Vol. 34, No. 1 (2024) | DOI: 10.37190/ord240110

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Operations Research and Decisions

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OPERATIONS RESEARCH AND DECISIONS QUARTERLY

Extended power hazard rate distribution and its application

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Abstract

A new model four-parameter model called the odd generalized exponential power hazard rate (OGE-PHR) distribution has been introduced. Some statistical properties for OGE-PHR are obtained. The moments, quantile, mode, reliability, and order statistics are discussed. Estimation of parameters, maximum likelihood technique is employed. Two real data sets are discussed with applications.

Keywords: *odd generalized exponential family, power hazard rate distribution, data analysis, probability models*

1. Introduction

Statistical distributions are very useful in describing and predicting real phenomena. Therefore, many distributions have been used and developed in various branches of applied science (engineering, finances, medicine, etc.) to fit different types of real natural phenomena. This has motivated researchers to seek and develop new and more flexible distribution. Therefore, attention has, recently, been paid to improving existing distributions or suggesting new flexible distributions which can fit any kind of data with any degree of complexity.

A generalization of the exponential distribution (GED) proposed Gupta and Kundu [6] as follows

$$F(x, \nu, \kappa) = (1 - \exp\{-\nu x\})^{\kappa}, \ \nu, \kappa > 0, x \geq 0 \tag{1}$$

the shape and scale parameters are κ, ν, respectively.

Based on the GED and a baseline distribution Taher et al. [24] proposed an odd generalized exponential family (OGE) as follows.

Received 5 December 2022, accepted 15 November 2023, published online 28 March 2024
ISSN 2391-6060 (Online)/

The costs of publishing this issue have been co-financed by the program *Development of Academic Journals* of the Polish Ministry of Education and Science under agreement RCN/SP/0241/2021/1

Let $G(x,\delta)$, $\overline{G}(x,\delta)$ be the cumulative distribution function (CDF) and survival function (SF) of the baseline distribution, (δ is a vector of $p(\geq 1)$ parameters), then OGE-family has the following CDF

$$F(x,\nu,\kappa,\delta) = \left(1 - \mathrm{e}^{-\nu\frac{G(x,\delta)}{\overline{G}(x,\delta)}}\right)^{\kappa}, \ \nu,\ \kappa > 0, x \geq 0 \tag{2}$$

The probability density function (PDF) to (2), is

$$f(x;\nu,\kappa,\delta) = \kappa\nu\,\frac{h_0(x;\delta)}{\overline{G}(x;\delta)}\,\mathrm{e}^{-\nu\frac{G(x,\delta)}{\overline{G}(x;\delta)}}\left(1 - \mathrm{e}^{-\nu\frac{G(x;\delta)}{\overline{G}(x;\delta)}}\right)^{\kappa-1} \tag{3}$$

where $h_0(x;\delta)$ is a hazard rate function (HRF) of a bassline distribution. The OGE family has $p+2$ unknown parameters.

The proposed family was applied when the baseline distribution is Weibull, Fréchet, and normal distribution to suggest OGE-Weibull (OGE-W), OGE-Fréchet (OGE-Fr), and OGE-normal (OGE-N) distributions [24]. Luguterah [14] introduced the OGE-Rayleigh (OGE-R) distribution and showed some of its statistical properties. Rosaiah et al. [21] developed the OGE-log logistic (OGE-LL) distribution while Damcese [2] developed and studied the OGE-Gompertz (OGE-G) distribution. Mustafa et al. [19] used the flexible Weibull distribution as a baseline to obtain the odd generalized exponential flexible Weibull (OGE-FW) distribution. Sarhan and Mustafa [22] applied the odd generalized exponential technique on the two-parameter bathtub hazard-shaped distribution to obtain a new distribution. They introduced some statistical properties, estimated the parameters for this extension by using Maximum likelihood and Bayes' techniques, and discussed some applications.

Mugdadi [17] proposed a new two-parameter distribution called the power hazard rate distribution (PHRD). It is an alternative to the Weibull, Rayleigh, and exponential distributions. The PDF, CDF, reliability function (RF), and HRF, for the PHR distribution are given as

$$g(x;\omega,\gamma) = \omega x^{\gamma}\mathrm{e}^{-\frac{\omega}{\gamma+1}x^{\gamma+1}}, \ x \geq 0 \tag{4}$$

$$G(x;\omega,\gamma) = 1 - \mathrm{e}^{-\frac{\omega}{\gamma+1}x^{\gamma+1}} \tag{5}$$

$$\overline{G}(x;\omega,\gamma) = \mathrm{e}^{-\frac{\omega}{\gamma+1}x^{\gamma+1}} \tag{6}$$

$$h_0(x;\omega,\gamma) = \omega x^{\gamma} \tag{7}$$

where $\omega > 0$, $\gamma > -1$ are the scale and shape parameters.

The PHRD has monotonic HRF: (i) PHRD has increasing HRF, when $\gamma > 0$, (ii) for $-1 < \gamma < 0$, PHRD has decreasing HRF and (iii) for $\gamma = 0$, PHRD has constant HRF. So, this distribution has been set out to be handy for modeling the data in engineering, medical and many more. Also, PHRD is very adjustable, it contains some special models: (i) when $\omega = 1/\gamma^2$ and $\gamma = 1$, PHRD refers to Rayleigh (γ), (ii) when $\gamma = \omega - 1$, PHRD reduces to Weibull $(\omega, 1)$, (iii) when $\gamma = 0$, PHRD is exponential distribution with $1/\omega$.

Several authors studied PHRD. Kinaci [12] discussed the stress-strength reliability model for PHRD. The estimation of parameters of a distribution has a power hazard function discussed by Ismail [7], the Bayes estimation of PHRD based on complete and Type II censored samples studied by Mugdadi and

Min [18]. El-Sagheer [4] introduced the estimations of the parameters for PHRD based on a progressive Type II censoring scheme while the parameter estimation based on record data from PHRD explained by Tarvirdizade and Nematollahi [25]. The inference on $P(X > Y)$ based on record values from PHR distribution introduced by Tarvirdizade and Nematollahi [26]. Bayesian and non-Bayesian approaches to the lifetime performance index with the Type II progressive censored sample of PHRD studied by Al-Saghir et al. [5]. A Type II adaptive progressive censored scheme was used to study the estimation of shape and scale parameters, RF, HRF for PHRD, by Al-Morshedy et al. [3].

Khan [10] derived the recurrence relations for single and product moments of generalized order statistics from the PHFD. Khan and Mustafa [9] obtained the weighted power hazard rate distribution with applications and derived some properties of this generalization. Khan and Mustafa [11] introduced another extension form PHRD. This extension called the transmuted PHRD and discussed its properties and some applications. Mustafa and Khan [20] introduced another extension from the PHRD, called the length-biased PHRD. They studied some properties and applications.

