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On biconnected sets with dispersion points

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CHAPTER I

§ 1. Introduction

A set X is said to be *dispersed* if it contains no connected subset(¹), i.e. if every component of X is a single point.

A point p of a connected set X is said to be its *dispersion point* if $X-(p)$ contains no connected subset. A set X is called *pulverable* if it is connected and contains a dispersion point p. The set $X-(p)$ and every set homeomorphic with $X-(p)$ are called *pulverized sets.* Every pulverized set, as containing no connected subset, is a dispersed set. The converse is not true, of course.

Above forty years ago Knaster and Kuratowski introduced [6] the notion of biconnected set. They have used the term *biconnected* to denote such a connected set as cannot be decomposed into two non-intersecting connected subsets. Each of biconnected sets constructed by Knaster and Kuratowski contains a dispersion point. Kline has proved [4] that every connected set contains at most one such point. It follows that every pulverable set P contains exactly one dispersion point. Nevertheless, Miller has shown [12] that, if the continuum hypothesis is true, there exists a biconnected set which contains no dispersion point.

In virtue of the definition, a biconnected set with dispersion point is pulverable. The converse is also true, because every connected subset of a pulverable set P must contain its dispersion point (see Lemma 4.1, p. 15), and consequently a pulverable set cannot be a sum of two non-intersecting and connected subsets.

We shall use the term "pulverable set" instead of the name "biconnected set with dispersion point". Consequently, we shall use the short term "pulverized set" instead of the long one "a set homeomorphic with a biconnected set with dispersion point, whose dispersion point is removed". In this way by means of the special name of *pulverable sets* we distinguish a large subfamily of biconnected sets; this is in accordance with its importance and its characteristic structure, which differs from the

(¹) In this paper a point set X is said to be *connected* if it contains at least two points and is not a sum of two non-void separated subsets.

structure of biconnected sets containing no dispersion point. To the family of pulverized sets belongs, for instance (see Roberts [13]), the important set of all rational points (i.e. points with rational coordinates only) of the Hilbert space (²).

It seems to be advantageous to get examine more closely the structure of pulverable and that of pulverized sets. This is the scope of the present paper.

Throughout the paper all sets (spaces) are separable metric sets. By virtue of the Urysohn metrization theorem all this amounts to an investigations of subsets of the Hilbert cube.

The paper consists of 4 chapters.

Chapter I is an introductory one. It contains an introduction (§ 1), preliminary notions and properties (§ 2), some properties of the new notion of relative quasicomponent (§ 3), and some elementary properties of pulverable sets (§ 4).

Chapter II contains some results on the structure of pulverable sets. § 5 deals with their connected subsets. Miller has proved ([12], theorem 4, p. 125) that if B is a biconnected set containing no dispersion point, and if T is a finite subset of B, then the set $B-T$ is connected. Theorem 5.3 completes this result as follows: if P is a pulverable set, a its dispersion point, and T a finite subset of $P-(a)$, then the set $P-T$ is also connected. Given any family of pulverable sets, the main theorem of § 6 permits to construct a new pulverable set (Theorem 6.1). Corollary 6.2 asserts that for every family of at most $2^{\aleph_0}$ pulverable sets there exists a pulverable set being a union of homeomorphic images of these sets. § 7 deals with quasicomponents of pulverized sets. A *quasicomponent* of a point p in a set X is the common part of all closed-open subsets of X containing p. In other words, it is a set of all points $q \in X$ such that the set X is connected between p and q. It follows at once that every closed-open subset of a set X is a union of some of its quasicomponents. Theorem 7.1 contains a topological characterization of pulverized sets; the remaining theorems are concerned, among others, with the power and dimension properties.

Chapter III contains solutions of problems concerned with continuous mappings (§ 8), certain minimal properties (§ 9) and σ-connectivity of pulverable sets (§ 10). For instance, every connected set can be obtained as a continuous image of some pulverable set (Theorem 8.5).

Chapter IV is the most extensive of all. It deals with the examples and theirs constructions. § 11 contains two lemmas on some decompositions of a segment $\mathscr{I}$ and Cantor set $\mathscr{C}$. These two lemmas are based upon

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(²) Erdős has proved [1] that this set has dimension 1, otherwise than in Euclidean spaces and in the Hilbert cube $\mathscr{I}^{\aleph_0}$ ([10], I, p. 87), where it has dimension 0 (see for instance [3]).

continuum hypothesis and they will be applied to the construction of example 3 only. No other theorem or example in this is based on these lemmas or on the continuum hypothesis; they are effective. Finally, § 12 contains five constructions.

I owe my very warm thanks to Professor Dr. Bronisław Knaster who contributed to my investigations by many valuable suggestions.

Notation

The notions and notation are derived from books [5] and [11]. Besides, I shall use in this paper the following notation:

$\mathscr{C}$ — Cantor set,

$\mathscr{M}$ — Cantor fan, i.e. the union $\bigcup_{\tau\in\mathscr{C}} L(\tau)$ of all segments $L(\tau)$ of ends $(\tau, 0)$ and $(1/2, 1/2)$, where $\tau\in\mathscr{C}$.

$\mathscr{I}^{\aleph_0}$ — Hilbert cube,

P — a pulverable set,

a — a dispersion point of P,

ψ — a continuous function mapping a set $P-(a)$ into the Cantor set $\mathscr{C}$ (i.e. $\psi[P-(a)]\subset\mathscr{C}$) in such a way that each counter-image under it of a point is a quasicomponent of $P-(a)$ ("quasikomponententreue Abbildung") (³).

$\varphi_M(N)$ — a continuous function mapping a closed subset M of N in one point not belonging to $\varphi(N-M)$, and such that $\varphi|N-M$ is a homeomorphism (identification of a closet M of N to a point) (⁴).

§ 2. Preliminary notions and properties

We begin with elementary and partially known lemmas concerning the subsets of any topological space.

LEMMA 2.1. *If W is open in T and T is open in X, then W is open in X.*

LEMMA 2.2. *If W is closed in T and T is closed in X, then W is closed in X.*

Both these lemmas are known ([10], I, p. 26).

LEMMA 2.3. *Let Z be a subset of a topological space X. If a subset W of $X-Z$ is closed in $X-Z$, and if*

$$\overline{W}\cap Z = 0, \tag{1}$$

then W is closed in X.

(³) See an equivalent definition, p. 8.

(⁴) See $\varphi_F(X\cup F)$, p. 10.

In fact, we infer from the hypotheses $W = \overline{W} \cap (X-Z)$ and (1) that $\overline{W} = \overline{W} \cap [(X-Z) \cup Z] = \overline{W} \cap (X-Z) \cup \overline{W} \cap Z = W$.

Lemma 2.4. *Let Z be a subset of a topological space X. If a subset W of $X-Z$ is open in $X-Z$, and if*

$$W \cap \bar{Z} = 0, \tag{2}$$

then W is open in X.

In fact, since the sets W and Z are disjoint, we have $X-W = [X-(W \cup Z)] \cup Z$. Hence, by hypotheses $(X-Z)-W = \overline{(X-Z)-W} \cap (X-Z)$ and (2) it follows that

$$\begin{aligned} \overline{X-W}-(X-W) &= \overline{X-W} \cap W = \overline{[X-(W \cup Z)] \cup Z} \cap W \\ &= [\overline{X-(W \cup Z)} \cup \bar{Z}] \cap W = \overline{X-(W \cup Z)} \cap W \cup W \cap \bar{Z} \\ &= \overline{X-(W \cup Z)} \cap W = \overline{X-(W \cup Z)} \cap W \cap [(X-Z) \cup Z] \\ &= \overline{X-(W \cup Z)} \cap (X-Z) \cap W \cup \overline{X-(W \cup Z)} \cap W \cap Z \\ &= [(X-Z)-W] \cap W \cup \overline{X-(W \cup Z)} \cap W \cap Z = 0. \end{aligned}$$

It means that $\overline{X-W} \subset X-W$.

Lemma 2.5. *Let Z be a subset of a topological space X. If a subset T of $X-Z$ is closed in $X-Z$, and a subset W of $T \cup Z$ is closed in $T \cup Z$, and if*

$$\overline{W} \cap Z = 0, \tag{3}$$

then W is closed in X.

In fact, the hypotheses $W = \overline{W} \cap (T \cup Z)$ and (3) imply

$$W = \overline{W} \cap T, \tag{4}$$

and the hypothesis $T = \bar{T} \cap (X-Z)$ implies $\overline{W} \cap T = \overline{W} \cap \bar{T} \cap (X-Z)$. Therefore, in view of (4) and of the inclusion $W \subset T$, we obtain from the last equality $W = \overline{W} \cap (X-Z)$. It remains to quote Lemma 2.3.

Lemma 2.6. *Let Z be a subset of a topological space X. If a subset T of $X-Z$ is open in $X-Z$, and a subset W of $T \cup Z$ is open in $T \cup Z$, and if*

$$W \cap \bar{Z} = 0,$$

then W is open in X.

In fact, by the hypothesis $(T \cup Z)-W = \overline{(T \cup Z)-W} \cap (T \cup Z)$ and by the identity $(T \cup Z)-W = (T-W) \cup (Z-W)$ we have $(T-W) \cup (Z-W) = \overline{(T-W) \cup (Z-W)} \cap (T \cup Z) = \overline{T-W} \cap T \cup \overline{Z-W} \cap T \cup \overline{T-W} \cap Z \cup \overline{Z-W} \cap Z$. Multiplying by T, we easily see that $T-W = \overline{T-W} \cap T \cup \overline{Z-W} \cap T$ in view of $T \cap Z = 0$. Hence, a fortiori, $T-W \supset \overline{T-W} \cap T$.

It means that W is open in T, and therefore W is open in $X-Z$ by Lemma 2.1, because T is open in $X-Z$. It remains to quote Lemma 2.4.

We infer from Lemmas 2.1 and 2.2 that

LEMMA 2.7. *A quasicomponent Q of a topological space X which is open and contains at least two points is connected.*

Proof. If we suppose the contrary, the quasicomponent Q is a sum of two non-void and separated subsets $Q = Q_1 \cup Q_2$, i.e. Q_1 and Q_2 are non-void, disjoint and closed-open in Q. As a quasicomponent, Q is closed in X ([10], II, p. 93), and, by hypothesis, Q is open in X. Therefore, by virtue of Lemmas 2.1 and 2.2, the non-void sets Q_1 and Q_2 are closed-open in X, which contradicts to Q being a quasicomponent.

Now, taking into account the above seven lemmas, we prove some simple properties of closed-open subsets of subsets of a connected space, and, in particular, of quasicomponents of subsets of a connected space.

LEMMA 2.8. *Let A be a non-void subset of a connected space S. Then for every subset H of $S-A$ non-void and closed-open in $S-A$ we have $A \cap \bar{H} \cup \bar{A} \cap H \neq 0$.*

Proof. If

$$A \cap \bar{H} \cup \bar{A} \cap H = 0, \tag{5}$$

then by Lemmas 2.3 and 2.4 the set H is closed-open in S. We have then a decomposition $S = H \cup (S-H)$ into two subsets of S non-void, disjoint and closed-open in S. Since by hypothesis the space S is connected and the set H is non-void, we have $S-H = 0$, whence $S \subset H$, and therefore $A \subset H$, contrary to (5) and to the hypothesis $A \neq 0$.

LEMMA 2.9. *Let A be a non-void subset of a connected space S. Then for every quasicomponent Q of $S-A$ open in $S-A$ we have $A \cap \bar{Q} \cup \bar{A} \cap \cap Q \neq 0$.*

In fact, the inequality follows from Lemma 2.8, because every quasicomponent Q of the set $S-A$ is closed in it ([10], II, p. 93).

LEMMA 2.10. *Let A be a non-void subset of a connected space S. Then every quasicomponent Q of $S-A$ not nowhere dense in S has a positive dimension (in each of its interior points).*

Proof. Since by hypothesis $\overline{S-\bar{Q}} \neq S$, we have $\overline{S-\bar{Q}} \neq S$ by $Q = \bar{Q}$ ([10], II, p. 93). Hence $\mathrm{Int}(Q) = S-\overline{S-Q} \neq 0$. Let $p \in \mathrm{Int}(Q)$. If $\dim_p Q = 0$, then there exists a neighbourhood $U \subset \mathrm{Int}(Q)$ of the point p, boundary of which is void, contrary to hypothesis that the set S is connected.

Consider now any continuous function f on a subset $S-A$ of a connected space S, carrying $S-A$ into the Cantor set $\mathscr{C}$. Every set $f^{-1}(\tau)$, where $\tau \in f(S-A)$, is then ([10], I, p. 74) a common part of a sequence of closed-open subsets of $S-A$. In fact, let $\{\Delta_n\}_{n=1,2,\dots}$ be a sequence of closed-open subsets of the Cantor set $\mathscr{C}$ such that $(\tau) = \bigcap_{n=1}^{\infty} \Delta_n$. Therefore $\Delta_n \cap f(S-A)$ is for $n = 1, 2, \dots$ a closed-open subset of $f(S-A)$ and $f^{-1}(\tau) = \bigcap_{n=1}^{\infty} f^{-1}[\Delta_n \cap f(S-A)]$ ([10], I, p. 17, formula 7a). In particular, it may occur that $f^{-1}(\tau)$ is a quasicomponent of the set $S-A$. At any rate $f^{-1}(\tau)$ is a closed subset of the set $S-A$.

There exists (see [10], II, p. 93) a continuous function defined on $S-A$, the range of which is a subset of the Cantor set $\mathscr{C}$, and such that counter-image of every point under it is a quasicomponent of the set $S-A$ ("quasikomponententreue Abbildung"). This function is denoted by ψ, according to the list of notation (see p. 5). Hence

(10) $$\psi\colon S-A \to \mathscr{C},$$

(11) $$S-A = \bigcup_{\tau\in\psi(S-A)} \psi^{-1}(\tau),$$

(12) $\psi^{-1}(\tau)$ is a quasicomponent of a set $S-A$ for each $\tau\in\psi(S-A)$.

LEMMA 2.11. *Let A be a connected subset of a connected space S. Then for every continuous function $f\colon S-A \to \mathscr{C}$ and every closed-open subset Δ of the Cantor set $\mathscr{C}$ such that $\Delta \cap f(S-A) \neq 0$, the set*

(13) $$A \cup \bigcup_{\tau\in\Delta\cap f(S-A)} f^{-1}(\tau)$$

is connected.

For the set $\bigcup_{\tau\in\Delta\cap f(S-A)} f^{-1}(\tau)$ is closed-open in $S-A$ as a counter-image under continuous function f of a set $\Delta \cap f(S-A)$ closed-open in $f(S-A)$ ([10], I, p. 74, formulae (3) and (4)). Hence set (13) is connected ([10], II, p. 83).

LEMMA 2.12. *Let A be a connected subset of a connected space S. Then for every continuous function $f\colon S-A \to \mathscr{C}$ and every subset Δ of the Cantor set $\mathscr{C}$, the set*

(14) $$A \cup \bigcup_{\tau\in f(S-A)-\Delta} f^{-1}(\tau)$$

is connected.

Proof. Let us choose for each point $\tau\in f(S-A)-\Delta$ a closed-open in the Cantor set $\mathscr{C}$ subset $\Gamma(\tau)$ of $\mathscr{C}$ such that

(15) $$\tau\in\Gamma(\tau) \subset \mathscr{C}-\Delta.$$

By virtue of Lemma 2.12 the set

$$A \cup \bigcup_{\eta\in\Gamma(\tau)\cap f(S-A)} f^{-1}(\eta)$$

is connected and therefore the set

$$A \cup \bigcup_{\tau \in f(S-A)-\bar{\Delta}} \; \bigcup_{\eta \in \Gamma(\tau) \cap f(S-A)} f^{-1}(\eta) \tag{16}$$

is connected too ([10], II, p. 82). It remains to verify the identity of the sets (14) and (16).

Since $\Gamma(\tau)$ is a neighbourhood of the point τ, closed-open in $\mathscr{C}$ and fulfilling (15), where $\tau \in f(S-A)-\bar{\Delta}$, we have

$$f^{-1}(\tau) \subset \bigcup_{\eta \in \Gamma(\tau) \cap f(S-A)} f^{-1}(\eta),$$

whence

$$\bigcup_{\tau \in f(S-A)-\bar{\Delta}} f^{-1}(\tau) \subset \bigcup_{\tau \in f(S-A)-\bar{\Delta}} \; \bigcup_{\eta \in \Gamma(\tau) \cap f(S-A)} f^{-1}(\eta). \tag{17}$$

Conversely, by (15) we have

$$\Gamma(\tau) \cap f(S-A) \subset f(S-A)-\bar{\Delta}$$

for each point $\tau \in f(S-A)-\bar{\Delta}$. Thus

$$\bigcup_{\eta \in \Gamma(\tau) \cap f(S-A)} f^{-1}(\eta) \subset \bigcup_{\tau \in f(S-A)-\bar{\Delta}} f^{-1}(\tau),$$

whence

$$\bigcup_{\tau \in f(S-A)-\bar{\Delta}} \; \bigcup_{\eta \in \Gamma(\tau) \cap f(S-A)} f^{-1}(\eta) \subset \bigcup_{\tau \in f(S-A)-\bar{\Delta}} f^{-1}(\tau). \tag{18}$$

By virtue of inclusions (17) and (18) sets (14) and (16) are identical.

LEMMA 2.13. *Let A be a closed and connected subset of a connected space S, let f be a continuous function mapping $S-A$ into the Cantor set $\mathscr{C}$, and let Δ be a set of all $\tau \in f(S-A)$ for which $\dim f^{-1}(\tau) > 0$. Then the set $f(S-A)-\bar{\Delta}$ is either void or uncountable.*

Proof. By the definition of Δ, we have $\dim f^{-1}(\tau) = 0$ for each $\tau \in f(S-A)-\bar{\Delta}$. If the set $f(S-A)-\bar{\Delta}$ were non-void and at most countable, we should have ([10], I, p. 176)

$$\dim\Big[\bigcup_{\tau \in f(S-A)-\bar{\Delta}} f^{-1}(\tau)\Big] = 0. \tag{19}$$

But, by virtue of Lemma 2.12, the set $A \cup \bigcup_{\tau \in f(S-A)-\bar{\Delta}} f^{-1}(\tau)$ is connected; therefore it has a positive dimension in each point ([10], II, p. 80), and so has the set $\bigcup_{\tau \in f(S-A)-\bar{\Delta}} f^{-1}(\tau)$, because the set A is closed. Thus (19) is impossible.

Lemma 2.13 is also valid for $f = \psi$. Namely

LEMMA 2.14. *Let A be a closed and connected subset of a connected space S. If every quasicomponent of the set $S-A$ is 0-dimensional, then the family of all quasicomponents of $S-A$ is uncountable.*

The last group of lemmas pertains to the so-called *identification*.

For any $X \subset \mathcal{I}^{\aleph_0}$ and any closed $F \subset \mathcal{I}^{\aleph_0}$ there exists a function φ_F (identification), defined on $X \cup F$ and having the following properties ([10], I, p. 138):

(20) φ_F is a continuous function,

(21) $\varphi_F|X-F$ is a homeomorphism,

(22) $\varphi_F(F)$ is a point not belonging to the set $\varphi_F(X-F)$.

These properties imply two following lemmas:

LEMMA 2.15. $\varphi_F(X \cup F)-\varphi_F(X-F) = \varphi_F(F)$.

In fact, by (22), $\varphi_F(F) \epsilon \varphi_F(X \cup F)-\varphi_F(X-F)$, and by [10] we have

$$\varphi_F(X \cup F)-\varphi_F(X-F) \subset \varphi_F(X \cup F)-[\varphi_F(X)-\varphi_F(X-F)]$$
$$= \varphi_F(X) \cup \varphi_F(F)-\{[\varphi_F(X) \cup \varphi_F(F)]-\varphi_F(F)\} = \varphi_F(F)$$

([10], I, p. 17, formulae 1 and 3).

LEMMA 2.16. *If for every non-void and closed-open subset H of a set X*

$$\bar{H} \cap F \neq 0, \tag{23}$$

then the set $\varphi_F(X \cup F)$ is connected.

Proof. The case $X \subset F$ is trivial, because $\varphi_F(X \cup F) = \varphi_F(F)$ is a point. Considering the case $X-F \neq 0$ suppose that

$$\varphi_F(X \cup F) = M \cup N, \quad M \neq 0 \neq N, \tag{24}$$

$$\bar{M} \cap N \cup \bar{M} \cap N = 0, \tag{25}$$

$$\varphi_F(F) \epsilon M. \tag{26}$$

We have then by virtue of (25) and (26),

$$\bar{N} \cap \varphi_F(F) = 0. \tag{27}$$

It follows ([10], II, p. 17 and 74) by (20) that $\varphi_F^{-1}[\bar{N} \cap \varphi_F(F)] = \varphi_F^{-1}(\bar{N}) \cap \varphi_F^{-1}\varphi_F(F) \supset \overline{\varphi_F^{-1}(N)} \cap \varphi_F^{-1}\varphi_F(F)$, whence, by (27), $\overline{\varphi_F^{-1}(N)} \cap \varphi_F^{-1}\varphi_F(F) = 0$, and therefore by $F \subset \varphi_F^{-1}\varphi_F(F)$ ([10], I, p. 17, (11))

$$\overline{\varphi_F^{-1}(N)} \cap F = 0. \tag{28}$$

The set N is, by (24) and (25), closed-open in $\varphi_F(X \cup F)$ and therefore by (20) we infer ([10], I, p. 74, formulae (3) and (4)) that the set $\varphi_F^{-1}(N)$ is closed-open in $\varphi_F^{-1}\varphi_F(X \cup F) = X \cup F$. Hence, by (28), the set $\varphi_F^{-1}(N) \subset X$ is closed-open in X. Supposition $N \neq 0$ implies $\varphi_F^{-1}(N) \neq 0$. Applying now (23) to the set $H = \varphi_F^{-1}(N)$ we have $\overline{\varphi_F^{-1}(N)} \cap F \neq 0$, contrary to (28).

