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A discussion on the optimality of bulk entry queue with differentiated hiatuses

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Abstract

We consider Markovian differentiated hiatuses queues with bulk entries. With the help of the matrix geometric method, we discuss the stability condition for the existence of the steady-state solution of our model and we obtain the stationary system size by using a probability generating function. The stochastic decomposition form of stationary system size and the waiting time distribution of an arbitrary beneficiary are also analysed. Furthermore, we perform the expense analysis using the particle swarm optimization technique and we obtain the optimality of service rate and hiatus rate. Finally, we study the effects of changes in the parameters on some important performance measures of the system through numerical observations.

Keywords: differentiated hiatus times, steady state probabilities, waiting time distribution, optimization of expenses

1. Introduction

Owing to wide applications of vacation queues in the field of production systems, inventory models, communication models, computer networks (see, e.g., Doshi [8]) have been studied by many researchers. To know more about vacation queues and their applications, the readers may vide the survey work of Teghem [24] and the monographs by Takagi [23], and Tian and Zhang [25].

Since our intent is to study vacation queueing models with bulk arrivals, we cite here only works related to vacation models with bulk arrivals. The study on queues with multiple vacations, where beneficiaries entries are bulk, were initiated by Baba [1]. Takagi [23] first suggested a variant of vacation policy, which is a generalization of the multiple and single vacation. Lee et al. [15] analysed the steady-state solution of Markovian bulk arrival queues with N-Policy and multiple vacations, where the service times follow general distributions. Ke and Chu [13] studied the variant policy for a bulk arrival queueing systems. Ke [12] analysed operating characteristics of bulk arrival variant vacation queues, where the service times follow general distributions. Ke et al. [14] analyzed the performance measures of batch entry $M^{[X]}/G/1$ vacation queues, where the server is unreliable.

Bouchentouf and Medjahri [6] discussed the performance evaluation of the feedback queue with differentiated vacations and balking. Bouchentouf and Guendouzi [3] derived a steady-state solution of the

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feedback queue associated with differentiated vacations, impatient customers and vacation interruptions. Bouchentouf et al. [5] analyzed the queue in which interruptions occurs when the server is on differentiated working vacation and also the customers lost their patience. Bouchentouf and Guendouzi [4] studied the steady-state solution of single server bulk arrival variant vacation queues with Bernoulli feedback and impatient beneficiaries.

Ibe and Isijola [11] first analysed a novel type of vacation scheme in queues in which the server is permitted to take two different vacations and the span of the second type of vacations is shorter than the span of the first type of vacations. The time-dependent solution of the queues analyzed in Ibe and Isijola [11] is carried out in Vijayashree and Janani [27]. Subsequently, Suranga and Liu [22] explored the time-dependent solution with the impact of client impatience on the model of Ibe and Isijola [11]. Recently, Suranga et al. [21] have studied the transient solution of multiple differentiated vacation queues with impatient clients with an application to a scenario that arises in IEEE 802.16E power saving mechanism. Vadivukarasi et al. [26] analysed the optimization of $M/M/1/N$ queue with differentiated vacations. Gupta and Kumar [10] studied the retrial queue with differentiated vacations and state dependent arrivals. Xu et al. [28] discussed the steady-state behaviour of batch customers entry single server working vacation queues. Yu et al. [30] analyzed the steady-state solution for the $GI^{[x]}/M^b/1/L$ queues with partial batch rejection and multiple working vacations. A discussion on the steady-state solution of discrete-time batch arrival working vacation queues can be seen in Li et al. [16]. The steady-state solution of single server Markovian queues, where the server follows multiple working vacations was studied by Baba [2]. Gao and Yin [9] studied Geo$^X/G/1$ queues with working vacations and vacation interruptions. Luo et al. [17] analysed the optimality of expense model for working vacation queues. Ye [29] investigated the queues where the arrivals are bulk and server is permitted to follow two-stage vacations policy.

In this paper, we extend the model analysed in Ibe and Isijola [11] by incorporating the possibility of beneficiaries' entry as a batch. Using particle swarm optimization, we present a discussion on the optimality for the model. The model discussed in this work helps us to study numerous real life queueing situations, in order to improve the service process. Moreover, to the best of our knowledge, queueing systems with bulk arrival and differentiated vacations has not been studied in the literature. This paper makes a contribution in this sense.

The rest of this paper is organized as follows. A detailed mathematical description of the $M^X/M/1$ queue is shown in Section 2. Practical motivation of the concerned model is given in Section 3. In Section 4, we establish the steady-state solution of the model. In Sections 5 and 6, we evaluate the stability condition and its stochastic decomposition form, respectively. The LST of the waiting time distribution of an arbitrary beneficiary is obtained in Section 7. The discussion on the optimality of the model is studied in Section 8. The impacts of the parameters on the essential performance measures and the conclusion notes are presented in Section 9 and 10, respectively.

2. The system description

Consider an $M^X/M/1$ differentiated hiatus queue. The mathematical description of this queue is as follows.

- Clients enter into the system in batches, according to a Poisson process with rate λ. Let the batch size of clients arrival be a random variable B with the probability distribution function $Pr(B = i) = b_i$, $i \geq 1$, and the probability generating function $B(z) = \sum_{i=1}^{\infty} b_i z^i$.
- We assume that the duration of the service to the clients has an exponential distribution with parameter μ.
- Once the system becomes empty, after a non-zero busy period, the server will take a hiatus of type I. When the duration of the hiatus of type I completes, if there are clients in the system, the server

starts to provide the service immediately. Otherwise, the server decides to take a hiatus of type II, which is of shorter duration than the duration of the hiatus of type I. On return from the hiatus of type II, if the server finds one or more clients in the system, the server provides service for them. If the server finds zero clients presence in the system on return from the hiatus of type II, it proceeds for another hiatus of type II and follows this approach until it finds at least one beneficiary waited for the service upon returning from a hiatus of type II. We assume that the duration of hiatus of type I and hiatus of type II have exponential distributions with parameters α_1 and α_2, respectively, and $\alpha_1 < \alpha_2$.