In this article, an odd generalized exponential power hazard rate (OGE-PHR) distribution is presented. It is organized as follows: The characteristics of the OGE-PHR distribution are presented in Section 2. Some statistical measures and properties are discussed in Section 3. The distribution of the order statistics is embodied in Section 4. The parameters of the proposed model are estimated in Section 5. Two real data sets are considered for comparison with existing distributions in Section 6. Finally, a conclusion is introduced in Section 7.

2. The OGE-PHR distribution

In this section, the four parameters OGE-PHR(ν, ω, γ, κ) distribution is studied. From (2) using (5) and (6), the CDF of OGE-PHR is obtained as follows

$$F(x,\ \xi) = \left(1 - \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa}, \quad \nu,\ \omega,\ \kappa > 0,\ \gamma > -1,\ x \geq 0 \tag{8}$$

The corresponding PDF, SF, and HRF can be derived as

$$f(x,\ \xi) = \kappa\nu\omega x^{\gamma}\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\left(1 - \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa-1} \tag{9}$$

$$\overline{F}(x,\ \xi) = 1 - \left(1 - \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa} \tag{10}$$

$$h(x,\ \xi) = \frac{\kappa\nu\omega x^{\gamma}\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\left(1 - \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa-1}}{1 - \left(1 - \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa}}. \tag{11}$$

where ω, γ are two additional shape parameters.

The plots for PDF and HRF of OGE-PHRD are displayed in Figures 1 and 2. We can extract that the PDF is heavily skewed (left/right) and near symmetric while HRF is increasing or bathtub-shaped. These properties will allow OGE-PHRD to be suitable for fitting different types of data sets.

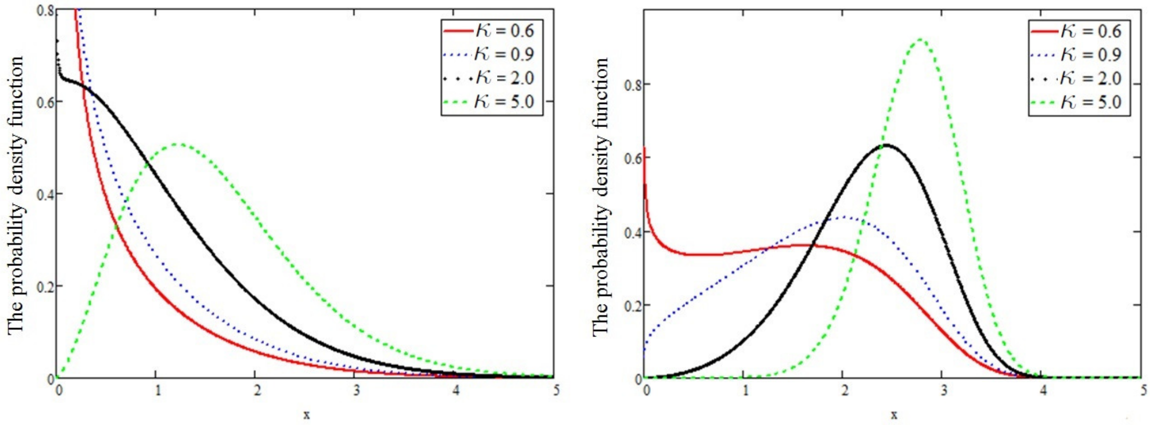

Figure 1. The PDF of OGE-PHR $(\nu, \omega, \gamma, \kappa)$ for $\nu = 0.3$, $\omega = 0.6$ and different κ when $\gamma = -0.6 < 0$ (left panel) and $\gamma = 0.4 > 0$ (right panel)

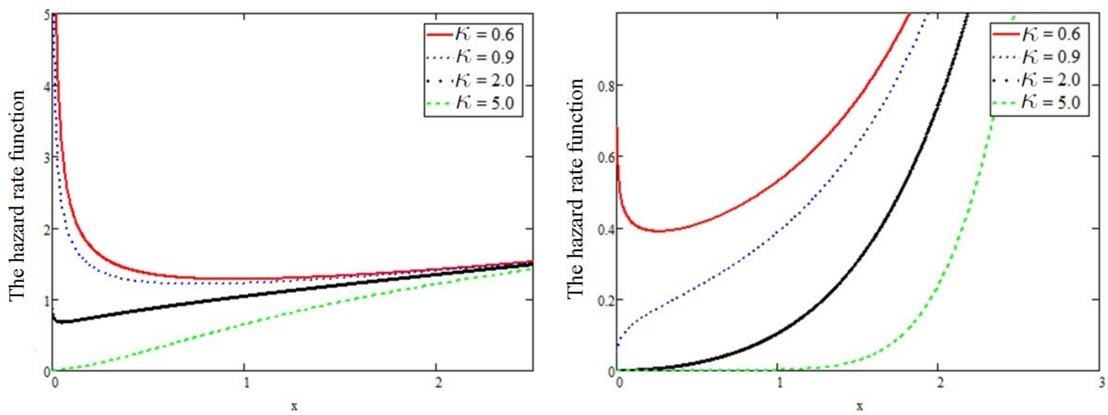

Figure 2. The HRF of OGE-PHR $(\nu, \omega, \gamma, \kappa)$ for $\nu = 0.3$, $\omega = 0.6$ and different κ when $\gamma = -0.6 < 0$ (left panel) and $\gamma = 0.4 > 0$ (right panel)

Linear combination (9) can be expressed as an odd exponential PHRD. For $\kappa > 0$,

$$(1-u)^{\kappa-1} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma(\kappa)}{\Gamma(\kappa-\ell)\ell!}\, u^{\ell} \tag{12}$$

$\Gamma(.)$ is the gamma function. Using (9), that can easily be verified

$$S_0(x;\nu,\omega,\gamma) = \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)} < 1$$

Applying (12), we get

$$f(x;\xi) = \kappa \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma(\kappa)}{\Gamma(\kappa-\ell)\ell!} f_0(x;(\ell+1)\nu,\omega,\gamma) \tag{13}$$

where $f_0(x;\nu,\omega,\gamma)$ and $S_0(x;\nu,\omega,\gamma)$ are PDF and SF of OE-PHRD.

The properties of the OGE-PHRD can be obtained depending on the OE-PHRD by equation (13).

Reliability interpretation. If there are κ independent and identical items each with a distribution of OE-PHR connected in a parallel system. The lifetime of this system can be interpreted as OGE-PHR, when κ is an integer.

3. Statistical properties

In this section, we discuss some important statistical and mathematical properties of the OGE-PHR distribution such as ordinary moments, mode, quantiles, skewness and kurtosis, and order statistics.

3.1. Moments

Moments are used to understand various characteristics of a frequency distribution. These have been applied to obtain a mean, variance, in addition to some measures, such as skewness and kurtosis.