§ 3. Relative quasicomponents

The new notion of relative quasicomponent has some interesting properties which will be proved in this paragraph and will be made use of in the proof of basic property of Example 3 (p. 41) which is one of principal results of this paper.

Let A be a subset of a topological space X and let $p \in \bar{A}$. A subset H of X is said to be an *envelope of* p if H is closed in X and $(p) \cup A \cap H$ is open in $(p) \cup A$. Since H is closed in X, $(p) \cup A \cap H$ is also closed, and therefore closed-open, in $(p) \cup A$. Now a *quasicomponent of the point* p *of the set* A *relatively to the space* X is a common part of all envelopes of p. We shall denote this quasicomponent by $Qc_p(A, X)$.

It is obvious that in the case $A = X$ the relative quasicomponents are identical with quasicomponents.

Relative quasicomponents may intersect and even one of them may be contained in the other. For example, let X be the square of opposite vertices $(0, 0)$ and $(2, 2)$ and let A be the union of two sequences of segments $I_n = \{(x, y): x = 1 - 1/n,\ 0 \leqslant y \leqslant 1\}$ and $J_n = \{(x, y): x = 1 + 1/n,\ 1 \leqslant y \leqslant 2\}$. The limit segments $I_0 = \{(x, y): x = 1,\ 0 \leqslant y \leqslant 1\}$ and $J_0 = \{(x, y): x = 1,\ 1 \leqslant y \leqslant 2\}$ do not belong to A. It is easy to see that the relative quasicomponent of the point $(1,0)$ is the segment I_0, that of the point $(1,2)$ — the segment J_0, and that of the point $(1,1)$ — the union $I_0 \cup J_0$ (see Fig. 1).

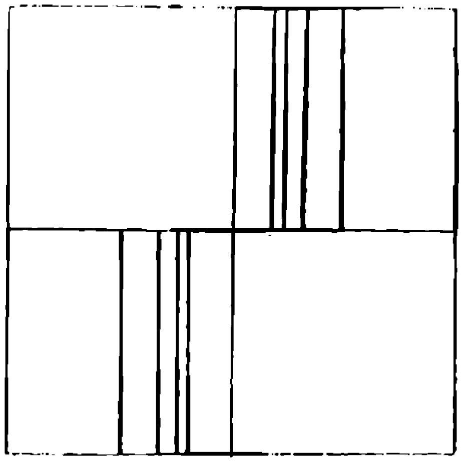

Fig. 1

In view of the Lindelöf theorem ([10], I, p. 131) every relative quasicomponent $Qc_p(A, X)$ is a common part of a sequence of envelopes of p, and since the common part of two envelopes of p is an envelope of p, we may assume that $Qc_p(A, X)$ is a common part of a *descending sequence* of envelopes of p.

Recall that a space X is said to be *peripherically compact* if and only if for every point $x \in X$ there exist arbitrarily small neighbourhoods of x the boundaries of which are compact.

LEMMA 3.1. *Let A be a subset of a peripherically compact space X and let $p \in \bar{A}$. If*

$$(1) \qquad (p) = \bigcap_{n=1}^{\infty} H_n,$$

where $\{H_n\}_{n=1,2,\ldots}$ *is a descending sequence of envelopes of* p, *then there exists a descending sequence* $\{H_n^*\}_{n=1,2,\ldots}$ *of envelopes of* p *such that* $\delta(H_n^*) < 1/n$ *for* $n = 1, 2, \ldots$

Proof. Let K_n be a neighbourhood of the point p such that the boundary $\mathrm{Fr}(K_n)$ is compact and $\delta(K_n) < 1/n$. For every $n = 1, 2, \ldots$ there exists m_n such that

$$\mathrm{Fr}(K_n) \cap H_m = 0 \quad \text{for any} \quad m > m_n \tag{2}$$

(because in the contrary case we should have $\mathrm{Fr}(K_n) \cap H_m \neq 0$ for $m = 1, 2, \ldots$, whence $\mathrm{Fr}(K_n) \cap \bigcap_{m=1}^{\infty} H_m \neq 0$ in view of Cantor's theorem ([10], II, p. 5), but the last inequality is impossible in view of (1) and of the definition of K_n). We may assume $m_{n+1} > m_n$ for $n = 1, 2, \ldots$, of course.

Putting for $n = 1, 2, \ldots$

$$H_n^* = H_{m_n+1} \cap K_n \tag{3}$$

we see that $p \in H_n^*$. Since $H_n^* = H_{m_n+1} \cap \overline{K_n}$ by virtue of (2), H_n^* is a closed subset of X $(n = 1, 2, \ldots)$. And since $(p) \cup A \cap H_{m_n+1}$ is, by hypothesis, an open subset of $(p) \cup A$, and K_n is an open subset of X, so by (3) the set $(p) \cup A \cap H_n^*$ is an open subset of $(p) \cup A$ $(n = 1, 2, \ldots)$. Moreover, $\delta(H_n^*) < 1/n$ and, by the hypothesis that $H_{n+1} \subset H_n$ and by the assumption that $m_{n+1} < m_n$, we easily obtain from (3) that $H_{n+1}^* \subset H_n^*$ for $n = 1, 2, \ldots$ Hence the sequence $\{H_n^*\}_{n=1,2,\ldots}$ satisfies the lemma.

THEOREM 3.2. *Let* A *be a subset of a peripherically compact space* X *and let* $p \in \bar{A}$. *A necessary and sufficient condition that* $\dim_p[(p) \cup A] = 0$ *is*

$$(p) = Qc_p(A, X). \tag{4}$$

Proof. If $\dim_p[(p) \cup A] = 0$, then there exist arbitrarily small closed-open subsets G of $(p) \cup A$ containing p. We infer then from equality $[(p) \cup A] \cap \bar{G} = G$ that G is an envelope of p. Hence, by the definition of relative quasicomponent, $Qc_p(A, X) \subset \bar{G}$, and consequently (4).

Conversely, hypothesis (4) implies that $(p) = \bigcap_{n=1}^{\infty} H_n$, where $\{H_n\}_{n=1,2,\ldots}$ is a descending sequence of envelopes of p. By virtue of Lemma 3.1 we can assume that $\lim_{n\to\infty} \delta(H_n) = 0$. Hence there exist arbitrarily small closed-open subsets $(p) \cup A \cap H_n$ of $(p) \cup A$ containing p. It means that $\dim_p[(p) \cup A] = 0$.

THEOREM 3.3. *If* A *is a subset of a compact space* X *and* $p \in \bar{A}$, *then* $Qc_p(A, X)$ *is a continuum.*

Proof. Suppose that

$$Qc_p(A, X) = M \cup N, \tag{5}$$

where M and N are non-void, closed in X, and disjoint. Then there exists an open subset G of X such that

$$p \in M \subset G \quad \text{and} \quad \bar{G} \cap N = 0. \tag{6}$$

By the definition of relative quasicomponent we have

$$Qc_p(A, X) = \bigcap_{n=1}^{\infty} H_n, \tag{7}$$

where $\{H_n\}_{n=1,2,\dots}$ is a descending sequence of envelopes of p. We shall show that

$$H_n \cap \mathrm{Fr}(G) \neq 0 \quad \text{for} \quad n = 1, 2, \dots \tag{8}$$

For supposing $H_n \cap \mathrm{Fr}(G) = 0$ for some $n = n_0$ we have $H_{n_0} \cap \bar{G} = H_{n_0} \cap G$ in view of $\mathrm{Fr}(G) = \bar{G} - G$, and since, by hypothesis, H_{n_0} is closed in X, so $H_{n_0} \cap G$ is also closed in X. Moreover, $(p) \cup A \cap (H_{n_0} \cap G)$

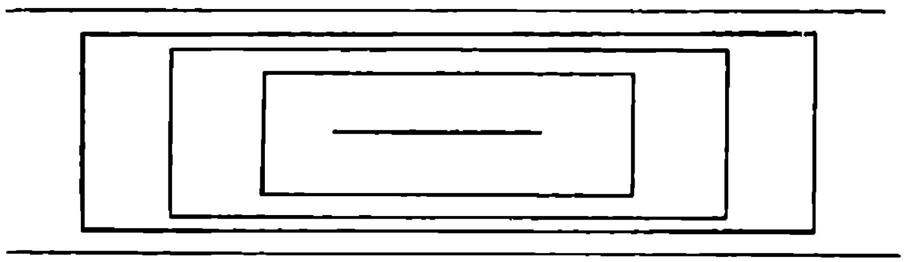

Fig. 2

is open in $(p) \cup A$, because, by hypothesis, $(p) \cup A \cap H_{n_0}$ is open in $(p) \cup A$, and G is open in X. Hence $H_{n_0} \cap G$ is an envelope of p and therefore, by the definition of relative quasicomponent, we have $Qc_p(A, X) \subset H_{n_0} \cap G$. But multiplying this inclusion by N we infer, by (5) and (6), that $N = 0$, contrary to the hypothesis that $N \neq 0$. The formula (8) is proved.

Since $\mathrm{Fr}(G)$ is compact as a closed subset of a compact space, formula (8) implies, by virtue of Cantor's theorem ([10], II, p. 5), the inequality $\bigcap_{n=1}^{\infty} [H_n \cap \mathrm{Fr}(G)] \neq 0$, i.e. by (7) the inequality $Qc_p(A, X) \cap \mathrm{Fr}(G) \neq 0$, which contradicts (5) and (6).

Hence $Qc_p(A, X)$ is connected.

Remark. Theorem 3.3 is not true in the Euclidean plane $\mathscr{E}^2$. For let $A = \bigcup_{n=1}^{\infty} A_n$, where A_n is a boundary of a rectangle of vertices $(-n, -1+1/n)$, $(-n, -1-1/n)$, $(n, 1-1/n)$, and $(n, -1+1/n)$ (see Fig. 2).

The relative quasicomponent $Qc_p(A, \mathcal{E}^2)$ of the point $p = (0,1)$ is the union of two straight lines $y=1$ and $y=-1$.

THEOREM 3.4. *If A is a subset of a topological space X and $p \in A$, then $A \cap Qc_p(A, X) \subset Qc_p(A, A)$.*

Proof. If $q \in A - Qc_p(A, A)$, then there exists a closed-open subset G of A such that $p \in G$ and $q \in A - \bar{G}$. But $p \in \bar{G}$, G is a closed subset of X, and in view of equality $\bar{G} \cap A = G$ the set $\bar{G} \cap A$ is open in A. It means that G is an envelope of p and therefore, by the definition of relative quasicomponent, we have $Qc_p(A, X) \subset \bar{G}$. Hence and from the assumption $q \in A - \bar{G}$ we infer that $q \in A - Qc_p(A, X)$. So we have proved the inclusion $A - Qc_p(A, A) \subset A - Qc_p(A, X)$ obviously equivalent to the thesis of our lemma.

LEMMA 3.5. *If $W \subset A \subset \mathscr{C} \times \mathscr{I}$ and for some $\tau_0 \in \mathscr{C}$*

(9) *each point of the set $(\tau_0 \times \mathscr{I}) \cap W$ is a point of condensation of this set,*

(10) *for each quasicomponent Q of A the set $(\tau_0 \times \mathscr{I}) \cap W \cap Q$ is of power at most $\aleph_0$,*

then the set of all quasicomponents of A relatively to $\mathscr{C} \times \mathscr{I}$ consisting of one point only and contained in $(\tau_0 \times \mathscr{I}) \cap W$ is dense in $(\tau_0 \times \mathscr{I}) \cap W$.

Proof. Let us remove from the segment $\tau_0 \times \mathscr{I}$ the closure of each component of the set $(\tau_0 \times \mathscr{I}) - W$. Denoting the remaining set by B we easily see that $(\tau_0 \times \mathscr{I}) - B$ consists of an at most countable set of segments, each of which has by its definition and by (9) at most ends common with W, and of points not belonging to W. The set B differs then from the set $(\tau_0 \times \mathscr{I}) \cap W$ for at most countable set and therefore by virtue of (9) the set B is a dense subset of $(\tau_0 \times \mathscr{I}) \cap W$.

It follows from (9) by the definition of the set B that if $b \in B$ and if I is a segment contained in $\tau_0 \times \mathscr{I}$, one end of which is b, then b is a point of condensation of the set $I \cap W$. Therefore, by virtue of the hypotheses $W \subset A$ and (10), there exist arbitrarily small segments J contained in $\tau_0 \times \mathscr{I}$ such that $b \in J$ and the ends of J belong to $A - Qc_b(A, A)$. In view of Theorems 3.3 and 3.4 we have then $Qc_b(A, \mathscr{C} \times \mathscr{I}) \subset J$, whence by the definition of relative quasicomponent we infer that $(b) = Qc_b(A, \mathscr{C} \times \mathscr{I})$.

Hence B is a dense subset of $(\tau_0 \times \mathscr{I}) \cap W$ and for each point $b \in B$ the relative quasicomponent $Qc_b(A, \mathscr{C} \times \mathscr{I})$ consists of the point b only.

THEOREM 3.6. *If A is a subset of $\mathscr{C} \times \mathscr{I}$ such that*

(11) *the set $\{p : \dim_p A = 0\}$ is nowhere dense in A,*

(12) *for each open subset U of $\mathscr{C} \times \mathscr{I}$, if $A \cap U \neq 0$, then there exists $2^{\aleph_0}$ points $\tau \in \mathscr{C}$ such that $(\tau \times \mathscr{I}) \cap A \cap U$ is of power $2^{\aleph_0}$,*

then A contains $2^{\aleph_0}$ quasicomponents of power $2^{\aleph_0}$.

Proof. Suppose that the family of quasicomponents of power $2^{\aleph_0}$ of the set A is at most countable and denote the union of all quasicomponents of this family by E. Since every quasicomponent of A is contained in only one segment $\tau\times\mathscr{I}$, so by (12) for each open subset U of $\mathscr{C}\times\mathscr{I}$, if $A\cap U\neq 0$, then there exists $2^{\aleph_0}$ points $\tau\in\mathscr{C}$ such that $(\tau\times\mathscr{I})\cap E=0$ and $(\tau\times\mathscr{I})\cap A\cap U$ is of power $2^{\aleph_0}$. This easily implies that the set W of points $\tau\times p\in A$, such that $(\tau\times\mathscr{I})\cap E=0$ and $\tau\times p$ is a point of condensation of the set $(\tau\times\mathscr{I})\cap A$, is dense in A and satisfies hypotheses (9) and (10) of Lemma 3.5 for all $\tau_0\in\mathscr{C}$ such that $(\tau_0\times\mathscr{I})\cap E=0$ and $(\tau_0\times\mathscr{I})\cap A$ is of power $2^{\aleph_0}$. Hence, by Lemma 3.5, the set of all quasicomponents of A relatively to $\mathscr{C}\times\mathscr{I}$ is dense in W and therefore in A, and so, by Theorem 3.2, the set of all points $p\in A$ such that $\dim_p A=0$ is dense in A, contrary to (11).

Thus the theorem is proved.

PROBLEM. Is it possible to replace hypothesis (11) of Theorem 3.6 by a weaker one: that $\dim A \;\; 1$?

§ 4. Elementary properties of pulverable sets

According to the list of notations (see p. 5) let P be a pulverable set and a its dispersion point. Therefore

LEMMA 4.1. *Every connected subset of P contains its dispersion point.*

In other words,

LEMMA 4.2. *Every quasicomponent of the set $P-(a)$ is a dispersed set.*

Lemma 2.8 also holds for P and a instead of S and A. Thus we obtain following lemma:

LEMMA 4.3. *If H is a closed-open subset of $P-(a)$, then $a\in\overline{H}$.*

And by the theorem of decomposition ([10], II, p. 83) we have

LEMMA 4.4. *If H is a closed-open subset of $P-(a)$, then the set $H\cup(a)$ is pulverable.*

Lemmas 2.7 and 2.9 imply

LEMMA 4.5. *No one of the quasicomponents of the set $P-(a)$ is open in $P-(a)$.*

For if some quasicomponent Q of the set $P-(a)$ is open in $P-(a)$, then we have $a\in\overline{Q}$ by Lemma 2.9. Hence Q contains at least two points, and therefore by virtue of Lemma 2.7 it is also connected. But it is impossible in view of $Q\subset P-(a)$.

LEMMA 4.6. *The set $\psi[P-(a)]$ is a dense in itself subset of the Cantor set $\mathscr{C}$.*

Indeed, if a point $\tau\in\psi[P-(a)]$ is an isolated point of the set $\psi[P-(a)]$, and thus open in it, then by (12) of § 2 the quasicomponent $\psi^{-1}(\tau)$ is open in $\psi^{-1}\psi[P-(a)]=P-(a)$ ([10], I, p. 74, formula (3)), contrary to Lemma 4.5.

CHAPTER II

§ 5. Connected subsets of pulverable sets

As mentioned above (Lemma 4.1), every connected subset of P contains its dispersion point a. It implies (see also [6], p. 216) that every connected subset of P is pulverable, and thus we receive, by applying Lemmas 2.11 and 2.12 for $S = P$ and $A = (a)$, the two following results:

LEMMA 5.1. *If $f: P-(a) \to \mathscr{C}$ is a continuous function, and Δ is a closed-open subset of $\mathscr{C}$ such that $\Delta \cap f[P-(a)] \neq 0$, then the set*

$$(a) \cup \bigcup_{\tau \in \Delta \cap f[P-(a)]} f^{-1}(\tau)$$

is pulverable and a is its dispersion point.

LEMMA 5.2. *If $f: P-(a) \to \mathscr{C}$ is a continuous function and Δ is a subset of $\mathscr{C}$ such that $f[P-(a)]-\overline{\Delta} \neq 0$, then the set*

$$a \cup \bigcup_{\tau \in f[P-(a)]-\overline{\Delta}} f^{-1}(\tau)$$

is pulverable and a is its dispersion point.

THEOREM 5.3. *If $a \in F = \overline{F} \cap P$, then the set $\varphi_F(P)$, obtained from P by identification of its subset F, is pulverable and $\varphi_F(F)$ is its dispersion point.*

Proof. Since, by (20) of § 2, the function φ_F is continuous, the set $\varphi_F(P)$ is connected. The set $P-(a)$ is dispersed and so is its subset $P-F$; therefore, by (21) of § 2, the set $\varphi_F(P-F)$ is also dispersed. By virtue of Lemma 2.15 the connected set $\varphi_F(P)$ differs from the dispersed set $\varphi_F(P-F)$ for a point $\varphi_F(F)$; consequently, this point is a dispersion point of the set $\varphi_F(P)$. Hence the set $\varphi_F(P)$ is by the definition pulverable and $\varphi_F(F)$ is its dispersion point.

According to Miller's theorem ([12], p. 125, th. 4) if B is a biconnected set not possessing a dispersion point, and if M is a finite subset of B, then the set $B-M$ is connected. We have the following theorem:

THEOREM 5.4. *If P is a pulverable set, a its dispersion point, and M a finite subset of $P-(a)$, then the set $P-M$ is connected.*

Proof. Suppose that for some finite $M \subset P-(a)$ the set $P-M$ is not connected, i.e.

(1) $$P-M = M_1 \cup M_2,$$

where M_1 and M_2 are non-void and separated. In view of another theorem of Miller ([12], p. 125, th. 3) the set $M_i \cup M$ contains a connected subset $\mathscr{C}_i$ ($i = 1$ and 2), which implies

(2) $$a \in C_1 \cap C_2$$

by Lemma 1.1. But $C_1 \cap C_2 \subset (M_1 \cup M) \cap (M_2 \cup M) = M_1 \cap M_2 \cup M_1 \cap \cap M \cup M_2 \cap M \cup M = M$ by the hypotheses $M_1 \cap M_2 = 0$ and (1), whence $C_1 \cap C_2 \subset M \subset P-(a)$, contrary to (2).

§ 6. Summation theorem

THEOREM 6.1. *Let P_τ be, for each $\tau \in T$, a pulverable set with dispersion point a_τ, and let F be a closed subset of the Hilbert cube $\mathscr{I}^{\aleph_0}$. If*

(1) *the set $\bigcup_{\tau \in T} [P_\tau - (a_\tau)]$ is dispersed,*

(2) $\bigcup_{\tau \in T} [P_\tau - (a_\tau)] - F \neq 0$,

(3) *$\overline{H}_1 \cap F \neq 0$ for every closed-open subset H_1 of $P_\tau - (a_\tau)$ and for every $\tau \in T$,*

then the set

$$\varphi_F\{F \cup \bigcup_{\tau \in T} [P_\tau - (a_\tau)]\}$$

is pulverable, and $\varphi_F(F)$ is its dispersion point.

Proof. By virtue of Lemma 2.16 we have

(4) $$\varphi_F\{\bigcup_{\tau \in T} [P_\tau - (a_\tau)] \cup F\} = \varphi_F\{\bigcup_{\tau \in T} [P_\tau - (a_\tau)] - F\} \cup \varphi_F(F).$$

We infer from (1) and (2), by virtue of (21) of § 2, that the first member on the right side of equality (4) is dispersed. By the definition of pulverable set (see p. 3) it remains then to prove that the union of the first and the second (i.e. the point $\varphi_F(F)$) member is connected.

