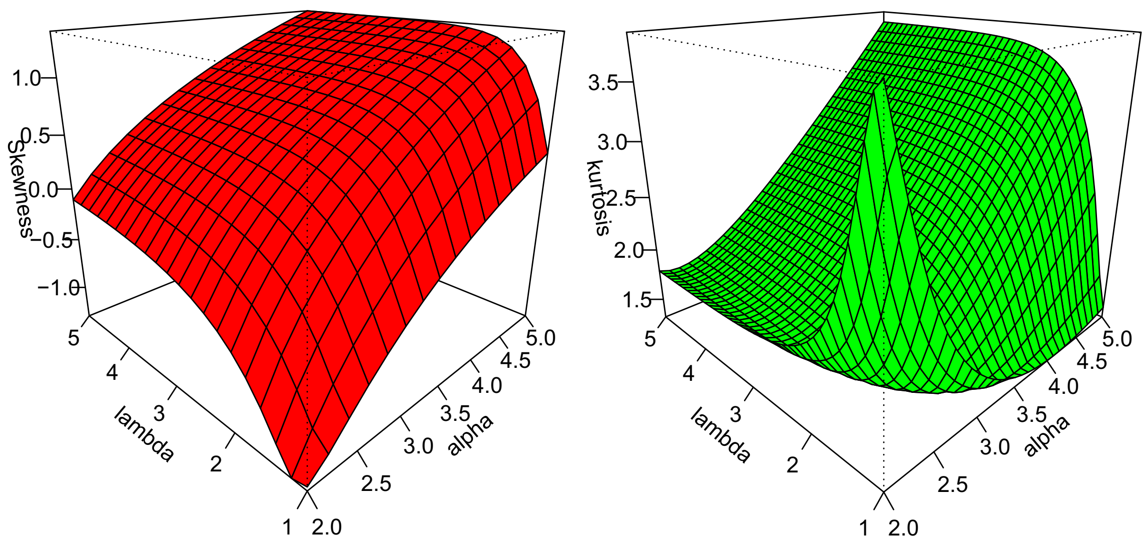

Figure 1. Graphs of the Bowley skewness and Moors kurtosis of X

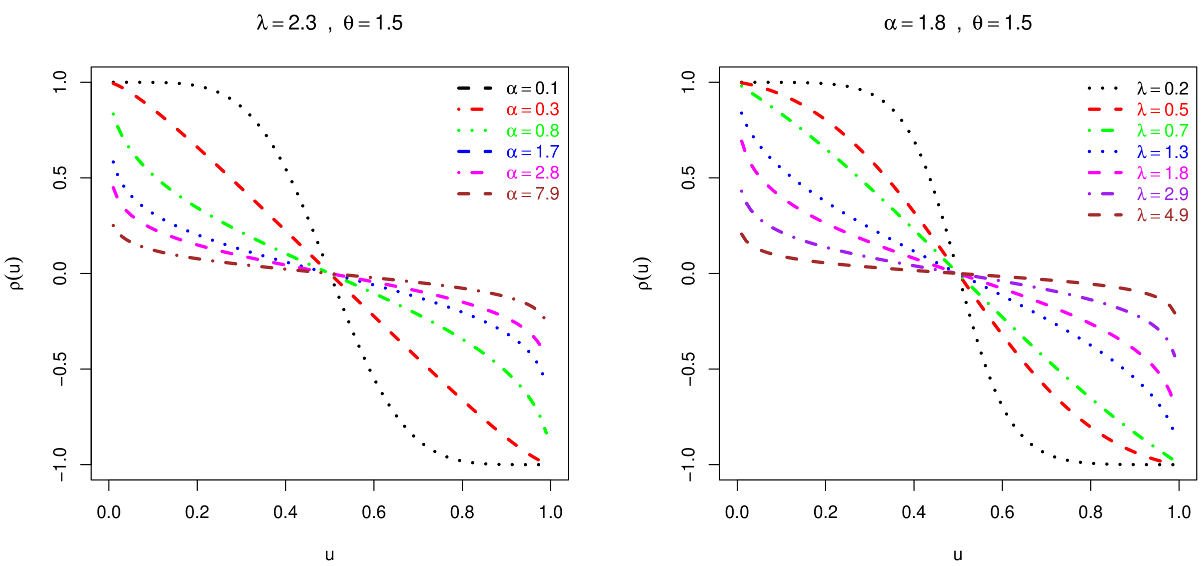

Figure 2. MacGillivray skewness of FLD for selected parameters combinations of X

Figure 3 shows the smallest and the largest order statistics for some parametric values and depends on $E(X_{1:n})$ and $E(X_{n:n})$, respectively. As the values of α increase, the median line moves to a much more central position. Similarly, the lower and upper record values are illustrated in Figure 4.

