ON DYNAMIC CREDIBILITY MODELS

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Elżbieta Ferenstein

Warsaw University of Technology, Poland
Polish-Japanese Academy of Information Technology, Poland
e-mail: elzbieta.ferenstein@pja.edu.pl

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Abstract: The Bühlamnn-Straub credibility model is extended for risk profiles varying with time. This is a special case of an evolutionary credibility model with risk parameter changing with time according to an unobserved sequence of random variables. The exact formulas of Bayes premiums are obtained for models in which the probability distributions of risk profiles switch to others at random unobserved time periods with known distributions. In particular, for exponential type class of distributions with conjugate priors, the Bayes premium is obtained recursively from a non-linear multidimensional Kalman type filter.

Keywords: Bayes premium, Bühlmann-Straub model, credibility model, exponential dispersion models, dispersion distribution, conjugate priors.

1. Introduction

The problem of finding Bayes insurance premiums in Bühlmann-Straub type models with risk parameter distributions varying with time is considered here. The author assumed that *I* independent contracts from an inhomogeneous portfolio are observed during T periods. Hence there are given random variables (r.v's) X_{it}, where $i = 1, 2, \dots, I$ and $t = 1, 2, .., T$. i denotes the individual contract (risk) and t is the time period. X_{it} represents the claim size (number of claims) related to risk i at t. The goal of an insurer is to predict the value of a r. v. $X_{i,T+1}$ or to estimate the net individual premium for contract *i* and period $T + 1$. In the Bühlmann model, for each contract i, the joint distribution of $X_{it}, t = 1, 2, \dots, T + 1$, depends on an unobserved random variable Θ_i, called the risk parameter (risk profile), interpreted as an unobserved random feature of policyholder *i* which determines the probability distribution of X_{it} at any time *t*. The basic assumption of the model is the conditional independence of X_{it},

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$t = 1, 2, .., T$ given Θ_i, for each $i = 1, 2, ..., I$, and this distribution does not vary with the passage of time. Moreover, the independence of contracts means that the vector-valued random variables related to each contract in a portfolio are independent. In the paper the author relaxes the former and finds Bayes premiums under a weaker assumption with the practical meaning that the a priori distribution of a risk profile changes at an unobserved random time N to the other known distribution. In such case the insurer does not know the true distribution of a risk profile. The paper analyses two variations of Bühlmann-Straub models with switching risk profiles at N. The portfolio model analysed in this paper may be considered as a dynamic extension of the claim dependence with common effects in the credibility models presented in Yeo and Valdez (2006), which derived Bayes premiums for normal common effects. The credibility premiums for such models were found in Wen, Wu and Zhou (2009).

The modern credibility theory starts with the seminal papers by Bühlmann (1967) and Bühlmann and Straub (1970). Credibility models were then generalised and analysed by many other authors. Jewell (1975) introduced hierarchical credibility, and Hachemeister (1975) regression credibility models. There are many papers addressing the time dependence issue in credibility models which is very natural in insurance practice; Jewell (1975) called them evolutionary credibility models. Such models for claim amounts with time risk profiles varying over time were investigated by Gerber and Jones (1975), Jewell (1975, 1976), Sundt (1981, 1983), Kremer (1982). Evolutionary (also called dynamic) credibility models for claim numbers were considered by many authors, among them Albrecht (1985), Bolancé, Guillen and Pinquet (2003), Bolancé et al. (2007), Purcaru and Denuit (2002, 2003). Various time dependence ideas are discussed in the textbook by Denuit et al. (2005). There are many papers dealing with credibility models for exponential dispersion distributions and their conjugate priors, in which case Bayes premiums are credibility premiums, i.e. they are convex combinations of theoretical and empirical means.

The textbook by Bühlmann and Gisler (2005), devoted entirely to credibility models, is a rich source of references. Goovaerts et al. (1990), Sundt (1983), Denuit et al. (2005, 2007) present the main non-life insurance issues and actuarial modelling of dependent risks. Schmidt (1998) gives a survey of Bayesian models in actuarial mathematics. In classical credibility modelling, the insurer's goal is to estimate the individual mean claim amount or claim frequency. There are other risk characteristics of interests, e.g. Pitselis (2013) developed quantile credibility models in which quantiles are embedded in Bühlmann (1967)

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credibility and Hachemeister (1975) regression credibility models, while Kim and Jeon (2013) analysed credibility theory based on trimming.

In this paper the author presents an evolutionary model which can be called a random sequence (e.g. Markov) modulated credibility model described in Section 2. Each policyholder of an heterogeneous portfolio is characterized by a set of risk profiles. At each time period of an insurance contract only one risk profile is active and it is chosen according to the value of the current state of a random environment (e.g. the state of an unobserved Markov chain). In Section 3, Bayes premium is obtained in cases when there are only two possible risk profiles with corresponding individual claim amount (frequency) distributions, which means that there are two possible credibility models. Initially the contract is fully characterized by one model, then, with the passage of time, at a random time period there is a change of this model to the other one. One obtains the Bayes premium depending on Bayes premiums for both credibility models and the probability distribution of the moment of switching the models. The paper also considers the simpler case when the change is possible only at the determined time period. In Section 4, precise forms of Bayes premiums are obtained when in both credibility models the claim amounts and risk profiles have distributions from dispersion exponential family and its conjugate priors, respectively. Some of the above Bayes premiums were calculated in Chowańska (2012).