3. Practical motivations

A practical justification of the proposed model arises from a situation associated with a bank clerk supporting daily banking activities, where the beneficiary entry may be more than one. A primary task of the clerk is to handle customers and their financial transactions. Once completing the needs of all the beneficiaries who are waiting in the line, the clerk decides to take a hiatus for a refreshment of having tea or coffee. After returns from the hiatus, if there is at least one beneficiary waiting to receive service, the bank clerk starts to provide the service immediately. If there is no waiting beneficiary while returning from the hiatus, the bank clerk decides to take a hiatus to do one of the secondary tasks like data entry, maintaining customer and transaction records, counting cash, verifying financial information without making any errors, cash management procedures, etc. After completing a secondary task, if there is at least one beneficiary waiting to receive service, the bank clerk starts to provide the service immediately, otherwise the clerk decides to do another secondary task. The clerk continues this process until he finds at least one beneficiary in the waiting line.

Another real time application of the system under discussion is as follows. In manufacturing environments, where assemblers work with automated systems and complex robotics, their main task is to produce components by assembling parts and subassemblies. After completing the main task, if no assembling parts are available to produce components, the assembler immediately decides to take a hiatus to have a tea/coffee. This hiatus may be taken as a hiatus of type I (of longer duration). After returning from the hiatus of type I, if any parts are available to assemble, the assembler immediately starts the main task, otherwise he decides to clean and maintain the work area and equipment, including tools. This hiatus period may be taken as a hiatus of type II. After completing the cleaning process, if any parts are available to assemble, the assembler immediately decides to start the main task, otherwise he decides to take another hiatus of type II to conduct quality control checks. The assembler continues this process and takes a hiatus of type II to check the supply levels, place orders, verify receipt of supplies, complete production and quality forms, and etc., until he/she find parts to assemble.

4. Steady state analysis

Let $L(t)$ be the number of beneficiaries present in the system at time t and

$$S(t) = \begin{cases} 2 & \text{if the server is busy} \\ 0 & \text{if the server is in type I hiatus} \\ 1 & \text{if the server is in type II hiatus} \end{cases}$$

Then the process $\{(S(t), L(t))\}$ is a two dimensional Markov process with the state space

$$\{(0,0)\} \cup \{(1,0)\} \cup \{(s,l) : l \geq 1,\ s = 0,1,2\}$$

Let

$$p_l = \lim_{t\to\infty} \text{Prob}\{S(t)=2, L(t)=l\},\ l\geq 1$$
$$q_{i,l} = \lim_{t\to\infty} \text{Prob}\{S(t)=i, L(t)=l\},\ i=0,1,\ l\geq 0$$

The inter-arrival times, service times and hiatus times are mutually independent. Moreover, a first in first out (FIFO) order is followed in the service providing process.

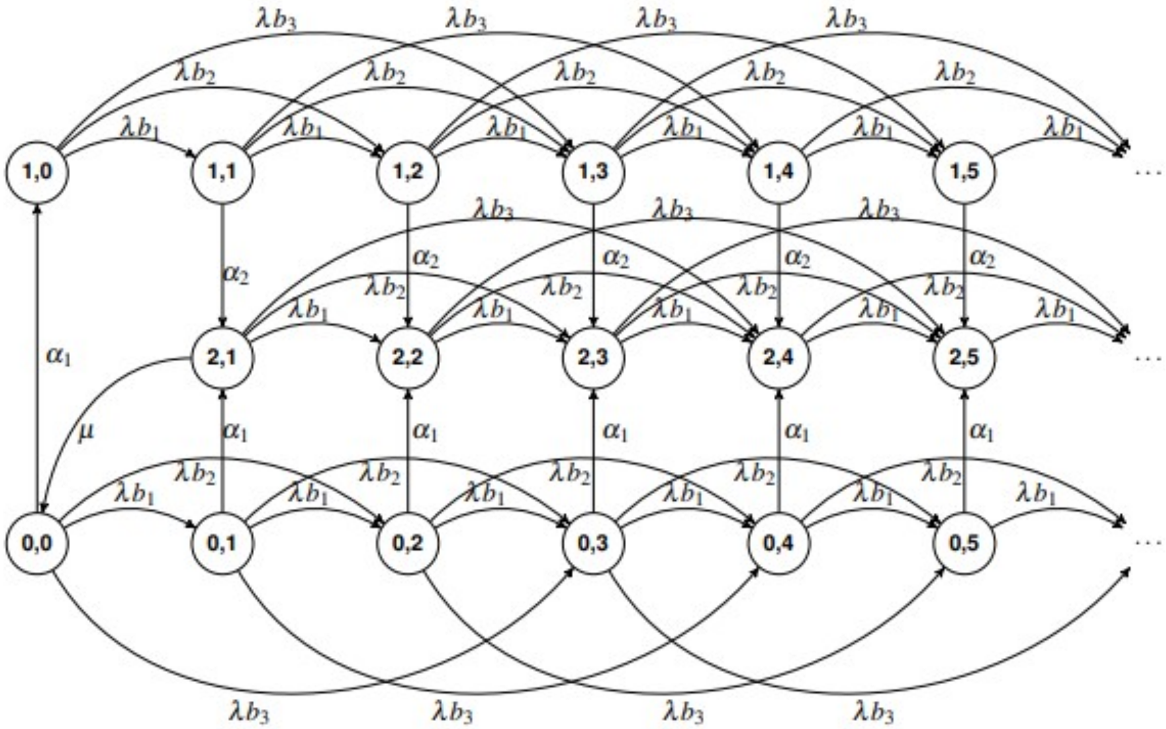


Figure 1. State transition diagram for the case $b_1+b_2+b_3=1$

According to the Markov theory, the steady-state balance flow equations of the assumed model are the following.