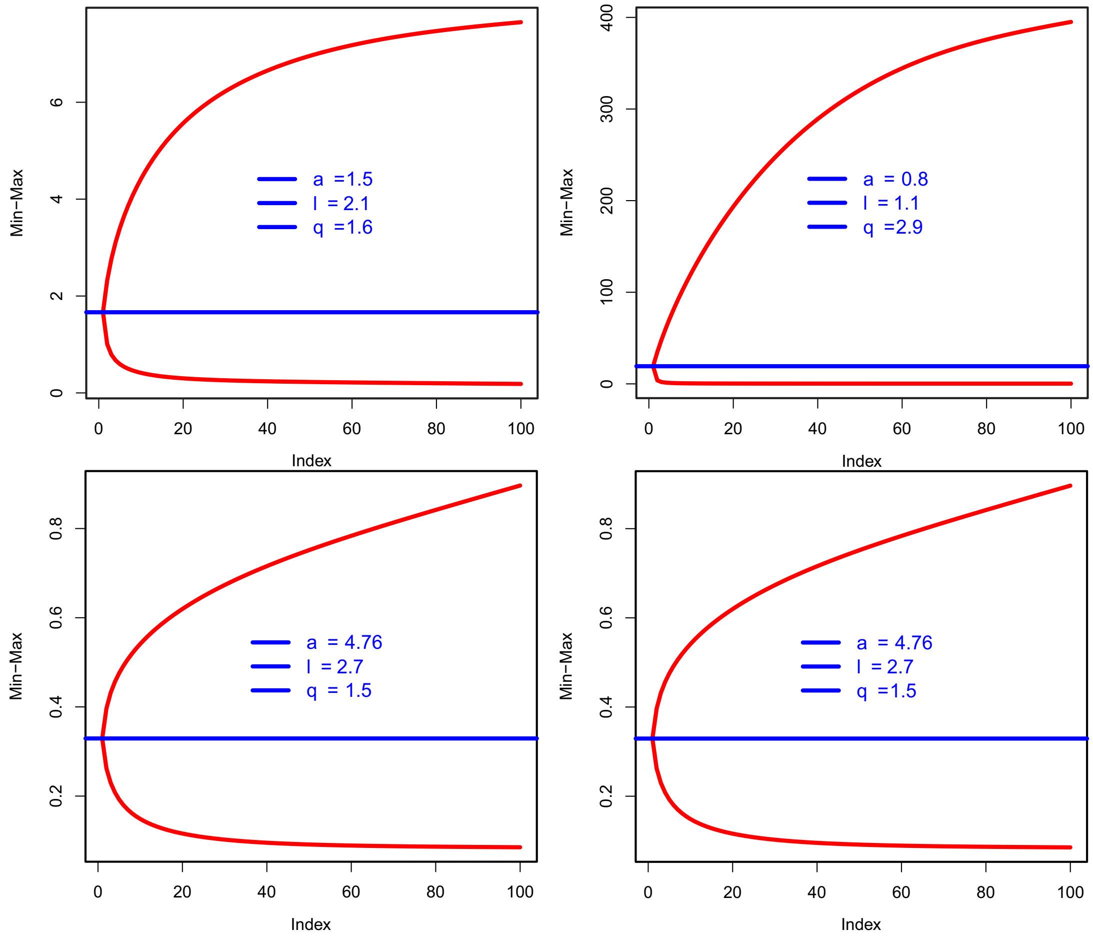

Figure 3. Min-Max plots of FLD for some parametric combinations

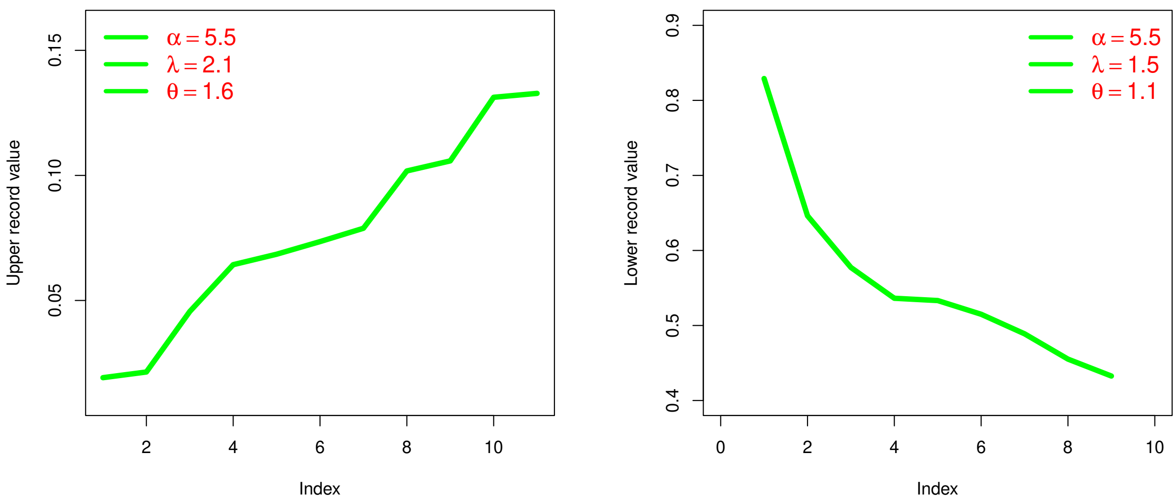

Figure 4. Plots of the upper and lower record value of FLD

An entropy of a continuous random variable X with density function $f_X(x)$ is defined as [16]

$$H(X) = -\int_0^\infty f_X(x) log f_X(x) dx. \tag{5}$$

Laz and Rathie have discussed analytical expressions for univariate distribution [10], Nadarajah and Zografos [13]. Also, the information properties of order statistics were studied by Wong and Chen [18], Park [14], and Ebrahimi et al. [6]. The measure given in equation (5) is not suitable for measuring the uncertainty of a component with information only about its current age. A more realistic approach which takes the use of age into account is described by Ebrahimi [6] and is defined as follows:

$$H(X) = -\int_{t}^{\infty} \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)} dt \tag{6}$$

It is obvious that for $t = \infty$, equation (6) is reduced to equation (5). In many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. Based on this idea, di Crescenzo and Longobardi [5] develop the concept of past entropy over $(0, t)$.

If X denotes the lifetime of a component, then the past entropy of X is defined by

$$H^0(X) = -\int_{t}^{\infty} \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)} dt \tag{7}$$

For $t = 0$, equation (7) is reduced to equation (5).

The rest of the paper is structured as follows: in Section 2, the distribution of order statistics based on FLD is derived with mathematical proofs; Section 3 comprises the single and product moments with explicit expressions derived both mathematically and empirically for the distribution of order statistics; the practical applicability of the proposed distribution based on entropy measures is discussed in Section 4. Finally, Section 5 consists of conclusive remarks and future directions related to the proposed model.