2. The portfolio model

We consider a dynamic model of a portfolio of I individual contracts observed during T time periods. For each contract i, random vector $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{iT})$ denotes the contract history, X_{it} is the claim amount (frequency) for the period t, and our interest is to predict $X_{i,T+1}$ for each i on the basis of all observed vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_T$ and to establish an insurance premium for each policyholder.

Suppose that d-variate random vector $\boldsymbol{\Theta}_i = (\Theta_{i1}, \ldots \Theta_{id})$ is an unobserved individual random characteristic of contract i. Moreover, there is an unobserved overall portfolio random characteristic $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_t, \ldots)$, a sequence of random variables with values in set $S = \{1, \ldots, d\}$. Y_t is a random factor influencing claim amount of each contract i at t so that the conditional distribution of X_{it} given $\boldsymbol{\Theta}_i, \boldsymbol{Y}$ is the conditional distribution of X_{it} given Θ_{iY_t}. Let us denote $\boldsymbol{\Theta} = (\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, \ldots, \boldsymbol{\Theta}_I)$ and $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_{T+1})$, and assume that the following conditions hold:

C1. $\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, \ldots, \boldsymbol{\Theta}_I, \boldsymbol{Y}$ are independent random sequences with known distributions.

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C2. Conditional cumulative distribution function of $\boldsymbol{X}$ given $\boldsymbol{\Theta} = \boldsymbol{\theta}, \boldsymbol{Y} = \boldsymbol{y}$ is of the form

$$F_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{y}) = \prod_{i=1}^{I} F_{\boldsymbol{X}_i|\boldsymbol{\Theta}_i,\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{\theta}_i,\boldsymbol{y}).$$

C3. Conditional distribution functions of $\boldsymbol{X}_i$ given $\boldsymbol{\Theta}_i = \boldsymbol{\theta}_i, \boldsymbol{Y} = \boldsymbol{y}$ are as follows

$$F_{\boldsymbol{X}_i|\boldsymbol{\Theta}_i,\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{\theta}_i,\boldsymbol{y}) = \prod_{t=1}^{T+1} F_{X_{it}|\Theta_{iY_t},Y_t}\left(x_t \middle| \theta_{i,y_t}, y_t\right), \text{ i} = 1, 2,..., \text{I}.$$

C4. For each *i* and *t*, $E|X_{it}| < \infty$.

Condition C3 means that for each contract *i* random claim amounts for subsequent time periods $X_{i1}, \dots, X_{iT}, X_{i,T+1}$ are conditionally, given $\boldsymbol{\Theta}_i = \boldsymbol{\theta}_i, \boldsymbol{Y} = \boldsymbol{y}$, independent and for each time period *t* the conditional distribution of X_{it} given $\boldsymbol{\Theta}_i = \boldsymbol{\theta}_i, \boldsymbol{Y} = \boldsymbol{y}$ is the conditional distribution of X_{it} given $\Theta_{iY_t} = \theta_{iy_t}, Y_t = y_t$. Therefore, the current state Y_t determines the risk profile Θ_{iY_t} of the contract.

Let us note that if we assume additionally that $\boldsymbol{Y}$ is a Markov chain and $(X_{it}|\boldsymbol{\Theta}_i, \boldsymbol{Y}) = \mu\left(\Theta_{i,Y_t}\right)$, $Var(X_{it}|\boldsymbol{\Theta}_i, \boldsymbol{Y}) = \sigma^2\left(\Theta_{i,Y_t}\right)/w_{it}$, then the model may be called the Markov modulated version of the Bühlmann--Straub credibility model.

Remark 1. Suppose that d = 1 and $Y_t = \Lambda, t = 1,2, \dots, T + 1$. Then, Conditions C2 to C4 become Assumptions A2, A4 and A5 formulated in Yeo and Valdez (2006). If additionally one assumes that the considered above distributions have probability density functions, then our portfolio model coincides with the one described in Yeo and Valdez (2006).

Next, we examine net type premiums:

$\mu_{it}\left(\Theta_{i,Y_t}, Y_t\right) = E\left(X_{it}\middle|\Theta_{i,Y_t}, Y_t\right)$ – the individual net premium,

$P_{i,T+1}^{B} = E\left(X_{i,T+1}\middle|\boldsymbol{X}_1, \dots, \boldsymbol{X}_T\right)$ – Bayes premium,

$P_{i,T+1}^{LB}$ – credibility premium (best linear predictor of $X_{i,T+1}$).

Remark 2. If **Y** is a Markov chain, then $\left\{\mu_{it}\left(\Theta_{i,Y_t}, Y_t\right)\right\}$ is the Markov process, $i = 1,2, \dots, I$.