Let H be any closed-open subset of $\bigcup_{\tau \in T} [P_\tau - (a_\tau)]$. There exists an index $\tau \in T$ such that $H \cap [P_\tau - (a_\tau)] \neq 0$; the set $H_1 = H \cap [P_\tau - (a_\tau)]$ is closed-open in $P_\tau - (a_\tau)$. We infer from this and from (3) that $\overline{H}_1 \cap F \neq 0$, and consequently $\overline{H} \cap F \neq 0$, whence by Lemma 2.16 follows the connectivity of the set (4).

COROLLARY 6.2. *For every family of pulverable sets P_τ, where $\tau \in T$ and $\overline{\overline{T}} \leqslant 2^{\aleph_0}$, there exists a pulverable set P which is a union of homeomorphs of P_τ such that each two meet only in the dispersion point of P and beyond this point are separated.*

Proof. By hypothesis the set T is equivalent to some subset of the Cantor set $\mathscr{C}$. Therefore we can assume that $T \subset \mathscr{C}$. Denote by $\mathscr{I}_\tau^{\aleph_0}$ the subset of the Hilbert cube $\mathscr{I}^{\aleph_0}$ consisting of all points $(x_1, x_2, x_3, \ldots)$ such that $x_1 = \tau$. Of course, the set $\mathscr{I}_\tau^{\aleph_0}$ is homeomorphic to Hilbert cube $\mathscr{I}^{\aleph_0}$. Then place separately every set P_τ into $\mathscr{I}_\tau^{\aleph_0}$ in such a way that the dispersion point a_τ of P_τ be the point $(\tau, 1/2, 1/2, \ldots)$. We can do it because by hypothesis $P_\tau \subset \mathscr{I}^{\aleph_0}$ (see p. 4), and $\mathscr{I}^{\aleph_0}$ is homeomorphic with $\mathscr{I}_\tau^{\aleph_0}$. Moreover, since cubes $\mathscr{I}^{\aleph_0}$ are components of the compact set $\bigcup_{\tau\in\mathscr{C}} \mathscr{I}_\tau^{\aleph_0}$ and $P_\tau - (a_\tau) < \mathscr{I}_\tau^{\aleph_0}$ for every $\tau \in T$, the union $\bigcup_{\tau\in T} [P_\tau - (a_\tau)]$ is not connected between any two elements of it. And since, moreover, the sets $P_\tau - (a_\tau)$ are dispersed, hypothesis (1) of Theorem 6.1 is satisfied. Now let F be the subset of Hilbert cube $\mathscr{I}^{\aleph_0}$ consisting of all points $(x, 1/2, 1/2, \ldots)$, where $0 \leqslant x \leqslant 1$. Therefore $a_\tau \in F$ for every $\tau \in T$. Hence, by Lemma 4.3, hypothesis (3) of Theorem 6.1 is also satisfied. Finally, we have $F \cap \bigcup_{\tau\in T} [P_\tau - (a_\tau)] = 0$, whence $\bigcup_{\tau\in T} [P_\tau - (a_\tau)] - F \neq 0$, i.e. hypothesis (2) of Theorem 6.1. Thus, in view of the Summation Theorem 6.1, the set $P = \varphi_F\{F \cup \bigcup_{\tau\in T} [P_\tau - (a_\tau)]\}$ is pulverable and $a = \varphi_F(F)$ is the dispersion point of P.

Finally, we have $a = \varphi_F(F) \in \varphi_F(P_\tau)$ for each $\tau \in T$, and since the sets $P_\tau - (a_\tau) \subset I_\tau^{\aleph_0} - F$ and $P_\tau - (a_\tau) \subset I_{\tau'}^{\aleph_0} - F$ are separated if $\tau \neq \tau'$, so by (21) of § 2 the sets $\varphi_F[P_\tau - (a_\tau)]$ and $\varphi_F[P_{\tau'} - (a_{\tau'})]$ are separated too.

Remarks. If $\overline{\overline{T}} = 1$, i.e. in the case of only one pulverable set P, we can choose by Theorem 6.1 another dispersion point. Namely, it is sufficient to add to the set $P - (a)$ a closed set F which is not separated in respect to any closed-open and non-void subset of $P - (a)$, and then to identify F to one point.

If $F = a_\tau$ for $\tau \in T$, i.e. if all pulverable sets P_τ have a common dispersion point, the Theorem 6.1 be comes a summation theorem for such pulverable sets.

In a general case we can take the set $\overline{\bigcup_{\tau\in T} (a_\tau)}$ as a set F. The hypothesis (3) is then realized by virtue of Lemma 4.3.

§ 7. Quasicomponents of pulverized sets

Now we investigate the topological structure of pulverized sets. We begin with a characterization of these sets.

Introducing the notion of spread set ("ensemble diffus"), Knaster characterized topologically the sets homeomorphic with a connected set without one point (see [8]). Adding to the above characterization a supplementary condition we easily come to a characterization of pulverized sets.

A set G is said to be *spread in* X if G has common points with both elements of every decomposition of X into two non-void and separated subsets ([8], p. 310).

THEOREM 7.1. *A set X is pulverized if and only if*

(1) *there exists in X a decreasing sequence of subsets, closed and spread in X, the common part of which is void,*

(2) *the set X is dispersed.*

Proof. Sufficiency. In view of the theorem proved in [8] it follows by (1) that the set X is homeomorphic with a connected set Z without one point. Therefore by (2) the set X is pulverized by definition (see p. 3).

Necessity. If X is homeomorphic with a pulverized set, then the theorem in [8] implies (1), and Lemma 4.2 implies (2).

THEOREM 7.2. *If P is a pulverable set, a its dispersion point, and $M \subset \mathscr{C}$, then*

(3) *the set $P_M = \varphi_{\mathscr{C}\times a}\{(\mathscr{C}\times a)\cup M\times[P-(a)]\}$ is pulverable and $a_M = \varphi_{\mathscr{C}\times a}(\mathscr{C}\times a)$ is its dispersion point,*

(4) *the function $\varphi_{\mathscr{C}\times a}$ is a homeomorphism on $M\times[P-(a)]$ and $\varphi_{\mathscr{C}\times a}\{M\times\times[P-(a)]\} = P_M-(a_M)$,*

(5) *every quasicomponent of $P_M-(a_M)$ is a set $\varphi_{\mathscr{C}\times a}(\tau\times Q)$, where $\tau\in M$, and Q is a quasicomponent of the pulverized set $P-(a)$.*

Proof. We have (3), because the hypotheses of the Summation Theorem 6.1 hold here. Indeed, the set $M\times[P-(a)] = \bigcup_{\tau\in M}\{\tau\times[P-(a)]\}$ is dispersed, whence (1) of § 6. Putting $F = \mathscr{C}\times a$ we see that $(\mathscr{C}\times a)\cap\{M\times[P-(a)]\} = 0$, i.e.

$$(6)\qquad M\times[P-(a)]-\mathscr{C}\times a = M\times[P-(a)],$$

whence $M\times[P-(a)]-\mathscr{C}\times a \neq 0$, i.e. (2) of § 6. By the inclusion $M\times a\subset\mathscr{C}\times a$ and Lemma 4.3 we then have for every $\tau\in M$ and for every closed-open subset H_1 of $\tau\times[P-(a)]$ the inequality $\overline{H}_1\cap(\mathscr{C}\times a)\neq 0$, whence (3) of § 6.

We have also (4), because by (20) of § 2 (for $X = M\times[P-(a)]$ and $F = \mathscr{C}\times a$) and (6) the function $\varphi_{\mathscr{C}\times a}$ is a homeomorphism on $M\times\times[P-(a)]$. By Lemma 2.15,

$$\varphi_{\mathscr{C}\times a}\{M\times[P-(a)]\cup\mathscr{C}\times a\}-\varphi_{\mathscr{C}\times a}\{M\times[P-(a)]-\mathscr{C}\times a\} = \varphi_{\mathscr{C}\times a}(\mathscr{C}\times a),$$

whence by (6)

$$\varphi_{\mathscr{C}\times a}\{M\times[P-(a)]\cup\mathscr{C}\times a\}-\varphi_{\mathscr{C}\times a}\{M\times[P-(a)]\} = \varphi_{\mathscr{C}\times a}(\mathscr{C}\times a).$$

Since the diminuend is here a subset of the subtrahend,

$$\varphi_{\mathscr{C}\times a}\{M\times[P-(a)]\cup\mathscr{C}\times a\}-\varphi_{\mathscr{C}\times a}(\mathscr{C}\times a) = \varphi_{\mathscr{C}\times a}\{M\times[P-(a)]\},$$

i.e. by notation assumed in (3)

$$P_M-(a_M) = \varphi_{\mathscr{C}\times a}\{M\times[P-(a)]\}.$$

To prove (5) we first show that

(7) every quasicomponent of $M\times[P-(a)]$ is a set $\tau\times Q$, where $\tau\in M$ and Q is a quasicomponent of the pulverized set $P-(a)$.

In fact, every one of the sets $\tau\times Q$ is obviously a homeomorph of the quasicomponent Q, and since $Q\subset P-(a)$, so $\tau\times Q$ is contained in one quasicomponent of $M\times[P-(a)]$ only. To prove (7) it is sufficient now to show that for every set $\tau_1\times Q_1$, where Q_1 is a quasicomponent of $P-(a)$ and $\tau_1\neq\tau$ or $Q_1\neq Q$, there exists a closed-open subset H of $M\times\times[P-(a)]$ such that

(8) $$\tau\times Q\subset H \quad\text{and}\quad (\tau_1\times Q_1)\cap H = 0.$$

If $\tau_1\neq\tau$, then there exists a closed-open subset Δ of M such that $\tau\in\Delta$ and $\tau_1\in\mathscr{C}-\Delta$. The set $H=\Delta\times[P-(a)]$ is evidently a closed-open subset of $M\times[P-(a)]$, for which (8) is true.

And if $Q_1\neq Q$, then by the definition of quasicomponent there exists a closed-open subset H_1 of $P-(a)$, which contains Q and is disjoint with Q_1. But therefore the set $H=M\times H_1$ is a closed-open subset of $M\times\times[P-(a)]$, which contains $\tau\times Q$ and is disjoint with $\tau_1\times Q_1$, and consequently, for which (8) is true.

Thus we have proved (7), whence, by (4), we have (5).

THEOREM 7.3. *Under the hypotheses of Theorem 7.2, if moreover $\bar{M}=\mathscr{C}$, then every quasicomponent of the pulverized set $P_M-(a_M)$ is nowhere dense in P_M.*

Proof. By (5) of Theorem 7.2 every quasicomponent of the set $P_M-(a_M)$ is a set $\varphi_{\mathscr{C}\times a}(\tau\times Q)$, where $\tau\in M$ and Q is a quasicomponent of $P-(a)$.

We first show that the quasicomponent $\varphi_{\mathscr{C}\times a}(\tau\times Q)$ is a boundary set in $P_M-(a_M)$. In fact, M being by hypothesis dense in $\mathscr{C}$, let $\{\tau_n\}_{n=1,2,\ldots}$ be a sequence of different points of M convergent to τ. Therefore $\{\tau_n\times Q\}$ is a sequence of disjoint sets convergent (topologically in $M\times[P-(a)]$) to $\tau\times Q$. By (4) of Theorem 7.2, $\{\varphi_{\mathscr{C}\times a}(\tau_n\times Q)\}$ is then a sequence of disjoint sets convergent in P_M-a_M to the quasicomponent $\varphi_{\mathscr{C}\times a}(\tau\times Q)$.

As a quasicomponent of the set P_M-a_M it is closed in it ([10], II, p. 93), and also, as just proved, boundary in it. It means that the quasicomponent $\varphi_{\mathscr{C}\times a}(\tau\times Q)$ is nowhere dense in $P_M-(a_M)$, and *a fortiori* nowhere dense in P_M.

THEOREM 7.4. *If a cardinal number $\mathfrak{m}$ is such that $\aleph_0\leqslant\mathfrak{m}\leqslant 2^{\aleph_0}$, then there exists a pulverized set $P_M-(a_M)$ consisting of $\mathfrak{m}$ quasicomponents, each of which is nowhere dense in P_M.*

Proof. We shall use a pulverized set $P-(a)$ consisting of $\aleph_0$ quasicomponents. Such sets exist (see Examples 1 and 2, p. 34). Let M be a dense subset of the Cantor set $\mathscr{C}$ such that $\overline{\overline{M}} = \mathfrak{m}$. The hypotheses of Theorems 7.2 and 7.3 are then satisfied. Let

$$P_M = \varphi_{\mathscr{C}\times a}\{\mathscr{C}\times a \cup M \times [P-(a)]\}.$$

By (3) of Theorem 7.2 the set P_M is pulverable and $a_M = \varphi_{\mathscr{C}\times a}(\mathscr{C}\times a)$ is its dispersion point.

And by (5) of the same theorem the pulverized set $P_M-(a_M)$ consists of $\overline{\overline{M}}\cdot\aleph_0$ quasicomponents, because by (4) $\varphi_{\mathscr{C}\times a}$ is a homeomorphism. Now $\overline{\overline{M}}\cdot\aleph_0 = \mathfrak{m}\cdot\aleph_0 = \mathfrak{m}$.

Finally, by virtue of Theorem 7.3, every quasicomponent of the pulverized set $P_M-(a_M)$ is nowhere dense in P_M.

THEOREM 7.5. *A cardinal number* $\mathfrak{m}$ *is the power of family of all quasicomponents of a pulverized set if and only if* $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$.

Proof. Since by virtue of (12) of § 2 the counter-images of points of the set $\psi[P-(a)]$ are quasicomponents of the set $P-(a)$, the power of family of all quasicomponents of the set $P-(a)$ is equal to the power of the set $\psi[P-(a)]$. In view of Lemma 4.6 we have then $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$.

Conversely, if a cardinal number $\mathfrak{m}$ is such that $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$, then there exists by Theorem 7.4 a pulverized set consisting of $\mathfrak{m}$ quasicomponents (even nowhere dense).

THEOREM 7.6. *If* $\mathfrak{m} = \aleph_0$, *then there exist three classes of pulverized sets consisting of* $\mathfrak{m}$ *quasicomponents, namely: all nowhere denses, some nowhere denses and some not nowhere denses, and all not nowhere denses; and if* $\mathfrak{m} > \aleph_0$, *then there exist the two first classes only.*

Proof. The pulverized sets of the first class exist by virtue of Theorem 7.4 for any cardinal number $\mathfrak{m}$ such that $\aleph_0 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$.

The pulverized sets of the second class for these cardinal numbers can be constructed as follows:

Let us take in the cube $\mathscr{I}^{\aleph_0}$ a pulverable set P_1 with a dispersion point a_1 and such that $P_1-(a_1)$ consists of $\mathfrak{m}$ quasicomponents each of which is nowhere dense in P_1 (such P_1 exists by Theorem 6.4); let us also take a pulverable set P_2 (constructed in Example 1) with a dispersion point a_2 and such that $P_2-(a_2)$ consists of $\aleph_0$ quasicomponents, not one of which is nowhere dense in P_2. Of course, we may assume that the sets P_1 and P_2 are separated. Then the set $\varphi_{a_1\cup a_2}(P_1\cup P_2)$, by virtue of the Summation Theorem 6.1 and Lemma 4.3, is a pulverable set with a dispersion point $\varphi_{a_1\cup a_2}(a_1\cup a_2)$. The sets P_1 and P_2 being separated, we infer by (20) of § 2 that the quasicomponents of the set $\varphi_{a_1\cup a_2}[P_2\cup P_2-(a_1\cup a_2)]$ are homeomorphic with the quasicomponents of the sets $P_1-(a_1)$ and $P_2-(a_2)$. This implies by Lemma 2.15 that the set

$\varphi_{a_1\cup a_2}[P_1 \cup P_2 - (a_1 \cup a_2)] = \varphi_{a_1\cup a_2}(P_1 \cup P_2) - \varphi_{a_1\cup a_2}(a_1 \cup a_2)$ consists of $\mathfrak{m} + \aleph_0 = \mathfrak{m}$ quasicomponents. Moreover, each of the not nowhere dense quasicomponents remains not nowhere dense, and this means that the set $\varphi_{a_1\cup a_2}(P_1 \cup P_2)$ belongs to the second class.

A pulverized set of third class for $\mathfrak{m} = \aleph_0$ is constructed in Example 1 (see p. 34), and for $\mathfrak{m} > \aleph_0$ pulverized sets of the third class do not exist, because every subset of the Hilbert cube $\mathcal{I}^{\aleph_0}$ is separable.

From Lemma 2.10 we deduce at once

THEOREM 7.7. *Every quasicomponent of a pulverized set $P-(a)$, which is not nowhere dense in P, has a positive dimension.*

Similarly, it follows at once from Lemma 2.14, by substitution $S = P$ and $A = (a)$, that

THEOREM 7.8. *If all quasicomponents of the pulverized set $P-(a)$ are 0-dimensional, then the family of quasicomponents is uncountable.*

CHAPTER III

§ 8. Continuous images of pulverable sets

LEMMA 8.1. *If S is a connected set, $p \in S$ and $\Delta \subset \mathscr{C}$, then the set*

$$(1) \qquad (\Delta \times S) \cup (\mathscr{I} \times p)$$

is connected.

In fact, since $\tau \times p \in \mathscr{I} \times p$ and $p \in S$ imply $\tau \times p \in \tau \times S$, then not one of connected sets $\tau \times S$, where $\tau \in \Delta$, is separated with a segment $\mathscr{I} \times p$, and consequently, the set (1) is connected ([10], II, p. 82).

LEMMA 8.2. *If S is a connected set, $p \in S$, and a set D disjoint with $\mathscr{I} \times p$ disconnects the cube $\mathscr{I}^{\aleph_0}$ between some points of the set $Z = (\mathscr{C} \times S) \cup (\mathscr{I} \times p)$, then the set*

$$(2) \qquad \{\tau : \tau \in \mathscr{C}, (\tau \times S) \cap D \neq 0\}$$

contains an open subset of the Cantor set $\mathscr{C}$.

For otherwise, if the complement Δ of the set (2) with respect to the Cantor set $\mathscr{C}$ were dense in $\mathscr{C}$, the subset $(\Delta \times S) \cup (\mathscr{I} \times p)$ of $Z - D$ would be dense in $Z - D$, and as, by Lemma 8.1, this subset is connected, the set $Z - D$ must be also connected ([10], II, p. 83), a contradiction.

LEMMA 8.3. *If S is a connected set and $p \in S$, then $\mathscr{C} \times S \subset \overline{\mathscr{C} \times S - \mathscr{I} \times p}$.*

Proof. It is sufficient to prove that the rest, i.e. the set $\mathscr{C} \times S - (\mathscr{C} \times S - \mathscr{I} \times p) = (\mathscr{C} \times S) \cap (\mathscr{I} \times p)$ is contained in $\overline{\mathscr{C} \times S - \mathscr{I} \times p}$. But

$$(3) \qquad (\mathscr{C} \times S) \cap (\mathscr{I} \times p) = \mathscr{C} \cap \mathscr{I} \times S \cap (p) = \mathscr{C} \times p,$$

because, by hypothesis, $\mathscr{C} \subset \mathscr{I}$ and $(p) \subset S$. Now, as connected by hypothesis, the set S is dense in itself, whence $(p) \subset \overline{S - (p)}$, and therefore $\mathscr{C} \times p \subset \mathscr{C} \times \overline{S - (p)} = \overline{\mathscr{C} \times [S - (p)]} = \overline{\mathscr{C} \times S - \mathscr{C} \times p} = \overline{\mathscr{C} \times S - (\mathscr{C} \times S) \cap (\mathscr{I} \times p)} = \overline{\mathscr{C} \times S - \mathscr{I} \times p}$, where the last but one sign of equality follows by (3).

LEMMA 8.4. *If g is a continuous function defined on a compact set X and $A_n \subset X$ for $n = 1, 2, \ldots$, then $g(\operatorname{Ls}_{n \to \infty} A_n) = \operatorname{Ls}_{n \to \infty} g(A_n)$.*

Proof. The left side is contained in the right. For if $g(x) \in g(\operatorname{Ls}_{n\to\infty} A_n)$, i.e. $x \in \operatorname{Ls}_{n\to\infty} A_n$, then by the definition of Ls ([10], I, p. 243) there exists a sequence of points $\{x_{k_n}\}_{n=1,2,\ldots}$ such that

$$(4) \qquad x_{k_n} \in A_{k_n}, \quad \text{where} \quad k_1 < k_2 < k_3 < \ldots,$$

$$(5) \qquad x = \lim_{n\to\infty} x_{k_n}.$$

We have $g(x_{k_n}) \in g(A_{k_n})$ for $n = 1, 2, \ldots$ by (4), and in view of the continuity of the function g we have also $g(x) = \lim_{n\to\infty} g(x_{k_n})$ by (5). Therefore $g(x) \in \operatorname{Ls}_{n\to\infty} g(A_{kn})$, whence ([10], I, p. 243, formula 5) $g(x) \in \operatorname{Ls}_{n\to\infty} g(A_n)$.