$$(\lambda+\mu)p_1 = \alpha_1 q_{0,1} + \alpha_2 q_{1,1} + \mu p_2 \tag{1}$$

$$(\lambda+\mu)p_n = \alpha_1 q_{0,n} + \alpha_2 q_{1,n} + \mu p_{n+1} + \lambda\sum_{k=1}^{n-1} b_k\, p_{n-k},\ n\geq 2 \tag{2}$$

$$(\lambda+\alpha_1)q_{0,0} = \mu p_1 \tag{3}$$

$$(\lambda+\alpha_1)q_{0,1} = \lambda b_1 q_{0,0} \tag{4}$$

$$(\lambda+\alpha_1)q_{0,n} = \lambda b_n q_{0,0} + \lambda\sum_{k=1}^{n-1} b_k\, q_{0,n-k},\ n\geq 2 \tag{5}$$

$$\lambda q_{1,0} = \alpha_1 q_{0,0} \tag{6}$$

$$(\lambda+\alpha_2)q_{1,1} = \lambda b_1 q_{1,0} \tag{7}$$

$$(\lambda+\alpha_2)q_{1,n} = \lambda\sum_{k=1}^{n} b_k\, q_{1,n-k},\ n\geq 2 \tag{8}$$

Let $P(z) = \sum\limits_{n=1}^{\infty} p_n z^n$ and $Q_i(z) = \sum\limits_{n=1}^{\infty} q_{i,n} z^n,\ i = 0, 1$, be the partial probability generating functions. Multiplying (1) and (2) by appropriate z^n and adding them, we get

$$(\lambda+\mu)\sum_{n=1}^{\infty} p_n z^n = \mu\sum_{n=1}^{\infty} p_{n+1} z^n + \alpha_1\sum_{n=1}^{\infty} q_{0,n} z^n + \alpha_2\sum_{n=1}^{\infty} q_{1,n} z^n + \lambda\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} b_k p_{n-k} z^n$$

$$P(z) = \frac{z[\alpha_1 Q_0(z) + \alpha_2 Q_1(z) - \mu p_1]}{(\lambda+\mu)z - \mu - \lambda z B(z)} \tag{9}$$

In the same way, from (4) and (5) we obtain

$$(\lambda+\alpha_1)\sum_{n=1}^{\infty} q_{0,n} z^n = \lambda\sum_{n=1}^{\infty} b_n z^n q_{0,0} + \lambda\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} b_k q_{0,n-k} z^n$$

$$Q_0(z) = \frac{\lambda B(z) q_{0,0}}{\lambda(1-B(z)) + \alpha_1} \tag{10}$$

From (7) and (8) we get

$$(\lambda+\alpha_2)\sum_{n=1}^{\infty} q_{1,n} z^n = \lambda\sum_{n=1}^{\infty} b_n z^n q_{1,0} + \lambda\sum_{n=2}^{\infty}\sum_{k=1}^{n-1} b_k q_{1,n-k} z^n$$

$$Q_1(z) = \frac{\lambda B(z) q_{1,0}}{\lambda(1-B(z)) + \alpha_2} \tag{11}$$

As $\lim_{z\to 1} P(z)$, we have

$$\begin{aligned} P(1) &= \lim_{z\to 1}\frac{z[\alpha_1 Q_0'(z) + \alpha_2 Q_1'(z)] + [\alpha_1 Q_0(z) + \alpha_2 Q_1(z) - \mu p_1]}{(\lambda+\mu) - \lambda z B'(z) - \lambda B(z)} \\ &= \frac{\alpha_1 Q_0'(1) + \alpha_2 Q_1'(1) + [\alpha_1 Q_0(1) + \alpha_2 Q_1(1) - \mu p_1]}{\mu - \lambda B'(1)}, \\ &= \frac{\frac{\lambda B'(1)}{\alpha_1} + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\lambda+\alpha_2}{\lambda+\alpha_1}\right) B'(1)}{1-\rho} p_0,\ \text{if } \rho = \frac{\lambda B'(1)}{\mu} \end{aligned}$$

Note that positive limiting probability $P(1)$ exists only if $\rho < 1$. From the law of total probability, we have

$$p_1 = \frac{1-\rho}{\frac{\lambda B'(1)}{\alpha_1} + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\lambda+\alpha_2}{\lambda+\alpha_1}\right) B'(1) + \frac{\lambda\mu}{\alpha_1(\lambda+\alpha_1)}(1-\rho) + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\mu}{\lambda+\alpha_1}\right)(1-\rho) + \frac{\mu(1-\rho)}{\lambda}} \tag{12}$$

Remark 1. If the entry of beneficiaries into the system is restricted to 1, then (12) becomes

$$p_1 = \frac{\frac{(\mu-\lambda)}{\mu}}{\left(\frac{\mu}{\alpha_1}\right) + \left(\frac{\alpha_1\mu}{\alpha_2\lambda}\right)\left(\frac{\lambda+\alpha_2}{\lambda+\alpha_1}\right)} \tag{13}$$

The above expression for p_1 is in agreement with $p_{1,0}$ given in [11], after making appropriate notations and assumptions.