Remark 3. Under Assumptions C1 to C4 we have

$$P_{i,T+1}^{B} = E\left(\mu_{i,T+1}\left(\Theta_{i,Y_{T+1}}, Y_{T+1}\right)\middle|\boldsymbol{X}_1, \dots, \boldsymbol{X}_T\right).$$

The above holds due to the tower property of the conditional expectation.

Recently, the credibility idea was applied for other claim characteristics, for instance for trimmed means in Kim and Jeon (2013), or quantiles in Pitselis (2013).

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3. Model with switching distributions

In this section the author obtains Bayes premiums for two specific models of the portfolio from Section 2, and considers one particular contract *i*. Thus index *i* will be omitted. The premium for X_{T+1} based on $X_1, \dots, X_T$ is of interest here.

Model 1. Suppose that the risk profile of the contract is a bivariate random variable $\boldsymbol{\Theta} = (\Theta_1, \Theta_2)$, with independent coordinates Θ_1, Θ_2, and their cumulative distribution functions U_1, U_2, have densities u_1, u_2. Let $F_1(x|\theta)$, $F_2(x|\theta)$, $x \in \mathbb{R}$, be cumulative distribution functions of claim sizes given $\Theta_1 = \theta, \Theta_2 = \theta$, respectively. Let *N* be a random variable taking values in the set of natural numbers with the probability distribution function $P(N = n) = g_n, n = 1,2 \dots$, and let $G_n = g_1 + \cdots + g_n$, $\bar{G}_n = 1 - G_n$. Assume that *N*, Θ are independent. Let $Y_t := \mathbb{I}(N \leq t)$, $t = 1,2, \dots$, and $\mathbb{I}(A)$ denote the indicator function of event *A*.

Assume that

C5. $P(X_t \leq x|\boldsymbol{\Theta}, Y_t) = (1 - Y_t)F_1(x|\Theta_1) + Y_t F_2(x|\Theta_2)$.

According to C5, *N* may be interpreted as the switching time that is beginning from the *N*-th time period when there is a change of risk profile Θ_1 into Θ_2, and the probability distribution of the claim amounts $F_1(x|\theta)$ changes into $F_2(x|\theta)$, under various values of risk profiles *θ*. Conditions C1 to C3 under the assumptions of Model 1 are fulfilled. C3 has a simpler form:

$$F_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Y}}(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{y}) = \prod_{t=1}^{T+1}[(1 - y_t)F_1(x_t|\theta_1) + y_t F_2(x_t|\theta_2)].$$

C4 is expressed as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x| dF_i(x|\theta) dU_i(\theta) < \infty, i = 1,2.$$

Remark 4. Model 1 may be modified in such a way that at moment *N* of switching, there is random choice of a risk profile with the corresponding distribution of the claim amounts, from the given set of profiles and distributions of the claim amounts.

Model 2. Assume that the assumptions of Model 1 are fulfilled, apart from the assumption on $\boldsymbol{Y} = (Y_1, Y_2, \dots, Y_{T+1}, \dots)$. Let $Y_t = \Lambda$, $t = 1, 2, \dots$, and Λ is the Bernoulli distributed random variable with probability law $p(1) = \pi = 1 - p(0), \pi \in (0,1)$. Then, Condition C5 becomes

C6. $P(X_t \leq x|\boldsymbol{\Theta}, \Lambda) = (1 - \Lambda)F_1(x|\Theta_1) + \Lambda F_2(x|\Theta_2)$.

Denote $\mu_i(\theta) = \int_{-\infty}^{\infty} x dF_i(x|\theta)$, $i = 1,2$, $x_{k:t} = (x_k, \dots, x_t)$,

$$X_{k:t} = (X_k, \dots, X_t), k = 1, 2, \dots, t \leq T + 1.$$

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One can derive the Bayes premiums for Models 1 and 2. From C5 and C6 one obtains the individual premiums for $T + 1$, respectively:

$$\mu(\boldsymbol{\Theta}, \boldsymbol{Y}) = (1 - Y_{T+1})\mu_1(\Theta_1) + Y_{T+1}\mu_2(\Theta_2),$$

$$\mu(\boldsymbol{\Theta}, \boldsymbol{\Lambda}) = (1 - \Lambda)\mu_1(\Theta_1) + \Lambda\mu_2(\Theta_2),$$

and the Bayes premiums:

$$P_{T+1}^B = E((1 - Y_{T+1})\mu_1(\Theta_1)|X_{1:T}) + E(Y_{T+1}\mu_2(\Theta_2)|X_{1:T}),$$

$$P_{T+1}^B = E\big((1 - \Lambda)\mu_1(\Theta_1)\big|X_{1:T}\big) + E(\Lambda\mu_2(\Theta_2)|X_{1:T}).$$