Conversely, the right side is contained in the left. For if $y \in \operatorname{Ls}_{n\to\infty} g(A_n)$, then, by the definition of Ls, there exists a sequence of points $\{y_{k_n}\}_{n=1,2,\ldots}$ such that

$$(6) \qquad y_{k_n} \in g(A_{k_n}), \quad \text{where} \quad k_1 < k_2 < k_3 < \ldots,$$

and $y = \lim_{n\to\infty} y_{k_n}$. Since the set X is compact and the function g is continuous, we have ([10], II, p. 36)

$$(7) \qquad \operatorname{Ls}_{n\to\infty} g^{-1}(y_{k_n}) \subset g^{-1}(y).$$

We infer from (6) that $g^{-1}(y_{k_n}) \cap A_{k_n} \neq 0$ for $n = 1, 2, \ldots$ The non-void subsets $\overline{g^{-1}(y_{k_n}) \cap A_{k_n}}$ of the compact set X contain a convergent subsequence ([10], II, p. 21). Therefore $\operatorname{Ls}_{n\to\infty} \overline{g^{-1}(y_{k_n}) \cap A_{kn}} \neq 0$, whence ([10], I, p. 243, formula 1) $\operatorname{Ls} g^{-1}(y_{k_n}) \cap A_{k_n} \neq 0$. Hence there exists a convergent sequence of points $\{x_{m_{k_n}}\}_{n=1,2,\ldots}$ such that

$$(8) \qquad x_{m_{k_n}} \in g^{-1}(y_{m_{k_n}}) \cap A_{m_{k_n}}, \quad \text{where} \quad m_{k_1} < m_{k_2} < \ldots$$

Let

$$(9) \qquad x = \lim_{n\to\infty} x_{m_{k_n}}.$$

We conclude from (9), (8), and (7) that

$$(10) \qquad x \in g^{-1}(y).$$

We have $x \in \operatorname{Ls}_{n\to\infty} A_{m_{k_n}}$ by (8) and (9), and then ([10], I, p. 243, formula 5) $x \in \operatorname{Ls}_{n\to\infty} A_n$. Hence, in view of (10), $y = g(x) \in g(\operatorname{Ls}_{n\to\infty} A_n)$.

THEOREM 8.5. *Every connected set S is a continuous image of a pulverable set P.*

Proof. Let $p \in S$, $Z = (\mathscr{C} \times S) \cup (\mathscr{I} \times p)$, and let $r(x)$ be the projection, parallel to the x_1-axis, of a point x onto the hyperplane $x_1 = 0$. Then

$$r(\mathscr{C} \times S) = 0 \times S, \tag{11}$$

$$\mathscr{I} \times p = r^{-1}(0 \times p). \tag{12}$$

The construction of the pulverable set P, of which S has to be a continuous image, will consist in a selection of a point or a pair of points from some (not necessarily from all) sets $\tau \times S$, where $\tau \in \mathscr{C}$, in such a way that $(0 \times S) - (0 \times p)$ will be the image of the set of selected points under projection r, the set of selected points will be dispersed, but its union with the segment $\mathscr{I} \times p$ will be connected. The set P will arise from the latter by identification of $\mathscr{I} \times p$.

For this purpose let $\mathscr{C} = \bigcup_{0 \leqslant \eta \leqslant 1} \mathscr{C}_\eta$ be a decomposition of the Cantor set $\mathscr{C}$ into $2^{\aleph_0}$ subsets disjoint and dense in $\mathscr{C}$ (5), and let $\{D_\eta\}_{0 \leqslant \eta \leqslant 1}$ be the family of all sets disconnecting the cube $\mathscr{I}^{\aleph_0}$ between some points of Z and disjoint with $\mathscr{I} \times p$, i.e.

$$\bigcup_{0 \leqslant \eta \leqslant 1} D_\eta \cap (\mathscr{I} \times p) = 0. \tag{13}$$

The set (2), as containing by Lemma 8.2 an open subset of $\mathscr{C}$, has common points with each $\mathscr{C}_\eta$ in view of its density in $\mathscr{C}$; then

$$(\mathscr{C}_\eta \times S) \cap D_\eta \neq 0 \quad \text{for} \quad 0 \leqslant \eta \leqslant 1. \tag{14}$$

Let E_1 be the union of points selected one by one from each non-void set of the sets $(\tau \times S) \cap D_\eta$ for $\tau \in \mathscr{C}_\eta$ and $0 \leqslant \eta \leqslant 1$; then

(15) $E_1 \cap (\tau \times S)$ for each $\tau \in \mathscr{C}$ is void or consists of a single point,

$$E_1 \cap D_\eta \neq 0 \quad \text{for} \quad 0 \leqslant \eta \leqslant 1, \tag{16}$$

and $E_1 \subset \bigcup_{0 \leqslant \eta \leqslant 1} D_\eta$, whence by (13)

$$E_1 \cap (\mathscr{I} \times p) = 0. \tag{17}$$

In view of (16) and (13)

(18) the set $E_1 \cup (\mathscr{I} \times p)$ is connected.

Furthermore, if $q \in (\mathscr{C} \times S) - (\mathscr{I} \times p)$, then the boundary of any arbitrarily small neighbourhood of q is one of disconnecting sets D_η in view of their definition. Therefore, by (16), the point q is a cluster point of the set E_1. Hence by Lemma 8.3

(19) E_1 is a dense subset of $\mathscr{C} \times S$.

(5) Such decompositions exist. It is sufficient to take the decomposition (which is effective, see [6], p. 254) of a segment $[0, 1]$ into $2^{\aleph_0}$ subsets disjoint and dense in $[0, 1]$ and then the counter-images of these sets, under a well known "fonction scalariforme" of Cantor.

Finally,

(20) $$0\times p\in\mathscr{I}^{\aleph_0}-r(E_1).$$

For if $0\times p\in r(E_1)$, then $r^{-1}(0\times p)\cap E_1\neq 0$, which implies by (12) the inequality $(\mathscr{I}\times p)\cap E_1\neq 0$, contrary to (17).

By (19) we have $E_1\subset\mathscr{C}\times S$, whence we infer by (11) and (20) that $r(E_1)\subset(0\times S)-(0\times p)$. If

(21) $$[(0\times S)-(0\times p)]-r(E_1)\neq 0,$$

then we make up the set E_1 by a set E_2 in such a way that

(22) $$r(E_1\cup E_2)=(0\times S)-(0\times p).$$

Namely, let us order to each point $q\in[(0\times S)-(0\times p)]-r(E_1)$ in a one-to-one way a point $\tau_q\in\mathscr{C}$, and thus a point $q'\in\tau_q\times S$ such that $r(q')=q$. Put

$$E_2=\{q'\colon q'\in\tau_q\times S,\ q\in[(0\times S)-(0\times p)]-r(E_1),\ r(q')=q\}.$$

If (21) is not true we have $E_2=0$, of course. Thus we have in every case

(23) $$E_2\subset\mathscr{C}\times S,$$

(24) $E_2\cap(\tau\times S)$ for each $\tau\in\mathscr{C}$ is void or consists of a single point.

It follows by (12) from $q\in(0\times S)-(0\times p)$ that $r^{-1}(q)\cap r^{-1}(0\times p)=r^{-1}(q)\cap(\mathscr{I}\times p)=0$, whence $E_2\cap(\mathscr{I}\times p)=0$ by the definition of E_2. Hence we have by (17)

(25) $$(E_1\cup E_2)\cap(\mathscr{I}\times p)=0.$$

It follows from (23) and (19) that $E_1\cup E_2\subset\mathscr{C}\times S$, and since the set $\mathscr{C}\times S$ is not connected between any pair of its subsets $\tau_1\times S$ and $\tau_2\times S$, where $\tau_1\in\mathscr{C}$, $\tau_2\in\mathscr{C}$ and $\tau_1\neq\tau_2$, so by (15) and (24)

(26) the set $E_1\cup E_2$ is dispersed.

Now put

27) $$P=\varphi_{\mathscr{I}\times p}[E_1\cup E_2\cup(\mathscr{I}\times p)].$$

By (20) of § 2 and by (26) the set P is connected ([10], II, p. 80). Moreover, this set differs by Lemma 2.15 from the set $\varphi_{\mathscr{I}\times p}[E_1\cup E_2-(\mathscr{I}\times p)]$ for a point

(28) $$a=\varphi_{\mathscr{I}\times p}(\mathscr{I}\times p),$$

and the set $\varphi_{\mathscr{I}\times p}[E_1\cup E_2-(\mathscr{I}\times p)]$ is dispersed by (25) and (26), because $\varphi_{\mathscr{I}\times p}$ is by (20) of § 2 a homeomorphism on $X-F=E_1\cup E_2-(\mathscr{I}\times p)$. Hence, in view of the definition of pulverable set (see p. 3), the set P is pulverable and a is its dispersion point.

Finally, we show that the function $r\varphi_{\mathscr{I}\times p}^{-1}$ maps continuously the set P onto the set $0\times S \underset{\text{top}}{=} S$. Indeed, we have, by (25), $E_1\cup E_2-(\mathscr{I}\times p) = E_1\cup E_2$, whence by Lemma 2.15 and (12)

$$\begin{aligned}\varphi_{\mathscr{I}\times p}(E_1\cup E_2) &= \varphi_{\mathscr{I}\times p}[E_1\cup E_2-(\mathscr{I}\times p)]\\ &= \varphi_{\mathscr{I}\times p}[E_1\cup E_2\cup(\mathscr{I}\times p)]-\varphi_{\mathscr{I}\times p}(\mathscr{I}\times p) = P-(a),\end{aligned}$$

and therefore $\varphi_{\mathscr{I}\times p}^{-1}$ is by (21) of § 2 a homeomorphic mapping $P-(a)$ onto $E_1\cup E_2$. Hence, by virtue of (22), the function $r\varphi_{\mathscr{I}\times p}^{-1}$ maps in a continuous way the set $P-(a)$ onto the set $(0\times S)-(0\times p)$. And for the point a we have by (28) and (25) $\varphi_{\mathscr{I}\times p}^{-1}(a) = \mathscr{I}\times p$, whence $r\varphi_{\mathscr{I}\times p}^{-1}(a) = 0\times p$ by (12), and consequently $r\varphi_{\mathscr{I}\times p}^{-1}(P) = 0\times S$.

It remains to prove that this mapping is continuous at the point a. Let $a = \lim\limits_{n\to\infty} x_n$. Then $\operatorname{Ls}\varphi_{\mathscr{I}\times p}^{-1}(x_n)\subset\varphi_{\mathscr{I}\times p}^{-1}(a) = \mathscr{I}\times p$, because the function $\varphi_{\mathscr{I}\times p}$ is continuous by (19) of § 2. In view of the compactness of the set $X = \bigcup\limits_{n=1}^{\infty}\varphi_{\mathscr{I}\times p}^{-1}(x_n)\cup\operatorname{Ls}\limits_{n\to\infty}\varphi_{\mathscr{I}\times p}^{-1}(x_n)$ following from the last inclusion, and in view of the continuity of the function r, we infer by Lemma 8.4 that $\operatorname{Ls}\limits_{n\to\infty} r\varphi_{\mathscr{I}\times p}^{-1}(x_n) = r[\operatorname{Ls}\limits_{n\to\infty}\varphi_{\mathscr{I}\times p}^{-1}(x_n)]\subset r(\mathscr{I}\times p)$, whence, by (12), $\operatorname{Ls}\limits_{n\to\infty} r\varphi_{\mathscr{I}\times p}^{-1}(x_n)\subset 0\times p$, i.e. $\lim\limits_{n\to\infty} r\varphi_{\mathscr{I}\times p}^{-1}(x_n) = 0\times p$. The continuity of the function $r\varphi_{\mathscr{I}\times p}^{-1}$ at the point a in thus proved.

§ 9. Minimal sets

A pulverable set P with a dispersion point a will be called *minimal* if every quasicomponent of $P-(a)$ is a single point. It can be easily proved that the pulverable G_δ set constructed by Knaster and Kuratowski in [7] is minimal. Two years later Wilder [15] constructed a minimal set and proved that it is minimal. Later on Knaster [9] constructed minimal sets of any finite dimension. All these examples are effective.

As Roberts [13] has proved the set of rational points in Hilbert space is homeomorphic with a plane pulverized set, which becomes a minimal set upon the addition of its dispersion point. Then the question arises: does every pulverable set contain a minimal subset? (Knaster). If the continuum hypothesis is true, the answer is negative; there exists (see Example 3, p. 41) a pulverable set such that every pulverized subset of it contains $2^{\aleph_0}$ quasicomponents of power $2^{\aleph_0}$. Nevertheless, for every pulverized set we shall construct a compactification such that choosing in a suitable manner one point from each component of this compactification we obtain a plane pulverized set, which becomes a minimal set upon the addition of its dispersion point. For this purpose we construct a compactification of $P-(a)$ containing the set $\mathscr{C}\times\mathscr{I}$ in such a way that each two segments $\tau_1\times\mathscr{I}$ and $\tau_2\times\mathscr{I}$ of this set, where $\tau_1\neq\tau_2$, lie in different components of this compactification.

Let P be any pulverable set, a its dispersion point and let $I_\tau^{\aleph_0}$ be the set of all points $(x_1, x_2, \ldots)$ of the Hilbert cube such that $x_1=\tau$. Since every set $I_\tau^{\aleph_0}$ is homeomorphic with the Hilbert cube $I^{\aleph_0}$ we can assume that $P\subset I_0^{\aleph_0}$.

Let

(1) $$a=(0, a_2, a_3, \ldots),$$

and let g denotes a function $g: P-(a)\to\mathscr{I}^{\aleph_0}$ defined by the formula

(2) $$g(0, x_2, x_3, \ldots)=\left(\sin\frac{1}{\varrho(a, q)}, x_2, x_3, \ldots\right)$$

for $q=(0, x_2, x_3, \ldots)\in P-(a)$, where $\varrho(a, q)$ is the distance between two points a and q. Of course,

(3) g is a homeomorphism on $P-(a)$.

LEMMA 9.1. *Let H be a non-void and closed-open subset of $P-(a)$ and T a segment $-1\leqslant x_1\leqslant 1$ of x_1-axis. Then $T\times(a_2, a_3, \ldots)\subset\overline{g(H)}$.*

Proof. Let $p=(x_1, a_2, a_3, \ldots)\in T\times(a_2, a_3, \ldots)$. We have to find a sequence of points

(4) $$p_n=(0, x_2^n, x_3^n, \ldots)\in H,\quad \text{where}\quad n=1, 2, \ldots,$$

such that $p=\lim_{n\to\infty} g(p_n)$, i.e. by (2), such that ([10], I, p. 86)

(5) $$x_1=\lim_{n\to\infty}\sin\frac{1}{\varrho(a, p_n)},$$

(6) $$a_i=\lim_{n\to\infty} x_i^n\quad \text{for}\quad i=2, 3, \ldots$$

Since $x_1\in T$, there exists an angle α such that $-\pi/2\leqslant\alpha\leqslant\pi/2$ and

(7) $$x_1=\sin\alpha.$$

Denote by K_n the sphere having a as a centre and a radius equal to $1/(\alpha+2n\pi)$. Beginning with some n_0 (we may assume $n_0=1$) we have $K_n\cap H\neq 0$ by connectivity of the set $(a)\cup H$ (see Lemma 4.4). Let $p_n\in K_n\cap H$ be an arbitrary point of this intersection for $n=1, 2, \ldots$ To prove (5) and (6) for the sequence $\{p_n\}_{n=1,2,\ldots}$ note that we have $\varrho(a, p_n)=1/(\alpha+2n\pi)$ by the definition of the sphere K_n.

Hence, firstly, we have $a=\lim_{n\to\infty} p_n$, which implies (6) by (1) and (4), and secondly, $\sin(1/\varrho(a, p_n))=\sin(\alpha+2n\pi)=\sin\alpha$ for $n=1, 2, \ldots$ whence $\sin(1/\varrho(a, p_n))=x_1$ by (7), and therefore (5).

LEMMA 9.2. *If H is a closed-open subset of $P-(a)$, then the set $\overline{g(\overline{H})}$ is connected.*

Proof. Firstly note that if G is non-void and closed-open subset of $g(H)$, then by (3) and Lemma 9.1

$$T\times(a_2, a_3, \ldots)\subset\bar{G}. \tag{8}$$

Now if $\overline{g(\overline{H})}=M\cup N$, where M and N are non-void, closed and disjoint, then we have equalities

$$\overline{M\cap g(H)}=M \quad \text{and} \quad \overline{N\cap g(H)}=N, \tag{9}$$

and therefore each of the two sets $M\cap g(H)$ and $N\cap g(H)$ is a non-void and a closed-open subset of $g(H)$. In view of (8) and (9) we have then inclusions $T\times(a_2, a_3, \ldots)\subset M$ and $T\times(a_2, a_3, \ldots)\subset N$. This is in contradiction with the assumption $M\cap N=0$.

Hence $\overline{g(\overline{H})}$ is connected.

THEOREM 9.3. *There exists a compactification of the set $P-(a)$ containing* (*topologically*) *the set $\mathscr{C}\times\mathscr{I}$ in such a way that each component of this compactification contains exactly one segment of this set.*

Proof. In view of (3) the set $g[P-(a)]$ is homeomorphic with the set $P-(a)$. Let h denote the function defined on $P-(a)$ by the formula

$$h(p)=\psi g(p)\times g(p) \quad \text{for each} \quad p\in P-(a).$$

The function h is a homeomorphism. Indeed, it is one-to-one, because $p\neq q$ implies $g(p)\neq g(q)$ by (3), whence $\psi g(p)\times g(p)\neq\psi g(q)\times g(q)$, i.e. $h(p)\neq h(q)$. Furthermore, $\psi g(p)$ being continuous as a superposition of two functions which are continuous, by (10) of § 2 and by (3) the function h is continuous too ([10], I, p. 86). Finally, $\lim_{n\to\infty}h(x_n)=h(x)$ implies $\lim_{n\to\infty}g(x_n)=g(x)$ ([10], I, p. 86), and the latter is equivalent to $\lim_{n\to\infty}x_n=x$. Hence the inverse function h^{-1} is also continuous.

Since h is a homeomorphism, $\overline{h[P-(a)]}$ is a compactification of the set $P-(a)$, and we have by (3) and Lemma 4.6

$$\overline{\psi g[P-(a)]}\underset{\text{top}}{=}\mathscr{C}. \tag{10}$$

And from the definition of h we infer that

$$\overline{h[P-(a)]}\subset\overline{\psi g[P-(a)]}\times\overline{g[P-(a)]},$$

and consequently that $\overline{h[P-(a)]} = \{\overline{\psi g[P-(a)]} \times \overline{g[P-(a)]}\} \cap \overline{h[P-(a)]}$, i.e.

(11) $$\overline{h[P-(a)]} = \bigcup_{\tau \in \overline{\psi g[P-(a)]}} \{\tau \times \overline{g[P-(a)]}\} \cap \overline{h[P-(a)]}.$$

Let $\{H_n^\tau\}_{n=1,2,\ldots}$ be for each $\tau \in \overline{\psi g[P-(a)]}$ a descending sequence of closed-open subsets of $P-(a)$ such that

(12) $\overline{\psi g(H_n^\tau)}$ is a closed-open subset of $\overline{\psi g[P-(a)]}$ for $n=1,2,\ldots$,

(13) $$(\tau) = \bigcap_{n=1}^{\infty} \overline{\psi g(H_n^\tau)}.$$

We shall prove that

(14) $\tau \times \bigcap_{n=1}^{\infty} \overline{g(H_n^\tau)} = \{\tau \times \overline{g[P-(a)]}\} \cap \overline{h[P-(a)]}$ for each $\tau \in \overline{\psi g[P-(a)]}$.

On the one hand, let $r \in \bigcap_{n=1}^{\infty} \overline{g(H_n^\tau)}$. Then $r = \lim_{n\to\infty} g(p_n)$ for a sequence of points $p_n \in H_n^\tau$ $(n=1,2,\ldots)$. Since $\psi g(p_n) \in \psi g(H_n^\tau)$ for $n=1,2,\ldots$ imply by virtue of (13) that $\lim_{n\to\infty} \psi g(p_n) = \tau$, then $\lim_{n\to\infty} [\psi g(p_n) \times g(p_n)] = \tau \times r$ ([10], I, p. 86), whence $\tau \times r \in \overline{h[P-(a)]}$. Hence $\tau \times \bigcap_{n=1}^{\infty} \overline{g(H_n^\tau)} \subset \overline{h[P-(a)]}$, and in view of the inclusion $\overline{g(H_n^\tau)} \subset \overline{g[P-(a)]}$ resulting by the hypothesis $H_n^\tau \subset P-(a)$ we infer that for $\tau \in \overline{\psi g[P-(a)]}$

(15) $$\tau \times \bigcap_{n=1}^{\infty} \overline{\psi g(H_n^\tau)} \subset \{\tau \times \overline{g[P-(a)]}\} \cap \overline{h[P-(a)]}.$$

On the other hand, if $\tau \times s \in \overline{h[P-(a)]}$, then there exists a sequence of points $q_n \in P-(a)$ such that $\tau \times s = \lim_{n\to\infty} [\psi g(q_n) \times g(q_n)]$. In particular, $\tau = \lim_{n\to\infty} \psi g(q_n)$. It follows by (12) and (13) that for $n=1,2,\ldots$ there exists an index m_n such that $\psi g(q_m) \in \psi g(H_n^\tau)$ for each $m \geqslant m_n$. We may assume that $m_n = n$, and therefore

(16) $$\psi g(q_n) \in \psi g(H_n^\tau) \qquad \text{for} \qquad n=1,2,\ldots$$

Since $g(H_n^\tau)$, as a closed-open subset of $g[P-(a)]$ by (3), is a union of quasicomponents of $g[P-(a)]$, we have, by the definition of the function ψ (see p. 8): $\psi^{-1}\psi g(H_n^\tau) = g(H_n^\tau)$ for $n=1,2,\ldots$, and consequently, in view of $\psi^{-1}\psi g(q_n) \subset \psi^{-1}\psi g(H_n^\tau)$ resulting from (16), $g(q_n) \in g(H_n^\tau)$ for $n=1,2,\ldots$ Therefore $\tau \times g(q_n) \in \tau \times g(H_n^\tau)$ for $n=1,2,\ldots$ and finally ([10], I, p. 245) $\tau \times q \in \tau \times \bigcap_{n=1}^{\infty} \overline{g(H_n^\tau)}$.