The rth moments, μ'_r, of OGE-PHRD can be represented as a linear combination of the moments of OE-PHRD, $\mu'_{r,\text{OE-PHR}}(\nu,\omega,\gamma)$, by using (13)

$$\mu'_r = \kappa \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\Gamma(\kappa)}{\Gamma(\kappa-\ell)\ell!}\, \mu'_{r,\text{OE-PHR}}(\nu,\omega,\gamma),\ \ell \leq \kappa, \quad r = 0,1,2,\cdots \tag{14}$$

Equation (14) has no closed-form solutions, therefore, this relation is not best used for OGE-PHRD moments.

3.2. The mode

The mode can be obtained by solving $f'(x) = 0$ concerning x

$$\begin{aligned}
&\omega\gamma\kappa\nu x^{\gamma-1} \mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}} \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)} \left(1-\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa-1} \\
&+\omega^2(\kappa-1)\kappa\nu^2 x^{2\gamma} \mathrm{e}^{\frac{2\omega}{\gamma+1}x^{\gamma+1}} \mathrm{e}^{-2\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)} \left(1-\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa-2} \\
&+\omega^2\kappa\nu x^{2\gamma} \mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}} \left(1-\nu \mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}\right) \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)} \\
&\left(1-\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x^{\gamma+1}}-1\right)}\right)^{\kappa-1} = 0
\end{aligned} \tag{15}$$

Equation (15) can be solved analytically in x to obtain the mode. A quick graphical test can be provided by plotting its left-hand side (Figure 3). The distribution does not have a mode when $\kappa < 2$.

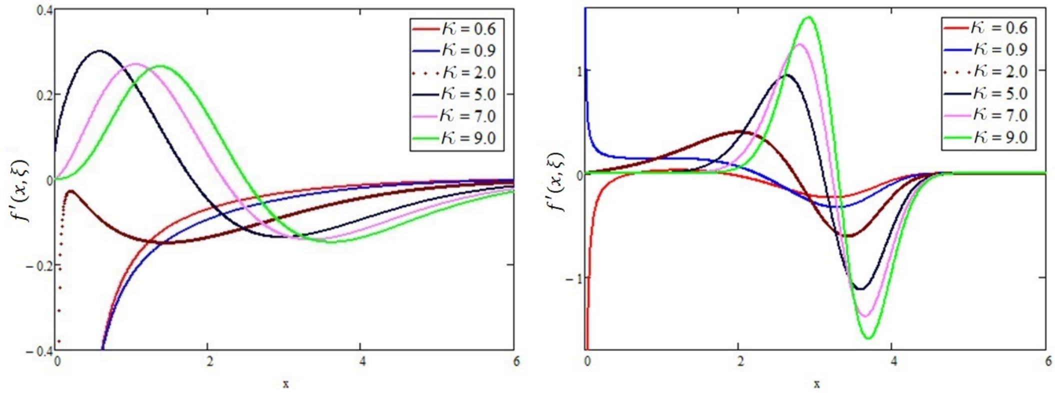

Figure 3. The $f'(x)$ of the OGE-PHR $(0.3,\ 0.6,\ \gamma,\ \kappa)$
when $\gamma = -0.6 < 0$ (left) and $\gamma = 0.4 > 0$ (right) at different values of κ

3.3. The quantiles

The quantiles of OGE-PHRD can easily be calculated in a simple and explicit form. Let $x_q, 0 < q < 1$, be the quantile of the OGE-PHR(ξ) distribution. By using (8), x_q can be obtained as follows

$$x_q = \left(\left(\frac{\gamma+1}{\omega} \right) \ln \left(1 - \frac{1}{\nu} \ln \left(1 - q^{1/\kappa} \right) \right) \right)^{1/(\gamma+1)} \tag{16}$$

From (16), some statistical measures for OGE-PHR distribution can be obtained as (i) by setting $q = 0.25$ in (16), the 1st quartile can be obtained, (ii) by setting $q = 0.5$, the median can be obtained, (iii) by setting $q = 0.75$, the 3rd quartiles can be obtained.

3.4. Skewness and kurtosis

Different approaches are available in the literature to obtain the skewness, sk, and kurtosis, ku, of a certain distribution. By using the moments, the sk and ku can be obtained as follows

$$sk = \frac{\mu_3}{\mu_2^{3/2}} \tag{17}$$

$$ku = \frac{\mu_4}{\mu_2^2} \tag{18}$$

where μ_k be the kth moments about the mean

$$\mu_k = E(X-\mu)^k = \sum_{r=0}^{k} \binom{k}{r} \mu'_r \mu_1'^{k-r}, \quad k = 1, 2, \cdots$$

where μ'_r is defined in (14).

There is an alternative technique to calculate sk and ku based on the quantiles. This technique is applied when a distribution has first moment only, as follows.

The Bowely's skewness [8] is

$$sk = \frac{x_{0.75} - 2x_{0.50} + x_{0.25}}{x_{0.75} - x_{0.25}} \tag{19}$$

The Moors kurtosis [16] is

$$ku = \frac{x_{0.875} - x_{0.625} + x_{0.375} - x_{0.125}}{x_{0.75} - x_{0.25}}. \tag{20}$$

Table 1 displays the basic statistical measures of OGE-PHRD. One can observe that the OGE-PHRD takes different shapes depending on its parameters.

Table 1. Basic statistics for OGE-PHRD at different values of ξ

$\xi = (\nu, \omega, \gamma, \kappa)$	μ'_1	μ'_2	Variance	Median	sk	ku
(0.3, 0.6, –0.6, 0.6)	0.649	1.339	0.917799	0.218	2.26630	9.214773
(0.3, 0.6, –0.6, 0.9)	0.888	1.928	1.139456	0.484	1.822111	6.924865
(0.3, 0.6, –0.6, 2.0)	1.511	3.76	1.476879	1.227	1.213787	4.677336
(0.3, 0.6, –0.6, 5.0)	2.386	7.254	1.561004	2.200	0.854139	3.903683
(0.3, 0.6, –0.6, 7.0)	2.722	8.932	1.522716	2.555	0.791604	3.825030
(0.3, 0.6, –0.6, 9.0)	2.972	10.315	1.482216	2.815	0.757271	3.803921
(0.3, 0.6, 0.4, 0.6)	1.609	3.562	0.973119	1.584	0.179431	2.083920
(0.3, 0.6, 0.4, 0.9)	1.947	4.626	0.835191	1.989	–0.07475	2.224691
(0.3, 0.6, 0.4, 2.0)	2.542	6.953	0.491236	2.594	–0.3194	2.748601
(0.3, 0.6, 0.4, 5.0)	3.044	9.506	0.240064	3.065	–0.29564	3.441356
(0.3, 0.6, 0.4, 7.0)	3.187	10.342	0.185031	3.199	–0.06711	1.721865
(0.3, 0.6, 0.4 9.0)	3.282	10.929	0.157476	3.289	–0.20243	4.347283

