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## Compact Abelian groups and extensions of Haar measures

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## Introduction

By a *topology in a group G* we mean a family $\tau$ of subsets of $G$ such that $G$ becomes a topological space with the class of open sets $\tau$ and such that the group operation $xy^{-1}$ is a continuous function of two variables [1]. In other words, a topology in a group $G$ is a topology in the set $G$ such that $G$ becomes a topological group.

We say that a topology $\tau$ in a group $G$ is *compact* if the topological space obtained by introducing the topology $\tau$ in $G$ is compact.

Usually a given infinite group admits more than one compact topology. Hence a natural question to ask what are the relations among different topologies in a given group.

In this paper we confine ourself to considering only special aspect of this rather wide problem. First of all we consider only Abelian groups and only compact topologies in them. In this case the problem of the algebraic structure of groups which admit compact topologies was formulated by I. Kaplansky [8] and solved in [4] and [5]. Here we give the description of the Abelian groups which have exactly one compact topology and those for which any two compact topologies define the same (up to continuous isomorphism) topological group. The theorems on the algebraic structure of compact groups as well as the theorems we have just mentioned will serve us in the proof of the main theorem of this paper (see section 2) concerning the relations among various invariant measures defined in a group $G$ on various $\sigma$-fields of subsets each being generated by a compact topology in $G$. The investigations of this kind had been started by J. Łoś and treated by several authors afterwards.

## 1. Preliminaries (topology & measure)

We reserve the following notation: $\{a_t: t\in T\}$ denotes the set of elements $a_t$, $t\in T$, $\langle a_t\rangle_{t\in T}$ the sequence of not necessarily distinct elements $a_t$, $t\in T$, $\mathrm{gp}\{a_t: t\in T\}$ the group generated by $a_t$, $t\in T$, and $m\{a_t: t\in T\}$ the modulo over the ring of $p$-adic integers generated by elements $a_t$, $t\in T$.

If $\varphi$ is a mapping of a set $A$ into $B$, then the image of an element

(1) We do not distinguish between *equivalent* topologies, i.e. between every two topologies such that for each set $V$ of one of them there exists a set $W$ of the other contained in $V$.

$a \in A$ by the mapping $\varphi$ we denote by $a\varphi$. We reserve the notation $\varphi(a)$ only for the case when $B$ is the set of real numbers.

If $A, B, C, \ldots$ are families of subsets of a set $X$, then by $[A, B, C, \ldots]_\sigma$ we denote the least $\sigma$-field containing all the families $A, B, C, \ldots$ We say that the sets belonging to the families $A, B, C, \ldots$ *generate* the $\sigma$-field $[A, B, C, \ldots]_\sigma$.

Let $\tau$ be a topology in a group $G$, then by $G_\tau$ we denote the topological group defined by $G$ and $\tau$.

LEMMA 1.1. *The family $T$ of the compact topologies of a group $G$ has cardinal $\leqslant 2^{\overline{\overline{G}}}$.*

Proof. If $G$ is finite, then there is nothing to prove. If $G$ is infinite and $\tau$ is a compact topology in $G$, then there exists a base $\mathscr{B} = \{U\}$ of sets belonging to $\tau$ such that

$$\tau = \{V: V = \bigcup_{U \in \Delta} U, \Delta \subset \mathscr{B}\}$$

and $\overline{\overline{G}} \geqslant \overline{\overline{\mathscr{B}}}$. Thus the cardinal of $T$ is not greater than the cardinal $\mathfrak{m}$ of the family of the families of cardinal $\leqslant \overline{\overline{G}}$ of subsets of $B$, whence $\mathfrak{m} = (2^{\overline{\overline{G}}})^{\overline{\overline{G}}} = 2^{\overline{\overline{G}}}$.