Thus we have also proved the inclusion inverse to (15) and therefore equation (14). It follows by (11) that

$$(17)\qquad \overline{h[P-(a)]} = \bigcup_{\tau\epsilon\overline{\psi g[P-(a)]}} [\tau\times\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}].$$

Since $H_{n+1}^{\tau}\subset H_n^{\tau}$ for each $\tau\epsilon\overline{\psi g[P-(a)]}$ and $n=1,2,\dots$ by hypothesis, so by Lemma 9.2 each set $\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}$ is a continuum ([10], II, p. 110). It follows by (10) that all members of the union (17) are components of the set $\overline{h[P-(a)]}$.

And since H_n^{τ} is by hypothesis closed-open subset of $P-(a)$ for each $\tau\epsilon\overline{\psi g[P-(a)]}$ and $n=1,2,\dots$, we have by Lemma 9.2 the inclusion $T\times(a_2,a_3,\dots)\subset\overline{g(H_n^{\tau})}$ for each $\tau\epsilon\overline{\psi g[P-(a)]}$ and $n=1,2,\dots$, whence $T\times(a_2,a_3,\dots)\subset\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}$, and consequently

$$(18)\qquad \bigcup_{\tau\epsilon\overline{\psi g[P-(a)]}}[\tau\times T\times(a_2,a_3,\dots)]\subset\bigcup_{\tau\epsilon\overline{\psi g[P-(a)]}}[\tau\times\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}].$$

But $\bigcup_{\tau\epsilon\overline{\psi g[P-(a)]}}[\tau\times T\times(a_2,a_3,\dots)]\underset{\text{top}}{=}\mathscr{C}\times\mathscr{I}$ by (10) and each set $\tau\times\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}$ is a component of $\overline{h[P-(a)]}$, hence by (18) each segment $\tau\times T\times(a_2,a_3,\dots)$ lies in the component $\tau\times\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}$ of the set $\overline{h[P-(a)]}$ and each component $\tau\times\bigcap_{n=1}^{\infty}\overline{g(H_n^{\tau})}$ of the set $\overline{h[P-(a)]}$ contains the segment $\tau\times T\times(a_2,a_3,\dots)$.

COROLLARY 9.4. *The compactification of the set $P-(a)$ constructed in Theorem 9.3 contains a pulverized set meeting each component of this compactification in exactly one point, i.e. such that upon the addition of its dispersion point it becomes a plane minimal set.*

Indeed, the set $\mathscr{C}\times\mathscr{I}$ contains a pulverized set [9] meeting each segment $\tau\times\mathscr{I}$ in exactly one point.

§ 10. Pulverability and σ-connectivity

We now deal with connexions between the notion of pulverable and that of σ-connected sets.

A set X is called *σ-connected* if there is no decomposition of it into a sequence of non-void, disjoint and closed in X subsets (see Lelek [11]).

Lelek [11] has proved that the set constructed in example α of the paper [6] is σ-connected, and he posed the problem: is every biconnected set σ-connected? ([11], P4, p. 267). The answer is negative even for pulverable sets. Namely

THEOREM 10.1. *There exist pulverable σ-connected sets and pulverable not σ-connected sets of any finite and infinite dimension.*

A construction of pulverable σ-connected sets B_n of any finite dimension n is described in Example 4 (p. 45), and a construction of pulverable but not σ-connected sets B'_n of any finite dimension n is described in Example 5 (p. 50). Applying Corollary 6.2 to the families $\{B_n\}_{n=1,2,\ldots}$ and $\{B'_n\}_{n=1,2,\ldots}$ we obtain pulverable sets B_∞ and B'_∞ of dimension ∞, the first of which is σ-connected and the second is not. The constituents of σ-decomposition of the set B'_∞ are: its dispersion point a'_∞ and all the constituents of σ-decompositions of the sets B'_n.

Since no pulverable set contains a continuum ([6], th. XIV, p. 216), Theorem 10.1 implies the following

COROLLARY 10.2. *For any dimension $n = 1, 2, \ldots, \infty$ there exists n-dimensional and σ-connected set containing no continuum.*

CHAPTER IV

§ 11. Two lemmas on decompositions

Both lemmas are intended for the construction of Example 3 (p. 41). We shall use in their proofs (but nowhere else) the continuum hypothesis.

Let $\mathcal{I}$ be the segment $0 \leqslant x \leqslant 1$ and let Ω be the smallest uncountable ordinal number.

LEMMA 11.1. *There exists a decomposition* $\mathcal{I} = \bigcup_{1\leqslant\sigma\leqslant\Omega} R_\sigma$ *such that the sets* R_σ *are disjoint, of power* $2^{\aleph_0}$, *dispersed, and perfect.*

Proof. Let f be a function transforming the segment $\mathcal{I}$ onto $\mathcal{I}^2$ in such a way that the image under it of any subsegment J of $\mathcal{I}$ such that

$$(1)\qquad J = \left\{x\colon \frac{k}{9^n} \leqslant x \leqslant \frac{k+1}{9^n}\right\}, \quad \text{where} \quad k = 0, 1, \ldots, 9^n-1,$$

is a solid square, the sides of which are parallel to the sides of $\mathcal{I}^2$. Such a function exists; for instance, see [14]. Let $\{x_\sigma\}_{\sigma<\Omega}$ be a sequence of all points of the segment $\mathcal{I}$. We show that the set R_σ defined by the formula $R_\sigma = f^{-1}(x_\sigma\times\mathcal{I})$ has announced properties. In fact, we have $x_\sigma \neq x_{\sigma'}$ for $\sigma\neq\sigma'$, whence $(x_\sigma\times\mathcal{I})\cap(x_{\sigma'}\times\mathcal{I}) = 0$. Therefore $f^{-1}[(x_\sigma\times\mathcal{I})\cap$ $\cap(x_{\sigma'}\times\mathcal{I})] = f^{-1}(x_\sigma\times\mathcal{I})\cap f^{-1}(x_{\sigma'}\times\mathcal{I}) = R_\sigma\cap R_{\sigma'} = 0$, i.e. each two distinct sets of the decomposition are disjoint. Since the function f is a one-to-one function besides of a countable set of points, and since the sets $x_\sigma\times\mathcal{I}$ are of power $2^{\aleph_0}$, then the sets R_σ are of the same power. Since, by hypothesis, the image of every subsegment of $\mathcal{I}$ contains a solid square, and the set $x_\sigma\times\mathcal{I}$ is a segment, i.e. does not contain any square, then $f^{-1}(x_\sigma\times\mathcal{I})$ contains no segment. Hence the sets R_σ are dispersed. Finally, every R_σ is a perfect set. Indeed, for each point $p\in R_\sigma = f^{-1}(x_\sigma\times\mathcal{I})$ and for each $n = 1, 2, \ldots$ there exists a segment J of shape (1) containing p. It follows that $f(J)\cap(x_\sigma\times\mathcal{I})-f(p)\neq 0$, whence $J\cap R_\sigma-(p)\neq 0$. Thus p is a cluster point of the set R_σ. This set is then dense in itself. The set $x_\sigma\times\mathcal{I}$ is closed and the function f is continuous; thus the set R_σ is closed too. Being closed and dense in itself it is a perfect set.

LEMMA 11.2. *There exists a decomposition* $\mathscr{C} = \bigcup_{1 \leq \xi < \Omega} C_\xi$ *such that the sets* C_ξ *are disjoint, countable and dense in* $\mathscr{C}$.

Proof. Let $\{\Gamma_k\}_{k=1,2,\dots}$ be the basis of $\mathscr{C}$ consisting of open sets. Choosing one point from each of these sets and denoting the set of such selected points by C_1' we obtain a countable set which is dense in $\mathscr{C}'$.

Using now the transfinite induction we shall define the set C_a' for $a < \Omega$ with the aid of sets C_ξ' disjoint, countable and dense in $\mathscr{C}$, where $\xi < a$. In view of $a < \Omega$ the set $\bigcup_{\xi < a} C_\xi'$ is countable. Then the sets $\Gamma_k - \bigcup_{\xi < a} C_\xi'$ are of power $2^{\aleph_0}$ for each $k = 1, 2, \dots$ Choosing one point from each of the latter sets and denoting the set of such selected points by C_a' we obtain a set C_a', disjoint with each C_ξ', where $\xi < a$, countable and dense in $\mathscr{C}$.

Thus we have defined a family of $2^{\aleph_0}$ sets C_ξ', disjoint, countable and dense in $\mathscr{C}$. Ordering the points of the set $\mathscr{C} - \bigcup_{1 \leq \xi < \Omega} C_\xi'$ in a finite or transfinite sequence $c_1, c_2, \dots, c_a, \dots$ put $C_\xi = C_\xi' \cup (c_\xi)$ if the point c_ξ exists, and $C_\xi = C_\xi'$ in the contrary case. The sets C_ξ are henceforth disjoint and have all other announced properties.

§ 12. Examples

EXAMPLE 1. *A pulverized set* $P - (a)$ *consisting of* $\aleph_0$ *quasicomponents not one of which is nowhere dense in* P.

We shall use in the construction the example α from paper [6] (pp. 241-244) of a pulverable set S having $(1/2, 1/2)$ as a dispersion point and lying in the Cantor fan $\mathscr{M}$.

First of all in a solid square K of the plane Oxy having the opposite vertices $(0, 0)$ and $(2, 2)$ we define two auxiliary sets M_0 and M_1. For this purpose let C_0 be the left half of the Cantor set $\mathscr{C}$, i.e. lying in the segment $0 \leq x \leq 1/3$, and C_1 — its right half, i.e. lying in the segment $2/3 \leq x \leq 1$. Furthermore, let $L_0(x)$ be the polygonal line with vertices $(\frac{1}{2}, 2)$, $\left(\frac{5x}{2(1+3x)}, \frac{1+6x}{1+3x}\right)$ and $\left(2, \frac{1+6x}{1+3x}\right)$, and $L_1(x)$ — the polygonal line with vertices $(\frac{1}{2}, 2)$, $\left(\frac{16x-2}{6x+9}, \frac{12x-8}{6x+9}\right)$ and $\left(2, \frac{12x-8}{6x+9}\right)$. Put

(1) $$M_0 = \bigcup_{\tau \in C_0} L_0(\tau), \quad M_1 = \bigcup_{\tau \in C_1} L_1(\tau), \quad a = (\tfrac{1}{2}, 2)$$

(see Figs. 3 and 4). It is easily seen that for $j, j' = 0, 1$

(2) $$L_0(\tau_0) \cap L_1(\tau_1) - (a) \neq 0 \quad \text{for each} \quad \tau_0 \in \mathscr{C}_0 \text{ and } \tau_1 \in \mathscr{C}_1,$$

(3) $$\underset{\text{top}}{L_j(\tau) \cap M_{j'} - (a) = \mathscr{C}} \quad \text{for each} \quad \tau \in \mathscr{C}_j \text{ and } j \neq j',$$

(4) $M_j-(a)$ is not connected between any pair of polygonal lines $L_j(\tau')$ and $L_j(\tau'')$, where $\tau' \neq \tau''$ and $\tau', \tau'' \in \mathscr{C}_j$.

Obviously, the sets M_0 and M_1 are both homeomorphic with $\mathscr{M}$, and so they contain the pulverable sets $S_0 \subset M_0$ and $S_1 \subset M_1$ with the same dispersion point a. For each $\tau \in \mathscr{C}$, the counter-image of $L_0(\tau)$ is the set $L(3\tau)$, and of $L_1(\tau)$ the set $L(3\tau-2)$. Since each segment $L(\tau)$ contains a quasicomponent of the set $S-(1/2, 1/2)$ and this set is dense in $L(\tau)$,

(5) every polygonal line $L_j(\tau)$, where $\tau \in \mathscr{C}_j$, contains a dense in it quasicomponent of the set $S_j-(a)$.

Let us add to each of sets S_0 and S_1 the set $M_0 \cap M_1$. Then the set

(6) $P_j = S_j \cup M_0 \cap M_1$ is pulverable, and a is its dispersion point.

In fact, by evident inclusions $S_j \subset P_j \subset M_j$ and by the equality $\overline{S}_j = M_j$, which follows from (5), the set P_j is connected ([10], II, p. 83).

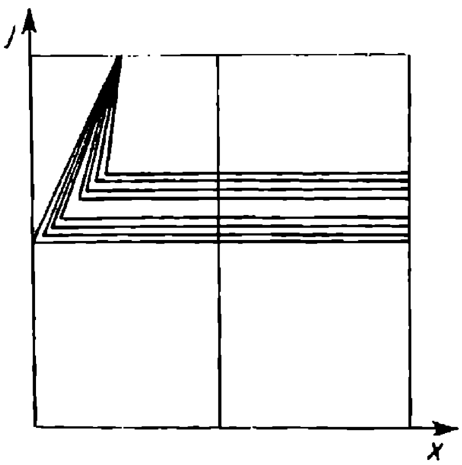

Fig. 3. The set M_0

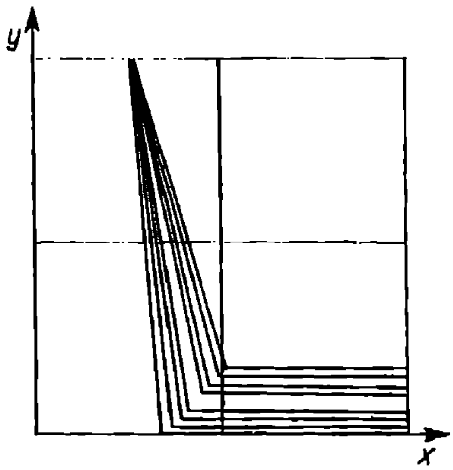

Fig. 4. The set M_1

By virtue of the definition of pulverable set (see p. 3) it remains to prove that a is a dispersion point of both sets P_j, i.e. that each quasicomponent of $P_j-(a)$ is a dispersed set. For this purpose we first show that

(7) every quasicomponent of the set $P_j-(a)$ is dense in the polygonal line in which it lies.

By the inclusion $S_j-(a) \subset P_j-(a)$ every quasicomponent of the set $P_j-(a)$ contains a quasicomponent of the set $S_j-(a)$, and by the inclusion $P_j-(a) \subset M_j-(a)$ and (4) every quasicomponent of the set $P_j-(a)$ is contained in some polygonal line $L_j(\tau)$, where $\tau \in C_j$. In view of the above two premises and (5) we get (7).

Now let Q be a quasicomponent of the set $P_j-(a)$. In virtue of (7) Q is a dense subset of the set $L_j(\tau)-(a)$ for some $\tau \in C_j$, whence

$Q = P_j \cap L_j(\tau) - (a)$. By the definition of P_j and by (1) we have then $Q = P_j \cap L_j(\tau) - (a) = (S_j \cup M_0 \cap M_1) \cap L_j(\tau) - (a) = S_j \cap L_j(\tau) \cup L_j(\tau) \cap M_{j'} - (a)$, where $j \neq j'$, i.e. $Q = [S_j \cap L_j(\tau) - (a)] \cup [L_j(\tau) \cap M_{j'} - (a)]$.

In virtue of (5) the first member of this union is a quasicomponent of the pulverized set $S_j - (a)$ and, by Lemma 4.2, it is a dispersed set. And by (3) the second member is a closed and dispersed set. Therefore, the union of these sets is a dispersed set, because every dispersed subset of polygonal line is a boundary set, and the union of boundary and nowhere dense sets is a boundary set ([10], I, p. 37). (6) is thus proved.

Now we show that

(8) for each quasicomponent Q of the set $P_j - (a)$ and for each $\tau' \in C_{j'}$, where $j \neq j'$, we have $Q \cap L_{j'}(\tau') \neq 0$.

Indeed, we have by (7)

(9) $$Q = P_j \cap L_j(\tau) - (a) \quad \text{for some} \quad \tau \in C_j,$$

whence $Q \subset L_j(\tau) - (a)$. For any $\tau' \in C_{j'}$, where $j \neq j'$, we have then

(10) $$Q \cap L_{j'}(\tau') - (a) \subset L_j(\tau) \cap L_{j'}(\tau') - (a).$$

Since $L_j(\tau) \cap L_{j'}(\tau') - (a) \subset M_0 \cap M_1 - (a)$ by (1), it follows by (6) that $L_j(\tau) \cap L_{j'}(\tau') - (a) \subset M_0 \cap M_1 - (a)$, whence by (9) we have $L_j(\tau) \cap L_{j'}(\tau') - (a) \subset Q$. Thus $L_j(\tau) \cap L_{j'}(\tau') - (a) \subset Q \cap L_{j'}(\tau') - (a)$, which implies by inclusion (10) that $Q \cap L_{j'}(\tau') - (a) = L_j(\tau) \cap L_{j'}(\tau') - (a)$. From this equality and (2) follows at once (8).

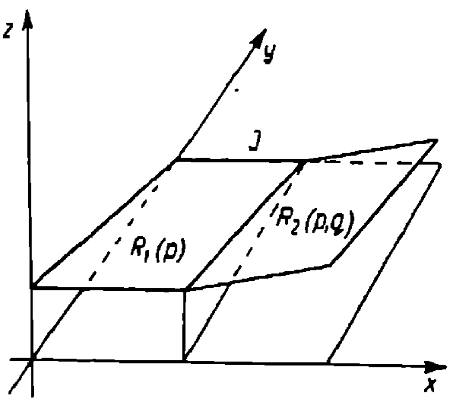

Fig. 5

The solid square K is a parallel to the z-axis projection of the union $R(p, q)$ of two rectangles $R_1(p)$ and $R_2(p, q)$ defined for each pair p, q of points of the z-axis as follows: $R_1(p)$ has vertices $(0, 2, 0)$, $(1, 2, 0)$, $(1, 0, p)$ and $(0, 0, p)$, and $R_2(p, q)$ vertices $(1, 2, 0)$, $(2, 2, q)$, $(1, 0, p)$ and $(2, 0, p+q)$. Thus for each $p \neq 0$ the rectangle $R_1(p)$ has exactly one side in common with the square K, namely the side J with end points $(0, 2, 0)$ and $(1, 2, 0)$, being the left half of the upper side of the square K, and for each $q \neq 0$ the rectangle $R_2(p, q)$ has exactly one vertex in common with the square K, namely the middle point $(1, 2, 0)$ of the upper side of K, and it has the whole side of ends $(1, 2, 0)$ and $(1, 0, p)$ in common with the rectangle $R_1(p)$ (see Fig. 5).

Let Z denote henceforth the z-axis, T the interval $0 < z < 1$ of Z, and W the set of rational points of Z. Put $W_0 = W \cap T$ and $W_1 = W' \cap T$, where W' is the set W translated under irrational number. Hence $\overline{W}_0 = \overline{T} = \overline{W}_1 = \overline{W_0 \cup W_1}$ and $W_0 \cap W_1 = 0$. Let $\mathscr{C}$ be the Cantor set lying in the interval T. The set of numbers $W_0 \cup W_1$ ordered according to their magnitude is then similar to the isolated set M of middle points of complementary intervals of $\mathscr{C}$ to the interval T. Let f be this similarity: $f(p) \in M$ for $p \in W_0 \cup W_1$.

For $p \in W_0 \cup W_1$ denote by $P_j(p)$ the parallel to z-axis projection of the pulverable set P_j onto $R[p, f(p)]$, and by $N_j(\tau, p)$ the similar projection of the polygonal line $L_j(\tau)$ onto the same set. Thus

(11) $P_j(p)$ is a homeomorph of P_j, and $a \in P_j(p)$,

(12) $P_j(p) - (a) \subset R[p, f(p)] - J$;

subsequently by (7)

(13) every quasicomponent Q of $P_j(p) - (a)$ is dense in the polygonal line $N_j(\tau, p)$ in which it lies,

and finally by (8)

(14) for each quasicomponent Q of $P_j(p) - (a)$ and for each $\tau' \in C_{j'}$, where $j \neq j'$, we have $Q \cap N_{j'}(\tau', p) \neq 0$.

We deduce from (13) and (14) that

(15) for each quasicomponent Q of $P_j(p) - (a)$ and for each quasicomponent Q_n of $P_{j'}(p_n) - (a)$, where $j \neq j'$, $\lim\limits_{n\to\infty} p_n = p$ implies $Q \cap \operatorname{Ls}\limits_{n\to\infty} Q_n \neq 0$.

In fact, it follows from (13) that

(16) $\overline{Q}_n = N_{j'}(\tau_n, p_n)$ for each $n = 1, 2, \ldots$ and some $\tau_n \in C_{j'}$.