3.5. Data simulation

A random sample from OGE-PHRD can be generated by using the following algorithm

1. Choose an appropriate value of m.
2. Select the parameter values $\xi = (\nu, \omega, \gamma, \kappa)$.
3. Generate $u_i \sim U(0, 1),\ i = 1, 2, dots, m$.
4. Use the following relation to get $x_i,\ i = 1, 2, \cdots, m$ from OGE-PHR $(\nu, \omega, \gamma, \kappa)$

$$x_i = \left(\left(\frac{\gamma + 1}{\omega} \right) \ln \left(1 - \frac{1}{\nu} \ln \left(1 - u_i^{1/\kappa} \right) \right) \right)^{1/(\gamma+1)}, i = 1, 2, \cdots, m. \tag{21}$$

By applying the above algorithm with $m = 1000$, the random sample from OGE-PHR$(0.3, 0.6, 0.4, 7)$ can be generated. Using $\xi = (0.3, 0.6, 0.4, 7)$, the actual PDF, CDF can be obtained. The estimated non-parametric PDF, CDF are calculated by using the generated sample.

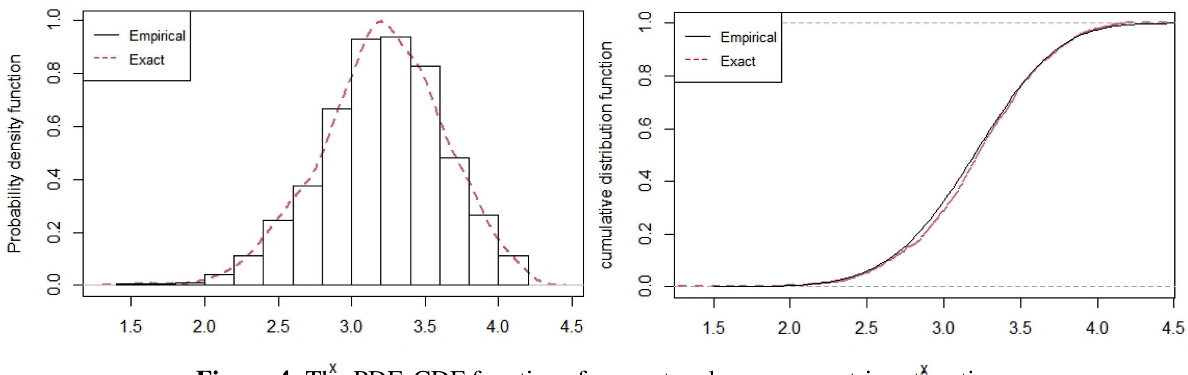

Figure 4. The PDF, CDF functions for exact and non-parametric estimation

Table 2. Some statistical measures

x	$x_{0.01}$	$x_{0.25}$	$x_{0.50}$	$x_{0.75}$	$x_{0.99}$	Mean	Variance
Exact	2.626000	2.903	3.199	3.484	4.13600	3.187	0.185031
Estimation	2.192562	2.944	3.218	3.489	4.08429	3.207	0.174180

Table 2 contains some numerical results, while Figure 4 displays the plots. There is a high agreement between estimated and exact measures and curves, therefore, the random number generation works well.

4. Order statistics

The ordered statistics play an important role in many areas of statistical theory. Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(m)}$ denote the order statistics for a random sample with size m from a population with CDF $F_X(x;\xi)$ and PDF, $f_X(x;\xi)$. The PDF of $X_{(i)}$ is given by

$$f_{(i)}(x;\xi) = \frac{1}{B(k, m-i+1)} \left(F(x;\xi)\right)^{i-1} \left(1-F(x;\xi)\right)^{m-i} f(x;\xi) \tag{22}$$

where $B(.,.)$ is the beta function.

Order statistics has an application in reliability theory, economics and finance. The order statistics $X_{(1)}$, $X_{(m)}$ are the lifetimes for the series and parallel systems with m components, respectively.

Since $0 < F(x;\xi) < 1$ for $x > 0$, then

$$\left(1-F(x;\xi)\right)^{m-i} = \sum_{\ell=0}^{m-i} \binom{m-i}{\ell} (-1)^{\ell} \left(F(x;\xi)\right)^{\ell}$$

Equation (22) can be rewritten as follows

$$f_{(i)}(x;\xi) = \sum_{\ell=0}^{m-i} \frac{(-1)^{\ell} m!}{i!(i-1)!(m-i-\ell)!} f(x;\xi) \left[F(x;\xi)\right]^{\ell+i-1} \tag{23}$$

The PDF for the order statistics of OGE-PHRD can be obtained by substituting (8) and (9) into (23).

5. Maximum likelihood estimation

The maximum likelihood method is used to estimate the unknown parameters for OGE-PHRD. The likelihood function based on a complete random sample, $x_1, x_2, \cdots, x_m$, with size m from OGE-PHRD, can be obtained as

$$L(\xi) = \prod_{\ell=1}^{m} \left(\kappa\nu\omega x_{\ell}^{\gamma} \mathrm{e}^{\frac{\omega}{\gamma+1} x_{\ell}^{\gamma+1}} \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1} x_{\ell}^{\gamma+1}} - 1\right)} \left(1 - \mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1} x_{\ell}^{\gamma+1}} - 1\right)}\right)^{\kappa-1} \right) \tag{24}$$

The log-likelihood function is

$$\begin{aligned}\mathcal{L}(\xi) = m\ln(\kappa\nu\omega)+\gamma\sum_{\ell=1}^{m}\ln(x_\ell)+\frac{\omega}{\gamma+1}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}-\nu\sum_{\ell=1}^{m}\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)\\ +(\kappa-1)\sum_{\ell=1}^{m}\ln\left(1-\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\right)\end{aligned} \tag{25}$$

The first derivatives (score functions) with respect to ν, ω, γ and κ can be obtained, respectively, as

$$\frac{\partial\mathcal{L}}{\partial\nu}=\frac{m}{\nu}-\sum_{\ell=1}^{m}\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)+(\kappa-1)\sum_{\ell=1}^{m}\left(\frac{\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1}{\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}-1}\right) \tag{26}$$

$$\frac{\partial\mathcal{L}}{\partial\omega}=\frac{m}{\omega}+\frac{1}{\gamma+1}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}-\frac{\nu}{\gamma+1}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}+\frac{\nu(\kappa-1)}{\gamma+1}\sum_{\ell=1}^{m}\left(\frac{x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}}{\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}-1}\right) \tag{27}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\gamma}=\sum_{\ell=1}^{m}\ln(x_\ell)+\frac{\omega}{(\gamma+1)^2}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}\left(-1+(\gamma+1)\ln(x_\ell)\right)-\frac{\omega\nu}{(\gamma+1)^2}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\\ \times\left(-1+(\gamma+1)\ln(x_\ell)\right)+\frac{(\kappa-1)\nu\omega}{(\gamma+1)^2}\sum_{\ell=1}^{m}\left(\frac{x_\ell^{\gamma+1}\left(-1+(\gamma+1)\ln(x_\ell)\right)\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}}{\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}-1}\right)\end{aligned} \tag{28}$$

$$\frac{\partial\mathcal{L}}{\partial\kappa}=\frac{m}{\kappa}+\sum_{\ell=1}^{m}\ln\left(1-\mathrm{e}^{-\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\right) \tag{29}$$

We obtain a system of non-linear equations by setting each of the equations (26)–(29) to zero. This system does not have an analytic solution. Some numerical programs can be used to solve it numerically.