2. Distribution of order statistics

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size from the flexible Lomax distribution and let $X_{1:n} \leq X_{2:n} \leq X_{3:n} \leq, \ldots, \leq X_{n:n}$ denotes the corresponding order statistics. Then the pdf of $X_{r:n}$, $1 \leq r \leq n$ is given by [2, 13]

$$f_{r:n}(x) = C_{r:n}(F(x))^{r-1}(1 - F(x))^{n-r} \tag{8}$$

where $C_{r:n} = \dfrac{n!}{(r-1)!(n-r)!}$.

The probability density functions of smallest $(r = 1)$ and largest $(r = n)$ order statistics can be easily obtained from equation (8) and are given, respectively, by

$$f_{1:n}(x) = n(1 - F(x))^{n-1} f(x)$$

$$f_{n:n}(x) = n(F(x))^{n-1} f(x)$$

Using equations (1) and (2), and taking $r = 1$ in equation (8) yields the pdf of the minimum order statistics for the flexible Lomax distribution

$$f_{1:n}(x) = \frac{n\alpha\lambda}{\theta^{\lambda}} x^{\lambda-1} \left(1 + \left(\frac{x}{\theta}\right)^{\lambda}\right)^{-n(\alpha+1)} \tag{9}$$

Similarly using equations (1) and (2), and taking $r = n$ in equation (8) yields the pdf of the largest order statistics for the flexible Lomax distribution

$$f_{n:n}(x) = \frac{n\alpha\lambda}{\theta^{\lambda}} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i x^{\lambda-1} \left(1 + \left(\frac{x}{\theta}\right)^{\lambda}\right)^{-\alpha(i+1)-1} \tag{10}$$

The joint pdf of $X_{r:s}$ and $X_{r:s}$ for $1 \leq r \leq s \leq n$ is given by [2]

$$f_{r:s:n}(x) = C_{r:s:n}(F(x))^{r-1}(F(y) - F(x))^{s-r-1}(1 - F(y))^{n-s} \tag{11}$$

for $-\infty \leq X \leq y \leq \infty$ where $C_{r:s:n} = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$

Following two theorems gives the distribution of the order statistics from the distribution.