Since the sequence of compact sets $\{\overline{Q}_n\}_{n=1,2,\ldots}$ contains a convergent subsequence $\{\overline{Q}_{m_n}\}_{n=1,2,\ldots}$ ([10], II, p. 21), $\lim\limits_{n\to\infty} p_n = p$ implies $\operatorname{Lim}\limits_{n\to\infty} \overline{Q}_{m_n} = N_{j'}(\tau, p)$ for some $\tau \in C_{j'}$.

Since $\operatorname{Lim}\limits_{n\to\infty} \overline{Q}_{m_n} = \operatorname{Lim}\limits_{n\to\infty} Q_{m_n}$ ([10], I, p. 245, formula 1), we conclude by (14) that $Q \cap \operatorname{Lim}\limits_{n\to\infty} Q_{m_n} \neq 0$.

Now we can define the set P. Namely, put

(17) $$P = \bigcup_{p \in W_0} P_0(p) \cup \bigcup_{p \in W_1} P_1(q).$$

First we show that

(18) the set $P - (a)$ is not connected between any pair of its subsets $P_j(p) - (a)$ and $P_{j'}(q) - (a)$, where $p \in W_j$, $q \in W_{j'}$, $p < q$, and $j \leqslant j'$.

Indeed, since the function f is a similarity between $W_0 \cup W_1$ and M, there exist two points: $r_1 \in Z - (W_0 \cup W_1)$ and $r_2 \in Z - M$, such that $p < r_1 < q$ and $f(p) < r_2 < f(q)$; moreover, for each pair of points $p, q \in W_0 \cup W_1$ the relation $p < q$ is equivalent to the relation $f(p) < f(q)$. In other words, the set $R(r_1, r_2)$ is disjoint with $R[s, f(s)] - J$ for each $s \in W_0 \cup W_1$ and lies between $R[p, f(p)] - J$ and $R[q, f(q)] - J$, if *between* is taken to mean that the union of two half-planes having the common segment $R_1(r_1) \cap R_2(r_1, r_2)$, the first of which contains $R_1(r_1)$ and the second $R_2(r_1, r_2)$, disconnects the space E^3 between $R[p, f(p)] - J$ and $R[q, f(q)] - J$. We have then (18) by (12).

Moreover, since M is isolated,

(19) the set $P_j(p)$ is not nowhere dense in P.

Now let H be an arbitrary open set in $P - a$ containing a quasicomponent Q of its subset $P_j(p) - (a)$. Then

(20) **there exists an interval G of z-axis Z containing the point p and such that for each $q \in W_{j'} \cap G$, where $j \neq j'$, the set H has common points with any quasicomponent of $P_{j'}(q) - (a)$.**

For supposing the contrary, i.e. that for each interval G of z-axis Z such that $p \in G$ there is a point $q \in W_j \cap G$ $(j \neq j')$ such that for some quasicomponent Q of $P_{j'}(q) - (a)$ we have $Q \cap H = 0$, we can take a sequence of intervals G_n such that $p \in \bigcup_{n=1}^{\infty} G_n$ and $\delta(G_n) < 1/n$, and consequently we can choose a point $q_n \in W_{j'} \cap G_n$ and a quasicomponent Q_n of $P_{j'}(q) - (a)$ such that

(21) $$H \cap Q_n = 0 \quad \text{for} \quad n = 1, 2, \ldots$$

We infer from (15) that $Q \cap \operatorname{Ls}_{n \to \infty} Q_n \neq 0$. But it is impossible in view of (21), because the set H is open and contains Q.

In particular, if H is simultaneously closed and open in $P - (a)$, then

(22) **there exists on z-axis an interval G containing the point p and such that for each $q \in W_{j'} \cap G$, where $j \neq j'$, we have $P_{j'}(q) - (a) \subset H$.**

In fact, since $P_{j'}(q) - (a) \subset P - (a)$, the set H is also closed-open in $P_{j'}(q) - (a)$, and has by (20) common points with each quasicomponent of $P_{j'}(q) - (a)$, where $q \in W_{j'} \cap G$ and $j \neq j'$, must contain it. Finally

(23) each of sets $P_j(p) - (a)$, where $p \in W_j$, is a quasicomponent of the set $P - (a)$.

By (18) each of quasicomponents of the set $P - (a)$ is contained in $P_j(p) - (a)$ for some $p \in W_j$. In order to prove the converse, i.e. that each set $P_j(p) - (a)$ is contained in some quasicomponent of $P - (a)$, we first show that the set $P - (a)$ is connected between each pair of points b_1 and b_2 of the set $P_j(p) - (a)$. For this purpose let Q_1 and Q_2 be two quasicomponents of $P_j(p) - (a)$ such that $b_1 \in Q_1$ and $b_2 \in Q_2$, let H_1 be a set closed-open in $P - (a)$ and containing Q_1, and similarly let H_2 be a set closed-open in $P-(a)$ and containing Q_2. Thus for each of the two pairs H_m, Q_m, where $m = 1$ and 2, the z-axis contains an interval G_m fulfilling (22). Since the set $W_{j'}$ is, by hypothesis, dense, $W_{j'} \cap G_1 \cap G_2 \neq 0$, whence

$$H_1 \cap H_2 \supset \{ \bigcup_{q \in W_{j'} \cap G_1} [P_{j'}(q)-(a)]\} \cap \{ \bigcup_{q \in W_{j'} \cap G_2} [P_{j'}(q)-(a)]\}$$
$$= \bigcup_{q \in W_{j'} \cap G_1 \cap G_2} [P_{j'}(q)-(a)] \neq 0.$$

It follows by $b_m \in H_m$ and in view of the free choice of sets H_m closed-open in $P-(a)$ that the set $P-(a)$ is connected between b_1 and b_2.

Now we can prove the properties of the set P.

First of all, the set P is pulverable and a is its dispersion point. Indeed, since, by (11), each subset $P_j(p)$ of P is connected, and each two subsets of P have the common point a, the union P of all these subsets of P is also connected ([10], II, p. 82). The point a is its dispersion point, because each subset $P_j(p)-(a)$ of P is dispersed, by (11), whence it follows by (18) that the set $P-(a)$ is also dispersed.

Thus set $P-(a)$ is pulverized.

The countability of the set $W_0 \cup W_1$ and (23) imply at once that the set $P-(a)$ has $\aleph_0$ quasicomponents.

Finally, by (19) and (23), each quasicomponent of $P-(a)$ is not nowhere dense in $P-(a)$.

EXAMPLE 2. *A pulverized set $P'-(a)$ consisting of $\aleph_0$ quasicomponents each of which is nowhere dense in P'* ([6]).

The set P' will be obtained from the set P defined in Example 1 by a continuous transformation g such that for each $p \in W_j$, where $j = 0$ or 1,

(1) $g[P_j(p)-(a)]$ is a quasicomponent of $g[P-(a)]$,

(2) $g[P_j(p)-(a)]$ is nowhere dense in $g[P-(a)]$,

(3) $g(P)$ is pulverable and a is its dispersion point,

(4) $g[P-(a)]$ consists of $\aleph_0$ quasicomponents.

For this purpose define g on the set $\bigcup_{p \in W_0 \cup W_1} R[p, f(p)]$ containing P as a function changing each set $R[p, f(p)]$ containing the quasicomponent $P_j(p)-(a)$ of $P-(a)$, where $p \in W_j$, into its projection parallel to z-axis, namely into the rectangle $R(p, 0)$:

$$g\{R[p, f(p)]\} = R(p, 0).$$

In other words, we straighten each right rectangle $R_2[p, f(p)]$ and we put it on the protraction of the left one. Since the set defining the

([6]) The existence of such a pulverable set was established in the proof of Theorem 7.4 (p. 20). Here an individual geometric example will be constructed by a simple modification of Example 1.

family of right rectangles is isolated, the function g, being continuous on each of these rectangles, is also continuous on their union. Consequently

$$(5) \qquad g[P-(a)] = g(P)-(a).$$

The images $R(p,0)$ or $R[p,f(p)]$ have now in common the whole edge of the ends $(0,2,0)$ and $(2,2,0)$. Denote this edge by E. Then for all $p \in W_j$

$$(6) \qquad g[P_j(p)-a] \subset R(p,0)-E,$$

$$(7) \qquad g[P_j(p)-(a)] \text{ is homeomorphic with } P_j(p)-(a).$$

Since the set $W_0 \cup W_1$ defining the union of the left sides $R_1(p)$ is dense in T, on account of which each one is a boundary set in their union, we now have the same for the right sides, i.e.

$$(8) \qquad g[P_j(p)-(a)] \text{ is a boundary set in } g[P-(a)].$$

At the same time, it is easy to prove, in a way analogous to that in the proof of property (17) of Example 1, that the set $g[P-(a)]$ is not connected between any pair of its subsets $g[P_j(p)-(a)]$, $g[P_{j'}(q)-(a)]$, where $p \in W_j$, $q \in W_{j'}$, $p \neq q$ and $j \leqslant j'$. Namely the plane containing $R(r_1,0)$ disconnects $\mathcal{E}^3$ between each pair of sets $R(p,0)-E$ and $R(q,0)-E$, where $p<r_1<q$; consequently it disconnects the space $\mathcal{E}^3$ between $g[P_j(p)-(a)]$ and $g[P_{j'}(q)-(a)]$ in virtue of (6).

Therefore the set $\bigcup_{p \in W_0 \cup W_1} R(p,0) = g\{\bigcup_{p \in W_0 \cup W_1} R[p,f(p)]\}$ is a union of a countable family of rectangles $R(p,0)$ continuous along the edge E and disjoint beyond it, and the set $g[P-(a)]$ is not connected between any pair of sets $g[P_j(p)-(a)]$ and $g[P_{j'}(q)-(a)]$, where $p \neq q$ and $j \leqslant j'$, i.e. any two points $p' \in g[P_j(p)-(a)]$ and $q' \in g[P_{j'}(q)-(a)]$ belong to different quasicomponents of $g[P-(a)]$. In other words, the continuous function g has the following property: if two points x and y belong to different quasicomponents of the set $P-(a)$, then their images $p' = g(x)$ and $q' = g(y)$ belong to different quasicomponents of $g[P-(a)]$. The inverse implication occurs too: if two points x and y belong to one quasicomponent $P_j(p)-(a)$ of $P-(a)$, then their images $g(x)$ and $g(y)$ belong to one quasicomponent of the set $g[P-(a)]$. For any continuous function has the following property: the image of any quasicomponent is always contained in some quasicomponent of the image. The above equivalence implies (1), whence (4) follows in view of countability of the set $W_0 \cup W_1$ running over by p. Property (2) follows then by (8) and (1), because every quasicomponent is a closed set ([10], II, p. 93).

Finally, each of the sets $P_j(p)-(a)$ is dispersed being a quasicomponent of the pulverized set $P-(a)$. Thus, by (7), the set $g[P_j(p)-(a)]$

is dispersed and, consequently, the set $g[P-(a)]$ is by (1) dispersed too. Since the set P is connected and the function g is continuous, the set $g(P)$ is connected. Hence by (5) the point a is its dispersion point, and therefore we have (3) by Lemma 3.1.

All the properties (1)-(4) of the set $g(P)$ are thus proved and it remains to denote this set by P'.

EXAMPLE 3. *A pulverable plane set P'' such that every subset B of $P''-(a)$ which has a dimension 1 in all points save a nowhere dense subset (which may be void) contains $2^{\aleph_0}$ quasicomponents of power $2^{\aleph_0}$. In particular, every pulverized subset of P'' contains $2^{\aleph_0}$ quasicomponents of power $2^{\aleph_0}$* (7).

We shall define P'' in the Cantor fan $\mathcal{M}$ by choosing some subset of each segment $L(\tau)$. We shall use the continuum hypothesis in this construction.

Let us take the decomposition $\bigcup_{1\leqslant\sigma<\Omega} R_\sigma$ of the segment $0\leqslant y\leqslant 1/2$ of y-axis into $2^{\aleph_0}$ disjoint, dispersed, and perfect sets, each of which is of power $2^{\aleph_0}$ (such a decomposition exists by Lemma 10.1), and the decomposition of the Cantor set $\mathcal{C}=\bigcup_{1\leqslant\xi<\Omega} C_\xi$ into $2^{\aleph_0}$ disjoint, countable subsets dense in $\mathcal{C}$ (such a decomposition exists by Lemma 10.2). Obviously, we have for each $\tau\in\mathcal{C}$

$$L(\tau)=\bigcup_{1\leqslant\sigma<\Omega} L(\tau)\cap(I\times R_\sigma), \tag{1}$$

where the members of the union are disjoint and non-void. Now let us order in a transfinite sequence $D_1, D_2, \ldots, D_\alpha, \ldots$, where $\alpha<\Omega$, all sets (we may confine ourselves to closed sets only, see [10], II, p. 97) disconnecting plane between some two points of the Cantor fan $\mathcal{M}$ and not containing the point $a=(1/2,1/2)$. It is easily seen that each of them must disconnect segments $L(\tau)$ for some open and hence closed-open set Δ of points $\tau\in\mathcal{C}$. Each of C_ξ is dense in $\mathcal{C}$; thus

$$\Delta\cap C_\xi\neq 0 \quad \text{for all} \quad \xi<\Omega. \tag{2}$$

Let σ_1 be the first index such that

$$D_1\cap[\bigcup_{\tau\in C_1} L(\tau)]\cap(I\times R_{\sigma_1})\neq 0.$$

Put $V_1=C_1$ and $W_1=R_{\sigma_1}$. Suppose that for each $\eta<\alpha$

$$D_\eta\cap\bigcup_{\beta<\alpha}\{[\bigcup_{\tau\in V_\beta} L(\tau)]\cap(I\times W_\beta)\}\neq 0, \tag{3}$$

(4) $V_\eta=W_\eta=0$, or $V_\eta=C_{\xi_\eta}$ and $W_\eta=R_{\sigma_\eta}$ for some $\xi_\eta<\Omega$ and $\sigma_\eta<\Omega$,

(7) Originally Example 3 was constructed in consideration of the property of its pulverized subsets formulated above. I an indebted to A. Lelek for calling my attention to the more general property of this example.

and that for each $\beta < \alpha$ and $\beta' < \alpha$ the inequality $\beta \neq \beta'$ implies the equalities

$$V_\beta \cap V_{\beta'} = 0, \tag{5}$$

$$W_\beta \cap W_{\beta'} = 0. \tag{6}$$

In order to define now the sets V_α and W_α, consider the two following cases:

$$D_\alpha \cap \bigcup_{\beta<\alpha} \{[\bigcup_{\tau\in V_\beta} L(\tau)] \cap (I \times W_\beta)\} \neq 0; \tag{I}$$

in this case put $V_\alpha = W_\alpha = 0$.

$$D_\alpha \cap \bigcup_{\beta<\alpha} \{[\bigcup_{\tau\in V_\beta} L(\tau)] \cap (I \times W_\beta)\} = 0; \tag{II}$$

in this case let C_{ξ_α} be the first set that differs from every V_β for $\beta < \alpha$ and at the same time such that

$$D_\alpha \cap \{\bigcup_{\tau\in C_{\xi_\alpha}} L(\tau) - \bigcup_{\beta<\alpha} (I \times W_\beta)\} \neq 0, \tag{7}$$

and let R_{σ_α} be the first set that differs from every W_β for $\beta < \alpha$ and such that

$$D_\alpha \cap [\bigcup_{\tau\in C_{\xi_\alpha}} L(\tau)] \cap (I \times R_{\sigma_\alpha}) \neq 0. \tag{8}$$

Put $V_\alpha = C_{\xi_\alpha}$ and $W_\alpha = R_{\sigma_\alpha}$.

It is easy to verify the properties (3)-(6) for $\alpha+1$ instead of α.

Also observe that (7) implies (8). In fact, for $\tau \in C_{\xi_\alpha}$ we have $D_\alpha \cap L(\tau) - \bigcup_{\beta<\alpha} (I \times W_\beta) \neq 0$. In virtue of (4) and (1) there exist indexes σ, different from σ_η, where $\eta < \alpha$, and such that $D_\alpha \cap L(\tau) \cap (I \times R_\sigma) \neq 0$.

It remains to prove the existence of indexes ξ_α fulfilling (7). For this purpose it is sufficient, by (4), to show that

$$D_\alpha \cap \{\bigcup_{\tau\in \mathscr{C} - \bigcup_{\beta<\alpha} V_\beta} L(\tau) - \bigcup_{\beta<\alpha} (I \times W_\beta)\} \neq 0. \tag{9}$$

Let Δ be a closed-open subset of $\mathscr{C}$, such that

$$D_\alpha \cap L(\tau) \neq 0 \quad \text{for each} \quad \tau \in \Delta. \tag{10}$$

Each of the sets $D_\alpha \cap [\bigcup_{\tau\in\Delta} L(\tau)] \cap (I \times W_\beta)$ is compact being a common part of compact sets. Let A_β be a projection of such a set from the point a onto the y-axis, i.e.

$$A_\beta = \{\tau : \tau \in \Delta,\ D_\alpha \cap L(\tau) \cap (I \times W_\beta) \neq 0\}. \tag{11}$$

The set A_μ is the compact too ([10], II, p. 11). Now, supposing that (9) is not true, i.e. that

$$D_\alpha \cap [\bigcup_{\tau\in\mathscr{C}-\bigcup\limits_{\beta<\alpha} V_\beta} L(\tau)] \subset \bigcup_{\beta<\alpha}(I\times W_\beta),$$

we have, by (10) and (11), $\Delta = \bigcup\limits_{\beta<\alpha} A_\beta \cup \Delta \cap \bigcup\limits_{\beta<\alpha} V_\beta$, where the set $\bigcup\limits_{\beta<\alpha} V_\beta$ is at most countable by the countability of the sets C_ξ and by (4). Then the compact set Δ is a union of an at most countable set $\Delta\cap\bigcup\limits_{\beta<\alpha} V_\beta$ and of a sequence of closed sets A_β. Baire category theorem ([10], I, p. 320) ensures that one of the latter sets, which is denoted by A_{β_0}, contains a subset B dense in a closed-open $\Delta' \subset \Delta$ subset of the Cantor set $\mathscr{C}$. Therefore we have (2) for Δ' instead of Δ. Thus $\bar{B} = \Delta'$, whence

(12) $$\Delta' \subset A_{\beta_0},$$

because the set A_{β_0} is closed.

As satisfying (2), each of the sets $\Delta'\cap C_{\xi_\beta} = \Delta'\cap V_\beta$ is non-void. Let then

(13) $$\tau\in\Delta'\cap V_{\beta_0}.$$

Therefore, by (12), we have $\tau\in A_{\beta_0}$, whence $D_\alpha\cap L(\tau)\cap(I\times W_{\beta_0}) \neq 0$ in view of (11), but it is in contradiction with (II) by virtue of (13).

The proof of (9) is thus complete and thereby the inductive definition of transfinite sequences $\{V_\alpha\}_{\alpha<\Omega}$ and $\{W_\alpha\}_{\alpha<\Omega}$ satisfying conditions (3)-(6) for each $\alpha<\Omega$ and each $\eta<\alpha$ is finished.

Now we can define the set P'' by the formula

(14) $$P'' = (a)\cup\bigcup_{\beta<\Omega}\{[\bigcup_{\tau\in V_\beta} L(\tau)]\cap(I\times W_\beta)\}.$$

We begin with the proof of the first property of the set P'', namely that the set P'' is pulverable and that a is its dispersion point.

For this purpose first note that the set $P''-(a)$ is dispersed. In fact, every set $L(\tau)\cap P''-(a)$ is, by (5) and (14), either void or equal to $L(\tau)\cap$ $\cap(\mathscr{I}\times W_\beta)$. The last set is dispersed by (4) and by the hypothesis that the sets R_σ are dispersed. Moreover, for each pair of points of P'' lying in two differents segments of the Cantor fan $\mathscr{M}$, there exists in the plane a straight line having only a as a common point with P'', and disconnecting the plane between the points of this pair.

In view of the definition of pulverable set (see p. 3) it remains to prove that the set P'' is connected. For this purpose we show that every closed set D disconnecting the plane between some two points of P'' has common points with P''. If $a\in D$, then $a\in D\cap P''$, because $a\in P''$

by (14). And if $a \in \mathscr{M} - D$, then $D = D_\eta$ for some $\eta < \Omega$, because in view of the inclusion $P'' \subset \mathscr{M}$ the set D disconnects also the Cantor fan $\mathscr{M}$. In this case $\eta < \alpha$ implies (3), whence we have by (14) the inequality $D \cap P'' \neq 0$ too.

Now we prove the second property of the set P'', that is that every subset B of $P'' - (a)$ which has dimension 1 in all points (save a nowhere dense subset which may be void) contains $2^{\aleph_0}$ quasicomponents of power $2^{\aleph_0}$.

We see that

$$(15) \qquad \{\tau\colon \tau \in \mathscr{C},\ L(\tau) \cap (\mathscr{I} \times z) \cap P'' \neq 0\} \subset V_\beta$$

for each $z \in W_\beta$. Indeed, supposing the contrary, we must have $\tau \in V_{\beta'}$, where $\beta \neq \beta'$, for some point $p \in L(\tau) \cap (\mathscr{I} \times z) \cap P''$, whence we infer by the relation $p \in P''$, (5), and (14), that $p \in L(\tau) \cap (\mathscr{I} \times W_{\beta'})$. Therefore it follows from $z \in W_\beta$ that $p \in L(\tau) \cap (\mathscr{I} \times W_{\beta'}) \cap (\mathscr{I} \times W_\beta)$ in spite of the equality $(\mathscr{I} \times W_{\beta'}) \cap (\mathscr{I} \times W_\beta) = \mathscr{I} \times W_{\beta'} \cap W_\beta = 0$ which follows from (6) by the supposition $\beta \neq \beta'$. Hence $\tau \in V_\beta$.