Since the MLEs cannot be obtained analytically, the actual distributions cannot be obtained to find the exact confidence intervals for the parameters. For large sample sizes, the MLE estimators have asymptotically multivariate normal distribution given by

$$(\hat{\nu},\hat{\omega},\hat{\gamma},\hat{\kappa})\sim N_4(\xi,\boldsymbol{V})$$

where $\boldsymbol{V}$ is the variance-covariance matrix [13].

$$\boldsymbol{V}=\begin{pmatrix}-\frac{\partial^2\mathcal{L}}{\partial\nu^2} & -\frac{\partial^2\mathcal{L}}{\partial\omega\partial\nu} & -\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\nu} & -\frac{\partial^2\mathcal{L}}{\partial\kappa\partial\nu}\\ -\frac{\partial^2\mathcal{L}}{\partial\nu\partial\omega} & -\frac{\partial^2\mathcal{L}}{\partial\omega^2} & -\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\omega} & -\frac{\partial^2\mathcal{L}}{\partial\kappa\partial\omega}\\ -\frac{\partial^2\mathcal{L}}{\partial\nu\partial\gamma} & -\frac{\partial^2\mathcal{L}}{\partial\omega\partial\gamma} & -\frac{\partial^2\mathcal{L}}{\partial\gamma^2} & -\frac{\partial^2\mathcal{L}}{\partial\kappa\partial\gamma}\\ -\frac{\partial^2\mathcal{L}}{\partial\nu\partial\kappa} & -\frac{\partial^2\mathcal{L}}{\partial\omega\partial\kappa} & -\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\kappa} & -\frac{\partial^2\mathcal{L}}{\partial\kappa^2}\end{pmatrix}^{-1}_{\xi=\hat{\xi}}$$

where

$$\frac{\partial^2\mathcal{L}}{\partial\nu^2} = -\frac{m}{\nu^2} - (\kappa-1)\sum_{\ell=1}^{m} \mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\Phi^2(x_\ell,\,\xi)$$

$$\frac{\partial^2\mathcal{L}}{\partial\omega\partial\nu} = -\frac{1}{\gamma+1}\sum_{\ell=1}^{m} x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}} + \frac{(\kappa-1)}{\gamma+1}\sum_{\ell=1}^{m}\Psi_1(x_\ell,\,\xi) - \frac{(\kappa-1)\nu}{\gamma+1}$$
$$\times\sum_{\ell=1}^{m}\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\Psi_2(x_\ell,\,\xi)$$

$$\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\nu} = -\frac{\omega}{(\gamma+1)^2}\sum_{\ell=1}^{m} x_\ell^{\gamma+1}\big(-1+(\gamma+1)\ln(x_\ell)\big)\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}$$
$$+\frac{(\kappa-1)\omega}{(\gamma+1)^2}\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)\Psi_1(x_\ell,\,\xi) - \frac{(\kappa-1)\omega\nu}{(\gamma+1)^2}$$
$$\times\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\Psi_2(x_\ell,\,\xi)$$

$$\frac{\partial^2\mathcal{L}}{\partial\kappa\partial\nu} = \sum_{\ell=1}^{m}\Phi(x_\ell,\,\xi)$$

$$\frac{\partial^2\mathcal{L}}{\partial\omega^2} = -\frac{m}{\omega^2} - \frac{\nu}{(\gamma+1)^2}\sum_{\ell=1}^{m} x_\ell^{2(\gamma+1)}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}} + \frac{(\kappa-1)\nu}{(\gamma+1)^2}\sum_{\ell=1}^{m} x_\ell^{\gamma+1}\Psi_1(x_\ell,\,\xi)$$
$$-\frac{(\kappa-1)\nu^2}{(\gamma+1)^2}\sum_{\ell=1}^{m} x_i^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\Psi_2(x_\ell,\,\xi)$$

$$\frac{\partial^2\mathcal{L}}{\partial\gamma\partial\omega} = \frac{1}{(\gamma+1)^2}\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)x_\ell^{\gamma+1} - \frac{\nu}{(\gamma+1)^3}$$
$$\times\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)(\gamma+1+\omega x_\ell^{\gamma+1})x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}} + \frac{(\kappa-1)\nu}{(\gamma+1)^2}$$
$$\times\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)\Psi_1(x_\ell,\,\xi) - \frac{(\kappa-1)\omega\nu^2}{(\gamma+1)^3}$$
$$\times\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\Psi_2(x_\ell,\,\xi)$$
$$+\frac{(\kappa-1)\omega\nu}{(\gamma+1)^3}\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)x_\ell^{\gamma+1}\Psi_1(x_\ell,\,\xi)$$

$$\frac{\partial^2\mathcal{L}}{\partial\kappa\partial\omega}=\frac{\nu}{\gamma+1}\sum_{\ell=1}^{m}\Psi_1(x_\ell,\,\xi)$$

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial\gamma^2}=&-\frac{2\omega}{(\gamma+1)^3}\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)x_\ell^{\gamma+1}+\frac{\omega}{\gamma+1}\sum_{\ell=1}^{m}\big(\ln(x_\ell)\big)^2x_\ell^{\gamma+1}\\&-\frac{\nu\omega^2}{(\gamma+1)^4}\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)^2x_\ell^{2(\gamma+1)}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\\&-\frac{\nu\omega}{(\gamma+1)^3}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\big(2-2(\gamma+1)\ln(x_\ell)+(\gamma+1)^2\ln(x_\ell)^2\big)\\&-\frac{(\kappa-1)\omega^2\nu^2}{(\gamma+1)^4}\sum_{\ell=1}^{m}\big(-1+\ln(x_\ell)\big)^2\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}\Psi_1^2(x_\ell,\,\xi)\\&+\frac{(\kappa-1)\omega\nu}{(\gamma+1)^3}\sum_{\ell=1}^{m}\big(2-2(\gamma+1)\ln(x_\ell)+(\gamma+1)^2\ln(x_\ell)^2\big)\Psi_1(x_\ell,\,\xi)\\&+\frac{(\kappa-1)\nu\omega^2}{(\gamma+1)^4}\sum_{\ell=1}^{m}x_\ell^{\gamma+1}\big(-1+(\gamma+1)\ln(x_\ell)\big)^2\left(1-\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\right)\Psi_1(x_\ell,\,\xi),\end{aligned}$$