5. The stability condition

Using the lexicographical order, the infinitesimal generator of the Markov process $\{(S(t), L(t))\}$ can be given in the Block-Jacobi matrix as

$$R = \begin{pmatrix} G_0 & G_1 & G_2 & G_3 & G_4 & \cdots \\ H_0 & F_1 & F_2 & F_3 & F_4 & \cdots \\ & F_0 & F_1 & F_2 & F_3 & \cdots \\ & & F_0 & F_1 & F_2 & \cdots \\ & & & F_0 & F_1 & \cdots \end{pmatrix}$$

where

$$G_0 = \begin{pmatrix} -(\lambda+\alpha_1) & \alpha_1 \\ 0 & \lambda \end{pmatrix}, G_i = \begin{pmatrix} 0 & \lambda b_i & 0 \\ 0 & 0 & \lambda b_i \end{pmatrix}, i \geq 1, H_0 = \begin{pmatrix} \mu & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} -(\lambda+\mu) & 0 & 0 \\ \alpha_1 & -(\lambda+\alpha_1) & 0 \\ \alpha_2 & 0 & -(\lambda+\alpha_2) \end{pmatrix}, F_0 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_i = \begin{pmatrix} \lambda b_{i-1} & 0 & 0 \\ 0 & \lambda b_{i-1} & 0 \\ 0 & 0 & \lambda b_{i-1} \end{pmatrix}, i \geq 2$$

Here, the matrix $E = \sum_{i=0}^{\infty} F_i$ is irreducible. From Neuts [19], we have the following theorem.

Theorem 1. The Markov process $\{S(t), L(t)\}$ is ergodic if and only if $\rho = \frac{\lambda \sum_{i=1}^{\infty} i b_i}{\mu} = \frac{\lambda b}{\mu} < 1$.

Proof. After some mathematical manipulations, the irreducible matrix E can be expressed as

$$\begin{pmatrix} 0 & 0 & 0 \\ \alpha_1 & -\alpha_1 & 0 \\ \alpha_2 & 0 & -\alpha_2 \end{pmatrix}$$

Assume δ to be the invariant vector of E, i.e. $\delta E = 0$ and $\delta e = 1$, where e denotes the column vector whose entries are all one. Here, the value of δ is $(0, 0, 1)$. From Neuts [19], $\{S(t), L(t)\}$ is ergodic if and only if

$$\delta \sum_{s=2}^{\infty} (s-1) F_s e < \delta F_0 e, \ \lambda b < \mu, \ \text{and } \lambda b/\mu < 1$$

In this case, the equation $\sum_{i=0}^{\infty} F_i K^i = 0$ has the minimal nonnegative solution of

$$K = \begin{pmatrix} k_1 & k_2 & k_3 \\ 0 & k_4 & k_5 \\ 0 & 0 & k_6 \end{pmatrix}$$

By solving the equation $\sum\limits_{i=0}^{\infty} F_i K^i = 0$, we get

$$
\begin{aligned}
&\mu - (\lambda+\mu)k_1 + \sum_{s=2}^{\infty} \lambda b_{s-1} k_1^s = 0 \\
&-(\lambda+\mu)k_2 + \sum_{s=2}^{\infty} \lambda b_{s-1} k_2 \sum_{l=0}^{s-1} k_1^l k_4^{s-l-1} = 0 \\
&-(\lambda+\mu)k_3 + \sum_{s=2}^{\infty} \lambda b_{s-1} \left\{ k_3 \sum_{l=0}^{s-1} k_1^l k_6^{s-l-1} + k_2 k_5 \sum_{l=0}^{s-2} k_1^l \sum_{m=0}^{s-l-2} k_4^m k_6^{s-l-m-2} \right\} = 0 \\
&\alpha_1 k_2 - (\lambda+\alpha_1)k_4 + \sum_{s=2}^{\infty} \lambda b_{s-1} k_4^s = 0 \\
&\alpha_1 k_3 - (\lambda+\alpha_1)k_5 + \sum_{s=2}^{\infty} \lambda b_{s-1} k_5 \sum_{l=0}^{s-1} k_4^l k_6^{s-l-1} = 0 \\
&\alpha_2 k_3 - (\lambda+\alpha_2)k_6 + \sum_{s=2}^{\infty} \lambda b_{s-1} k_6^s = 0
\end{aligned}
$$

By solving the above set of equations, we obtain the minimal nonnegative solution of $\sum\limits_{i=0}^{\infty} F_i K^i = 0$. □

6. Stochastic decomposition

In this section, we establish the stochastic decomposition form of stationary system size as the following theorem.

Theorem 2. If $\rho < 1$, then the stationary system length $R(z)$ can be decomposed into addition of two independent random variables

$$R(z) = R_\mu(z) + R_h(z)$$

where $R_\mu(z)$ is the steady-state queue length of the standard $M^X/M/1$ queues and $R_h(z)$ is the additional load in the queue length owing to the differentiated hiatuses and is given by

$$
R_h(z) = \frac{\alpha_1 B(z)\lambda(\lambda z\phi_2(z) + \alpha_2\phi_1(z) + \phi_1(z)\phi_3(z)) + \lambda^2 B(z)\phi_2(z)\phi_3(z) + (\lambda+\alpha_1)\phi_1(z)\phi_2(z)(\phi_3(z) - \lambda z)}{\lambda(\lambda+\alpha_1)\phi_1(z)\phi_2(z)(z-1)g}
$$

Here

$$
\begin{aligned}
\phi_1(z) &= \lambda(1 - B(z)) + \alpha_1 \\
\phi_2(z) &= \lambda(1 - B(z)) + \alpha_2 \\
\phi_3(z) &= (\lambda z(1 - B(z)) + \mu(z-1)) \\
\phi(z) &= \phi_1(z)\phi_2(z)\phi_3(z) \\
g &= \frac{1}{\frac{\lambda B'(1)}{\alpha_1} + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\lambda+\alpha_2}{\lambda+\alpha_1}\right) B'(1) + \frac{\lambda\mu}{\alpha_1(\lambda+\alpha_1)}(1-\rho) + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\mu}{\lambda+\alpha_1}\right)(1-\rho) + \frac{\mu(1-\rho)}{\lambda}}
\end{aligned}
$$