Note also that

(16) if Y is an open interval contained in the segment $0 \leqslant y \leqslant 1/2$ of y-axis, then there exists $2^{\aleph_0}$ sets W_β each of which contains $2^{\aleph_0}$ points of the interval Y.

Indeed, since Y is open, there exists a point $p \in P''$ the ordinate of which is less than any number of Y. Now, if $y_1 \in Y \cap R_\sigma$ for some $\sigma < \Omega$, then the segment $\mathscr{I} \times y_1$ disconnects the square $\mathscr{I}^2$ between the points $p \in P''$ and $a \in P''$, and therefore, P'' is connected as just proved, we have $(\mathscr{I} \times y_1) \cap P'' \neq 0$. It implies by definition (14) the existence of an index $\beta < \Omega$ such that $L(\tau) \cap (\mathscr{I} \times y_1) \cap P'' \neq 0$ for some $\tau \in V_\beta$, whence $y_1 \in W_\beta$ in view of (14) and (5). But the inequality $R_\sigma \cap W_\beta \neq 0$ implies, by (4), the equality $R_\sigma = W_\beta$. Since the sets R_σ are nowhere dense, every set W_β is also nowhere dense by (4) and then the Baire category theorem ([10], I, p. 320) ensures the existence of $2^{\aleph_0}$ sets W_β having common points with the interval Y. Each set R_σ is perfect, so by (4) each set $W_\beta \cap Y$ is of power $2^{\aleph_0}$. Thus we have proved (16).

Now let B be a subset of P'' such that $\dim_p B = 1$ for some $p \in B$. Then

(17) for any neighbourhood (in $\mathscr{M}$) U of p there exist $2^{\aleph_0}$ points $\tau \in \mathscr{C}$ such that $L(\tau) \cap B \cap U$ is of power $2^{\aleph_0}$.

In fact, let us choice a sequence $\{Y_n\}_{n=1,2,\ldots}$ of open intervals of y-axis and a sequence $\{\Delta\}_{n=1,2,\ldots}$ of closed-open subsets of the Cantor set $\mathscr{C}$ such that putting

$$U_n = (\mathscr{I} \times Y_n) \cap \bigcup_{\tau \in \Delta_n} L(\tau)$$

we have for $n = 1, 2, \ldots$

(18) $$p \in U_n \subset U,$$

(19) $$\delta(U_n) < 1/n.$$

If we suppose that (17) is not true, then by (18)

(20) there exists an at most countable set of points $\tau \in \mathscr{C}$ for which $L(\tau) \cap B \cap U_n$ is of power $2^{\aleph_0}$ $(n = 1, 2, \ldots)$.

The set $Y_n - (y_p)$, where y_p is the ordinate of p, is the union of open and disjoint intervals Y_n^1 and Y_n^2. And since to each W_β corresponds by (4) in a one-to-one manner the set $V_\beta \neq 0$, so for $n = 1, 2, \ldots$ and $i = 1, 2$ there exists by (20) a set $V_{\beta_n}^i$ such that $W_{\beta_n}^i \cap Y_n^i$ is of power $2^{\aleph_0}$ and such that for each $\tau \in V_{\beta_n}^i$ the set $L(\tau) \cap B \cap U_n$ is at most of power $\aleph_0$. It follows by the countability of the set $V_{\beta_n}^i$ that for some point $d_n^i \in W_{\beta_n}^i \cap Y_n^i$ the segment $\mathscr{I} \times d_n^i$ is disjoint with the set $\bigcup_{\tau \in V_{\beta_n}^i} L(\tau) \cap B \cap U_n$, whence $(\mathscr{I} \times d_n^i) \cap B \cap U_n = 0$ by (15). Choosing now for $n = 1, 2, \ldots$ two points $b_n^1, b_n^2 \in \mathscr{I} - \mathscr{C}$ such that $b_n^1 < \tau_0 < b_n^2$ and $\overline{b_n^1 b_n^2} \cap \mathscr{C} \subset \Delta_n$, where $\tau_0 \in \mathscr{C}$, $p \in L(\tau_0)$ and $\overline{b_n^1 b_n^2}$ is a segment of ends b_n^1 and b_n^2, we easily see that the quadrangle Q_n, vertices of which are common points of segments $\mathscr{I} \times d_n^i$ and $L(b_n^j)$ $(i, j = 1, 2)$, satisfies conditions $p \in Q_n \subset U_n$ and $\mathrm{Fr}(Q_n) \cap B \cap U_n = 0$ for every $n = 1, 2, \ldots$ Then $B \cap Q_n$ is a closed-open subset of B, contains p, and the diameter of $B \cap Q_n$ is, by (19), less than $1/n$ $(n = 1, 2, \ldots)$. Hence $\dim_p B = 0$, contrary to hypothesis.

Thus we have proved (17).

The spaces $\mathscr{M} - (a)$ and $(\mathscr{C} \times \mathscr{I}) - (\mathscr{C} \times e)$, where e is one of the ends of $\mathscr{I}$, are topologically equivalent, of course. Hence the second property of the set P'' formulated above is a simple consequence of (17) and Theorem 3.6.

EXAMPLE 4. *A pulverable and σ-connected set P_n of any finite dimension $n = 1, 2, \ldots$*

Let notation $\mathscr{C} = P \cup Q$, $L(\tau)$ and S mean the same as in paper [6] (see [6], example α, pp. 241-244). Namely, $P \subset \mathscr{C}$ is the set of end-points of intervals of $\mathscr{I} - \mathscr{C}$, $Q = \mathscr{C} - P$, $L(\tau)$ is a segment joining the point $a_1 = (1/2, 1/2)$ to the point $\tau \in \mathscr{I}$, and $S \subset \mathscr{M}$ is a pulverable set consisting of all points of $L(\tau)$ with rational ordinate if $\tau \in P$, and irrational ordinate if $\tau \in Q$.

The construction of P_n will be inductive and based upon the following property of S:

(1) for every countable set $B \subset Q$ the set $S \cap \bigcup_{\tau \in \mathscr{C} - B} L(\tau)$ is pulverable with dispersion point a_1 and at the same time this set is σ-connected.

The set $\mathscr{C}-B$ is $F-F_\sigma = G_\delta$. We may then apply to it the Baire category theorem. Therefore the proofs that $S \cap \bigcup_{\tau\in\mathscr{C}-B} L(\tau)$ is pulverable and σ-connected are quite similar to those that the set S is pulverable ([6], pp. 241-244) and σ-connected ([11], pp. 274-276); it is sufficient to substitute $\mathscr{C}-B$ for $\mathscr{C}$ and $Q-B$ for Q.

Now let B be a fixed countable subset of Q, dense in $\mathscr{C}$. Denote by $f_y(x)$ the scalariforme function which maps the Cantor set $\mathscr{C}_y$ lying on the segment with end-points $(y/2, y)$ and $\big((1-y)/2, y\big)$ onto this segment ([10], I, p. 236). Therefore the function $F(x, y)$ defined by the formula

$$F(x, y) = \big(f_y(x), y\big) \quad \text{for} \quad (x, y)\in\mathscr{M}$$

sticks segments $L(\tau_1)$ and $L(\tau_2)$ of Cantor fan $\mathscr{M}$ which have lower end points τ_1 and τ_2 belonging to $\mathscr{C}$ and bounding an interval contained in $\mathscr{I}-\mathscr{C}$. Hence $F(x, y)$ is a continuous function mapping the Cantor fan $\mathscr{M}$ onto the triangle of base $\mathscr{I}$ and vertex a_1. Let T^2 be this triangle. Notice the properties of the function $F(x, y)$ resulting from its definition:

(2) $F(x, y)$ is continuous on $\mathscr{M}$,

(3) $F(x, y)$ is a homeomorphism on each $L(\tau)$,

(4) $F[L(\tau)] = L[F(\tau, 0)]$ for each $\tau\in\mathscr{C}$,

(5) the set $F(B, 0)$ is dense in $\mathscr{I}$,

(6) if τ_1 and τ_2 are end-points of an interval of $\mathscr{I}$ complementary to the Cantor set $\mathscr{C}$, then

$$F[S\cap L(\tau_1)-(a_1)] = F[S\cap L(\tau_2)-(a_1)],$$

(7) $$F^{-1}[L(\tau)-(a_1)] = \begin{cases} [L(\tau_1)-(a_1)]\cup[L(\tau_2)-(a_1)] & \text{if} \quad \tau\in F(P, 0), \\ L(\tau')-(a_1) & \text{if} \quad \tau\in F(Q, 0), \end{cases}$$

where τ_1 and τ_2 are the end points of the same complementary interval of $\mathscr{C}$ to $\mathscr{I}$, and $\tau'\in Q$.

We show that the set

(8) $$P_1 = F[S\cap \bigcup_{\tau\in\mathscr{C}-B} L(\tau)]$$

has the following properties:

(9) P_1 is dense in the triangle T^2,

(10) P_1 is pulverable and a_1 is its dispersion point,

(11) P_1 is σ-connected,

(12) $\dim P_1 = 1$.

Indeed, $P_1 \subset \bigcup_{\tau\in\mathscr{C}-B} L[F(\tau, 0)]$ by (8) and (4), i.e.

(13) $$P_1 \subset \bigcup_{\tau\in\mathscr{I}-F(B,0)} L(\tau),$$

whence we have (9) by the countability of the set $F(B, 0)$.

In view of (1), (2), and (8), the set P_1 is connected ([10], II, p. 80). Thus to prove that it is pulverable it is sufficient, in view of the definition of pulverable set (see p. 3), to show that a_1 is the dispersion point of P_1, i.e. that the set $P_1 - (a_1)$ is dispersed. Applying succesively (8) and (7) we have

$$F^{-1}[P_1 \cap L(\tau) - (a_1)] = F^{-1}(P_1) \cap F^{-1}[L(\tau) - (a_1)]$$
$$= \begin{cases} [S \cap L(\tau_1) - (a_1)] \cup [S \cap L(\tau_2) - (a_1)] & \text{if} \quad \tau \in F(P, 0), \\ S \cap L(\tau') - (a_1) & \text{if} \quad \tau \in F(Q, 0), \end{cases}$$

whence by virtue of (6)

$$(14) \qquad P_1 \cap L(\tau) - (a_1) = \begin{cases} F[S \cap L(\tau_1) - (a_1)] & \text{if} \quad \tau \in F(P, 0), \\ F[S \cap L(\tau') - (a_1)] & \text{if} \quad \tau \in F(Q, 0), \end{cases}$$

where τ_1 and τ_2 are the end points of the same interval of $\mathscr{I} - \mathscr{C}$. Since both sets appearing in brackets in formula (14) are dispersed by Lemma 4.2 as quasicomponents of the pulverized set $S - (a_1)$, the set $P_1 \cap L(\tau) - (a_1)$, by (3) and (14), is also dispersed for each $\tau \in \mathscr{I} - F(B, 0)$.

We have $P_1 - (a_1) = \bigcup_{\tau \in \mathscr{I} - F(B,0)} P_1 \cap L(\tau) - (a_1)$ by (13). In order to prove (10) it remains to show that the set $P_1 - (a_1)$ is not connected between any two points $p \in P_1 \cap L(\tau) - (a_1)$ and $q \in P_1 \cap L(\tau') - (a_1)$, where $\tau < \tau'$. In fact, there exists by (5) a point $b \in F(B, 0)$ such that $\tau < b < \tau'$. Consequently, the straight line containing the segment $L(b)$ disconnects the plane between p and q, and by (13) it is disjoint with $P_1 - (a_1)$. Hence the property (10) is proved.

The property (11) follows from (1) and (2) ([11], T 1, p. 266).

The property (12) follows from two premises: firstly, the set P_1, as a connected one, has a positive dimension ([10], II, p. 80), and secondly, the set $\bigcup_{\tau \in \mathscr{I} - F(B,0)} L(\tau)$, which contains the set P_1 by (13), is at most 1-dimensional, as a boundary set in the triangle T^2 by (5) ([10], II, p. 353).

Thus we have proved all the properties (9)-(12) of P_1. Before defining now the sets P_n for $n > 1$ we construct some auxiliary sets in $\mathscr{E}^{n+1}$.

Let $a_n = (1/2, 1/2, \ldots, 1/2) \in E^{n+1}$. Denote by T_p^n, where $p \in \mathscr{I}$, the least convex set containing the base $p \times \mathscr{I}^{n-1}$ and the vertex a_n, and by T^{n+1} the least convex set containing the base $\mathscr{I}^n$ and the same vertex. So T_p^2 is the triangle in $\mathscr{E}^3$ with vertices $(p, 0, 0)$, $(p, 1, 0)$, and $a_2 = (1/2, 1/2, 1/2)$, and T^3 is a pentahedron, also in $\mathscr{E}^3$, of base $\mathscr{I}^2$ and vertex a_2.

Obviously, we have for each $n = 1, 2, \ldots$

$$(15) \qquad T^{n+1} = \bigcup_{p \in \mathscr{I}} T_p^n,$$

and for each $p \in \mathcal{I}$

(16)
there exists a homeomorphism g_p^n of T^n onto T_p^n such that $g_p^n(a_{n-1}) = a_n$.

Let W be the set of rational points of the segment $\mathcal{I}$ and let

$$Z^n = \bigcup_{p \in \mathcal{I} - W} T_p^n. \tag{17}$$

We show that

(18) $Z^n - (a_n)$ is not connected between any pair of its subsets $T_p^m - (a_n)$ and $T_q^m - (a_n)$, where $p < q$.

In fact, for each $s \in W$ such that $p < s < q$, the hyperplane containing $s \times \mathcal{I}^{n-1}$ and a_n disconnects $\mathcal{E}^{n+1}$ between the sets $T_p^m - (a_n)$ and $T_q^m - (a_n)$ and is disjoint with $Z^n - (a_n)$.

It is obvious that the set Z is a boundary set in $\mathcal{E}^{n+1}$ and hence ([10], II, p. 353) that

$$\dim Z^n < n+1. \tag{19}$$

Let V be a dense and countable subset of $\mathcal{I} - W$. It follows from the density of V and from (15) that

$$\overline{\bigcup_{p \in V} T_p^m} = T^{m+1}. \tag{20}$$

Now let $S(C)$ and M mean the same as in paper [9], namely let $S(C) = \bigcup_{\tau \in \mathscr{C}} S$, where $S(\tau)$ is an n-dimensional sphere, the diameter of which is the segment of x_1-axis with end points $(\tau, 0, \ldots, 0)$ and $p = (3, 0, \ldots, 0)$, and let M be a dense subset of $S(C)$ having in each point dimension 1 and meeting each sphere $S(\tau)$ in exactly one point. Now applying the results of paper [9] we show that

(21) the set $\bigcup_{p \in \mathcal{I} - (W \cup V)} T_p^m$ contains a subset A_n such that $\dim A_n = n$ and that the intersection of $A_n \cap T_p^m$ is for each $p \in \mathcal{I} - (W \cup V)$ either void or a single point.

Indeed, in view of Alexandroff-Hausdorff theorem ([10], I, p. 335) the set $\mathcal{I} - (W \cup V)$ contains a Cantor set C^*. Let us embed by homeomorphism g the set $\bigcup_{\tau \in C^*} T_\tau^n$ into the bundle of spheres $S(C)$ in such a way that $g(\bigcup_{\tau \in C^*} T_\tau^n)$ contains an open subset of $S(C)$. By virtue of formula (23) from paper [9] the set $A_n' = M \cap g(\bigcup_{\tau \in C^*} T_\tau^n)$ is non-void; since by the same paper the set M is n-dimensional in each of its points, it follows that $\dim g^{-1}(A_n') = n$. Evidently, homeomorphism g maps each T_τ^n, where $\tau \in C^*$, in only one sphere $S(\tau)$ of the bundle $S(C)$. Therefore, by (20) and (21) from [9], the set $A_n' \cap g(T_\tau^n)$ is void or a single point. To complete the proof of (21) it remains to denote $g^{-1}(A_n')$ by A_n.

Now proceed by induction. Assume that the set P_{n-1} has the following properties:

(22) P_{n-1} is dense in T^n,

(23) P_{n-1} is pulverable and a_{n-1} is its dispersion point,

(24) P_{n-1} is σ-connected,

(25) $\dim P_{n-1} = n-1$.

These properties are true for $n = 2$ in virtue of (9)-(12).

Defining the set P_n by the formula

(26) $$P_n = \bigcup_{p\in V} g_p^n(P_{n-1}) \cup A_n,$$

we shall show that it has the properties (22)-(25) for n instead of $n-1$.

For this purpose first observe that, by (22) and (16), $\bigcup_{p\in V} g_p^n(P_{n-1}) \subset \bigcup_{p\in V} T_p^n$, and, by (21), $A_n \subset \bigcup_{p\in \mathscr{I}-(W\cup V)} T_p^n$, whence by (26) and (17) we infer that

(27) $$P_n \subset Z^n.$$

In virtue of (22) and (16)

(28) the set $g_p^n(P_{n-1})$ is dense in T_p^n,

whence, by (20),

(29) the set $\bigcup_{p\in V} g_p^n(P_{n-1})$ is dense in T^{n+1}.

It follows from (21) and (15) that

(30) $$A_n \subset T^{n+1}.$$

Furthermore,

(31) the set $P_n-(a_n)$ is dispersed.

For if $p\in V$, then $P_n \cap T_p^n-(a_n)$, by (26), (21), and (16), is identical with $g_p^n(P_{n-1})-(a_{n-1})$. But the latter is dispersed being an image under homeomorphism g_p^n of the set $P_{n-1}-(a_{n-1})$, which is dispersed by (23). And if $p\in\mathscr{I}-(W\cup V)$, then by (21), (16) and (26) the set $P_n\cap T_p^n-(a_n)$ is void or consists of a single point. It follows by (27) and (18) that $P_n-(a_n)$ does not contain connected subsets; that is (31).

Finally

(32) the set P_n is connected.

In fact, each two connected sets $g_p^n(P_{n-1})$ and $g_q^n(P_{n-1})$ have by (16) a common point a_n for $p, q\in V$, and so the set $\bigcup_{p\in V} g_p^n(P_{n-1})$ is connected ([10], II, p. 82). Hence and from (29), (30) and (26) follows (32) ([10], II, p. 83).

Now we can prove the properties (22)-(25) of the set P_n.

Property (22) follows from (29), (30), and (26).

Property (23) follows from (31) and (32) by the definition of pulverable set (see p. 3).

To prove property (24) suppose that there exists a decomposition $P_n = \bigcup_{k=1}^{\infty} F_k$ into sets F_k closed in P_n, non-void and disjoint. Since for each $p \in V$, $g_p^n(P_{n-1})$ is connected and σ-connected, by (24) and (16), it must lie in one costituent of this decomposition. But all these sets have a common point a_n and therefore they must be contained in the same constituent of decomposition, for instance in F_1. And since F_1 is closed, $P_n = F_1$ by (29), (30), and (26), whence $F_2 = F_3 = \ldots = 0$, contrary to supposition.

Finally, we infer from (26) and (27) that $A_n \subset P_n \subset Z^n$, whence $n \leqslant \dim P_n < n+1$ by (21) and (19).

Example 5. *A pulverable and non σ-connected set P'_n of any finite dimension $n = 1, 2, \ldots$*

Let $\mathscr{C} = (0) \cup \bigcup_{k=1}^{\infty} C_k$ be a decomposition of the Cantor set $\mathscr{C}$, where C_k is a subset of $\mathscr{C}$ lying in the segment $2/3^k \leqslant x \leqslant 1/3^{k-1}$. Denote by $J(a, \beta)$ an interval $a < y < \beta$ of y-axis, and by F_0 the segment $0 \leqslant x \leqslant 1$ of the straight line $y = 1$.

Putting, generally, $\mathrm{Int}_X(A) = X - \overline{X - A}$ for $A \subset X$, we define an operation γ on the set $A_0 = \mathscr{C} \times \mathrm{Int}_{\mathscr{I}}(\mathscr{I})$. Namely, let for $m = 1, 2, \ldots$

$$C_m^1 = C_m \times \overline{J\left(\frac{1}{2^{m+1}}, \frac{2^{m+1}-1}{2^{m+1}}\right)},$$

$$D_{2m-1}^1 = C_m \times J\left(\frac{2^{m+1}-1}{2^{m+1}}, 1\right),$$

$$D_{2m}^1 = C_m \times J\left(0, \frac{1}{2^{m+1}}\right),$$

$$\gamma(A_0) = \bigcup_{m=1}^{\infty} C_m^1 \cup \bigcup_{m=1}^{\infty} D_m^1$$

(see Fig. 6). We have then between A_0 and each D_m^1 a geometrical similarity without a change of the order of the coordinates of points.

