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Generalization of stable preference ordering towards ideal solution approach for working with imprecise data

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Abstract

When solving real-world decision-making problems, it is important to deal with imprecise quantitative values modeled by numerical intervals. Although a different extension of the multi-criteria decision-making methods could deal with intervals, many of them are complex and lack such properties as robustness to rank reversal. We present an extension of the stable preference ordering towards ideal solution (SPOTIS) rank reversal free method to deal with imprecise data. This extension of SPOTIS is also rank reversal-free. It offers a new efficient approach for solving multi-criteria decision-analysis problems under imprecision and can use different metrics of distance between intervals. The proposed approach is compared to the popular interval technique for order preference by similarity to ideal solution) extension and performs very similarly to it. We also show on a practical example that the interval TOPSIS approach is not robust to rank reversal, contrary to our new SPOTIS extension approach, which offers a stable decision-making behavior.

Keywords: *interval values, MCDA, SPOTIS, decision-making*

1. Introduction

Interval data could appear in every part of the decision-making problem. Intervals could appear in the data due to imprecise measurements, the uncertain nature of the data, missed data, and others. In this case, the decision maker should handle the imprecision of the data to solve the problem. By eliminating imprecision, the decision maker loses important information about alternatives. Therefore, to properly deal with imprecise data, the interval extensions of the multi-criteria decision-analysis (MCDA) methods should be used.

For solving multi-criteria decision-making problems in a crisp environment, one could use a wide range of different methods, such as TOPSIS (technique fororder preference by similarity to ideal solu-

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tion), VIKOR (visekriterijumska optimizacija i kompromisno resenje, in Serbian), ELECTRE (eLimination and choice translating reality), and other [47]. Some of those methods are more popular in the domain and have different extensions over time, such as extensions for the interval or fuzzy calculations. However, with all those existing methods, it is hard to pick one that is most suitable for the actual problem and will provide the most reliable solution. The different methods provide results that could differ significantly [36]. Another problem is a rank reversal (RR) paradox, which appears when the number of alternatives in a set changes. Most of the popular MCDA methods are susceptible to it.

In 2020, Dezert et al. proposed a stable preference ordering towards ideal solution (SPOTIS) approach, which has several advantages over the classical MCDA method. It is easy to use and robust to RR phenomena by design [7]. Although the SPOTIS method is relatively new, the usability and reliability of this method have already been proved in different scientific works [40, 48]. In 2021, Shekhovtsov et al. proposed an approach that allows the use of the SPOTIS method with interval data. However, this approach has several flaws and cannot easily process interval criteria weights and interval data bounds. In this paper, we introduce a new generalized approach to dealing with interval data with the SPOTIS method. The proposed approach could be used with interval criteria weights and data bounds in order to provide robust ranking in a full imprecise framework modeled by numerical intervals.

The main contribution is a new generalization of the SPOTIS method for working with imprecise data. The approach proposed in this paper is completely resistant to RR phenomena, builds a full model of the decision problem, and allows the use of different functions to determine the distance between interval values. If the distance function works as a generalization on degenerated intervals, the method will coincide with the original SPOTIS method. This work is an extension of the initial research presented in [38]. This new approach is more general because it allows using $\mathbb{IR}$ intervals in each part of a decision problem, for example, in a decision matrix, in criteria bounds, and in criteria weights. It significantly improves the previously proposed extension, which can process only the interval decision matrix, not the weights or criteria bounds. To prove the superiority of the proposed approach, we compare it with the interval TOPSIS (technique for order preference by similarity ideal solution) method [15].

The remainder of the paper is organized as follows: Section 2 contains the literature review on recent trends in interval data MCDA methods and a short review of their practical applications. Section 3 describes how the interval arithmetic works and introduces all functions required to understand the proposed approach. Section 4 introduces three variations of the proposed interval SPOTIS method and contains numerical examples that show how the calculations are performed. Section 5 contains the study case on the data from [15]. In this section, our new proposed method is compared to the interval TOPSIS approach. In Section 5.1, we prove on a practical example that interval SPOTIS resisted rank reversal (RR) phenomena and interval TOPSIS does not resist RR. We also show the expanded example in Section 5.2. This example demonstrates how the proposed method is performed in the case where interval values are involved in every part of the decision problem. Section 6 contains a discussion of the results, and Section 7 contains conclusions and directions for future works.

2. Literature review

Different researchers propose extensions of the popular MCDA methods to deal with imprecise values, such as interval values. Table 1 contains references for a literature review of recent trends in interval-valued extensions of MCDA methods. One of the first interval methods was the TOPSIS extension proposed by [15]. The I-TOPSIS method proposed by Jahanshahloo et al. allows the use of an interval decision matrix with crisp criteria weights and obtains crisp results, which are easy to compare.

Table 1. Extensions of the MCDA methods for imprecise data

Acronym	Extended form	Decision matrix	Weights	Results	Reference
I-TOPSIS	Interval technique for order preference by similarity to ideal solution	✓	×	×	[15]
COPRAS	Complex proportional assessment	✓	×	×	[51]
VIKOR	Visekriterijumska optimizacija i kompromisno resenje (in Serbian)	✓	×	✓	[37]
ELECTRE	Elimination and choice translating reality	✓	✓	–	[44]
MOORA	multi-objective optimization on the basis of ratio analysis	✓	×	×	[42]
DI-TOPSIS	Direct interval technique for order preference by Ssimilarity to ideal solution	✓	×	×	[9]
EDAS	Evaluation based on distance from average solution	✓	×	×	[43]
COMET	Characteristic objects method	✓	–	✓	[34]
ELECTRE--IDAT	Elimination and choice translating reality -interval data and target-based	✓	×	×	[14]
CODAS	Combinative distance-based assessment	✓	×	×	[25]
MARCOS	Measurement of alternatives and ranking according to compromise solution	✓	×	✓	[23]

Checkmark represented as an interval, × as crisp numbers, – as other (details in the text)

Another extension of the TOPSIS method, called DI-TOPSIS, was also proposed by [9]. In 2008, Zavadskas et al. proposed a COPRAS-G extension of the COPRAS method to select effective dwelling house walls under an interval environment [51]. In 2009, Sayadi et al. proposed an extension of the VIKOR method for interval values [37]. This approach is interesting because the final preferences of the alternatives are presented as intervals. However, the weights were crisp values. Another extension of the MCDA method in the interval environment is the extension of the ELECTRE method proposed by Vahdani et al. in 2010. This extension allows interval weights, however the results are presented as an outranking matrix, and building a full ranking could be a problem here [44]. There are also extensions for such methods as MOORA proposed by [42], and EDAS which is proposed by [43]. Both extensions allow for interval values only in the decision matrix and return crisp preference values. Both papers use a numerical example from [52]. Another interesting method that has an interval extension is a COMET [34]. Because of how the COMET method works, the COMET's extension does not use any weights, and the final preferences are represented as intervals. In 2019, Jahan et al. presented another extension for the ELECTRE method called the ELECTRE-IDAT method [14]. This extension allows target-based decision-making under an interval environment. Unlike the previous ELECTRE extension, this extension results in crisp preference values, which are easier to rank, but do not allow the use of interval weights. The most recent works on extending the MCDA method to deal with intervals were done by Mathew et al. and Liu. Matthew and Thomas[25] extend the CODAS method to address intervals to select the best flexible manufacturing system among available alternatives. CODAS extension does not accept interval weights and returns crisp preference values. In 2023, Liu proposed an extension for the MARCOS method to select the best renewable desalination. The resulting preference values are presented as intervals, therefore an additional method should be used to rank them.

Interval extensions of multi-criteria decision-making are often used in different fields. For example, [10] used the PIVN-AHP (parametric form of interval number analytic hierarchy process) to evaluate criteria weight for the problem of evaluating performance for the requirement of a school teacher. The AHP interval was also used to evaluate criteria weights in [21]. Another extension that is used very often is Jahanshahloo's interval TOPSIS. For example, in [27], the authors used interval TOPSIS to select doffing tube components for rotor-spun yarn weft knitting fabrics. Interval TOPSIS, DI-TOPSIS, and VIKOR methods were used [2] to select the material family for capacitor applications. COPRAS-G method was used to evaluate investment projects under interval data [33]. The interval extensions of the MCDA method are also useful in group decision-making problems. For example, [50], and [1] used Jahanshahloo's interval TOPSIS to solve supplier selection problems.