Proposition 1. Suppose that the assumptions of Model 1 are fulfilled and there exist densities $f_1(\cdot\,|\theta), f_2(\cdot\,|\theta)$ of the cumulative distribution functions $F_1(\cdot\,|\theta), F_2(\cdot\,|\theta)$. Then, the Bayes premium is of the form

$$P_{T+1}^B = A_0\mu_1^B(X_{1:T}) + \sum_{k=1}^T A_k\mu_2^B(X_{k:T}) + A_{T+1}^0\mu_2^0,$$

where $\mu_2^0 = E\mu_2(\Theta_2)$ and, for i = 1,2, $\mu_i^B(X_{1:T}) = E_i(\mu_i(\Theta_i)|X_{1:T})$

is the Bayes premium calculated for the Bühlmann model with characteristics $\Theta_i, F_i(x|\theta)$,

$$A_0 = \bar{G}_{T+1}\frac{f_1(X_{1:T})}{f(X_{1:T})}, A_k = \frac{g_k f_1(X_{1:k-1})f_2(X_{k:T})}{f(X_{1:T})}, A_{T+1}^0 = g_{T+1}\frac{f_1(X_{1:T})}{f(X_{1:T})},$$

$k = 1, 2, \dots, T + 1$, where

$f_i(x_{k:t})$ denotes the joint density of the cumulative distribution function

$$F_i(x_{k:t}) = \int_{-\infty}^{\infty}\prod_{j=k}^t F_i\big(x_j\big|\theta\big)\,dU_i(\theta), f_i(x_{1:0}) = 1, f_i(x_{T+1:T}) = 1$$

and $f(x_{1:T})$ is the joint density of $X_{1:T}$:

$f(x_{1:T}) = \sum_{n=1}^T g_n f_1(x_{1:n-1})f_2(x_{n:T}) + \bar{G}_T f_1(x_{1:T})$.

Proof. The proof is presented in Appendix.

Example 1. Suppose that N is geometrically distributed, i.e. $g_k = pq^{k-1}, k = 1,2, \dots$, and, for i = 1, 2, Θ_i has Gamma distribution with the shape and scale parameters α_i, β_i, respectively, $F_i(x|\theta) = 1 - e^{-\theta x}$, $x > 0$. Then,

$$\mu_1^B(X_{1:T}) = \frac{\beta_1 + \sum_{j=1}^T X_j}{\alpha_1 + T - 1}, \mu_2^B(X_{k:T}) = \frac{\beta_2 + \sum_{j=k}^T X_j}{\alpha_2 + T - k}, \mu_2^0 = \frac{\beta_2}{\alpha_2 - 1},$$

Note that $\bar{G}_T = q^T$, therefore we get the Bayes premium in Proposition 1 with

$$A_0 = q^{T+1}\Big/\left(q^T + \sum_{n=1}^T pq^{n-1}\frac{f_1(X_{1:n-1})f_2(X_{n:T})}{f_1(X_{1:T})}\right),$$

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$$A_k = pq^{k-1} f_1(X_{1:k-1}) f_2(X_{k:T}) / \Big(q^T f_1(X_{1:T}) + \sum_{n=1}^{T} pq^{n-1} f_1(X_{1:n-1}) f_2(X_{n:T}) \Big),$$

$$A_{T+1}^0 = pq^{-1} A_0,$$

where for i = 1, 2 and any r – dimensional vector $(z_1, z_2, \ldots, z_r)$ densities $f_i(\cdot)$ are as follows

$$f_i(z_1, z_2, \ldots, z_r) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \cdot \frac{\Gamma(\alpha_i + r)}{\left(\beta_i + \sum_{j=1}^{r} z_j\right)^{\alpha_i + r}}.$$

Note that the Bayes premium in Example 1 cannot be expressed in a recursive form with updated observations of claim sizes, although in both Bühlmann models the Bayes premiums are credibility premiums. The reason is that $g_k > 0$ for any k. In the following example we assume that there is only one possible moment K of switching the distributions.

Example 2. Suppose that the assumptions of Example 1 are fulfilled, apart from the distribution of the switching time period N. Let $K \leq T$ be the fixed and known moment of a possible change in the distribution of claim sizes within the observation period, i.e. $g_K = p \in (0,1)$, $\bar{G}_{T+1} = 1 - p$.

Now, applying Proposition 1 one obtains the Bayes premium as follows

$$P_{T+1}^B = A_0 \mu_1^B(X_{1:T}) + A_K \mu_2^B(X_{K:T}),$$

where

$$A_0 = (1-p) \frac{f_1(X_{1:T})}{f(X_{1:T})}, A_K = p \frac{f_1(X_{1:K-1}) f_2(X_{K:T})}{f(X_{1:T})},$$

$f(x_{1:T}) = (1-p) f_1(x_{1:T}) + p f_1(x_{1:K-1}) f_2(x_{K:T})$, f_i, $i = 1{,}2$, are as in Example 1.

Let us examine closer the form of the Bayes premium in Example 2. Denote $x_{\cdot 1:r} = \sum_{j=1}^{r} x_j$. Then, for $T > K$, we obtain

$$P_{T+1}^B = \Phi_{T+1}(X_{\cdot 1:T}, X_{\cdot 1:K-1}) = \Phi_{T+1}(X_{\cdot 1:T-1}, X_T, X_{\cdot 1:K-1}),$$
$$P_T^B = \Phi_T(X_{\cdot 1:T-1}, X_{\cdot 1:K-1}),$$

where the precise forms of Φ_{T+1}, Φ_T may be obtained from the formulas for the densities f_1, f_2, and for the Bayes premiums μ_1^B, μ_2^B, which are functions of the appropriate sums of the observed claims $\{X_t\}$. Therefore, the subsequent premiums may be updated recursively in a non-linear form. Thus we have a form of a non-linear Kalman filter. It is possible to express P_{T+1}^B as the non-linear function of P_T^B, $X_T, X_{\cdot 1:K-1}$.