Proof. From the analysis carried out in Section 3, the PGF of stationary queue length is

$$\begin{aligned} R(z) &= P(z) + Q_0(z) + Q_1(z) + q_{0,0} + q_{1,0} \\ &= \frac{z[\alpha_1 Q_0(z) + \alpha_2 Q_1(z) - \mu p_1]}{(\lambda+\mu)z - \mu - \lambda z B(z)} + \frac{\lambda B(z) q_{0,0}}{\lambda(1-B(z)) + \alpha_1} + \frac{\lambda B(z) q_{1,0}}{\lambda(1-B(z)) + \alpha_2} + \frac{\mu p_1}{\lambda} \end{aligned}$$

After some mathematical manipulations, the above equation can be expressed as

$$R(z) = \frac{\mu(1-\rho)(z-1)}{\lambda z(1-B(z)) + \mu(z-1)} R_h(z) \tag{14}$$

where

$$R_h(z) = \frac{\alpha_1 B(z)\lambda(\lambda z\phi_2(z) + \alpha_2\phi_1(z) + \phi_1(z)\phi_3(z)) + \lambda^2 B(z)\phi_2(z)\phi_3(z) + (\lambda+\alpha_1)\phi_1(z)\phi_2(z)(\phi_3(z) - \lambda z)}{\lambda(\lambda+\alpha_1)\phi_1(z)\phi_2(z)(z-1)g}$$

Here

$$\begin{aligned} &\phi_1(z) = \lambda(1-B(z)) + \alpha_1 \\ &\phi_2(z) = \lambda(1-B(z)) + \alpha_2 \\ &\phi_3(z) = (\lambda z(1-B(z)) + \mu(z-1)) \\ &\phi(z) = \phi_1(z)\phi_2(z)\phi_3(z) \\ &g = \frac{1}{\frac{\lambda B'(1)}{\alpha_1} + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\lambda+\alpha_2}{\lambda+\alpha_1}\right) B'(1) + \frac{\lambda\mu}{\alpha_1(\lambda+\alpha_1)}(1-\rho) + \left(\frac{\alpha_1}{\alpha_2}\right)\left(\frac{\mu}{\lambda+\alpha_1}\right)(1-\rho) + \frac{\mu(1-\rho)}{\lambda}} \end{aligned}$$

Now $\lim_{z\to 1} R_h(z) = 1$ reveals that $R_h(z)$ is the PGF of the extra queue length occurring in the system owing to the differentiated hiatuses. □

7. Waiting time distribution

In this section, we obtain the LST of the waiting time distribution of an arbitrary beneficiary.

Theorem 3. If $\rho = \lambda b/\mu < 1$, then

$$\begin{aligned} W^*(s) = &\left(\frac{1-B(G^*(s))}{b(1-G^*(s))}\right)\left(\left(\frac{\alpha_1}{s+\alpha_1}\right)[Q_0(G^*(s)) \right. \\ &\left. + q_{0,0}] + \left(\frac{\alpha_2}{s+\alpha_2}\right)[Q_1(G^*(s)) + q_{1,0}] + P(G^*(s))\right) \end{aligned}$$

where $W^*(s)$ is the LST of the stationary waiting time of an arbitrary beneficiary.

Proof. Here we assume the following five viable cases.

Case 1. A bunch of beneficiaries, including the labelled beneficiary, enter into the system when the system is in the first hiatus state $(0,k), k \geq 1$. As there are k beneficiaries in front of the service provider, the waiting time of the labelled beneficiary is the sum of the service times of k beneficiaries who are outside of his/her batch and the sum of the service times of all beneficiaries who are stand in front of him/her in the batch. Let $r_l, l = 0, 1, \ldots$, be the probability that the labelled beneficiary stand in the jth position within his/her batch. Using the results from renewal theory [7], we have

$$r_l = \frac{1}{b}\sum_{n=l}^{\infty} b_n$$

Since there is no service during the first hiatus time, after the end of this hiatus time all the beneficiaries are served in the regular busy period. The LST of the service time distribution in a regular busy period is

$$G^*(s) = \frac{\mu}{s+\mu}$$

The waiting time of the labelled beneficiary in the state $(0,k)$, denoted by $W_{0,k}$, has the LST

$$\begin{aligned} W^*_{0,k}(s) &= \sum_{l=1}^{\infty} r_l \left(\frac{\alpha_1}{s+\alpha_1}\right) \{G^*(s)\}^{k+l-1} \\ &= \left(\frac{\alpha_1}{s+\alpha_1}\right) \sum_{l=1}^{\infty} \frac{1}{b} \sum_{m=l}^{\infty} b_m G^*(s)^{k+l-1} \\ &= \frac{1}{b}\left(\frac{\alpha_1}{s+\alpha_1}\right) \sum_{m=1}^{\infty} b_m G^*(s)^k \left\{\frac{1-G^*(s)^m}{1-G^*(s)}\right\} \\ &= \frac{1}{b}\left(\frac{\alpha_1}{s+\alpha_1}\right) \left\{\frac{G^*(s)^k - B\left(G^*(s)\right) G^*(s)^k}{1-G^*(s)}\right\} \end{aligned}$$

From this

$$\sum_{k=1}^{\infty} q_{0,k} W^*_{0,k}(s) = \frac{1}{b}\left(\frac{\alpha_1}{s+\alpha_1}\right) \left\{\frac{Q_0\left(G^*(s)\right)\left(1-B\left(G^*(s)\right)\right)}{1-G^*(s)}\right\} \tag{15}$$

Case 2. A bunch of beneficiaries join the waiting line together with the labelled beneficiary when the system is in the state $(1,k), k \geq 1$, that is the server is on type II hiatus. As discussed in the case 1, if the labelled beneficiary is in the jth position of his batch, the LST of waiting time of the labelled beneficiary is given by

$$\frac{1}{b}\left(\frac{\alpha_2}{s+\alpha_2}\right) \left\{\frac{G^*(s)^k - B\left(G^*(s)\right) G^*(s)^k}{1-G^*(s)}\right\}$$