Besides that, there are many other works that investigate the uncertainty or imprecise data in multi-criteria decision making. They often use very advanced mathematical algorithms and, therefore, are very difficult to understand and use in many cases. For example, in 2022, Diao et al. showed how to use [8] spherical fuzzy sets for group decision making. Dembczyński et al. demonstrated the usefulness of rough sets for decision-making under imprecise data [4]. There are also other researchers who have proposed their own methodologies to handle uncertainty and imprecision in the data to make robust and effective decisions [22, 26, 49].

Although different proposed extensions could operate on imprecise data (i.e., numerical intervals), there is no simple but robust method that will be resistant to the RR paradox and deal with interval values in complex decision-making problems. The interval extension of the SPOTIS proposed in this paper fills this gap because of its universality, simplicity, and robustness to rank reversal.

3. Preliminaries

3.1. Definition of the interval

A closed interval $\mathbf{x}$ is defined as

$$\mathbf{x} = [\underline{x}, \bar{x}] = \{x | \underline{x} \leq x \leq \bar{x} \land \underline{x}, \bar{x}, x \in \mathbb{R}\} \tag{1}$$

where $\underline{x} = \inf(\mathbf{x})$ is the infimum of $\mathbf{x}$ and $\bar{x} = \sup(\mathbf{x})$ is the supremum of $\mathbf{x}$. Any real number $x \in \mathbb{R}$ can be expressed as the degenerate interval $\mathbf{x} = [x, x]$. A non-degenerate interval is called a proper interval.

3.1.1. Basic arithmetic operations

Interval arithmetic (IA) is an arithmetic defined on intervals of $\mathbb{IR}$. Its development started mainly with Moore's works [28–31], however, there are also other scientists who developed it. The basic arithmetic operations on closed intervals are defined as follows:

- Addition $\mathbf{x}+\mathbf{y}=[\underline{x}+\underline{y},\bar{x}+\bar{y}]$
- Subtraction $\mathbf{x}-\mathbf{y}=[\underline{x}-\bar{y},\bar{x}-\underline{y}]$. In particular, $-\mathbf{x}=[-\bar{x},-\underline{x}]$, because $-\mathbf{x}=[0,0]-[\underline{x},\bar{x}]$.
- Multiplication $\mathbf{x}\cdot\mathbf{y}=[\min\{S_\times(\mathbf{x},\mathbf{y})\},\max\{S_\times(\mathbf{x},\mathbf{y})\}]$, where $S_\times(\mathbf{x},\mathbf{y})\triangleq\{\underline{x}\underline{y},\underline{x}\bar{y},\bar{x}\underline{y},\bar{x}\bar{y}\}$ is the set of all possible products of endpoints of $\mathbf{x}$ and $\mathbf{y}$.

  In particular, $-\mathbf{x}=[-\bar{x},-\underline{x}]$ because $-\mathbf{x}=[-1,-1]\times[\underline{x},\bar{x}]=[\underline{x},\bar{x}]\times[-1,-1]$.
- Division $\mathbf{x}/\mathbf{y}=[\min\{S_\div(\mathbf{x},\mathbf{y})\},\max\{S_\div(\mathbf{x},\mathbf{y})\}]$, if $0\notin\mathbf{y}$ and where

  $$S_\div(\mathbf{x},\mathbf{y})\triangleq\{\underline{x}/\underline{y},\underline{x}/\bar{y},\bar{x}/\underline{y},\bar{x}/\bar{y}\}$$

  is the set of all possible divisions of endpoints of $\mathbf{x}$ and $\mathbf{y}$.

  If $0\in\mathbf{y}$ then the division by $\mathbf{y}$ can be handled with more effort using extended interval arithmetic [16, 28] not detailed in this paper.

Algebraic properties such as associativity, commutativity and neutral elements hold for $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{IR}$:

- Associativity $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ and $(\mathbf{x}\mathbf{y})\mathbf{z}=\mathbf{x}(\mathbf{y}\mathbf{z})$.
- Commutativity $(\mathbf{x}+\mathbf{y})=(\mathbf{y}+\mathbf{x})$ and $(\mathbf{x}\mathbf{y})=(\mathbf{y}\mathbf{x})$.
- Neutral element (addition): $\mathbf{0}+\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{x}$, where $\mathbf{0}\triangleq[0,0]$, $\mathbf{0}\cdot\mathbf{x}=\mathbf{x}\cdot\mathbf{0}=\mathbf{0}$
- Neutral element (multiplication) $\mathbf{1}\cdot\mathbf{x}=\mathbf{x}\cdot\mathbf{1}=\mathbf{x}$, where $\mathbf{1}\triangleq[1,1]$.

The distributivity law does not hold for proper intervals; however, sub-distributivity low does: $\forall\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{IR}$, $\mathbf{x}(\mathbf{y}+\mathbf{z})\subseteq\mathbf{x}\mathbf{y}+\mathbf{x}\mathbf{z}$. Proper intervals also do not have additive or multiplicative inverses.

3.1.2. Basic interval functions

To understand how the interval arithmetic SPOTIS extension works, a definition of the absolute value of the interval should be provided. Other interval functions can be found elsewhere [28, 41].

Absolute value [41]:

$$|\mathbf{x}|=\begin{cases}[|\bar{x}|,|\underline{x}|], & \text{if } \bar{x}\le 0\\ [|\underline{x}|,|\bar{x}|], & \text{if } \underline{x}\ge 0\\ [0,\max\{|\underline{x}|,|\bar{x}|\}], & \text{if } \underline{x}<0 \text{ and } \bar{x}>0\end{cases}\tag{2}$$

3.2. Distances between intervals

There are many different ways to define the distance between two intervals. For this research, we chose three of them to compare how they perform in the interval SPOTIS method.

Hausdorff distance between two intervals $\mathbf{x}$ and $\mathbf{y}$ is generally defined as $d_H(\mathbf{x},\mathbf{y}) = \max_{x\in\mathbf{x}}\{\min_{y\in\mathbf{y}} d(x,y)\}$. In other words, it is a maximum distance between $x \in \mathbf{x}$ to its nearest point $y \in \mathbf{y}$. $d(x,y)$ could be an any metric, such as L_1 or L_2, etc. In this paper, we will use the L_1 Hausdorff metric defined by

$$d_H(\mathbf{x},\mathbf{y}) = \max\{|\underline{x} - \underline{y}|, |\bar{x} - \bar{y}|\} \tag{3}$$

Example 1. Hausdorff distance between $\mathbf{x} = [1,3]$ and $\mathbf{y} = [2,5]$:

$$d_H([1,3],\ [2,5]) = 2 \tag{4}$$

Example 2. Hausdorff distance between $\mathbf{x} = [4,4]$ and $\mathbf{y} = [1,1]$, i.e., intervals $\mathbf{x}$ and $\mathbf{y}$ are degenerate:

$$d_H([4,4],\ [1,1]) = 3 \tag{5}$$

In the second example, we calculated the distance between 1 and 4 and got three as a result, which is expected.

Wasserstein's distance was successfully used in the framework of belief functions to solve decision-making problems under uncertainty [5, 12, 13]. Wasserstein's distance between $\mathbf{x}, \mathbf{y} \in \mathbb{IR}$ is

$$d_W(\mathbf{x},\ \mathbf{y}) \triangleq \sqrt{\left[\frac{\underline{x}+\bar{x}}{2} - \frac{\underline{y}+\bar{y}}{2}\right]^2 + \frac{1}{3}\left[\frac{\bar{x}-\underline{x}}{2} - \frac{\bar{y}-\underline{y}}{2}\right]^2}, \tag{6}$$

which corresponds to Mallows' distance [24] between two probability distributions when we assume that each interval is the support of a uniform distribution.

Example 1. Wasserstein's distance between $\mathbf{x} = [1,3]$ and $\mathbf{y} = [2,5]$ is

$$d_W([1,3],\ [2,5]) \approx 1.5275 \tag{7}$$

The result here differs from the result obtained using the Hausdorff distance. It is because the Hausdorff distance is L_1 and the Wasserstein's is L_2.