Theorem 1. Let $f(x)$ and $F(x)$ be the cdf and pdf of the flexible Lomax distribution. Then the density function of the rth order statistics say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = C_{r:s:n}\frac{n\alpha\lambda}{\theta^{\lambda}} \sum_{i=0}^{r-1} \binom{r-1}{i} (-1)^i x^{\lambda-1} \left(1 + \left(\frac{x}{\theta}\right)^{\lambda}\right)^{-\alpha(n+i-r+1)-1} \tag{12}$$

Proof. First it should be noted that equation (8) can be written as

$$f_{r:n}(x) = C_{r:n} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i (F(x))^{r+i-1} f(x) \tag{13}$$

The proof follows by substituting equations (1) and (2) into equation (13).

Theorem 2. Let $X_{r:n}$ and $X_{s:n}$ for $1 \leq r \leq s \leq n$ be the rth and sth order statistics from the flexible Lomax distribution. Then the joint pdf of $X_{r:n}$ and $X_{s:n}$ is given by

$$\begin{aligned} f_{r:s:n}(x) = (\alpha\lambda)^2 C_{r:s:n} &\sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} \sum_{k=0}^{s-r-1-i+j} \sum_{l=0}^{r+i-1} \\ &\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \binom{s-r-1}{i} \binom{n-s}{j} \binom{s-r-1-i+j}{k} \binom{r+i-1}{l} \\ &\times (-1)^{i+j+k+l+m+n-2} \frac{mn}{\theta^{\lambda(m+n+2)}} x^{\lambda(m+1)-1} y^{\lambda(n+1)-l} \end{aligned}$$

Proof. Another form of representing equation (11) is as follows:

$$\begin{aligned} f_{r:s:n}(x) = C_{r:s:n} &\sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} \binom{s-r-1}{i} \binom{n-s}{j} (-1)^{i+j} \\ &\times (F(y))^{s-r-1-i+j} (F(x))^{r+i-1} f(x) f(y) \end{aligned} \tag{14}$$

The proof immediately follows by substituting equations (1) and (2) into equation (11).

3. Single and product moments

In this section, we derive explicit expressions from the FLD for both single and product moments of order statistics.

Theorem 3. Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from the distribution and let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics. Then kth moment of the rth order statistics for $k = 1, 2, \ldots$ denoted by $\mu_{r:n}^{(k)}$ is given by

$$\mu_{r:n}^{(k)} = \theta^k \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i+\frac{k}{\lambda}-1} \frac{\beta(n-r-\frac{i}{\alpha}+\frac{1}{\alpha}+1; r)}{\beta(r; n-r+1)} \tag{15}$$

where $\beta(..;..)$ is the beta function.

Proof. We know that

$$\mu_{r:n}^{(k)} = \int_0^\infty x^k f_{r:n}(x)dx, \quad \mu_{r:n}^{(k)} = C_{r,n} \int_0^\infty x^k (F(x))^{r-1}(1-F(x))^{n-r} f(x)dx \tag{16}$$

Now substituting equations (1) and (2) into equation (16), yields equation (15).

Theorem 3 can be exploited to derive the mean, variance and other related measures of the rth order statistics. For example, when $k = 1$, we can obtain the mean of the rth order statistics as follows:

$$\mu_{r:n}^{(1)} = \theta \sum_{i=0}^{\infty} (-1)^{i+\frac{1}{\lambda}-1} \frac{\beta(n-r-\frac{i}{\alpha}+\frac{1}{\alpha}+1; r)}{\beta(r; n-r+1)} \tag{17}$$

For $k = 2$, one can get the second order moment of the rth order statistics as

$$\mu_{r:n}^{(2)} = \theta^2 \sum_{i=0}^{\infty} (-1)^{i+\frac{2}{\lambda}-1} \frac{\beta(n-r-\frac{i}{\alpha}+\frac{1}{\alpha}+1; r)}{\beta(r; n-r+1)} \tag{18}$$

Therefore, the variance of the rth order statistics can be obtained easily by using the equation

$$Var(X_{r:n}) = \mu_{r:n}^{(2)} - (\mu_{r:n}^{(1)})^2$$

$$\begin{aligned} Var(X_{r:n}) =& \theta^2 C_{r;n} \left(\sum_{i=0}^{\infty} (-1)^{i+\frac{2}{\lambda}-1} \beta(n-r-\frac{i}{\alpha}+\frac{1}{\alpha}+1; r) \right. \\ & \left. - C_{r;n} \Big(\sum_{i=0}^{\infty} (-1)^{i+\frac{1}{\lambda}-1} \beta(n-r-\frac{i}{\alpha}+\frac{1}{\alpha}+1; r) \Big)^2 \right) \end{aligned}$$