Here $W_{1,k}$ and $W^*_{1,k}(s)$ denote labelled beneficiary's waiting time conditioned that a batch of beneficiaries arrive in the state $(1,k)$ and its LST, respectively. Then

$$\sum_{k=1}^{\infty} q_{1,k} W^*_{1,k}(s) = \frac{1}{b}\left(\frac{\alpha_2}{s+\alpha_2}\right) \left\{\frac{Q_1\left(G^*(s)\right)\left(1-B\left(G^*(s)\right)\right)}{1-G^*(s)}\right\} \tag{16}$$

Case 3. A bunch of beneficiaries join the queue including the labelled beneficiary when the service provider is in the regular busy period in which k beneficiaries are already in the system. The labelled beneficiary's waiting time W_μ has the LST

$$\begin{aligned} W^*_{\mu}(s) &= \sum_{l=1}^{\infty} r_l \{G^*(s)\}^{k+l-1} \\ &= \sum_{l=1}^{\infty} \frac{1}{b} \sum_{m=l}^{\infty} b_m G^*(s)^{k+l-1} \\ &= \frac{1}{b} \sum_{m=1}^{\infty} b_m G^*(s)^k \left\{\frac{1-G^*(s)^m}{1-G^*(s)}\right\} \\ &= \frac{1}{b} \left\{\frac{G^*(s)^k - B\left(G^*(s)\right) G^*(s)^k}{1-G^*(s)}\right\} \end{aligned}$$

From this

$$\sum_{k=1}^{\infty} p_k W^*_{\mu}(s) = \frac{1}{b} \left\{\frac{P\left(G^*(s)\right)\left(1-B\left(G^*(s)\right)\right)}{1-G^*(s)}\right\} \tag{17}$$

Case 4. A batch of beneficiaries including the labelled beneficiary arrive when the system is in the state $(0,0)$, that is, the server is on type I hiatus and empty system. As there is no beneficiaries in front of the batch of beneficiaries in the system. The labelled beneficiary's waiting time, $W_{0,0}$, is equal to the sum of service times of beneficiaries infront of his/her position within the batch and it has the LST

$$\begin{aligned} W_{0,0}^*(s) &= \sum_{l=1}^{\infty} r_l \left(\frac{\alpha_1}{s+\alpha_1}\right) \{G^*(s)\}^{l-1} \\ &= \left(\frac{\alpha_1}{s+\alpha_1}\right) \sum_{l=1}^{\infty} \frac{1}{b} \sum_{m=l}^{\infty} b_m G^*(s)^{l-1} \\ &= \frac{1}{b}\left(\frac{\alpha_1}{s+\alpha_1}\right) \sum_{m=1}^{\infty} b_m \left\{\frac{1-G^*(s)^m}{1-G^*(s)}\right\} \\ &= \frac{1}{b}\left(\frac{\alpha_1}{s+\alpha_1}\right) \left\{\frac{1-B\left(G^*(s)\right)}{1-G^*(s)}\right\} \end{aligned} \quad (18)$$

Case 5. A batch of beneficiaries including the labelled beneficiary arrive in the second hiatus state $(1,0)$. Therefore, the labelled beneficiary's waiting time conditioned that a batch of beneficiaries arrive in the state (1,0), denoted by $W_{1,0}$, has the LST

$$\begin{aligned} W_{1,0}^*(s) &= \sum_{l=1}^{\infty} r_l \left(\frac{\alpha_2}{s+\alpha_2}\right) \{G^*(s)\}^{l-1} \\ &= \frac{1}{b}\left(\frac{\alpha_2}{s+\alpha_2}\right) \left\{\frac{1-B\left(G^*(s)\right)}{1-G^*(s)}\right\} \end{aligned} \quad (19)$$

The waiting time distribution $W^*(s)$ is defined by

$$W^*(s) = \sum_{k=1}^{\infty} q_{0,k} W_{0,k}^*(s) + \sum_{k=1}^{\infty} q_{1,k} W_{1,k}^*(s) + \sum_{k=1}^{\infty} p_k W_{\mu}^*(s) + q_{0,0} W_{0,0}^*(s) + q_{1,0} W_{1,0}^*(s)$$

From equations (15)-(19), we have

$$\begin{aligned} W^*(s) &= \frac{1}{b}\left(\frac{\alpha_1}{s+\alpha_1}\right)\left\{\frac{\left(1-B\left(G^*(s)\right)\right)}{1-G^*(s)}\right\}\left[Q_0\left(G^*(s)\right)+q_{0,0}\right] + \frac{1}{b}\left\{\frac{P\left(G^*(s)\right)\left(1-B\left(G^*(s)\right)\right)}{1-G^*(s)}\right\} \\ &\quad + \frac{1}{b}\left(\frac{\alpha_2}{s+\alpha_2}\right)\left\{\frac{\left(1-B\left(G^*(s)\right)\right)}{1-G^*(s)}\right\}\left[Q_1\left(G^*(s)\right)+q_{1,0}\right] \\ &= \left(\frac{1-B(G^*(s))}{b(1-G^*(s))}\right)\left(\left(\frac{\alpha_1}{s+\alpha_1}\right)\left[Q_0(G^*(s))+q_{0,0}\right] \right. \\ &\quad \left. + \left(\frac{\alpha_2}{s+\alpha_2}\right)\left[Q_1(G^*(s))+q_{1,0}\right] + P(G^*(s))\right) \end{aligned}$$

□

Remark 2. The waiting time W is seen by the labelled beneficiary to consist of two independent waiting times, W_μ and W_h. That is

$$W^*(s) = W_\mu^*(s) W_h^*(s)$$

where

$$\begin{aligned} W_\mu^*(s) &= \left(\frac{1-B(G^*(s))}{b(1-G^*(s))}\right) \\ W_h^*(s) &= \left(\left(\frac{\alpha_1}{s+\alpha_1}\right)\left[Q_0(G^*(s))+q_{0,0}\right] + \left(\frac{\alpha_2}{s+\alpha_2}\right)\left[Q_1(G^*(s))+q_{1,0}\right] + P(G^*(s))\right) \end{aligned}$$

and $W_{\mu}^{*}(s)$ is the LST of waiting time of the labelled beneficiary in $M^X/M/1$, [18](equation (6.7.8)), and $W_h^*(s)$ is the LST of additional waiting time to the labelled beneficiary caused by differentiated hiatus times.