It was proved in [7] that

1.2. *For each compact topology $\tau$ in a group $G$ there exists a family $\mathscr{B}$ such that each open set is a union of sets of the family $\mathscr{B}$ and that $2^{\overline{\overline{\mathscr{B}}}} = \overline{\overline{G}}$.*

By the *class of Baire sets* defined by a compact topology $\tau$ in a group $G$ we mean the least $\sigma$-field of subsets of $G$ containing the closed (compact) subsets $C$ of the form

$$C = \bigcap_{i=1}^{\infty} V_i \quad \text{for some } V_i \in \tau.$$

We denote it by $\mathscr{B}_\tau$.

LEMMA 1.3. *For each compact topology $\tau$ in an infinite group $G$ we have $\overline{\overline{\mathscr{B}}}_\tau \leqslant \overline{\overline{G}}$.*

Proof. For $\tau$ we select the family $\mathscr{B}$ the existence of which 1.2 asserts. If $C$ is one of the compact sets that generates $\mathscr{B}_\tau$, then

$$C = \bigcap_{n=1}^{\infty} \bigcup_{\lambda} U_\lambda^{(n)} = \bigcap_{n=1}^{\infty} \bigcup_{\lambda \in \delta} U_\lambda^{(n)},$$

where $U_\lambda^{(n)} \in \mathscr{B}$ and $\delta$ is a finite set. Hence the family of the compact sets generating $\mathscr{B}_\tau$ has cardinal at most $\overline{\overline{\mathscr{B}}}^{\aleph_0}$ and hence

$$\overline{\overline{\mathscr{B}}}_\tau = (\overline{\overline{\mathscr{B}}}^{\aleph_0})^{\aleph_0} = \overline{\overline{\mathscr{B}}}^{\aleph_0} \leqslant 2^{\overline{\overline{\mathscr{B}}}} = \overline{\overline{G}},$$

as required.

By the *Haar measure* induced by a compact topology $\tau$ in a group $G$ we mean the regular invariant measure $\mu_\tau$ defined on the $\sigma$-field of Baire sets $\mathscr{B}_\tau$, such that for each $V \in \tau \cap \mathscr{B}_\tau$ we have $\mu_\tau(V) > 0$ and $\mu_\tau(G) = 1$.

For each compact topology $\tau$ of a group $G$ the Haar measure exists and is unique (see e.g. [3], p. 263).

For each compact topology $\tau$ in a group $G$ we denote by $\mathscr{B}_\tau^*$ the family of the Baire sets of positive Haar measure $\mu_\tau$.

LEMMA 1.4. *For each compact topology $\tau$ in an infinite group $G$ every set $M$ of the family $\mathscr{B}_\tau^*$ has the cardinal $\overline{\overline{M}} = \overline{\overline{G}}$.*

The proof follows immediately from the well known theorem saying that the set $MM^{-1}$ contains a set $V$ belonging to $\tau$. Since $\tau$ is a compact topology, $G = a_1V \cup \ldots \cup a_kV$ for some $a_1, \ldots, a_k \in G$. Hence $\overline{\overline{G}} = \overline{\overline{V}} \leqslant \overline{\overline{M}} \leqslant \overline{\overline{G}}$.

For a group and a compact topology $\tau$ in it we denote by $\boldsymbol{B}_\tau$, and call the *class of Borel sets*, the least $\sigma$-field containing $\tau$. Evidently $\mathscr{B}_\tau \subset \boldsymbol{B}_\tau$. It is well known (see e. g. [3], p. 289) that the Haar measure $\mu_\tau$ can be extended uniquely from $\mathscr{B}_\tau$ to $\boldsymbol{B}_\tau$ and that for each set $A \in \boldsymbol{B}_\tau$ there is a set $M \in \mathscr{B}_\tau$ such that $A \subset M$ and $\mu_\tau(M \setminus A) = 0$.