$$\frac{\partial^2\mathcal{L}}{\partial\kappa\partial\gamma}=\frac{\omega\nu}{(\gamma+1)^2}\sum_{\ell=1}^{m}\big(-1+(\gamma+1)\ln(x_\ell)\big)\Psi_1(x_\ell,\,\xi),\quad\frac{\partial^2\mathcal{L}}{\partial\kappa^2}=-\frac{m}{\kappa^2}$$

and

$$\Psi_j(x_\ell,\,\xi)=x_\ell^{\gamma+1}\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\left(\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)}-1\right)^{-j},\ j=1,2$$

$$\Phi(x_\ell,\,\xi)=\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}-1\right)\left(\mathrm{e}^{\nu\left(\mathrm{e}^{\frac{\omega}{\gamma+1}x_\ell^{\gamma+1}}\right)-1}-1\right)^{-1}$$

A $100(1-\alpha)\%$ confidence interval for $\xi=(\nu,\,\omega,\,\gamma,\,\kappa)$, can be approximated by

$$\hat{\nu}\pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\nu})},\quad\hat{\omega}\pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\omega})},\quad\hat{\gamma}\pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\gamma})},\quad\text{and}\quad\hat{\kappa}\pm z_{\frac{\alpha}{2}}\sqrt{var(\hat{\kappa})}$$

where $z_{\frac{\alpha}{2}}$ is the upper $100\frac{\alpha}{2}$th percentile of $N(0,1)$, and $var(\hat{\xi}_i)$ is the diagonal ith element in $\boldsymbol{V}$.

6. Numerical applications

In this section, two real data sets using the OGE-PHR $(\nu,\,\omega,\,\gamma,\,\kappa)$model will be analyzed and compared with PHRD. Some criteria are chosen, these are K-S, R^2, RMSE, AIC, AICC, BIC and, HQIC [1, 23].

- K-S – the Kolmogorov-Smirnov test

$$K - S = \sup_x |F_m(x) - \hat{F}(x)|$$

- R^2 – the determination coefficient

$$R^2 = \frac{\sum_{\ell=1}^{m} \left(\hat{F}(x_\ell) - \overline{F}\right)^2}{\sum_{\ell=1}^{m} \left(\hat{F}(x_\ell) - \overline{F}\right)^2 + \sum_{\ell=1}^{m} \left(F_m(x_\ell) - \hat{F}(x_\ell)\right)^2}$$

- RMSE – the root mean square error

$$\text{RMSE} = \left(\frac{1}{m}\sum_{\ell=1}^{m} \left(F_m(x_\ell) - \hat{F}(x_\ell)\right)^2\right)^{1/2}$$

- AIC – the Akaike information criterion

$$\text{AIC} = 2k - 2\mathcal{L}$$

- AICC – the Akaike information criterion with correction

$$\text{AAIC} = \text{AIC} + \frac{2k(k+1)}{(m-k+1)}$$

- BIC – the Bayesian information criterion

$$\text{BIC} = k - 2\mathcal{L} + \ln(m)$$

- HQIC – the Hannan-Quinn information criterion

$$\text{HQIC} = -2\mathcal{L} + 2k\ln(\ln(m))$$

$\hat{F}(x), F_m(x)$ are estimated and empirical CDF, and k, m are the number of parameters and values of data, respectively.

$$\overline{F}(x) = \frac{1}{m}\sum_{\ell=1}^{m} \hat{F}(x_\ell), \quad F_m(x) = \frac{1}{m}\sum_{\ell=1}^{m} I(x_{(\ell)} \le x)$$

and

$$I(x_{(\ell)} \le x) = \begin{cases} 1, & \text{if } x_{(\ell)} \le x \\ 0, & \text{otherwise} \end{cases}$$

If the data have (i) larger values of R^2 and p-value, and (ii) lower values of criteria and K-S and RMSE, our model is best for the data.

Example 6.1. The data set below is taken from [15]. The data comprises 100 observations on the breaking stress of carbon fibers. Table 3 gives MLEs of parameters, K-S, and corresponding p-value for OGE-

0.920	1.196	1.244	1.351	1.477	1.544	1.616	1.777	1.944	2.046
0.928	1.213	1.259	1.359	1.48	1.552	1.617	1.794	1.972	2.059
0.997	1.215	1.261	1.388	1.489	1.556	1.628	1.799	1.984	2.111
0.997	1.212	1.263	1.408	1.501	1.562	1.684	1.806	1.987	2.165
1.061	1.220	1.276	1.449	1.507	1.566	1.711	1.814	2.020	2.686
1.117	1.224	1.310	1.449	1.515	1.585	1.718	1.816	2.030	2.778
1.162	1.225	1.321	1.450	1.530	1.586	1.733	1.828	2.029	2.972
1.183	1.228	1.329	1.459	1.530	1.599	1.738	1.830	2.035	3.504
1.187	1.237	1.331	1.471	1.533	1.602	1.743	1.884	2.037	3.863
1.192	1.240	1.337	1.475	1.544	1.614	1.759	1.892	2.043	5.306

PHR, PHR, and some of the OGE-distributions. Table 4 contains the values of $\mathcal{L}$, some values of proposal criteria. Based on Tables 3 and 4, we can conclude that OGE-PHRD suggests a better fit to the data.

Table 3. MLEs of $\nu, \omega, \gamma, \kappa$, K-S and p-value

Model	$\hat{\nu}$	$\hat{\omega}$	$\hat{\gamma}$	$\hat{c}$	$\hat{\kappa}$	K-S	p-value
OGE-PHR	9.926	0.174	0.025	–	15.892	0.1254222	0.0798707
PHR	–	0.521	1.632	–	–	0.1951352	0.0008231
OGE-RD	0.024	1.51	–	–	0.236	0.4663114	0.000
OGE-W(ν)	0.024	0.84	–	–	1.415	0.8845954	0.000
OGE-E	6.475	0.246	–	–	15.915	0.1255142	0.0772553
OGE-W	0.057	3.143	0.354	–	6.94	0.1615050	0.0095914
OGE-G	0.00112	2.766	–	-0.206	0.129	0.5535101	0.000
OGE-FW	7.28	0.124	4.974	–	0.680	0.2221729	0.000

Table 4. The $\mathcal{L}$, AIC, AICC, BIC, RMSE and R^2.