The third and fourth order moments of the rth order statistic, $\mu_{r:n}^{(3)}$ and $\mu_{r:n}^{(4)}$, can be obtained in similar ways. The mean, variance and other statistical measures of the extreme order statistics are always of great interest. Taking $r = 1$, one can obtain the mean of the smallest order statistics:

$$\mu_{1:n}^{(1)} = n\alpha\theta \sum_{i=0}^{\infty} (-1)^{i+\frac{1}{\lambda}-1} \frac{1}{n\alpha - i + 1}$$

Also, the second-order moment of the smallest-order statistics can be obtained as follows:

$$\mu_{1:n}^{(2)} = n\alpha\theta^2 \sum_{i=0}^{\infty}(-1)^{i+\frac{2}{\lambda}-1}\frac{1}{n\alpha - i + 1}$$

Therefore, the variance of the smallest order statistics is

$$Var(X_{1:n}) = \mu_{1:n}^{(2)} - [\mu_{1:n}^{(1)}]^2$$

$$= n\alpha\theta^2 \left(\sum_{i=0}^{\infty}(-1)^{i+\frac{2}{\lambda}-1}\frac{1}{n\alpha - i + 1} - n\alpha \left(\sum_{i=0}^{\infty}(-1)^{i+\frac{1}{\lambda}-1}\frac{1}{n\alpha - i + 1} \right)^2 \right)$$

Similarly, the mean, the second order moment and hence the variance of the largest order statistics $(r = n)$ is given by

$$\mu_{n:n}^{(1)} = n\theta \sum_{i=0}^{\infty}(-1)^{i+\frac{1}{\lambda}-1}\beta\left(\frac{1}{\alpha} - \frac{i}{\alpha} + 1; n\right)$$

and

$$\mu_{n:n}^{(2)} = n\theta^2 \sum_{i=0}^{\infty}(-1)^{i+\frac{2}{\lambda}-1}\beta\left(\frac{1}{\alpha} - \frac{i}{\alpha} + 1; n\right)$$

Therefore, the variance of the largest order statistics is

$$Var(X_{n:n}) = \mu_{n:n}^{(2)} - [\mu_{n:n}^{(1)}]^2$$

$$Var(X_{n:n}) = n\theta^2 \left(\sum_{i=0}^{\infty}(-1)^{i+\frac{2}{\lambda}-1}\beta\left(\frac{1}{\alpha} - \frac{i}{\alpha} + 1; n\right) - n \left(\sum_{i=0}^{\infty}(-1)^{i+\frac{1}{\lambda}-1}\beta\left(\frac{1}{\alpha} - \frac{i}{\alpha} + 1; n\right) \right)^2 \right)$$

In Table 2, employing the result defined in equation (15), we compute the moments and other related measures for smallest order statistics for some values of $n = 10, 20, 30, 50$ for selected combinations of parameter of the density defined in equation (9). This table is the modified version of Table 1. It gives the empirical findings for the distribution for the kth smallest order statistics.

Table 2. Moments and related measures for smallest order statistics $(r = 1)$ for selected parameters at various sample sizes

Parameters $(\alpha,\ \lambda,\ \theta)$	n	$\mu_{1:n}^{(1)}$	$\mu_{1:n}^{(2)}$	$\mu_{1:n}^{(3)}$	$\mu_{1:n}^{(4)}$	$\mathbb{V}(X)$	$\mathbb{SD}(X)$	$\mathbb{C}oV$	$\mathbb{C}oS$	$\mathbb{C}oK$
	10	–0.182	0.364	–0.727	1.455	0.331	0.575	–3.162	8.100	9.100
$(0.9, 1.2, 2.0)$	20	–0.095	0.190	–0.381	0.762	0.181	0.426	–4.472	18.050	19.050
	30	–0.065	0.129	–0.258	0.516	0.125	0.353	–5.477	28.033	29.033
	50	–0.160	0.319	–0.638	1.277	0.294	0.542	–3.396	9.618	10.618
$(1.0, 0.5, 1.5)$	10	0.136	0.205	0.307	0.460	0.186	0.431	3.162	8.100	9.100
	20	0.071	0.107	0.161	0.241	0.102	0.319	4.472	18.050	19.050
	30	0.048	0.073	0.109	0.163	0.070	0.265	5.477	28.033	29.033
	50	0.120	0.180	0.269	0.404	0.165	0.406	3.396	9.618	10.618
$(0.89, 1.1, 2.5)$	10	–0.042	0.104	–0.260	0.651	0.102	0.320	–7.681	57.017	58.017
	20	–0.002	0.004	–0.010	0.024	0.004	0.062	-40.112	1.61×10^3	1.61×10^3
	30	-0.0001	0.0002	-0.0004	0.001	0.0002	0.013	-198.800	3.95×10^4	3.95×10^4
	50	6.969	17.423	43.557	108.8917	65.990	8.123	1.166	4.095	3.095