For each compact topology $\tau$ of a group $G$ we select a class $\mathscr{B}_\tau^{**}$ of subsets of $G$ defined as follows:

For each set $M \in \mathscr{B}_\tau^*$ we choose a class $S(M) = \{A\}$ of Borel subsets $A$ of the set $M$, such that $\overline{\overline{S(M)}} = \overline{\overline{M}} = \overline{\overline{G}}$ and such that for each $A \in S(M)$ we have $\mu_\tau(A) > 0$. We define $\mathscr{B}_\tau^{**}$ putting

$$\mathscr{B}_\tau^{**} = \bigcup_{M \in \mathscr{B}_\tau^*} S(M).$$

A measure $\mu$ is a *common extension* of a family of measures $\{\mu_t \colon t \in T\}$, where for each $t \in T$ the measure $\mu_t$ is defined on a $\sigma$-field $B_t$ of subsets of a set $X$, if $\mu$ is defined on a $\sigma$-field $B$ containing all the $\sigma$-fields $B_t$, $t \in T$, and for each $M \in B_t$ we have $\mu(M) = \mu_t(M)$.

A measure $\mu$ defined on a $\sigma$-field $B$ of subsets of a group $G$ is *invariant* if for each $M \in B$ and $a \in G$ the set $aM \in B$ and $\mu(aM) = \mu(M)$.

Suppose that $\{\mu_t \colon t \in T\}$ is a family of measures such that for each $t \in T$ the measure $\mu_t$ is defined on a $\sigma$-field $\mathscr{B}_t$ of subsets of a set $X$ and there exists a common extension $\mu$ of the measures $\mu_t$, $t \in T$.

We say that the measures $\mu_t$, $t \in T$, are *independent in the common extension* $\mu$, if for every finite subset $\delta$ of the set $T$ and sets $M_t$, $M_t \in \mathscr{B}_t$, $t \in \delta$, we have

$$\mu\Big(\bigcap_{t \in \delta} M_t\Big) = \prod_{t \in \delta} \mu(M_t).$$

A set $T$ of compact topologies in a group $G$ we call *semi-regular* if the family $\{\mu_\tau \colon \tau \in T\}$ has a common invariant extension.

A set $T$ of compact topologies in a group $G$ we call *regular* if the family $\{\mu_\tau: \tau \in T\}$ has a common invariant extension in which the measures $\mu_\tau$, $\tau \in T$, are independent.

Let $G$ be a group and let $\tau_1$ and $\tau_2$ be two compact topologies in it. The topological groups $G_{\tau_1}$ and $G_{\tau_2}$ are called *topologically isomorphic* if there exists an isomorphism

$$\varphi: G \xrightarrow{\text{onto}} G$$

such that $V\varphi \in \tau_2$ for any $V \in \tau_1$ and $U\varphi^{-1} \in \tau_1$ for any $U \in \tau_2$. We then say that the topologies $\tau_1$ and $\tau_2$ are *isomorphic* and we write

$$\tau_1 \varphi = \tau_2.$$

If $\tau$ is a compact topology in a group $G$ and $\varphi$ an automorphism of $G$, then

$$\mathscr{B}_\tau \varphi = \mathscr{B}_{\tau\varphi}, \quad \mathscr{B}_\tau^* \varphi = \mathscr{B}_{\tau\varphi}^*, \quad \boldsymbol{B}_\tau \varphi = \boldsymbol{B}_{\tau\varphi}.$$

It follows from the uniqueness of the Haar measure that for each $M \in \mathscr{B}_\tau$ we have

$$\mu_\tau(M) = \mu_{\tau\varphi}(M_\varphi).$$

Measures $\mu_\tau$, $\mu_{\tau\varphi}$ satisfying the above equality we call *isomorphic*. We write $\mu_\tau \varphi = \mu_{\tau\varphi}$.

If $T$ and $S$ are two sets of topologies of a group $G$ and for each topology of one of the sets $S$, $T$ there exists an isomorphic topology in the other, then we say that the *sets $S$ and $T$ are isomorphic*.