Model	$\mathcal{L}$	AIC	AICC	BIC	HQIC	RMSE	R^2
OGE-PHR	–64.5178	137.0356	137.4566	147.4563	141.2530	0.0041972	0.937793
PHR	–90.1492	184.2984	184.4221	189.5088	186.4072	0.1095762	0.768124
OGE-RD	–195.6188	397.2375	397.4875	405.0530	400.4006	0.0620341	0.139852
OGE-W(ν)	–109.0921	224.1842	224.4342	231.9997	227.3473	0.2864156	0.017589
OGE-E	–66.8960	139.7920	140.0420	147.6080	142.9550	0.0042795	0.937644
OGE-W	–74.4591	156.9182	157.3392	167.3389	161.1356	0.0079095	0.872042
OGE-G	–184.4214	376.8427	377.2638	387.2634	381.0602	0.0784983	0.077269
OGE-FW	–87.2450	182.4899	182.9110	192.9106	186.7074	0.0134347	0.746835

The corresponding $\boldsymbol{V}$ matrix is

$$\boldsymbol{V} = \begin{pmatrix} 29.952 & -0.424 & 0.582 & -5.748 \\ -0.424 & 6.133 \cdot 10^{-3} & -8.695 \cdot 10^{-3} & 0.133 \\ 0.582 & -8.695 \cdot 10^{-3} & 0.023 & -0.571 \\ -5.748 & 0.133 & -0.571 & 30.848 \end{pmatrix}$$

Then the 95% confidence interval for ν, ω, γ and κ for OGE-PHR distribution are (0, 20.65317), (0.02096, 0.32795), (–0.26929, 0.32023), and (5.0063, 26.77834), respectively. Figures 5 and 6 show that the likelihood function has a unique solution. For $\hat{\nu} = 9.926, \hat{\omega} = 0.174, \hat{\gamma} = 0.025$, and $\hat{\kappa} = 15.892$, the OGE-PHRD is right skewed ($sk = 0.09543 > 0$) and approximately symmetric. It is a platykurtic or short-tailed distribution ($ku = 1.2533 < 3$).

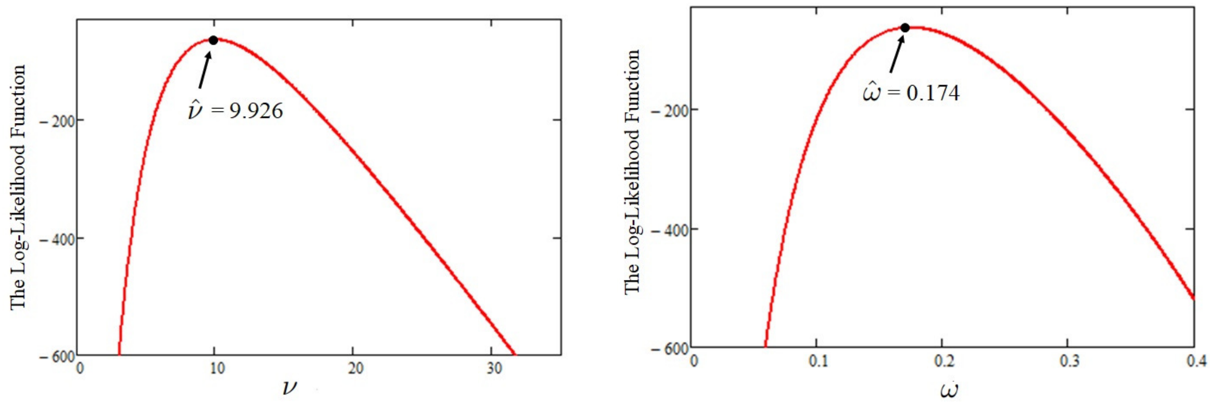

Figure 5. The log-likelihood function of ν and ω

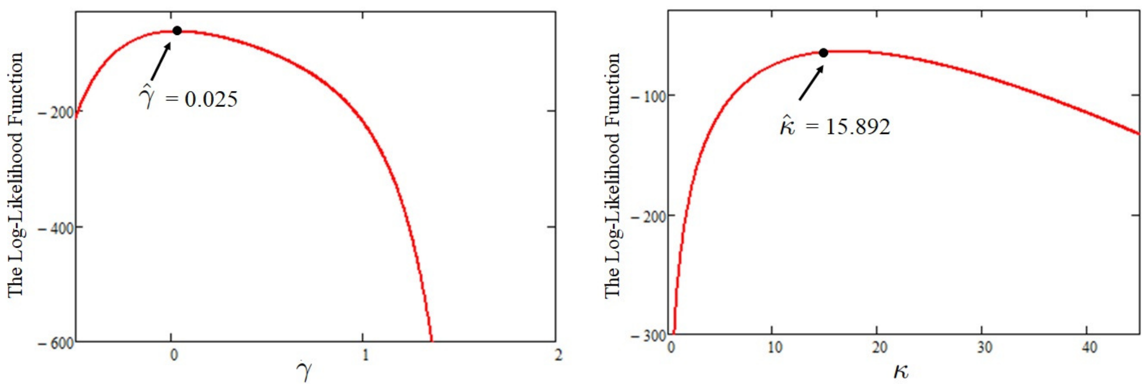

Figure 6. The log-likelihood function of γ and κ

Example 6.2. The following simulated data set is taken from [15]

1.014	1.271	1.292	1.409	1.501	1.579	1.684	1.757	1.916
1.081	1.272	1.304	1.426	1.506	1.581	1.691	1.800	1.972
1.082	1.275	1.306	1.459	1.524	1.591	1.704	1.806	2.012
1.185	1.276	1.355	1.460	1.526	1.593	1.731	1.867	2.592
1.223	1.278	1.361	1.476	1.535	1.602	1.735	1.876	3.197
1.248	1.286	1.364	1.481	1.541	1.666	1.747	1.878	4.121
1.267	1.288	1.379	1.484	1.568	1.670	1.748	1.910	2.456

The MLEs of $\nu, \omega, \gamma, \kappa$, K-S and p-value for OGE-PHR, PHR and some OGE-distributions are given in Table 5. Table 6 shows the numerical values of $\mathcal{L}$, AIC, AICC, BIC, HQIC, RMSE, and R^2.