8. Optimization analysis

In this section, we present an expense model by defining the total expense function, in which the service rate is the control variable. Our motto is to control these variables in order to reduce the total mean expense per quantity. The expense elements are defined per unit time as follows:

- $C_1 \equiv$ holding expense for each client seen in the system,
- $C_2 \equiv$ waiting expense when one client is waiting to receive the service,
- $C_3 \equiv$ expense for the period of server busy,
- $C_4 \equiv$ expense when the server is on first hiatus,
- $C_5 \equiv$ expense when the server is on second hiatus,
- $C_6 \equiv$ expense for service.

Here, for the numerical discussion, we take the service rate μ as the decision variable and with the above expense elements, the expense function TC as

$$TC = C_1E(N) + C_2E(W) + C_3P_B + C_4P_{V_1} + C_5P_{V_2} + C_6\mu$$

where P_B is the probability that the server is on busy state, P_{V_1} and P_{V_2} are the probabilities that the server is on the first and the second hiatus, respectively. Our expense reduction approach can be summarized mathematically as

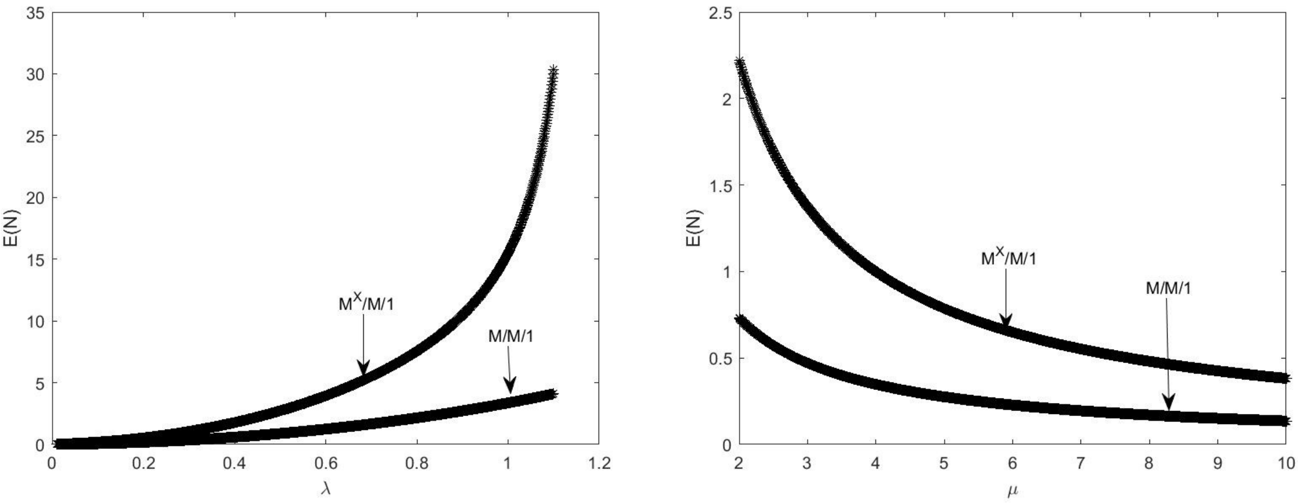
$$\text{Minimize} \quad TC(x)$$

where x is the optimum rate. The motto is to get the optimal service rate μ^* to cut down the total expense $TC(x)$. We solve the problem using Particle Swarm Optimization (PSO) and the results are presented in Section 8. For the concerned algorithm we refer to [20].

9. Numerical examples

In the experiment, we aim to study the behaviour of the mean number of customers in the system $E(N)$ against the arrival rate λ for two different values of average batch size $B'(1)$. For this study, we have taken $\mu = 2$, $\alpha_1 = 0.2$ and $\alpha_2 = 0.6$. In Figure 2a), it is evident that a surging trend is seen in the curves of $E(N)$ for the increase in the arrival rate. Also, we can see that the curve pertaining to $B'(1) = 1.5$ is above the curve pertaining to the single arrival system, that is $B'(1) = 1$. This is reasonable, since the increase of λ indicates the increase in the number of customers joining the queue, which will increase $E(N)$. In Figure 2b), we take the values of λ, α_1 and α_2 as 0.45, 0.2 and 0.6, respectively. As expected, the mean system size decreases as the service rate μ increases. The curve pertaining to $B'(1) = 1.5$ is above the curves pertaining to the remaining two average batch values due to the decrease in the average batch size.