Example 2. Wasserstein's distance between $\mathbf{x} = [4,4]$ and $\mathbf{y} = [1,1]$

$$d_W([4,4],\ [1,1]) = 3 \tag{8}$$

Similarly to the Hausdorff distance, the distance between two degenerate intervals works as expected, i.e., Wasserstein's distance between 1 and 4 is 3.

Euclidean distance between intervals assumes that intervals are represented by a point $(\underline{x}, \bar{x})$ on a plane. It is defined as the Euclidean distance between two points and was presented in [18]

$$d_E(\mathbf{x},\mathbf{y}) = \sqrt{(\underline{x} - \underline{y})^2 + (\bar{x} - \bar{y})^2} \tag{9}$$

Example 1. Euclidean distance between $\mathbf{x} = [1,3]$ and $\mathbf{y} = [2,5]$

$$d_E([1,3],\ [2,5]) \approx 2.2361 \tag{10}$$

In this case, the result is also different from the Hausdorff distance, because this distance is L_2.

Example 2. Euclidean distance between $\mathbf{x} = [4,4]$ and $\mathbf{y} = [1,1]$

$$d_E([4,4],\ [1,1]) \approx 4.2426 \tag{11}$$

When definition (9) is used for degenerated intervals, it does not work as one could expect. According to this definition, the distance between 1 and 4 is approximately 4.24.

3.3. Correlation coefficients

Weighted Spearman's rank correlation coefficient

For a sample of size N, the rank values x_i and y_i are defined by equation (12). In this approach, we consider that the positions at the top of both rankings are more important. The weight of significance is calculated for each comparison. It is the element that determines the main difference to Spearman's rank correlation coefficient, which examines whether the differences appeared and not where they appeared [32].

$$r_w = 1 - \frac{6\sum_{i=1}^{N}(x_i - y_i)^2((N - x_i + 1) + (N - y_i + 1))}{N^4 + N^3 - N^2 - N} \tag{12}$$

Rank similarity coefficient

. For a sample of size N, the rank values x_i and y_i are defined by equation (13) [35]. It is an asymmetric measure. The weight of a given comparison is determined based on the significance of the position in the first ranking, which is used as a reference ranking during the calculation.

$$WS = 1 - \sum_{i=1}^{N} 2^{-x_i} \frac{|x_i - y_i|}{\max(|x_i - 1|, |x_i - N|)} \tag{13}$$

3.4. Ordering interval preferences

Suppose that we have two intervals $\mathbf{x} = [\underline{x}, \bar{x}]$ and $\mathbf{y} = [\underline{y}, \bar{y}]$. Then according to [45], the degree of possibility $\mathbf{x} \geq \mathbf{y}$ is defined as $P(\mathbf{x} \geq \mathbf{y})$. In the literature, various mathematical definitions can be found [3, 11, 19, 20]. For example, Wang et al. [46] presented a simple equation (14) which provides the degree of possibility that one interval is greater than another. Currently, this approach seems to be the most popular in the literature

$$\mathrm{P}(\mathbf{x} \geq \mathbf{y}) = \frac{\max\left(0, \bar{x} - \underline{y}\right) - \max\left(\underline{x} - \bar{y}, 0\right)}{\bar{x} + \bar{y} - \underline{y} - \underline{x}} \tag{14}$$

We assume that we have N alternatives evaluated using the appropriate MCDA method. A suitable MCDA method must be applicable, correctly selected according to [47], and return the preference results in the form of intervals. The generalized SPOTIS method described in Section 3.1 will be used in the following.

As a result of the evaluation, we obtained preference intervals $\mathbf{p}$ for all alternatives, which can be written as $\mathbf{p}_i = [\underline{p}_i, \bar{p}_i]$, where $i = 1, \ldots, N$. Then, the possibility degree (*PD*) matrix with all values of the possible degree should be determined as follows (15):

$$PD = [P(\mathbf{p}_i \geq \mathbf{p}_j)]_{N \times N} \tag{15}$$

where $i = 1, \ldots, N$, $j = 1, \ldots, N$, and P is used one of the equation (14). Then we count the cumulative probability vector PR in according to (16):

$$PR_i = \sum_{j=1}^{N} PD_{ij} \tag{16}$$

Finally, the alternatives are ranked from the highest to the smallest value of PR_i, where the highest value means the maximum cumulative possibility degree.

However, when we use this approach with the proposed interval SPOTIS method, we need to order alternatives in ascending order, e.g., from the smallest (closer to the ideal solution point (ISP)) to the largest (farther from ISP).

Another interesting approach is to compare the intervals proposed by [6] and based on a geometrical interpretation of the intervals. However, in this paper, we will use Wang's possibility degree method because it is more popular in the literature.

3.5. Stable preference ordering towards ideal solution

The SPOTIS method is a relatively new MCDA method proposed by [7]. It is based on the concept of reference objects. Unlike other MCDA methods, such as TOPSIS, it uses arbitrary set decision problem bounds as reference objects. This way, the SPOTIS method is completely free from the rank-reversing paradox. Despite the novelty of the method, it has been used in multiple scientific works and was extended to work under fuzzy environment [39].

To apply this method, the expert should define data boundaries. For each criterion C_j the maximum $S_j^{\max}$ and minimum $S_j^{\min}$ bounds should be selected. Ideal Solution Point (ISP) S_j^* is defined as $S_j^* = S_j^{\max}$ for profit criterion and as $S_j^* = S_j^{\min}$ for cost criterion. The decision matrix is defined as $X = (x_{ij})_{m \times n}$, where x_{ij} is the attribute value of the ith alternative for jth criterion.

Step 1. Calculation of the normalized distances to the ISP

$$d_{ij}(A_i, S_j^*) = \frac{|S_{ij} - S_j^*|}{|S_j^{\max} - S_j^{\min}|} \tag{17}$$

Step 2. Calculation of weighted normalized distances $d(A_i, S^*) \in [0, 1]$

$$d(A_i, S^*) = \sum_{j=1}^{N} w_j d_{ij}(A_i, S_j^*) \tag{18}$$

Step 3. Determine the final ranking by ordering the alternatives by $d(A_i, S^*)$ values. Better alternatives have smaller values of $d(A_i, S^*)$.

To apply this MCDA method easily, one can use `pymcdm` library written in Python, which contains the implementation of the SPOTIS and some other popular MCDA methods [17].

4. Proposed approach

4.1. Interval decision matrix and interval ideal solution point

In order to work with interval values, we modified equation (19), so we could calculate the distance between S_{ij} and S_j^*, which are now interval values.

$$d_{ij}(A_i, s_j^*) = \frac{\left|S_{ij} - S_j^\star\right|}{\left|S_j^{\max} - S_j^{\min}\right|} \tag{19}$$

Step 1. Calculate the distances from every decision matrix interval value $\tilde{S}_{ij}$ from the interval ISP value $\tilde{S}_j^\star$. In the case of real-valued ISP, we could use degenerated interval $[x, x]$. Hausdorff's and Wasserstein's distance functions work well with them, too. It will be demonstrated in the Preliminaries section.

$$d_{ij}(A_i, \tilde{s}_j^*) = \frac{d(\tilde{S}_{ij}, \tilde{S}_j^\star)}{\left|S_j^{\max} - S_j^{\min}\right|}, \tag{20}$$

where $d(\cdot, \cdot)$ could be either Hausdorff, Wasserstein's distance, or any other distance between intervals.
Step 2. Aggregate these values using the weighted sum

$$d_i(A_i, \tilde{s}^*) = \sum_{j=1}^{N} w_j d_{ij}(A_i, \tilde{s}_j^*) \tag{21}$$

When using $d(\cdot, \cdot)$ which is not generalization and does not work properly for real numbers, we need to use equation (22) which is given later and represents bounds as degenerative interval values.

4.1.1. Numerical example 1

This numerical example demonstrates how the proposed extension can be applied to the interval decision matrix using the real-valued bounds and weights but with an interval ISP. Consider the decision matrix presented in Table 2 and the chosen interval ISP. The ISP was chosen as an interval between the highest or lowest value for each criterion and the chosen bound for this criterion. That is, for C_1 which is profit, we define ISP as $S_1^* = [11, 12]$, because 11 is the largest value for this criterion and $S_1^{\max} = 12$ is the upper bound for this criterion.