In the below Proposition 2 we arrive at the Bayes premium for Model 2.

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Proposition 2. Suppose that the assumptions of Model 2 are fulfilled and there exist densities $f_1(\cdot|\theta), f_2(\cdot|\theta)$ of cumulative distribution functions $F_1(\cdot|\theta), F_2(\cdot|\theta)$. Then, the Bayes premium is of the form

$$P_{T+1}^B = \frac{\pi f_1(X_{1:T})}{\pi f_1(X_{1:T})+(1-\pi)f_2(X_{1:T})}\mu_1^B(X_{1:T}) + \frac{(1-\pi)f_2(X_{1:T})}{\pi f_1(X_{1:T})+(1-\pi)f_2(X_{1:T})}\mu_2^B(X_{k:T}),$$

where $\mu_1^B(X_{1:T}) = E_1(\mu_1(\Theta_1)|X_{1:T})$, $\mu_2^B(X_{k:T}) = E_2(\mu_2(\Theta_2)|X_{k:T})$.

Proof of Proposition 2 is similar to the proof of Proposition 1, so it is omitted.

Remark 5. Observe that the Bayes premiums given in Example 2 and Proposition 2 have a similar structure. In Example 2 the moment of switching the distributions of risk profiles and claim amounts may occur with probability p at fixed period $1 \le K \le T$ or with probability $1-p$ is greater than $T+1$. In Proposition 2 the ‘nature’ decides on the above distributions before the observation period. One may combine Models 1 and 2, assuming that the switching may occur at moment *N* with the distribution law as below

$$P(N=0) = \pi = 1 - P(N>0),$$

$$P(N=k) = (1-\pi)g_k, \sum_{k=1}^{\infty} g_k = 1.$$

4. Model with switching for exponential families

In this section the author provides a formula for the Bayes premium in the model switching from one Bühlmann-Straub model to the other one, assuming that the claim amounts distributions, given fixed risk profiles, are exponential, and the risk profiles laws belong to the class of conjugate priors in both models, before and after switching.

Proposition 3. *Assume C1 to C5 and let distribution functions* $F_{it}(\cdot|\Theta_j), i = 1,2,$ have exponential type densities with respect to measure ν which is either the Lebesgue measure or the counting measure such that

$$f_{it}(x|\theta) = \exp\left[\frac{x\theta - b_i(\theta)}{\sigma_i^2/w_t} + c_i(x, \sigma_i^2/w_t)\right], x \in R.$$

Assume that the risk profiles before and after switching are real valued continuous random variables with densities, for *i* =1, 2,

$$u_{i\gamma_i}(\theta) = \exp\left[\frac{x_{0i}\theta - b_i(\theta)}{\tau_i^2} + d_i(x_{0i}, \tau_i^2)\right], \theta \in R,$$

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where $\gamma_i = \left(x_{0i}, \tau_i^2\right) \in \Gamma_i$ is fixed. Then, the Bayes premium is of the form

$$P_{T+1}^B = P_{1,T+1}^B + P_{2,T+1}^B + P_{3,T+1}^B,$$

$$\text{where} \quad P_{1,T+1}^B = \frac{\bar{G}_{T+1} f_1(X_{1:T})}{f(X_{1:T})}\left[\alpha_{1,1}\bar{X}_{1,T} + \left(1-\alpha_{1,1}\right)x_{01}\right],$$

$$P_{2,T+1}^B = \sum_{k=1}^{T} \frac{g_k f_1(X_{1:k-1}) f_2(X_{k:T})}{f(X_{1:T})}\left[\alpha_{2,k}\bar{X}_{k,T} + \left(1-\alpha_{2,k}\right)x_{02}\right],$$

$$P_{3,T+1}^B = g_{T+1}\frac{f_1(X_{1:T})}{f(X_{1:T})}x_{02},$$

and for $k = 1,2,\dots,T$, $i = 1,2$,

$$\bar{X}_{k,T} = \sum_{t=k}^{T} \frac{w_t}{w_{\cdot k,T}} X_t\,,\ w_{\cdot k,T} = \sum_{t=k}^{T} w_t\,,\ \alpha_{i,k} = \frac{w_{\cdot k,T}}{w_{\cdot k,T} + \sigma_i^2/\tau_i^2},$$

$$f_i(x_{k:t}) = \exp\left[-d_i\left(x_{0i}^{k,t}, \tau_{i,k,t}^2\right) + d_i\left(x_{0i}, \tau_i^2\right) + \sum_{r=k}^{t} c_i\left(x_r, \frac{\sigma_i^2}{w_r}\right)\right],\ x_{k:t} \in R^{t-k+1},$$