Table 5. The MLEs of $\nu, \omega, \gamma, \kappa$, K-S and p-value

Model	$\hat{\nu}$	$\hat{\omega}$	$\hat{\gamma}$	$\hat{c}$	$\hat{\kappa}$	K-S	p-value
OGE-PHR	17.2733	0.1168	0.3478	–	13.6484	0.1314074	0.21174254
PHR	–	0.5171	2.062	–	–	0.2051130	0.00841414
OGE-RD	0.0239	1.1978	–	–	0.2942	0.4765220	0.000
OGE-W(ν)	0.0239	1.0291	–	–	0.8037	0.7189660	0.000
OGE-E	0.9711	0.7229	–	–	6.0645	0.2028970	0.009463885
OGE-W	0.0293	3.7838	0.3475	–	7.8523	0.1545998	0.089370279
OGE-G	0.000524	1.7983	–	0.1077	0.0929	0.5841869	0.000
OGE-FW	5.3382	0.3036	5.7944	–	0.487	0.2771419	0.000

Table 6. The $\mathcal{L}$, AIC, AICC, BIC, RMSE and R^2

Model	$\mathcal{L}$	AIC	AICC	BIC	HQIC	RMSE	R^2
OGE-PHR	-28.9737	65.9474	66.6371	74.520	69.3191	0.00533555	0.91907098
PHR	-46.3669	96.7338	96.9338	101.020	98.4196	0.11953744	0.72326318
OGE-RD	-105.529	217.0581	217.464	223.4875	219.5868	0.06505871	0.17522668
OGE-W(ν)	-68.9706	143.9413	144.3481	150.3707	146.470	0.17540187	0.06630705
OGE-E	-40.9001	87.8001	88.2069	94.2295	90.3288	0.01382935	0.78986324
OGE-W	-34.4322	76.8645	77.5541	85.437	80.2361	0.00792246	0.86915008
OGE-G	-129.0352	266.0704	266.7601	274.643	269.4421	0.08237886	0.05252965
OGE-FW	-53.7568	115.5136	116.2033	124.0861	118.8852	0.02340996	0.55943517

From the results contained in Tables 5 and 6, it can be concluded that OGE-PHRD provides a better fit of the data.

Therefore, the matrix $\boldsymbol{V}$ is

$$\boldsymbol{V} = \begin{pmatrix} 123.324 & -0.742 & 1.242 & -6.723 \\ -0.742 & 4.612 \cdot 10^{-3} & -8.505 \cdot 10^{-3} & 0.1 \\ 1.242 & -8.505 \cdot 10^{-3} & 0.041 & -0.788 \\ -6.723 & 0.1 & -0.788 & 31.845 \end{pmatrix}$$

Then the 95% confidence interval for ν, ω, γ and κ for OGE-PHR distribution are and (0, 39.03937), (0, 0.24994), (-0.04803, 0.74368) and (2.58794, 24.70888), respectively.

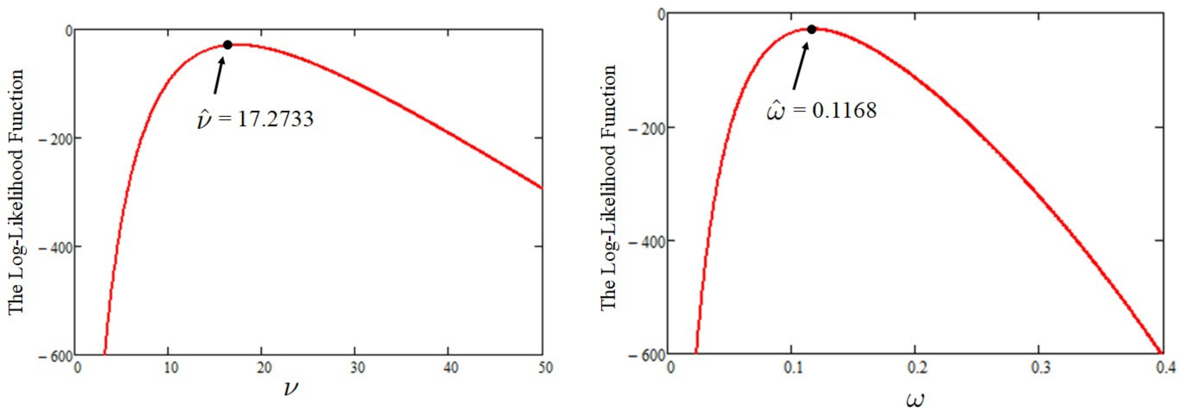

Figure 7. The log-likelihood function of ν and ω

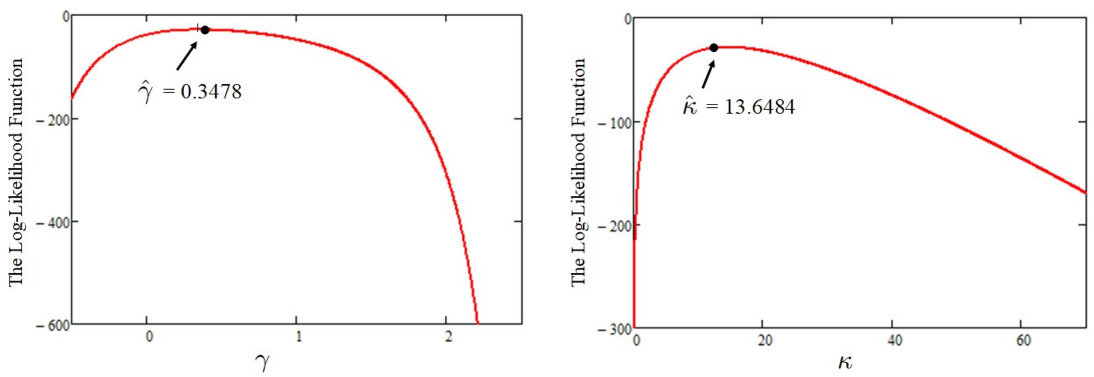

Figure 8. The log-likelihood function of γ and κ

Figures 7 and 8 show that the likelihood function has a unique solution. For $\hat{\nu} = 17.2733$, $\hat{\omega} = 0.1168, \hat{\gamma} = 0.3478$ and $\hat{\kappa} = 13.6484$, the OGE-PHRD is right skewed ($S_k = 0.07992 > 0$) and approximately symmetric. It is a platykurtic or short-tailed distribution ($K_u = 1.24823 < 3$).

7. Conclusions

The generalized continuous distributions have been widely studied in the literature. In this article, we have studied a new probability distribution with four parameters called odds generalized exponential–power hazard rate distribution. The proposed distribution contains some special distributions. The study of the characteristics of this distribution includes the study of these cases. We examined some properties of the proposal distribution. Various statistical properties of the new distribution such as moments, mode, quantity, skewness, kurtosis, and order statistics were derived. The parameters of the new distribution were estimated by the method of maximum likelihood.

A real data set was used to compare the distribution of OGE-PHR with the sub-models, PHR, OGE-RD, OGE-W(ν), OGE-E based on some statistical criteria such as K-S, AIC, AICC, BIC, HQIC, RMSE and R^2. It was shown that the new distribution can be used very effectively to provide the best fit compared to the sub-models. Also, we compared the OGE-PHR with other known odd generalized exponential distributions such as OGE-W, OGE-G and OGE-FW. Applications on the real data showed that the OGE-PHR was the best distribution for fitting these data sets compared with OGE-W, OGE-G and OGE-FW distributions. We conclude that this model provides consistently better fits than other special models. We hope that this generalization will include more applications in various fields of life.

Acknowledgement

The author is grateful to two anonymous reviewers for their valuable comments and suggestions made on the draft of this manuscript.

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