The criteria bounds $S_j^{\min}$ and $S_j^{\max}$ for this example are defined as real numbers and are presented in Table 3. We also show the criteria weights chosen for this example.

When we have defined the decision matrix, ISP, criteria bounds, and weight, we could apply equation (20) and calculate the normalized distances from the ISP $d_{ij}(A_i, s_j^*)$, which are presented in Table 4. In

Table 2. Decision matrix for numerical example 1 and chosen ISP ($\tilde{S}_j^*$)

	C_1	C_2	C_3
$\tilde{S}_j^*$	[11.0, 12.0]	[–6.0, –4.0]	[3.5, 5.0]
A_1	[10.5, 11.0]	[–4.0, –3.1]	[1.5, 1.8]
A_2	[–4.7, –4.0]	[–1.0, 0.5]	[3.4, 3.5]
A_3	[7.6, 8.3]	[–0.3, 0.5]	[1.1, 1.5]
A_4	[3.0, 3.3]	[7.0, 7.5]	[–5.5, –5.2]

Table 3. Criteria bounds and weights for numerical example 1

	C_1	C_2	C_3
$S_j^{\min}$	–5	–6	–8
$S_j^{\max}$	12	10	5
w_j	0.2	0.3	0.5

this and the following examples, we will use Wasserstein's distance between two real-valued intervals defined as (6).

Table 4. Normalized distances from the ISP: $d_{ij}(A_i, \tilde{s}_j^*)$

	C_1	C_2	C_3
A_1	0.0449	0.0928	0.2018
A_2	0.9324	0.2970	0.0689
A_3	0.2089	0.3195	0.2282
A_4	0.4913	0.7661	0.7389

Next, we could apply weights and calculate the aggregated normalized distances from the ISP using equation (21). Weighted normalized distances and aggregated values are presented in Table 5.

The values $d_i(A_i, \tilde{s}^*)$ are the final preference values for each alternative A_i. The alternative A_1 has a preference value of 0.1377, which means that it is the closest to the ISP. Therefore, A_1 is the best alternative. The full ranking is as follows:

$$A_1 > A_3 > A_2 > A_4$$

4.2. Interval decision matrix, ISP and bounds

To deal with interval-valued decision problem bounds, we should replace equation (20) with the new formula (22) described below.

Updated step 1. Calculate the distances from every decision matrix interval value $\tilde{S}_{ij}$ from the interval ISP value $\tilde{S}_j^{\star}$. Use one of the available distance functions $d(\cdot,\cdot)$ to calculate the distance between $\tilde{S}_j^{\max}$, $\tilde{S}_j^{\min}$ which are also interval values now.

$$d_{ij}(A_i, \tilde{s}_j^*) = \frac{d(\tilde{S}_{ij}, \tilde{S}_j^{\star})}{d(\tilde{S}_j^{\max}, \tilde{S}_j^{\min})}, \tag{22}$$

Table 5. Weighted normalized distances from the ISP with final preference value ($d_i(A_i, \tilde{s}^*)$)

	C_1	C_2	C_3	$d_i(A_i, \tilde{s}^*)$
A_1	0.0090	0.0278	0.1009	0.1377
A_2	0.1865	0.0891	0.0345	0.3100
A_3	0.0418	0.0958	0.1141	0.2517
A_4	0.0983	0.2298	0.3695	0.6976

Updated step 2. Normalized distance from decision matrix values to ISP $d_{ij}(A_i, \tilde{s}_j^*)$ is real precise values, because the result of the $d(\cdot, \cdot)$ function is a real value. Therefore, equation (21) used in **Step 2** remains the same.

4.2.1. Numerical example 2

This numerical example shows how to deal with decision problems with an interval decision matrix, interval ISP, and interval criteria bounds. A decision problem like this could appear when several experts cannot agree on the criteria bounds, so we can use an interval value here. The decision matrix and ISP are the same as in numerical Example 1 (Table 2).

In this example, we introduce the interval-valued criteria bounds presented in Table 6. They were created similarly to ISPs in numerical example 1. For example, the arbitrarily chosen upper bound for C_1 is $S_1^{\max} = 12$ and the largest value from the decision matrix for C_1 is 11, therefore $\tilde{S}_j^{\max} = [11, 12]$. In this table, we also provide a distance between upper and lower interval bounds calculated using Wasserstein's distance. This value is used as a denominator in equation (22) in Step 1 of the modified SPOTIS algorithm.

Table 6. Criteria bounds, normalization denominator and criteria weights

	C_1	C_2	C_3
$\tilde{S}_j^{\min}$	[–5.0, –4.7]	[–6.0, –4.0]	[–8.0, –5.5]
$\tilde{S}_j^{\max}$	[11.0, 12.0]	[7.5, 10.0]	[3.5, 5.0]
$d_W(\tilde{S}_j^{\max}, \tilde{S}_j^{\min})$	16.35	13.75	11.0
w_j	0.2	0.3	0.5

The next step is a calculation of the normalized distances from the ISP using equation (22). The results are presented in Table 7.

Table 7. Normalized distances from the ISP $d_{ij}(A_i, \tilde{s}_j^*)$

	C_1	C_2	C_3
A_1	0.0467	0.1079	0.2384
A_2	0.9694	0.3456	0.0815
A_3	0.2172	0.3717	0.2696
A_4	0.5108	0.8914	0.8730

After applying the weights from Table 6, we could calculate the final preference values calculated using equation (21). Table 8 contains weighted normalized distances from the ISP and the final preference value $d_i(A_i, \tilde{s}^*)$, which could be interpreted as a weighted distance from the ISP.

Table 8. Weighted normalized distances from the ISP with final preference value ($d_i(A_i, \tilde{s}^*)$)

	C_1	C_2	C_3	$d_i(A_i, \tilde{s}^*)$
A_1	0.0093	0.0324	0.1192	0.1609
A_2	0.1939	0.1037	0.0407	0.3383
A_3	0.0434	0.1115	0.1348	0.2898
A_4	0.1022	0.2674	0.4365	0.8061

In the second example, the alternative A_1 got a preference value of 0.1609. Therefore, it has first place in the ranking. The order of the other alternatives is the same as for the first example:

$$A_1 > A_3 > A_2 > A_4$$

Notice how we get the same rankings despite the preference values differing.

4.3. Interval decision matrix, ISP, bounds and weights

Updated step 1. The normalized distances from the ISP are calculated with equation (22). In this step, we do not use criteria weights.

Updated step 2. Criteria weights are used in this step. Because the criteria weights are interval, this step is modified. The value $d_{ij}(A_i, \tilde{s}_j^*) \in \mathbb{R}$ and $\tilde{w}_j \in \mathbb{IR}$. Therefore, the result of the computations in this step $\tilde{d}_i(A_i, \tilde{s}^*) \in \mathbb{IR}$.

$$\tilde{d}_i(A_i, \tilde{s}^*) = \sum_{j=1}^{N} \tilde{w}_j d_{ij}(A_i, \tilde{s}_j^*) \tag{23}$$

Interval weights should fulfill the following rule (24) [44]

$$\text{mid}\left(\sum_{j=1}^{M} \tilde{w}_j\right) = 1 \tag{24}$$

where $\text{mid}(\mathbf{x}) = \dfrac{\underline{x} + \bar{x}}{2}$ defines the middle point of the interval $\mathbf{x}$.

Alternatively, one could use a more admissible rule on the weights. For, it should be ensured that there should exist at least one precise weights vector in an imprecise weights vector, so the sum of these precise weights equals one. This rule is more general. However, the stability of this approach should be investigated in detail.

Those equations could also be used on $\mathbb{R}$ if they are represented as degenerative intervals. Hausdorff and Wasserstein's distances work as a generalization in this case.

4.3.1. Numerical example 3

In this simple numerical example, we demonstrate how to apply the last variation of the interval SPOTIS extension to a decision problem, which includes an interval decision matrix, interval criteria bounds, interval weights, and an interval ISP.