Put $A_1 = \gamma(A_0)$. Let us perform the operation γ on each of the elements D_m^1 of the set A_1 (see Fig. 7), and put

$$A_2 = \bigcup_{m=1}^{\infty} C_m^1 \cup \bigcup_{m=1}^{\infty} \gamma(D_m^1).$$

Ordering all first elements of the sets $\gamma(D_m^1)$ in a simple sequence without repeat $\{C_m^2\}_{m=1,2,\ldots}$, and the second ones in a simple sequence without repeat $\{D_m^2\}_{m=1,2,\ldots}$, we obtain

$$A_2 = \bigcup_{m=1}^{\infty} C_m^1 \cup \bigcup_{m=1}^{\infty} C_m^2 \cup \bigcup_{m=1}^{\infty} D_m^2.$$

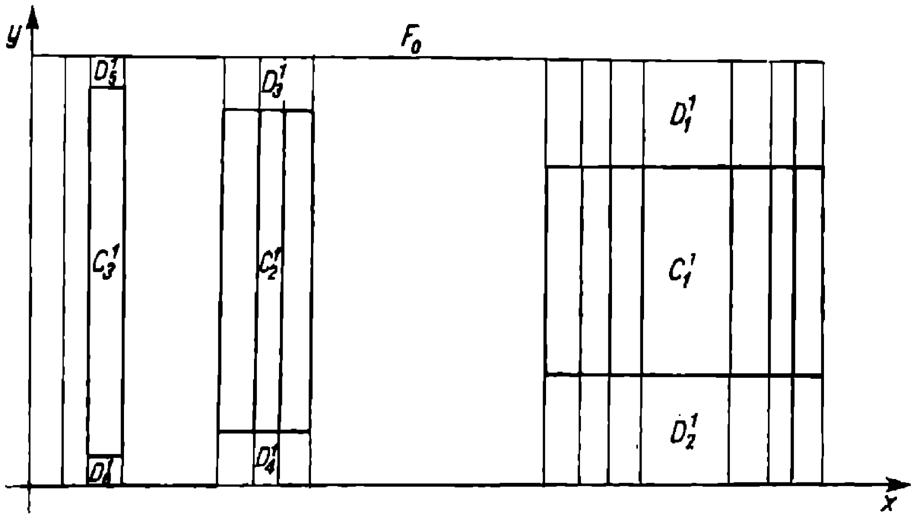

Fig. 6

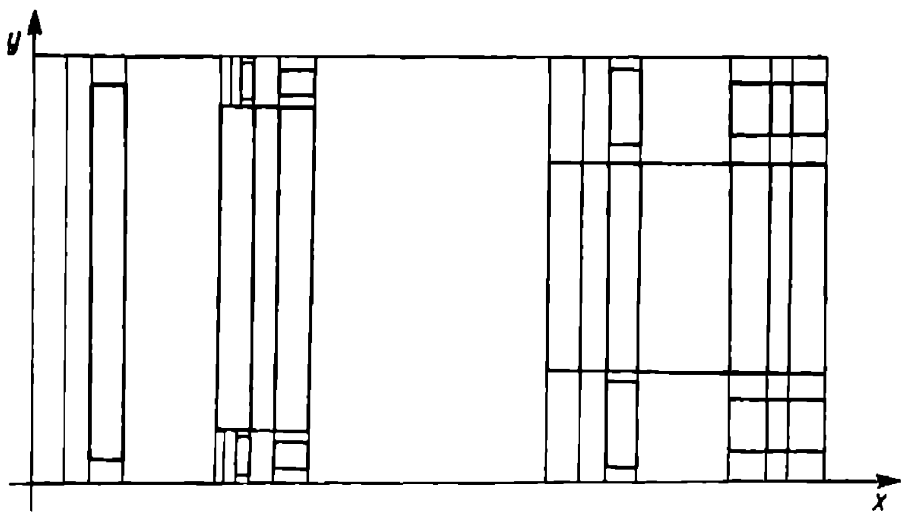

Fig. 7

Between A_0 and each of the sets D_m^2 we have again a geometrical similarity preserving the order of the ordinates. Therefore, we can perform on each of them the operation γ in the manner described above. Generally, put for $i = 1, 2, \ldots$

$$A_i = \bigcup_{j=1}^{i-1} \bigcup_{m=1}^{\infty} C_m^j \cup \bigcup_{m=1}^{\infty} \gamma(D_m^{i-1}) = \bigcup_{j=1}^{i} \bigcup_{m=1}^{\infty} C_m^j \cup \bigcup_{m=1}^{\infty} D_m^i.$$

So we have defined a sequence of sequences $\{\{C^j_m\}_{m=1,2,\ldots}\}_{j=1,2,\ldots}$ of sets C^j_m (disjoint as well for different j as for different m), each of which is homeomorphic with the set $\mathscr{C}\times\mathscr{I}$ (being similar to it) and disjoint with F_0. Now order all them including F_0 in a simple sequence without repeat $\{F_k\}_{k=0,1,2,\ldots}$. Thus

(1) $$\bigcup_{k=1}^{\infty} F_k \subset \mathscr{C}\times\mathscr{I},$$

(2) $$F_k \cap F_l = 0 \quad \text{for} \quad k \neq l \text{ and } k, l = 0, 1, 2, \ldots$$

Put

(3) $$F = \bigcup_{k=0}^{\infty} F_k,$$

(4) $$G_k = \mathrm{Int}_{\mathscr{C}\times\mathscr{I}}(F_k) \quad \text{for} \quad k = 1, 2, \ldots$$

Evidently, each G_k is homeomorphic with $\mathscr{C}\times\mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})$ (by similarity), and

(5) each segment $\tau\times\mathscr{I}$, where $\tau\in\mathscr{C}$, contains at most one component of G_k for $k = 1, 2, \ldots$

Denoting by W_k the set consisting of 4 vertices of the least rectangle containing F_k, we have

(6) $$W_k \subset F_k \quad \text{for} \quad k = 1, 2, \ldots$$

Now observe that

(7) if a set D disconnects $\mathscr{E}^2$ between some points of the set $(\mathscr{C}\times\mathscr{I})\cup F_0$ and if $D\cap F_0 = 0$, then there exists an open subset Γ of the Cantor set $\mathscr{C}$ such that D disconnects $\tau\times\mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})$ for each $\tau\in\Gamma$.

Furthermore,

(8) if τ is the right end point of an interval of $\mathscr{I}$ complementary to the Cantor set $\mathscr{C}$, and $p\in[\tau\times\mathrm{Int}_{\mathscr{E}^1}\mathscr{I})]-\bigcup_{k=1}^{\infty} W_k$, then there exists an open subset V of $\tau\times\mathscr{I}$ such that $p\in V\subset \underline{\lim}_{l\to\infty} G_{k_l}$ for a sequence of sets $\{G_{k_l}\}_{l=1,2,\ldots}$.

In fact, let V be the component of the set $\mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})-\bigcup_{k=1}^{\infty} W_k$ containing the point p; V is then an interval. Two cases are possible.

I. V is the left extreme component of D^j_m for some natural j and m. The set F contains by definition a sequence of sets $\{F_{k_l}\}_{l=1,2,\ldots}$ resulting from the operation γ on the set D^j_m; we have then $\overline{V} = \lim_{l\to\infty} G_{k_l}$ ([10], I, p. 245), whence obviously $V\subset \underline{\mathrm{Lim}}_{l\to\infty} G_{k_l}$.

II. V is the left extreme component of a F_k. Then, accordingly to (4), we have $V \subset G_k = \operatorname*{Lim}_{l\to\infty} G_{k_l}$, where $G_{k_l} = G_k$ for $l = 1, 2, \ldots$

The construction of P'_n will be based upon the following four properties ((9)-(12)) of the set F:

(9) The set F is connected.

In other words, if a set D disconnects the plane $\mathscr{E}^2$ between some two points of F, then $D \cap F \neq 0$. If $D \cap (F_0 \cup \bigcup_{k=1}^{\infty} W_k) \neq 0$ this inequality follows at once from (6) and (3). And if $D \cap (F_0 \cup \bigcup_{k=1}^{\infty} W_k) = 0$, then D disconnects $F \cap [\tau \times \mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})]$ by (7) together with its neighbourhood in $\mathscr{C} \times \mathscr{I}$ for some τ being the right end point of an interval of the set $\mathscr{I} - \mathscr{C}$. We infer from $D \cap \bigcup_{k=1}^{\infty} W_k = 0$ that $D \cap [\tau \times \mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})] \subset [\tau \times \mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})] - \bigcup_{k=1}^{\infty} W_k$. Therefore, for each $p \in D \cap [\tau \times \mathrm{Int}_{\mathscr{E}^1}(\mathscr{I})]$, there exists by (8) an open subset V of $\tau \times \mathscr{I}$ such that $p \in V \subset \operatorname*{Lim}_{l\to\infty} G_{k_l}$ for a sequence $\{G_{k_l}\}_{l=1,2,\ldots}$. There must then exist an index k_l such that $D \cap G_{k_l} \neq 0$, whence $D \cap F \neq 0$ by (4) and (3).

Evidently, the set $\bigcup_{k=1}^{\infty} W_k$ is countable, whence by (6) and (9) follows at once the second property:

(10) The set $\bigcup_{k=1}^{\infty} W_k$ is a boundary set in F.

From (9) and (10) follows the third property:

(11) If $F - \bigcup_{k=1}^{\infty} W_k = X \cup Y$, $X \neq 0 \neq Y$ and $X \cap \overline{Y} \cup \overline{X} \cap Y = 0$, then there exists a point $w \in \bigcup_{k=1}^{\infty} W_k \cap \overline{X} \cap \overline{Y}$.

Since each F_k, where $k = 0, 1, 2, \ldots$, is non-void and closed in F (being compact), we have by (2) the fourth and last property:

(12) The decomposition (3) is a σ-decomposition of F.

Thus the set F is not σ-connected.

Now put $F^1 = F$, and for $n = 2, 3, \ldots$

(13) $$F_k^n = F_k \times \mathscr{I}^{n-1}, \quad \text{where} \quad k = 0, 1, 2, \ldots,$$

(14) $$G_k^n = G_k \times \mathrm{Int}_{\mathscr{E}^{n-1}}(\mathscr{I}^{n-1}), \quad \text{where} \quad k = 1, 2, \ldots$$

It follows then by (4) that

(15) $$G_k^n \subset F_k^n \quad \text{for} \quad k = 1, 2, \ldots,$$

and by (5) that

(16) each set $\tau\times\mathscr{I}^n$, where $\tau\in\mathscr{C}$, contains at most one component of G_k^n for $k=1,2,\ldots$

We define now the set F^n putting

$$F^n=\bigcup_{k=0}^{\infty}F_k^n. \tag{17}$$

This definition is analogous to the definition (3) of the set F. Accordingly, properties (1) and (7)-(12) hold for F^n and $\mathscr{I}^n$, respectively, as they do for F and $\mathscr{I}$. Namely

$$\bigcup_{k=1}^{\infty}F_k^n\subset\mathscr{C}\times\mathscr{I}^n. \tag{18}$$

(19) If a set D disconnects $\mathscr{E}^{n+1}$ between some points of $(\mathscr{C}\times\mathscr{I}^n)\cup F_0^n$ and if $D\cap F_0^n=0$, then there exists an open subset Γ of the Cantor set $\mathscr{C}$ such that D disconnects $\tau\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ for each $\tau\in\Gamma$.

(20) If τ is the right end point of an interval of $\mathscr{I}$ complementary to the Cantor set $\mathscr{C}$, and $p\in[\tau\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)]-\bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1})$, then there exists an open subset V of $\tau\times\mathscr{I}^n$ such that $p\in V\subset\operatorname{Lim}_{l\to\infty}G_{k_l}^n$ for some sequence of sets $\{G_{k_l}^n\}_{l=1,2,\ldots}$.

(21) The set F^n is connected.

(22) The set $\bigcup_{k=1}^{\infty}W_k\times\mathscr{I}^{n-1}$ is a boundary set in F^n.

(23) If $F^n-\bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1})=X\cup Y$, $X\neq 0\neq Y$, $X\cap\bar{Y}\cup\bar{X}\cap Y=0$, then there exists a point $w\in\bigcup_{k=1}^{\infty}W_k$ such that $w\times\mathscr{I}^{n-1}\subset\bar{X}\cap\bar{Y}$.

(24) The decomposition (17) is a σ-decomposition of F^n.

Thus the set F^n is not σ-connected.

We say that a set $N\subset\mathscr{C}\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ *divides locally* the space $\mathscr{E}^{n+1}$ (see, for instance, "local coupure" in Knaster's paper [9]) provided that for every continuum $D\subset\mathscr{E}^{n+1}$ the condition

(i) the set $\{\tau\colon\tau\in\mathscr{C},\ D\cap[\tau\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)]\neq 0\}$ contains an open subset of the Cantor set $\mathscr{C}$

implies the inequality

$$D\cap N\neq 0. \tag{ii}$$

We show that

(25) the set $\mathscr{C}\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ contains a set N dividing locally the space $\mathscr{E}^{n+1}$ and meeting every component $\tau\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ of $\mathscr{C}\times\mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ exactly in one point.

Using, as in Example 4, the notations $S(\mathscr{C})$, M, and p, in the same sense as in [9], recall the proposition proved there that the set $S(\mathscr{C})-(p)$ contains a subset M dividing locally the space $\mathscr{E}^{n+1}$ and having exactly one common point with every component $S(\tau)-(p)$ of $S(\mathscr{C})-(p)$. Since there exists a homeomorphism g between $\mathscr{C}^{n+1}-(p)$ and an open subset containing $\mathscr{C}\times \mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ and mapping $S(\tau)-(p)$ onto $\tau\times \mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$, the set $N = g(M)$ has the property (25).

The similarity of the sets $\mathscr{C}\times \mathrm{Int}_{\mathscr{E}^n}(\mathscr{I}^n)$ and G_k^n can be extended onto $\mathscr{E}^{n+1}$; so (25) implies that

(26) each G_k^n contains N_k dividing locally $\mathscr{E}^{n+1}$ and having exactly one common point with each component of G_k^n.

Now we define the set P_n'

$$(27)\qquad P_n' = \varphi_{F_0^n}\Big(F_0^n \cup \bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k\Big).$$

First of all observe that $\bigcup_{k=1}^{\infty} W_k \subset \bigcup_{k=1}^{\infty} F_k^n$ by (6) and (23), and that $\bigcup_{k=1}^{\infty} N_k \subset \bigcup_{k=1}^{\infty} F_k^n$ by (26) and (15). Thus

$$(28)\qquad \bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k \subset \bigcup_{k=1}^{\infty} F_k^n,$$

whence by (24)

$$(29)\qquad \Big(\bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k\Big) - F_0^n = \bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k.$$

Furthermore,

(30) if the set $F^n - D \cap \bigcup_{k=1}^{\infty} (W_k\times \mathscr{I}^{n-1})$ is not connected and $D = \bar{D}$, then there exists a point $w \in \bigcup_{k=1}^{\infty} W_k$ such that $w\times \mathscr{I}^{n-1} \subset D$.

In fact, let $F^n - D \cap \bigcup_{k=1}^{\infty} (W_k\times \mathscr{I}^{n-1}) = X\cup Y$, where $X\neq 0 \neq Y$ and

$$(31)\qquad X\cap \bar{Y} \cup \bar{X}\cap Y = 0.$$

Therefore we have $\bar{X}\cap\bar{Y}\subset D$ and

$$F^n - \bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1}) = \Big[X - \bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1})\Big]\cap\Big[Y - \bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1})\Big].$$

Since D is closed by hypothesis, and since the sets X and Y are open in F^n by (31), $X - \bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1}) \neq 0$ and $Y - \bigcup_{k=1}^{\infty}(W_k\times\mathscr{I}^{n-1}) \neq 0$ by

(22). By (31) these sets are separated. It follows that in virtue of (23) there exists a point $w \in \bigcup_{k=1}^{\infty} W_k$ such that

$$w \times \mathscr{I}^{n-1} \subset \overline{X - \bigcup_{k=1}^{\infty} (W_k \times \mathscr{I}^{n-1})} \cap \overline{Y - \bigcup_{k=1}^{\infty} (W_k \times \mathscr{I}^{n-1})},$$

whence $w \times \mathscr{I}^{n-1} \subset \overline{X} \cap \overline{Y} \subset D$.

Finally, observe that

(32) the set $\bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k$ is dispersed.

Indeed, we have $\bigcup_{k=1}^{\infty} F_k^n \subset \mathscr{C} \times \mathscr{I}^n$ by (13) and (1), whence $\bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k \subset \mathscr{C} \times \mathscr{I}^n$ by (28). Then the set $\bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k$ is not connected between the points lying in two different components of the compact set $\mathscr{C} \times \mathscr{I}^n$, and each component $\tau \times \mathscr{I}^n$ of this compact set contains at most a countable subset of the set $\bigcup_{k=1}^{\infty} W_k$, because the last set is countable, and at most a countable subset of the set $\bigcup_{k=1}^{\infty} N_k$, because $\tau \in \mathscr{I}^n$ contains at most one point of each G_k^n in view of (16) and (26). Therefore, the set $(\tau \times \mathscr{I}^n) \cap (\bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k)$ is at most countable and *a fortiori* dispersed.

Putting

(33) $$L = F_0^n \cup \bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k$$

we have $L \subset \bigcup_{k=0}^{\infty} F_k^n$ by (28), whence we infer by (18) that

(34) $$L \subset (\mathscr{C} \times \mathscr{I}^n) \cup F_0^n.$$

(35) The set L is connected.

In other words, if a set D disconnects the space $\mathscr{E}^{n+1}$ between some points of L, then

(36) $$D \cap L \neq 0.$$

Indeed, we may assume D is closed ([10], II, p. 97) and bounded, hence a continuum ([2], p. 343). If $D \cap (F_0^n \cup \bigcup_{k=1}^{\infty} W_k) \neq 0$, then (36) follows directly from (33). And if

(37) $$D \cap (F_0^n \cup \bigcup_{k=1}^{\infty} W_k) = 0,$$

then by (30)

(38) the set $F^n - D \cap \bigcup_{k=1}^{\infty} (W_k \times \mathscr{I}^{n-1})$ is connected.

From (37) and (34) follows the existence of $\Gamma \subset \mathscr{C}$ fulfilling (19). Hence we see by (38) that there exists a point $\tau \in \Gamma$ being the right end point of an interval of $\mathscr{I}$ complementary to the Cantor set $\mathscr{C}$, such that D disconnects $\tau \times \mathscr{I}^n$ together with its neighbourhood in $\mathscr{C} \times \mathscr{I}^n$ and D does not meet $\bigcup_{k=1}^{\infty} (W_k \times \mathscr{I}^{n-1})$. Then for each point $p \in D \cap [(\tau \times \mathscr{I}^n) - \bigcup_{k=1}^{\infty} (W_k \times \mathscr{I}^{n-1})]$ there exists by (20) an open subset of $\tau \times \mathscr{I}^n$ containing p and contained in $\operatorname{Lim}_{l\to\infty} G^n_{k_l}$ for some sequence $\{G^n_{k_l}\}_{l=1,2,\ldots}$. Then there exists k_0 such that $D \cap (\tau \times \mathscr{I}^n) \cap G^n_{k_0} \neq 0$ for all τ running over an open subset $\Gamma_0 \subset \Gamma$ of the Cantor set $\mathscr{C}$, whence $N_{k_0} \cap D \neq 0$ by (26). Hence and from (33) we obtain inequality (36).

(39) The set L is not σ-connected, and F^n_0 is one of the constituents of its σ-decomposition.

In fact, we have $L \subset F^n$ by (33), (28), and (17), and L has by (6) and (24) common points with each element of the σ-decomposition (17) of F^n. Then $\bigcup_{k=0}^{\infty} (L \cap F^n_k)$ is a σ-decomposition of L. Finally, $F^n_0 \subset L$ by formula (33).

Now we can prove the properties of the set P'_n.

The set P'_n is pulverable. Indeed, it is connected in view of (33), (27), and (20) of § 2 ([10], II, p. 80), and the set $\varphi_{F^n_0}(\bigcup_{k=1}^{\infty} W_k \cup \bigcup_{k=1}^{\infty} N_k)$ is dispersed by (32), (29), and (21) of § 2. Since by virtue of Lemma 2.15, (29), and (27), the last set differs from the connected set P'_n for a point $\varphi_{F^n_0}(F^n_0)$, this point is a dispersion point of P'_n. Hence P'_n is pulverable by the definition of pulverable set (see p. 3).

The set P'_n is not σ-connected and its dispersion point is one of the constituents of its σ-decomposition, because the identification $\varphi_{F^n_0}$ of F^0_n, which is by (39) one of the constituents of the σ-decomposition of L, maps this σ-decomposition, by (27) and (33), onto the σ-decomposition of P'_n. Hence the set P'_n is not σ-connected and the point $\varphi_{F^n_0}(F^n_0)$ is one of the constituents of its σ-decomposition.

Finally, the set P'_n is n-dimensional. Indeed, the set N_k divides locally the space $\mathscr{E}^{n+1}$; thus N_k is at least n-dimensional (Mazurkiewicz theorem; see [10], II, p. 343). The set $(\mathscr{C} \times \mathscr{I}^n) \cup F^n_0$, as containing no solid $(n+1)$-dimensional sphere, is a boundary set in $\mathscr{E}^{n+1}$ and therefore at most

n-dimensional ([10], II, p. 353). Since $\bigcup\limits_{k=1}^{\infty} N_k \subset Z \subset (\mathscr{C}\times\mathscr{I}^n)\cup F_0^n$, by (33) and (34), then

$$n \leqslant \dim\,(\bigcup_{k=1}^{\infty} N_k) \leqslant \dim Z \leqslant \dim\,[(\mathscr{C}\times\mathscr{I}^n)\cup F_0^n] \leqslant n,$$

i.e. $\dim\,(\bigcup\limits_{k=1}^{\infty} N_k) = \dim Z = n$, whence also $\dim P_n' = n$, because, by (21) of § 2 and (29), the identification neither increases nor lowers dimension.

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