Theorem 4. Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from FLD and let $X_{1:n}$, $X_{2:n}$, $X_{3:n}, \ldots, X_{n:n}$ denote the corresponding order statistics.

Then for $1 \leq r \leq n$ we have the following moment relation:

$$\mu_{r:n}^{(k)} = \alpha\lambda \sum_{i=0}^{\infty} \frac{(-1)^{i-1}(n-r+1)}{\theta^{\lambda i}(k+\lambda i)} \left(\mu_{r:n}^{(k-\lambda i)} - \mu_{r-1:n}^{(k-\lambda i)}\right).$$

Proof. Using equations (4) and (16) gives

$$\mu_{r:n}^{(k)} = \int_{i=0}^{\infty} x^k (F(x))^{r-1}(1-F(x))^{n-r} f(x)dx$$

$$\mu_{r:n}^{(k)} = \alpha\lambda \sum_{i=0}^{\infty} \frac{(-1)^{i-1}}{\theta^{\lambda i}} C_{r:n} \int_{i=0}^{\infty} x^{k+\lambda i-1}(F(x))^{r}(1-F(x))^{n-r+1}dx$$

By using integration by parts, we easily obtain the desired result.

Theorem 5. for $1 \leq r \leq s \leq n \leq$, and $n \in N$, we have

$$\begin{aligned}\mu_{r:s:n}^{(k_1,k_2)} =& \alpha\lambda^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j-2}(n-s+1)}{\theta^{\lambda(i+j)}(k_1\lambda i, k_2+\lambda j)} \\ & \left(r\left(\mu_{r:s:n}^{(k_1\lambda i,k_2+\lambda j)} - \mu_{r:s-1:n}^{(k_1\lambda i,k_2+\lambda j)}\right) - n\left(\mu_{r-1:s-1:n-1}^{(k_1\lambda i,k_2+\lambda j)} - \mu_{r-1:s-2:n-1}^{(k_1\lambda i,k_2+\lambda j)}\right)\right)\end{aligned}$$

Proof: We start by noting that

$$\mu_{r:s:n}^{(k_1,k_2)} = C_{r:s:n} \int_0^{\infty}\int_x^{\infty} x^{k_1} y^{k_2} (F(x))^{r-1}(F(y)-F(x))^{s-r-1}(1-F(y))^{n-s} f(x)f(y)dydx$$

or

$$\mu_{r:s:n}^{(k_1,k_2)} = C_{r:s:n} \int_0^{\infty} x^{k_1}(F(x))^{r-1} f(x) I_X dx$$

where

$$I_X = \int_x^{\infty} y^{k_2}(F(y)-F(x))^{s-r-1}(1-F(y))^{n-s} f(y)dy \tag{19}$$

Applying equation (4) gives

$$I_X = \alpha\lambda \sum_{j=0}^{\infty} \frac{(-1)^{j-1}}{\beta^{\lambda j}} \int_x^{\infty} y^{k_2+\lambda j-1}(F(y)-F(x))^{s-r-1}(1-F(y))^{n-s+1}dy$$

Now, integrating by parts and then substituting I_X into equation (19) gives directly the desired result.

4. Entropy-based order statistics

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from a distribution $F_X(x)$ with density function $f(x)$ and let $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ denote the corresponding order statistics. Then the pdf of Y_r, $1 \leq r \leq n$, is given by

$$f_{Y_r}(y) = C_{r:n}(F_X(y))^{r-1}(1 - F_X(y))^{n-r}, \quad 0 \leq X \leq \infty$$

for $C_{r:n} = \dfrac{1}{\beta(r, n-r+1)} = \dfrac{n!}{(r-1)!(n-r)!}$ and $\beta(\ldots, \ldots)$ is the beta function as before.