The decision matrix for this example remains the same as for numerical examples 1 and 2 and is shown in Table 2. We also use the same interval criteria bounds as for the second example, but now weights are presented as interval values. Criteria bounds and the distance between them are shown in Table 9. We also present interval weights in this table. Instead of $w_1 = 0.2$ for C_1 we use $\tilde{w}_1 = [0.15, 0.25]$ and so on. The sum of the weights fulfills the property (24) presented by [44].

Table 9. Criteria bounds, normalization denominator and criteria weights

	C_1	C_2	C_3
$S_j^{\min}$	[–5.0, -4.7]	[–6.0, –4.0]	[–8.0, –5.5]
$S_j^{\max}$	[11.0, 12.0]	[7.5, 10.0]	[3.5, 5.0]
$\lvert S_j^{\max} - S_j^{\min}\rvert$	16.35	13.75	11.0
$\tilde{w}_j$	[0.15, 0.25]	[0.25, 0.35]	[0.45, 0.55]

After applying equation (22) to the decision matrix presented in Table 2, we obtain normalized distances from the ISP presented in Table 10. These distances are the same as for the second numerical example (Table 7) because we apply the same equation to the same interval decision matrix and the same interval bounds and interval ISP.

Table 10. Normalized distances from the ISP: $d_{ij}(A_i, \tilde{s}_j^*)$

	C_1	C_2	C_3
A_1	0.0467	0.1079	0.2384
A_2	0.9694	0.3456	0.0815
A_3	0.2172	0.3717	0.2696
A_4	0.5108	0.8914	0.8730

Next, we apply equation (23) to the normalized distances in order to obtain weighted normalized distances and then aggregate them. The resulting value is an interval because we multiply the normalized distance from the ISP, which is a real value, and the criterion weight, which is an interval. Therefore, we need to use a methodology that allows ordering preferences with interval values, for example, the one presented in Section 3.4.

Table 12 contains the possibility degree matrix calculated using equation (14), as well as cumulative probability degree. Because the possibility degree function determines the probability that interval A_i is greater than or equal to A_j, we should order alternatives by the cumulative probability degree ascending. Therefore, we have a ranking $A_1 > A_3 > A_2 > A_4$, which is the same ranking as for the other two numerical examples.

Table 11. Weighted normalized distances from the ISP with final preference value ($d_i(A_i, \tilde{s}^*)$)

	C_1	C_2	C_3	$d_i(A_i, \tilde{s}^*)$
A_1	[0.01, 0.01]	[0.03, 0.04]	[0.11, 0.13]	[0.14, 0.18]
A_2	[0.15, 0.24]	[0.09, 0.12]	[0.04, 0.04]	[0.27, 0.41]
A_3	[0.03, 0.05]	[0.09, 0.13]	[0.12, 0.15]	[0.25, 0.33]
A_4	[0.08, 0.13]	[0.22, 0.31]	[0.39, 0.48]	[0.69, 0.92]

Numbers are rounded up to two digits.

Table 12. Possibility degree matrix with all values $P(A_i \geq A_j)$

$A_i \backslash A_j$	A_1	A_2	A_3	A_4	$\sum_{j=1}^{N} PD_{ij}$
A_1	0.50	0.00	0.00	0.00	0.50
A_2	1.00	0.50	0.71	0.00	2.22
A_3	1.00	0.28	0.50	0.00	1.78
A_4	1.00	1.00	1.00	0.50	3.50

5. Study case

In this section, we present a study case previously presented in [15]. The data shows the comparison of fifteen bank branches, which corresponds to fifteen alternatives. Four financial ratios are used as criteria for these banks. We use these data to examine the applicability of the proposed method and compare it with the results presented in the original study.

Table 13 describes the four criteria. Criterion C_1 is a cost criterion, and the others are profit. Each criterion has a precise weight valued $w_j = 0.25$. In this experiment, we do not use interval weight because interval TOPSIS does not support them. Criteria bounds were determined based on the decision matrix by rounding values, that is, the criterion C_1 has the smallest value, 58.69. Therefore, $S_1^{\min} = 50$ was chosen as the lower bound for this criterion.

Table 13. Criteria description

C_j	Weight	Type	$S_j^{\min}$	$S_j^{\max}$	S_j^*
C_1	0.25	cost	50	3070	50
C_2	0.25	profit	100000	3200000	3200000
C_3	0.25	profit	400	55000	55000
C_4	0.25	profit	50	7000	7000

The decision matrix from [15] is presented in Table 14. All values in the presented decision matrix are interval values. The matrix contains fifteen alternatives from A_1 to A_{15} and four criteria described previously.

Next, we apply the proposed interval SPOTIS approach, that is, equations (20) and (21) to obtain preference values for those fifteen alternatives. We also use three different distance functions defined for $\mathbb{IR}$. Therefore, Table 15 contains four preference vectors P_X and ranking R_X, defined as follows:

Table 14. Decision matrix

A_i	C_1	C_2	C_3	C_4
A_1	[500.37, 961.37]	[2696995, 3126798]	[26364, 38254]	[965.97, 6957.33]
A_2	[873.7, 1775.5]	[1027546, 1061260]	[3791, 50308]	[2285.03, 3174.0]
A_3	[95.93, 196.39]	[1145235, 1213541]	[22964, 26846]	[207.98, 510.93]
A_4	[848.07, 1752.66]	[390902, 395241]	[492, 1213]	[63.32, 92.3]
A_5	[58.69, 120.47]	[144906, 165818]	[18053, 18061]	[176.58, 370.81]
A_6	[464.39, 955.61]	[408163, 416416]	[40539, 48643]	[4654.71, 5882.53]
A_7	[155.29, 342.89]	[335070, 410427]	[33797, 44933]	[560.26, 2506.67]
A_8	[1752.31, 3629.54]	[700842, 768593]	[1437, 1519]	[58.89, 86.86]
A_9	[244.34, 495.78]	[641680, 696338]	[11418, 24108]	[1070.81, 2283.08]
A_{10}	[730.27, 1417.11]	[453170, 481943]	[2719, 2955]	[375.07, 559.85]
A_{11}	[454.75, 931.24]	[309670, 342598]	[2016, 2617]	[936.62, 1468.45]
A_{12}	[303.58, 630.01]	[286149, 317186]	[14918, 27070]	[1203.79, 4335.24]
A_{13}	[658.81, 1345.58]	[321435, 347848]	[6616, 8045]	[200.36, 399.8]
A_{14}	[420.18, 860.79]	[618105, 835839]	[24425, 40457]	[2781.24, 4555.42]
A_{15}	[144.68, 292.15]	[119948, 120208]	[1494, 1749]	[282.73, 471.22]

- P_{SH}, R_{SH} – preference values and ranking for interval SPOTIS with Hausdorff's distance (3),
- P_{SW}, R_{SW} – preference values and ranking for interval SPOTIS with Wasserstein's distance (6),
- P_{SE}, R_{SE} – preference values and ranking for interval SPOTIS with Euclidean distance (9),
- P_T, R_T – preference values and ranking for interval TOPSIS ([15]).

Table 15. Preference values and rankings for interval SPOTIS with different distances and interval TOPSIS.