where $x_{0i}^{k,t} = \left(\bar{x}_{k,t} + \frac{x_{0i}\sigma_i^2}{w_{\cdot k,t}\tau_i^2}\right)\left(1 + \frac{\sigma_i^2}{w_{\cdot k,t}\tau_i^2}\right)^{-1}$, $\tau_{i,k,t}^2 = \sigma_i^2\left(w_{\cdot k,t} + \frac{\sigma_i^2}{\tau_i^2}\right)^{-1}$, $k \le t$,

$$f(x_{1:T}) = \sum_{n=1}^{T} g_n f_1(x_{1:n-1}) f_2(x_{n:T}) + \bar{G}_T f_1(x_{1:T}).$$

Proof. The proof is presented in Appendix.

Remark 6. Analysing the Bayes premium obtained in Proposition 3, one can conclude that, in general, there is no recursion form of the Bayes premium in Model 1, although there are recursions for the Bayes premiums in both possible models – before and after switching. To calculate P_{T+1}^B one needs all observations up to *T*. In Proposition 4 one obtains the recursive algorithm for P_{T+1}^B in Model 2. The algorithm is in the updated form – it uses the currently observed claim amount and some functions of previous observations, also obtained recursively. Thus one obtains Kalman-type non-linear filtering formulas.

Proposition 4. Assume Model 2 and let the densities $f_{it}(\cdot \mid \cdot)$, $u_{i\gamma_i}(\cdot)$, i =1,2, satisfy assumptions of Proposition 3. Then,

$$P_{T+1}^B = \frac{\pi}{\pi + (1-\pi)L_T} \cdot \mu_1^B(X_{1:T}) + \frac{1-\pi}{\pi + (1-\pi)L_T^{-1}} \cdot \mu_2^B(X_{k:T}),$$

where

$$L_T := \frac{f_2(X_{1:T})}{f_1(X_{1:T})} = \Phi\big(C_1(X_{1:T}), C_2(X_{1:T}), x_{01}(1,T), x_{02}(1,T), w_1(1,T), w_2(1,T)\big),$$

$$\Phi(C_1, C_2, x_1, x_2, w_1, w_2) \coloneqq$$

$$\exp\left[-d_2\left(\frac{x_2}{w_2}, \frac{1}{w_2}\right) + d_1\left(\frac{x_1}{w_1}, \frac{1}{w_1}\right) + C_2 - C_1 + d_2(x_{02}, \tau_2^2) - d_1(x_{01}, \tau_1^2)\right],$$

and, for i =1,2,

$$C_i(X_{1:T+1}) = C_i(X_{1:T}) + c_i\left(X_{T+1}, \frac{\sigma_i^2}{w_{T+1}}\right), C_i(X_{1:1}) = d_i\big(x_{0i}, \tau_i^2\big) + c_i\left(X_1, \frac{\sigma_i^2}{w_1}\right),$$

$$x_{0i}(1, T+1) = x_{0i}(1,T) + \frac{X_{T+1}w_{T+1}}{\sigma_i^2}, x_{0i}(1,1) = \frac{x_{0i}}{\tau_i^2} + \frac{X_1 w_1}{\sigma_i^2},$$

$$w_i(1, T+1) = w_i(1,T) + \frac{w_{T+1}}{\sigma_i^2}, w_i(1,1) = \frac{1}{\tau_i^2} + \frac{w_{T+1}}{\sigma_i^2}.$$

Proof. The proof is presented in Appendix.

Remark 7. The Bayes premium in Proposition 4 is obtained as a result of the non-linear multidimensional Kalman filter.

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Appendix

Proof of Proposition 1: Let us note that under the assumptions of the model we have

$$P(X_1 \leq x_1, \dots, X_{T+1} \leq x_{T+1} | N = n, \Theta_1 = \theta_1, \Theta_2 = \theta_2) = \prod_{k=1}^{n-1} F_1(x_k|\theta_1) \cdot \prod_{r=n}^{T+1} F_2(x_r|\theta_2),$$

where the first or second products on the right side of the above equality are supposed to be 1 if $n = 1$ or $n > T + 1$. Hence, before the moment of switching at N we have the Bühlmann model with parameters $\Theta_1, F_1(\cdot\,|\,\cdot)$ and starting from time period N it becomes the model with $\Theta_2, F_2(\cdot\,|\,\cdot)$. Individual premiums for both Bühlmann models are as follows

$$\mu_i(\Theta_i) = \int_{-\infty}^{\infty} x dF_i(x|\Theta_i), i = 1,2.$$

Thus, the individual premium in Model 1, assuming the general risk parameter (N, Θ_1, Θ_2), has the form

$$P_{T+1}^{ind} = \mathbb{I}(N > T + 1)\mu_1(\Theta_1) + \mathbb{I}(N \leq T + 1)\mu_2(\Theta_2),$$

and the Bayes premium is

$$P_{T+1}^{B} = P_{1,T+1}^{B} + P_{2,T+1}^{B},$$

where

$$P_{1,T+1}^{B} = E(\mathbb{I}(N > T + 1)\mu_1(\Theta_1)|X_{1:T}), P_{2,T+1}^{B} = E(\mathbb{I}(N \leq T + 1)\mu_2(\Theta_2)|X_{1:T}).$$