Further, let U be the uniform distribution defined over the unit interval. The order statistics of a sample taken randomly from a uniform distribution $U_1, U_2, U_3, \ldots, U_n$ are denoted by $W_1 \leq W_2 \leq \cdots \leq W_n$. The random variable W_r, $r = 1, 2, \ldots, n$ has a beta distribution with density function

$$g_r(w) = C_{r:n}(w)^{r-1}(1-w)^{n-r}, 0 \leq w \leq 1$$

In the following subsections, we derive the exact form of entropy, residual entropy and past residual entropy for the FL distribution based on order statistics.

4.1. Entropy

Using the transformation $W_r = F_X(Y_r)$, the entropies of order statistics can be computed by

$$H(Y_r) = H_n(Y_r) - \mathbb{E}_{g_r}(\log f_X(F_X^{-1}(W_r)) \tag{20}$$

where f_X is the probability density function of the random variable X and $H_n(W_r)$ denotes the entropy of the beta distribution and is given by

$$\begin{aligned} H_n(W_r) = &\log \beta(r, n-r+1) - (r-1)(\psi(r) - \psi(n+1)) \\ &- (n-r)(\psi(n-r+1) - \psi(n+1)) \end{aligned} \tag{21}$$

ψ is the digamma function and is defined by $\psi(\theta) = \left(\dfrac{d}{d\theta}\right) \log \Gamma(\theta)$.

Remark 4.1. For $r = 1$, i.e., the smallest order statistics and for $r = n$, i.e., the largest order statistics, it can be easily shown that

$$H_n(W_1) = H_n(W_n) = 1 - \log(n) - \frac{1}{n} \tag{22}$$

Remark 4.2. It should be noted that $\psi(n+1) - \psi(n) = \dfrac{1}{n}$.

Theorem 6. Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size n from FLD with a distribution function given in equation (2) and let $Y_1 \leq Y_2 \leq \cdots \leq Y_n$ denote the corresponding order statistics. Then the entropy of rth order statistics of FLD is given by

$$H_n(W_r) = \log\beta(r, n-r+1) - (r-1)(\psi(r) - \psi(n+1)) - (n-r)(\psi(n-r+1)$$
$$-\psi(n+1)) - \log\left(\frac{\alpha\lambda}{\theta}\right) - \alpha\left(1-\frac{1}{\lambda}\right)\sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^{i-1}}{\beta(r; n-r-1)} \tag{23}$$
$$\times\left(\psi(1) - \psi(1-\alpha(n-r+i+1))\right)\left(1+\frac{1}{\alpha}\right)[\psi(n-r+1) - \psi(n+1)]$$

Proof. Using equation (2) and the probability integral transformation $Yr = F^{-1}(W_r)$, one can easily arrive at

$$F^{-1}(W_r) = \theta((1-W_r)^{\frac{-1}{\alpha}} - 1)^{\frac{1}{\lambda}}$$

Therefore, after applying equation (20) we get the following:

$$\mathbb{E}_{g_r}(\log f_X(F_X^{-1}(W_r)) = -\log\left(\frac{\alpha\lambda}{\theta}\right) - \alpha\left(1-\frac{1}{\lambda}\right)\sum_{i=0}^{r-1}\binom{r-1}{i}\frac{(-1)^{i-1}}{\beta(r; n-r-1)}$$
$$\times\left(\psi(1) - \psi(1-\alpha(n-r+i+1))\right)\left(1+\frac{1}{\alpha}\right)(\psi(n-r+1) - \psi(n+1))$$

Corollary 1. For $r = 1$, i.e., smallest order statistics, we have

$$H(Y_1) = \left(1-\frac{1}{n}\right) + \log\left(\frac{\theta}{n\alpha\lambda}\right) + \left(1-\frac{1}{\lambda}\right)(\psi(1-\alpha n) + \gamma) + \frac{1}{n}\left(1-\frac{1}{\alpha}\right)$$

where $-\psi(1) = \gamma = 0.5772$ is the Euler constant.