A_i	P_{SH}	P_{SW}	P_{SE}	P_T	R_{SH}	R_{SW}	R_{SE}	R_T
A_1	0.4512	0.3037	0.4796	0.6991	2	1	1	1
A_2	0.6975	0.5588	0.8307	0.4747	9	6	7	6
A_3	0.5667	0.5466	0.7742	0.5065	4	5	5	4
A_4	0.8423	0.8108	1.1517	0.3260	14	14	14	14
A_5	0.6658	0.6596	0.9335	0.4401	7	9	9	9
A_6	0.4378	0.3835	0.5519	0.5640	1	2	2	2
A_7	0.5798	0.5129	0.7337	0.4969	5	4	4	5
A_8	0.9417	0.8777	1.2516	0.2198	15	15	15	15
A_9	0.6497	0.5898	0.8391	0.4746	6	8	8	8
A_{10}	0.7928	0.7656	1.0864	0.3578	13	13	13	13
A_{11}	0.7542	0.7266	1.0306	0.3967	11	10	10	11
A_{12}	0.6667	0.5751	0.8271	0.4747	8	7	6	7
A_{13}	0.7870	0.7570	1.0744	0.3613	12	12	12	12
A_{14}	0.5555	0.4676	0.6738	0.5365	3	3	3	3
A_{15}	0.7516	0.7429	1.0517	0.4115	10	11	11	10

Due to the difference in those distance functions, we also got different preference values and slightly different rankings. Table 15 contains the resulting preference vectors rounded up to four digits. In columns, P_{SH}, P_{SW}, P_{SE} a smaller value means a better alternative, and for the column, P_T it is reversed, i.e., better alternatives have a bigger preference value. Alternative A_1 got the first place in the rankings R_{SW}, R_{SE}, R_T, because all Wasserstein's and Euclidean distances are both L_2, as well as distance used in the interval TOPSIS method. This alternative got second in the ranking R_{SH}, which uses L_1 Hausdorff distance. Alternative A_6 got the first position in the R_{SH} ranking and the second position in other rankings. The third position in all rankings is taken by the alternative A_{14}. The ranking R_{SH} is more different from the other rankings because it is based on L_1 distance functions, and the others are based on L_2 distance functions. Rankings R_{SW}, R_{SE}, and R_T are more similar: the tail and head of those rankings are the same, and there are some changes in the middles of those rankings. The preference values P_{SE} were calculated using equation (22) due to how it handles degenerative intervals. If equation (20) was used, the preference values could be larger than 1.

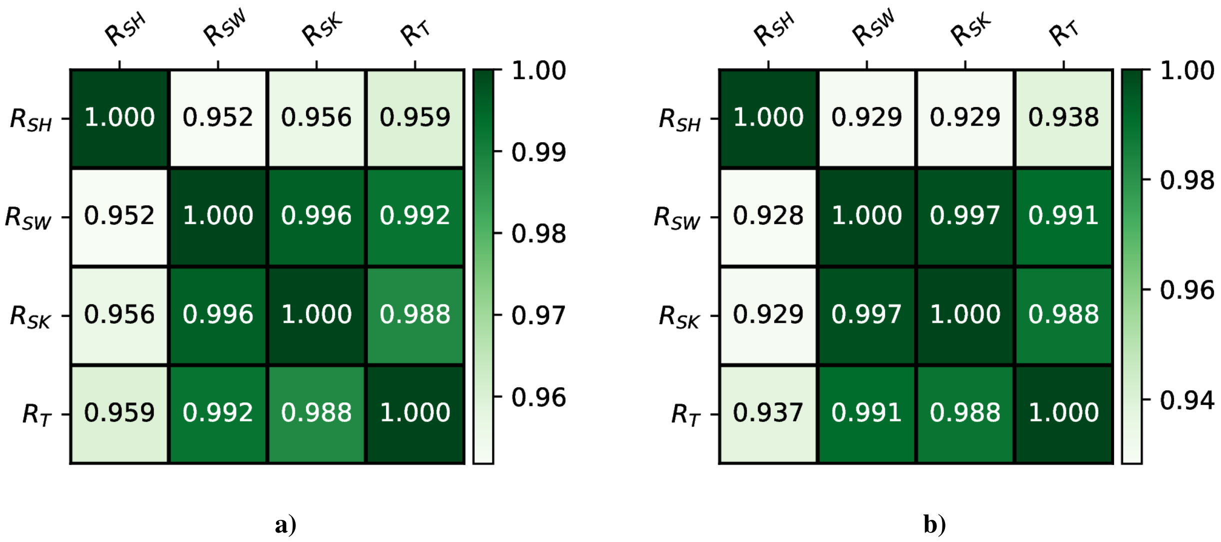

Figure 1. Heatmaps of weighted Spearman's r_w (a) and rank similarity *WS* (b) coefficients

To compare the obtained rankings numerically, we use two correlation coefficients described in Section 3.3: the weighted Spearman correlation coefficient (r_w), and the rank similarity coefficient (*WS*). The results of these comparisons are presented in Figures 1a) and 1b).

The heatmap in Figure 1a) presents r_w correlations between the ranking in Table 15. The range of this correlation value is $[-1, 1]$, where 1 means a strong correlation between two rankings. However, the rankings obtained are very similar according to this correlation coefficient. The rankings R_{SW}, R_{SE} and R_T have correlation values in the range $[0.988, 0.996]$, which means they are very similar. The correlation between R_{SH} ranking and the other ranking is lower because we have used L_1 Hausdorff distance.

Similar values can be seen on another heatmap, which is presented in Figure 1b). The range of these values is $[0, 1]$, which is not typical for correlation coefficients. The value 1 means perfect correlation, and 0 means lack of correlation between rankings. These values are very similar to the correlations r_w and show that the R_{SH} ranking is less correlated with the other three rankings. The rankings based on L_2 distance are more similar.

5.1. Rank reversal in the interval TOPSIS method

One of the advantages of the original SPOTIS method is that it is rank reversal free by design. Providing criteria bounds allows for building a complete decision problem model to order alternatives stably. The proposed interval extension of the SPOTIS method also has this property. In this section, we provide a simple experiment that demonstrates the RR behavior in the interval TOPSIS rankings and shows that the SPOTIS method is resistant to it in a practical example.

To explain the experiment clearly, we use Algorithm 1. First, we define the interval decision matrix for this experiment, as well as the criteria weights and the criteria types. The decision matrix X contains N alternatives. The next step is to evaluate the preference values for the complete decision matrix X using the chosen multi-criteria decision analysis method $eval()$ as specified in lines 1-2 of the Algorithm 1. Next, for each alternative A_i, we construct the decision matrix X', which contains all alternatives except A_i, which is specified on line 4. On lines 5-6 we evaluate the alternatives $N-1$ in the decision matrix X' to obtain the preference vector P' and then ranking R' which contains $N-1$ values. On lines 7-8, we construct the preference vector P'_{Full}, which contains the same values as P_{Full}, but without ith value, e.g., without preference for the alternative A_i. Then, we build the ranking R'_{Full} from P'_{Full}. The last thing is to calculate the correlation between these two rankings and write them as well as the correlation value.

Require: Interval Decision matrix X with N alternatives
Require: Weights W
Require: Criteria types T
1: $P_{Full} \leftarrow eval(X, W, T)$ {Evaluate alternatives and rank for full matrix X}
2: $R_{Full} \leftarrow rank(P_{Full})$
3: **for** i in $1, 2, 3, \ldots, N$ **do**
4: $X' \leftarrow X$ without alternative A_i
5: $P' \leftarrow eval(X', W, T)$
6: $R' \leftarrow rank(P')$
7: $P'_{Full} \leftarrow P_{Full}$ without ith value.
8: $R'_{Full} \leftarrow rank(P'_{Full})$
9: $cr \leftarrow correlation(R', R'_{Full})$
10: Write cr and R' in order to visualize later
11: **end for**

Algorithm 1: Pseudocode for the RR sensitivity analysis experiment

We conducted an experiment described with Algorithm 1 for interval TOPSIS [15] and for the proposed interval SPOTIS method for the data from the study case. The decision matrix contains the values presented in Table 14 and uses the criteria weights, types, and bounds defined in Table 13.

First, the procedure described with Algorithm 1 was applied for the interval TOPSIS method. Vector rankings R' and correlation values between them and the ranking created based on the full decision matrix were saved and then visualized in Figure 2. For the sake of readability, we only visualize the ranking in which the rank reversal paradox occurs. Each alternative is represented with a color line, which allows following it an easy way to see how the relative position of this alternative changes between subsequent rankings. The first and last columns show the full ranking, and the other columns show rankings created without certain alternatives. For example, column w/o A_1 represents the ranking without alternative A_1. Instead of the dot, which should represent A_1 in the column w/o A_1 dashed line is drawn. In the upper part of Figure 2, we add a visualization of the r_w correlation values between the following ranking.

When the relative order of the alternatives changes in the subsequent ranking, we observe a crossing of the lines on the visualization. For example, alternative A_3 forth position) was better than alternative

A_7 (fifth position) in the full ranking, then after removing A_1 from the set of alternatives, we observe that alternatives A_3 and A_7 change their relative order, and now A_7 is better. This means that a rank reversal paradox has occurred.