Denote

$$f(n, \theta_1, \theta_2 | x_{1:T}) = \frac{\partial^2}{\partial\theta_1\partial\theta_2} P(N = n, \Theta_1 \leq \theta_1, \Theta_2 \leq \theta_2 | X_{1:T} = x_{1:T}),$$

and for any k, $i = 1,2$, $f_i(x_1, \dots, x_k, \theta) = \prod_{r=1}^{k} f_i(x_r|\theta)u_i(\theta)$, $(x_1, \dots, x_k) \in R^k$, $\theta \in R^1$,

$$f_i(x_1, \dots, x_k) = \int_{-\infty}^{\infty} f_i(x_1, \dots, x_k, \theta) d\theta.$$

Note that the above density is the joint density of k claim sizes in the Bühlmann model with characteristics Θ_i and $F_i(\cdot\,|\,\cdot)$.

The joint conditional probability-density function of risk parameter (N, Θ_1, Θ_2), given the observed claim sizes $x_{1:T}$, is

$$f(n, \theta_1, \theta_2 | x_{1:T}) = \frac{g_n f_1(x_{1:T}, \theta_1) u_2(\theta_2)}{f(x_{1:T})}, \text{ for } n > T,$$

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$$f(n,\theta_1,\theta_2|x_{1:T}) = \frac{g_n f_1(x_{1:n-1},\theta_1) f_2(x_{n:T},\theta_2)}{f(x_{1:T})}, \text{for } n \le T,$$

where $f(x_{1:T}) = \sum_{n=1}^{T} g_n f_1(x_{1:n-1}) f_2(x_{n:T}) + \bar{G}_T f_1(x_{1:T}),\ x_{1:T} \in R^T$, is the density function of the observed claims $X_{1:T}$ during T time periods. In the above conditional densities the assumption of independence of N, Θ_1, Θ_2 was used Hence

$$E(\mathbb{I}(N > T+1)\mu_1(\Theta_1)|X_{1:T} = x_{1:T}) =$$

$$\sum_{n=T+2}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_1(\theta_1) f(n,\theta_1,\theta_2|x_{1:T}) d\theta_1 d\theta_2 =$$

$$\sum_{n=T+2}^{\infty} g_n \int_{-\infty}^{\infty} \mu_1(\theta_1) \frac{f_1(x_{1:T},\theta_1)}{f_1(x_{1:T})} d\theta_1 \cdot \frac{f_1(x_{1:T})}{f(x_{1:T})} = \bar{G}_{T+1} \cdot \mu_1^B(x_{1:T}) \cdot \frac{f_1(x_{1:T})}{f(x_{1:T})},$$

$$E(\mathbb{I}(N \le T+1)\mu_2(\Theta_2)|X_{1:T} = x_{1:T}) =$$
$$\sum_{n=1}^{T+1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_2(\theta_2) f(n,\theta_1,\theta_2|x_{1:T}) d\theta_1 d\theta_2 =$$

$$\sum_{n=1}^{T} g_n \int_{-\infty}^{\infty} \mu_2(\theta_2) \frac{f_2(x_{n:T},\theta_2)}{f_2(x_{n:T})} d\theta_2 \cdot \int_{-\infty}^{\infty} \frac{f_1(x_{1:n-1},\theta_1) f_2(x_{n:T})}{f(x_{1:T})} d\theta_1 +$$

$$g_{T+1} \int_{-\infty}^{\infty} \mu_2(\theta_2) u_2(\theta_2) d\theta_2 \cdot \frac{f_1(x_{1:T})}{f(x_{1:T})} =$$

$$\sum_{n=1}^{T} g_n \mu_2^B(x_{n:T}) \cdot \frac{f_1(x_{1:n-1}) f_2(x_{n:T})}{f(x_{1:T})} + g_{T+1} E[\mu_2(\Theta_2)] \cdot \frac{f_1(x_{1:T})}{f(x_{1:T})}.$$

The above derivations give us the Bayes premium in Proposition 1.

Proof of Proposition 3: We apply the form of Bayes premium from Proposition 1. First, to find $\mu_1^B(X_{1:T}), \mu_2^B(X_{k:T})$ we use the Bayes premiums for exponential classes of distributions and their conjugate priors derived in Bühlmann and Gisler (2005, 2.5.1). Hence, if there is no change of the model up to time period T we have

$$\mu_1^B(X_{1:T}) = \alpha_{1,1}\bar{X}_{1,T} + \left(1 - \alpha_{1,1}\right) x_{01},$$

where $\bar{X}_{1,T} = \sum_{k=1}^{T} \frac{w_k}{w.} X_k$, $\overline{w}. = \sum_{k=1}^{T} w_k$, $\alpha_{1,1} = \frac{w.}{w. + \sigma_1^2/\tau_1^2}$.