Corollary 2. For $r = n$, i.e., largest order statistics, we have

$$H(Y_n) = \left(1-\frac{1}{n}\right) + \log\left(\frac{\theta}{n\alpha\lambda}\right) + n\left(1-\frac{1}{\lambda}\right)\sum_{i=0}^{n-1}\binom{n-1}{i}$$
$$\times\frac{(-1)^i}{(i+1)}\left(\psi(1-\alpha(i+1)) + \gamma\right) - \left(1-\frac{1}{\alpha}\right)(\psi(n+1) + \gamma)$$

4.2. Residual entropy

Analogous to equation (6), the residual entropy of order statistics $X_{r;n}$ is given by

$$H(X_{r,n;t}) = -\int_t^{\infty}\frac{f_{r,n}(x)}{\bar{F}_{r,n}(t)}\log\frac{f_{r,n}(x)}{\bar{F}_{r,n}(t)}dt \tag{24}$$

The residual entropy of first-order statistics is obtained by substituting $r = 1$ and using the probability integral transformation $U = F_X(x)$ in equation (24). Then, we have

$$H(X_{1,n;t}) = \frac{n}{n-1} - \log n + \log \bar{F}(t) - \frac{n}{\bar{F}^n(t)} \int_t^\infty (1-u)^{n-1} \log(f(F^{-1}(u)))du \tag{25}$$

The residual entropy of the first-order statistics for MI distribution can be easily obtained by using equations (1), (2), and (3), and then put $F^{-1}(u) = \theta((1-u)^{\frac{-1}{\alpha}} - 1]^{\frac{1}{\lambda}}$ into equation (25)

$$H(X_{1,n;t}) = \frac{n}{n-1} - \log\left(\frac{n\alpha\lambda}{\theta}\right) - \log \bar{F}(t) - \left(1 - \frac{1}{\lambda}\right)\left((\bar{F}(t))^{\frac{1}{\alpha}} \log\left((\bar{F}(t))^{\frac{1}{\alpha}} - 1\right)\right)$$

$$\times \frac{n}{\bar{F}^n(t)} \sum_{i=0}^{\infty} \frac{\left((\bar{F}(t))^{\frac{1}{\alpha}} - 1\right)}{i} - \left(1 - \frac{1}{\alpha}\right)\left(\log \bar{F}(t) - 1\right)$$

where $F(t)$ and $\bar{F}(t)$ are the cumulative distribution function and survival function for FLD given by equations (2) and (3), respectively. The case for $r = n$ follows on similar lines.

4.3. Past residual entropy

Analogous to equation (7), the past residual entropy of the rth $f(F^{-1}(u))$ order statistics is defined as

$$H^0(X_{r,n;t}) = -\int_0^t \frac{f_{r,n}(x)}{F_{r,n}(t)} \log \frac{f_{r,n}(x)}{F_{r,n}(t)} dt \tag{26}$$

The past residual entropy of nth order statistics is obtained by substituting $r = n$ and using the probability integral transformation $U = F_X(x)$ in equation (27), we have

$$H^0(X_{n,n;t}) = \frac{n-1}{n} - \log n + \log(F(t)) - \frac{n}{F^n(t)} \int_t^\infty u^{n-1} \log(f(F^{-1}(u)))du \tag{27}$$

The past residual entropy of the nth order statistics for MI distribution can be easily obtained by using (1), (2), (3) and $f(F^{-1}(u)) = \dfrac{\alpha W_r^{1-\frac{1}{\alpha}}(W_r^{\frac{1}{\alpha}} - 1)}{(2W_r^{\frac{1}{\alpha}} - 1)}$ in equation (27)

$$H^0(X_{n,n;t}) = \frac{n}{n-1} - \log\left(\frac{n\alpha\lambda}{\theta}\right) - \log F(t) - \frac{n\alpha}{F^n(t)}\left(1 - \frac{1}{\lambda}\right) \sum_{i=0}^{n-1} \sum_{j=0}^{\infty} \binom{n-1}{i} (-1)^i$$

$$\times \left(\frac{(\bar{F}^{\frac{1}{\alpha}}(t))^{j-1}}{j-1} \log\left(\bar{F}^{\frac{1}{\alpha}}(t) - 1\right) - \frac{(\bar{F}^{\frac{1}{\alpha}}(t))^{j-1}}{(j-1)^2}\right)$$

$$\times \left(1 - \frac{1}{\alpha}\right) \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \left(\frac{(\bar{F}(t))^{i+1}}{(i+1)^2} - \frac{(\bar{F}(t))^{i+1}}{(i+1)} \log \bar{F}(t)\right)$$

The case for $r = 1$ follows on similar lines.

5. Conclusion

In this paper, we study the sampling distribution from the order statistics of FL distribution. Explicit mathematical expressions have been derived for applied purposes. Also, we consider the single and product moment of order statistics from FL distribution. We establish recurrence relations for single moments of order statistics. Also, we have derived the entropy, residual and past residual entropies for order statistics of the FL distribution. The distribution of order statistics of FLD may have significant applications in numerous disciplines such as goodness-of-fit testing, quality control, dependability analysis, and many other issues. Further, its utility can be fascinating to explore in the research of lifelong reliability theory.

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