The next interesting situation is how the alternatives A_9 and A_{12} behave in those rankings. Their relative positions change in several rankings. Another interesting alternative is alternative A_2, which was better or worse than alternatives A_9 and A_{12} in different rankings.

Figure 2. Visualisation that shows the rank reversal behavior in the interval TOPSIS method

Because of changes in the relative order of the alternatives when the interval TOPSIS method is used, we cannot be certain that when we add a new alternative or remove the one that is not relevant, we will obtain the same order for the unchanged set of the alternatives.

On the other hand, the SPOTIS method and proposed interval extension of the SPOTIS method are resistant to the rank reversal by design. We ran the experiment described with Algorithm 1 for the interval SPOTIS method with Wasserstein's distance. We used the same data as for the experiment with interval TOPSIS. On the visualization, we present the rankings, which were presented on the visualization of the interval TOPSIS method.

The visualization in Figure 3 follows the same rules as described for the visualization of the interval TOPSIS rankings. We can see that the lines that represent alternatives are mostly parallel and do not cross themselves. In this situation, when the alternative is removed (dashed lines), the lines touch but do not cross. For example, when we remove alternative A_6 from the evaluated decision matrix, alternative A_{14} takes the place of A_6, but the relative order of the alternatives is preserved. A similar situation could be observed in the columns w/o A_2 and w/o A_3. When the alternative A_2 is missed, the alternative A_3 takes its place in the ranking and the other way around. However, their relative positions to other alternatives never change.This experiment proves practically that there is no rank reversal in the SPOTIS method when the set of alternatives changes. That means that when we need to add a new alternative or remove the one that is no longer relevant, we are guaranteed to obtain stable ordering of the unchanged part of the alternatives' set.

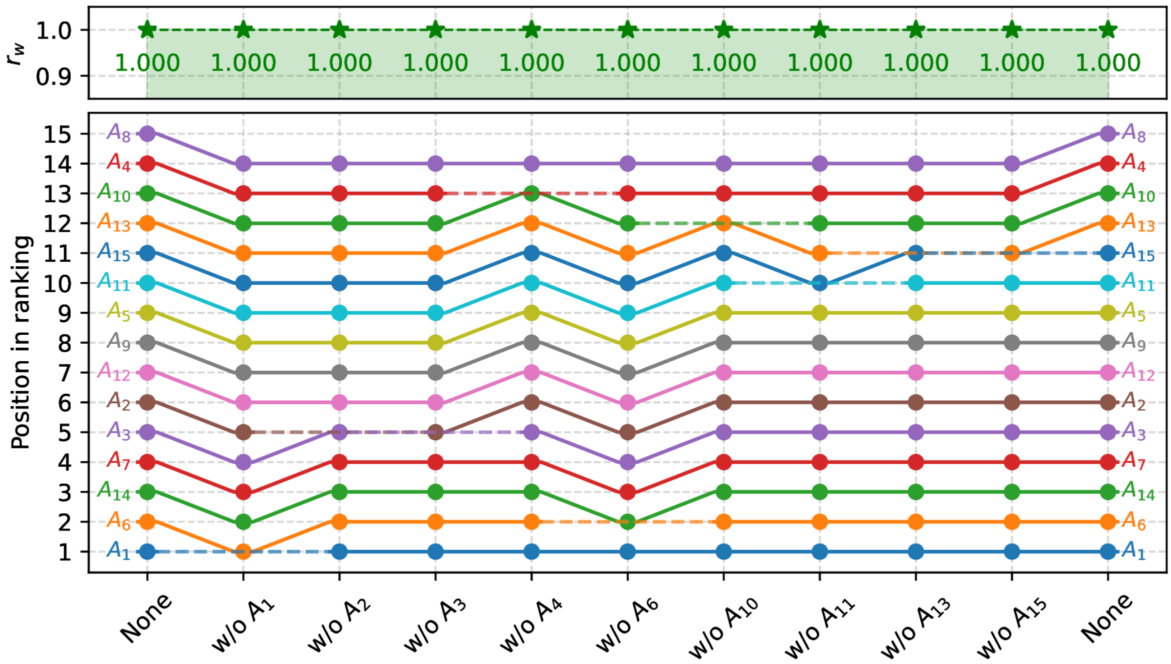

Figure 3. Visualisation that shows lack of the rank reversal when interval SPOTIS method with Wasserstein distance is used

5.2. Interval SPOTIS working with all imprecise data

To demonstrate the stability of the proposed interval SPOTIS method, we present another example in which we extend the problem presented in Section 5. We introduce the interval weights and interval criteria bounds defined similarly to the interval weights and bounds of the simple numerical examples in Section 4. In this example, we use the interval decision matrix from the study case presented in Table 14. Table 16 contains the description of the criteria used. The table was extended for the interval version of the criteria weights and criteria bounds.

The interval weights $\tilde{w}_j$ are small intervals around equal weights w_j, which fulfills the property (24), and the interval bounds $\tilde{S}_j^{\text{min}}$ and $\tilde{S}_j^{\text{max}}$ are made as an interval between some arbitrarily chosen maximum and minimum points and the highest and smallest values in the decision matrix. For example, $\tilde{S}_1^{\text{min}} = [50.0, 58.69]$, where 50 is an arbitrarily valued value and 58.69 is a real value from the decision matrix.

Table 16. Description of criteria with interval weights and interval bounds.

C_j	w_j	$\tilde{w}_j$	S_j^{min}	S_j^{max}	$\tilde{S}_j^{\text{min}}$	$\tilde{S}_j^{\text{max}}$	Type
C_1	0.25	[0.2, 0.3]	50	3700	[50.0, 58.69]	[3629.54, 3700.0]	cost
C_2	0.25	[0.2, 0.3]	100000	3200000	[100000.0, 119948.0]	[3126798.0, 3200000.0]	profit
C_3	0.25	[0.2, 0.3]	400	55000	[400.0, 492.0]	[50308.0, 55000.0]	profit
C_4	0.25	[0.2, 0.3]	50	7000	[50.0, 58.89]	[6957.33, 7000.0]	profit

Using equations described in Section 4 and the Wasserstein distance equation, we build four different rankings from these data using the following definitions:

- P_1, R_1. The decision matrix is interval-valued (Table 14). Other values are real, e.g., we use weight vector w_j and criteria bounds $S_j^{\min}$ and $S_j^{\max}$.
- P_2, R_2. Interval decision matrix (Table 14) and interval ISP which is defined based on interval bounds $\tilde{S}_j^{\min}$ and $\tilde{S}_j^{\max}$. However, weights and bounds remain real valued.
- P_3, R_3. As for P_2 and R_2, but the interval criteria bounds $\tilde{S}_j^{\min}$ and $\tilde{S}_j^{\max}$ are used.
- P_4, R_4. Each part of the decision problem defined as interval values, e.g. P_3 and R_3 but with interval weights $\tilde{w}_j$.

After applying the interval SPOTIS procedure (Section 4), we obtain four preference values and four rankings corresponding to different combinations of the real and interval-valued parts of the decision problem. Table 17 shows the resulting preference vectors and rankings. The interval ranking R_4 was made from the preferences P_4 by methods presented in Section 3.4. The obtained rankings R_1–R_4 are the same. Small changes in the definition of the decision problem likely will not introduce any change in the rankings obtained previously, despite the change in the preference vectors P_1–P_4.

Table 17. Preference values and rankings for four interval SPOTIS variations.