The Bayes premium for Model 2 based on observations $X_{k:T}$ is

$$\mu_2^B(X_{k:T}) = \alpha_{2,k}\bar{X}_{k,T} + \left(1 - \alpha_{2,k}\right) x_{02},$$

where $\bar{X}_{k,T} = \sum_{t=k}^{T} \frac{w_t}{w_{\cdot k,T}} X_t$, $w_{\cdot k,T} = \sum_{t=k}^{T} w_t$, $\alpha_{2,k} = \frac{w_{\cdot k,T}}{w_{\cdot k,T} + \sigma_2^2/\tau_2^2}$.

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Now it is sufficient to derive for i = 1, 2, and $1 \leq k \leq t \leq T$, the boundary densities $f_i(x_{k:t})$. Using the assumed densities in the formulation of Proposition 3 we obtain

$$f_i(x_{k:t}) = \int_{-\infty}^{\infty} \left(\prod_{r=k}^{t} \exp\left[\frac{x_r\theta - b_i(\theta)}{\sigma_i^2 / w_r}\right]\right) \exp\left[\frac{x_{0i}\theta - b_i(\theta)}{\tau_i^2}\right] d\theta \cdot \exp[C_i(x_{k:t})],$$

where $x_{k:t} \in R^{t-k+1}$, $C_i(x_{k:t}) = \sum_{r=k}^{t} c_i\left(x_r, \frac{\sigma_i^2}{w_r}\right) + d_i\left(x_{0i}, \tau_i^2\right)$, which may be rewritten as follows

$$f_i(x_{k:t}) = \int_{-\infty}^{\infty} \exp[x_{0i}(k,t)\theta - b_i(\theta) w_i(k,t)]\, d\theta \cdot \exp[C_i(x_{k:t})],$$

where we introduced notations:

$$x_{0i}(k,t) = \sum_{r=k}^{t} \frac{x_r w_r}{\sigma_i^2} + \frac{x_{0i}}{\tau_i^2}, w_i(k,t) = \sum_{r=k}^{t} \frac{w_r}{\sigma_i^2} + \frac{1}{\tau_i^2}.$$

Let us introduce new parameters $x_{0i}^{k,t}$ and $\tau_{i,k,t}^2$ of conjugate prior distribution as

$$x_{0i}^{k,t} = \left(\bar{x}_{k,t} + \frac{x_{0i}\sigma_i^2}{w_{\cdot k,t}\tau_i^2}\right)\left(1 + \frac{\sigma_i^2}{w_{\cdot k,t}\tau_i^2}\right)^{-1} = \frac{x_{0i}(k,t)}{w_i(k,t)},$$

$$\tau_{i,k,t}^2 = \sigma_i^2 \left(w_{\cdot k,t} + \frac{\sigma_i^2}{\tau_i^2}\right)^{-1} = \frac{1}{w_i(k,t)}, \gamma_{i,k,t} = \left(x_{0i}^{k,t}, \tau_{i,k,t}^2\right),$$

Then, $f_i(x_{k:t})$ can be rewritten as follows:

$$f_i(x_{k:t}) = \exp\left(-d_i\left(\gamma_{i,k,t}\right)\right) \cdot \int_{-\infty}^{\infty} u_{i\gamma_{i,k,t}}(\theta) d\theta \cdot \exp[C_i(x_{k:t})],$$

which completes the proof since the integral above is equal to 1.

Proof of Proposition 4: The formula for P_{T+1}^B follows directly from Proposition 3 and its proof since we may write

$$f_i(x_{1:T}) = \exp\left[-d_i\left(\frac{x_{0i}(1,T)}{w_i(1,T)}, \frac{1}{w_i(1,T)}\right) + d_i(x_{0i}, \tau_i^2) + \sum_{r=1}^{T} c_i\left(x_r, \frac{\sigma_i^2}{w_r}\right)\right].$$

Thus the definitions of L_T, and $x_{0i}(1,T)$, $w_i(1,T)$ give us the recursion described in the proposition. Finally, we obtain the desired Bayes premium, which is a non-linear function of the particular statistics obtained recursively in the updated forms.

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O DYNAMICZNYCH MODELACH WIARYGODNOŚCI

Streszczenie: Model Bühlmanna-Strauba rozszerzono o profile ryzyka zmieniające się z czasem. Jest to szczególny przypadek ewolucyjnego modelu wiarygodności ze zmiennym w czasie parametrem ryzyka zależnym od nieobserwowanego ciągu zmiennych losowych. Otrzymano dokładną postać składek bayesowskich dla modeli, w których rozkłady prawdopodobieństwa profili ryzyka zmieniają się w nieobserwowanych losowych momentach o znanych rozkładach. Przede wszystkim dla wykładniczych klas rozkładów ze sprzężonymi rozkładami *a priori* otrzymano składkę bayesowską rekurencyjnie jako pewien nieliniowy wielowymiarowy filtr Kalmana.

Słowa kluczowe: składka bayesowska, model Bühlmanna-Strauba, model wiarygodności, dyspersyjne modele wykładnicze, sprzężone rozkłady *a priori*.