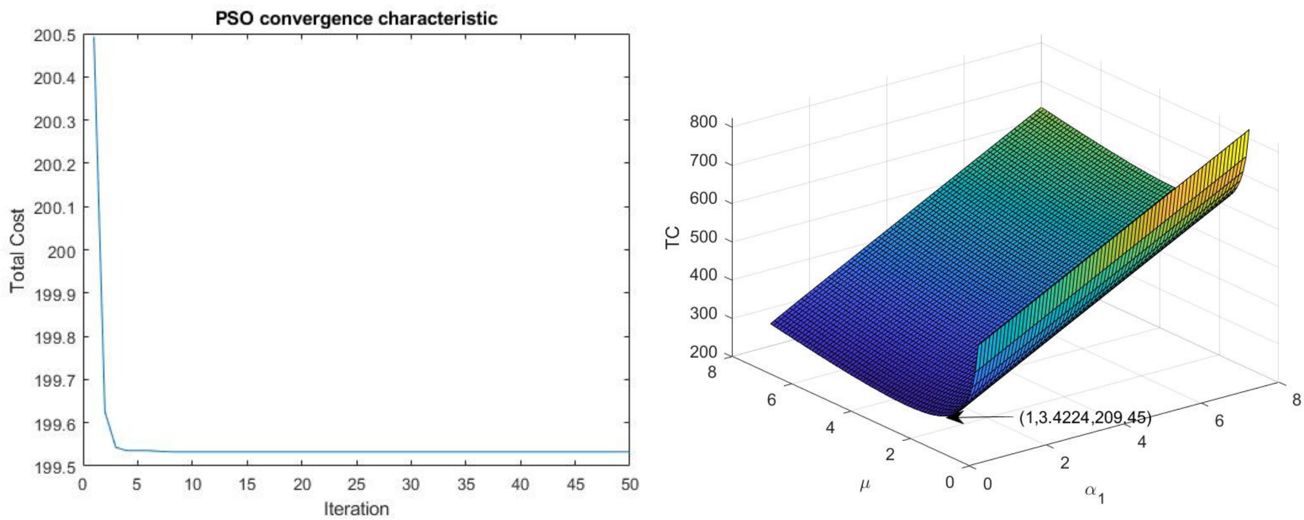
Now, we apply the PSO algorithm to optimize our expense function described in Section 7. Figure 3a) represents the number of iterations required to attain the optimum expense in PSO. Here, we take $C_1 = 10, C_2 = 15, C_3 = 60, C_4 = 30, C_5 = 41, C_6 = 53, \lambda = 0.45, \alpha_1 = 0.91$ and $\alpha_2 = 3.9$. The curve pertaining to the optimal expense attains the steady state after 12th iteration. Hence the minimum expected expense $TC = 199.5328$ is obtained at $\mu^* = 1.845$. For fixed values of $\lambda = 0.45$ and $\alpha_1 = 0.2$, Table 1 lists the optimum values of μ together with the mean system length $E(N)^*$, the mean waiting



a) b)

Figure 2. $E(N)$ against λ for $M/M/1$ and $M^X/M/1$ queue

time of a customer $E(W)^*$ and the expense function f^*. A decreasing pattern is observed in all four concerning metrics for the surge in α_2. Table 2 shows the optimum values of μ together with $E(N)^*$, $E(W)^*$ and f^*. A decreasing pattern is observed in all four parameters considered in Table 2, when the surge occurs in α_1. Also, note that the decreasing pattern seen in all the metrics discussed in Table 2 is drastic compared to the decreasing pattern observed in all the metrics discussed in Table 1.



a) b)

Figure 3. Total expense against: a) iterations, b) μ and α_1

In Figure 3b), we consider the parameters $C_1 = 21, C_2 = 15, C_3 = 42, C_4 = 31, C_5 = 24, C_6 = 18, \alpha_2 = 8.2$ to optimize the expense function. The curve reaches the steady state after the 8th iteration. Hence the minimum expected expense, $TC = 209.45$ is acquired at $\alpha_1 = 1, \mu = 3.4224$. Corresponding to swarm optimization, Figure 3b) demonstrates the effect of α_1 and μ on the expense function.

Table 1. Expense analysis for various values of α_2

$\lambda = 0.45, \alpha_1 = 0.2$				
α_2	μ^*	$E(N)^*$	$E(W)^*$	f^*
0.3	2.1975	2.0352	4.5228	247.0979
0.4	2.1864	2.0108	4.4684	245.2667
0.45	2.1838	2.0047	4.4549	244.7950
0.5	2.1804	2.0005	4.4454	244.4712
0.58	2.1806	1.9966	4.4369	244.1382
0.6	2.1804	1.9959	4.4359	244.0787
0.65	2.1800	1.9945	4.4322	243.9586
0.7	2.1797	1.9936	4.4301	243.8697
0.73	2.1796	1.9931	4.4292	243.8272
0.8	2.1795	1.9924	4.4276	243.7536

Table 2. Expense analysis for various values of α_1

$\lambda = 0.45,\ \alpha_2 = 3.9$				
α_1	μ^*	$E(N)^*$	$E(W)^*$	f^*
0.41	1.9483	1.5074	3.3498	212.7395
0.52	1.9048	1.4081	3.1291	206.9547
0.59	1.8867	1.3665	3.0366	204.6079
0.62	1.8805	1.3520	3.0044	203.8086
0.67	1.8715	1.3315	2.9588	202.6854
0.73	1.8628	1.3114	2.9142	201.6099
0.77	1.8579	1.3002	2.8894	201.0220
0.82	1.8526	1.2883	2.8630	200.4018
0.86	1.8490	1.2801	2.8446	199.9812
0.91	1.8450	1.2712	2.8248	199.5328

10. Conclusion

In this paper, we discussed an $M^X/M/1$ queue with differentiated hiatuses. We obtained the steady-state probabilities involving in our model through PGFs and stability condition. We also derived stochastic decomposition property of system size distribution. Waiting time distribution of an arbitrary beneficiary in the system is carried out. We also considered a cost optimization problem using particle swarm optimization (PSO). Further, we investigated the impact of parameters on the performance indices and the cost functions of the system through numerical arguments. The queueing model may be considered as a generalized of the model of discussed by Ibe [11]. The considered model can be further generalize with the following general type service times. Furthermore, the analysed model can be generalized with unreliable server and repair times.

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