A_i	P_1	P_2	P_3	P_4	R_1	R_2	R_3	R_4
A_1	0.3037	0.2876	0.2932	[0.2346, 0.3518]	1	1	1	1
A_2	0.5588	0.5424	0.5524	[0.4419, 0.6629]	6	6	6	6
A_3	0.5466	0.5317	0.5409	[0.4327, 0.6491]	5	5	5	5
A_4	0.8108	0.7961	0.8121	[0.6497, 0.9746]	14	14	14	14
A_5	0.6596	0.6449	0.6568	[0.5254, 0.7881]	9	9	9	9
A_6	0.3835	0.3677	0.3735	[0.2988, 0.4482]	2	2	2	2
A_7	0.5129	0.4972	0.5043	[0.4034, 0.6051]	4	4	4	4
A_8	0.8777	0.8630	0.8796	[0.7037, 1.0555]	15	15	15	15
A_9	0.5898	0.5745	0.5858	[0.4687, 0.7030]	8	8	8	8
A_{10}	0.7656	0.7509	0.7662	[0.6130, 0.9195]	13	13	13	13
A_{11}	0.7266	0.7119	0.7271	[0.5817, 0.8726]	10	10	10	10
A_{12}	0.5751	0.5598	0.5708	[0.4566, 0.6850]	7	7	7	7
A_{13}	0.7570	0.7422	0.7568	[0.6054, 0.9081]	12	12	12	12
A_{14}	0.4676	0.4517	0.4599	[0.3679, 0.5519]	3	3	3	3
A_{15}	0.7429	0.7282	0.7436	[0.5949, 0.8923]	11	11	11	11

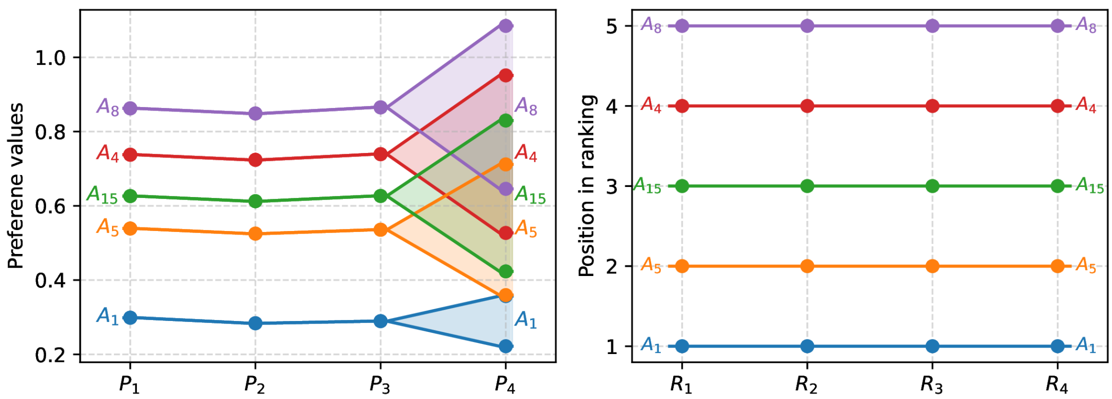

Figure 4. Preference values (left) and ranking positions changing (right) for the chosen alternatives

We also create a small visualization for the chosen alternatives, namely A_1, A_4, A_5, A_8, and A_{15} to demonstrate how exactly preference values generate a change from the vector P_1 to the vector P_4. We decided to show only those alternatives because the difference between them was sufficient to make the visualization readable. This visualization is presented in Figure 4. The left side of this figure shows how the preference values of those alternatives change and the right side shows how the rankings change. As shown in Table 17, the rankings are the same, which means that the relative order of the alternatives does not change.

6. Discussion

The results shown at the beginning of Section 5 show that the proposed interval SPOTIS approach is a useful tool for solving decision problems with interval values involved. We demonstrated how the proposed approach performs compared to the interval TOPSIS method on the data presented in [15]. We also demonstrate that the interval SPOTIS method could use different distance functions with very similar final results. The heatmaps shown in Figures 1a) and 1b) demonstrate how similar rankings are obtained using the interval SPOTIS procedure with different distance metrics and how they are similar to the ranking obtained using the interval TOPSIS approach. The ranking obtained using Wasserstein's distance and Euclidean distances are more similar to the interval TOPSIS ranking because these three rankings were obtained using L_2 distance metrics. The ranking built using the Hausdorff distance function is less similar because we use the L_1 Hausdorff distance function. This result shows the advantage of the proposed approach over the interval TOPSIS. interval SPOTIS is much simpler but performs similarly to the interval TOPSIS method. It also allows for the use of different distance functions to better fit the decision problem, which is not possible in the interval TOPSIS method.

Section 5.1 introduces the experiment, which allows one to study rank reversal phenomena by the MCDA methods. The flow of the experiment was explained with pseudo-code in Algorithm 1. The visualization presented in Figure 2 shows how the relative positions of the decision alternatives change when the set of alternatives changes. The comparison of 2 with 3 shows the changes in the same ranking for the interval SPOTIS method, and one can easily see how many times the decision alternatives change their positions when interval TOPSIS is used. Every change in the set of alternatives is a possible change in the domain of the decision problem when the alternatives are evaluated using the interval TOPSIS method. It implies that the ranking built using the interval TOPSIS method could not be used when the stability of the ranking is essential. The interval SPOTIS method allows for building a stable ranking, and the relative order of the alternatives will not change when a new one is added or when an old one is removed. This method builds a complete model of the decision problem, which makes the interval SPOTIS resistant to RR phenomena. The only important thing that the decision maker should remember is that the SPOTIS method, as well as its generalization, guarantees a stable ranking only if the criteria bounds remain unchanged. In this situation, when we need to change the problem domain, some alternatives can change their positions in the ranking. However, it only underlines the importance of the criteria bounds and the fact that they should be chosen very carefully.

In Section 5.2, we demonstrate how the interval SPOTIS method will perform on the example from [15] when intervals are used in every part of the decision problem. For this, we define interval weights and interval criteria bounds for these data. These values are presented in Table 16. We applied the interval SPOTIS to the four variations of this decision problem and got an identical ranking for every case. Data from Table 17 show that the proposed approach is stable and robust, in addition to the different parts of the decision problem becoming intervals. We could observe small changes in the preference values, but the ranking remains the same, and the relative positions of the alternatives do not change. This underlines the subsequent superiority of the interval SPOTIS method, namely its ability to use interval values in the decision matrix, weights, and criteria bounds, which is not possible when the interval TOPSIS method is used.

However, there is no ideal approach that will fit every MCDA problem. In addition to the advantages such as simplicity, resistance to the rank reversal phenomena, and the ability to use interval values in every part of the decision problem, the proposed approach also has some limitations. The interval SPOTIS approach requires criteria bounds to be defined before alternatives can be evaluated. In some cases, it could be difficult or impossible to determine or obtain the criteria bounds. In this case, the criteria bounds could be approximated in order to apply the interval SPOTIS method. Another limitation is the need to use an interval preference ordering method when interval criteria weights are used. One of the possible approaches is presented in this paper. However, the different ordering methods could introduce some complications and allow rank reversing in the case of broad intervals. The interval SPOTIS method uses one of the simplest approaches to handle uncertain data: the interval of the real values. There are also other approaches to handling uncertainty, which implies that methods that use them are more accurate.

7. Conclusion

In this paper, we present a simple way to evaluate decision alternatives in the interval domain. The proposed approach allows the use of $\mathbb{IR}$ intervals in a decision matrix, criteria bounds, weights, and ideal solution point. As shown in Section 5 and discussed in Section 6, the proposed approach builds the complete model of the decision problem, which implies that it resisted the rank reversal phenomenon. The proposed method is a simple and robust solution for evaluating decision alternatives with interval values. The example shown in Section 5 proves that the interval SPOTIS method performs similarly to the interval TOPSIS method but is much simpler. However, the proposed approach also has some small limitations: the decision maker should define the decision criteria bounds to use the proposed method. However, if there is a problem with criteria bounds determining, one could have an a priori idea of what these bounds should be and then make a sensitivity analysis on it. It is also required to use the interval preference ordering method to order the preferences in case of using interval weights. One of the interval ordering methods is presented in Section 3.4.

The proposed approach and the experiments presented create several interesting future research directions. The proposed interval SPOTIS could be a base for research on the following extensions of the SPOTIS method. The equations presented in this paper could be accommodated for other approaches that handle uncertain data, such as fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, and so on. The proposed approach could also be used in different practical decision problems requiring interval data to evaluate alternatives simply and stably. Another interesting research direction is to check the relative accuracy of the interval SPOTIS approach and other methods that perform on interval values. We will evaluate how the geometric method for interval preference ordering presented in [6] could be applied and evaluated in our future research works on SPOTIS.

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