If $T$ is a set of compact topologies in a group $G$ and $\Omega$ is an arbitrary set, then a mapping

$$(T\Omega): \Omega \to T$$

we call a *system* of topologies in the group $G$.

We say that a system $(T'\Omega)$ of topologies in a group $G$ is *isomorphic* with a system $(T''\Omega)$ if for each $\iota \in \Omega$ there exists an automorphism $\varphi_\iota$ of the group $G$ such that

$$\iota(T'\Omega)\varphi_\iota = \iota(T''\Omega) \quad \text{for any } \iota \in \Omega.$$

We say that a system $(T\Omega)$ of topologies in a group $G$ is *regular* if it is a one-to-one mapping and the set $T$ is regular.

If $G$ is a group, $\tau$ a topology in it and $\pi$ a homomorphism of $G$ onto a group $H$, then the family $\{V\pi: V \in \tau\}$ is a topology in $H$. We denote it by $\tau\pi$. If $\tau$ is compact, then $\tau\pi$ is also compact.

Let $G$ be a group, $\tau$ a topology in it and $H$ a subgroup of $G$ closed in the topology $\tau$. Then the family $\{V \cap H: V \in \tau\}$ is a topology in $H$; we denote it by $\tau \wedge H$. If $\tau$ is compact, then also $\tau \wedge H$ is compact.

Let $G$ be a compact topological group and $\nu$ its Haar measure. For any subsets $A$ of $G$ the outer measure $\nu^*(A)$ of $A$ is defined as the infimum of $\nu(M)$ for all Baire sets $M$ of $G$ such that $A \subset M$. In order that $\nu^*(A) = 1$ it is necessary and sufficient that $A \cap M \neq \emptyset$ for any Baire set $M$ with $\nu(M) > 0$. Let $A$ be a subset of $G$ with $\nu^*(A) = 1$. Let $\bar{\mathscr{B}}^* = \{M^*\}$ be the family of all subsets $M^*$ of $G$ of the form $M^* = A \cap M$, where $M$ is a Baire set. It is easy to see that $\bar{\mathscr{B}}^*$ is a $\sigma$-field of subsets of $A$ and that $\nu'(M^*)$ defined on $\bar{\mathscr{B}}^*$ by $\nu'(M^*) = \nu(M)$, where $M^* = A \cap M$ and $M$ is a Baire set, is a measure on $\bar{\mathscr{B}}^*$.

We close this section with a simple lemma which easily follows from well known theorems (see e. g. [3], p. 26-29).

LEMMA 1.5. *Let $E$ be a class of subsets of a set $X$. Suppose that $E$ contains a subclass $A$ such that*

(i) *for each $M \in A$ we have $X \setminus M \in A$,*

(ii) *for each finite collection of sets $M_1, \ldots, M_k$ of the class $A$ we have $\bigcap_{i=1}^{k} M_i \in E$,*

(iii) *for each finite collection of disjoint sets $M_1, \ldots, M_k$ of the class $E$ we have $\bigcup_{i=1}^{k} M_i \in E$,*

(iv) *if $M_1, M_2, \ldots$ is an ascending or descending sequence of sets belonging to the class $E$, then $\bigcup_{i=1}^{\infty} M_i \in E$ or $\bigcap_{i=1}^{\infty} M_i \in E$, respectively.*

*Then the $\sigma$-field generated by $A$ is contained in $E$.*

## 2. Problems and the theorem

In connection with his investigations on the foundations of the theory of probability J. Łoś has formulated the following problem:

*Given a group and two compact topologies $\tau_1, \tau_2$ in it. Does there exists a common extension of the two Haar measures $\mu_{\tau_1}$ and $\mu_{\tau_2}$?*

The answer to this problem is "*No*". P. Erdös and K. Urbanik proved that if $K$ is the group of rotations of the unite circle and $M$ a subset of $K$ such that $M$ is of the first category and has Lebesgue measure equal $2\pi$, then there exists an automorphism $\varphi$ of $K$ such that $M\varphi = K \setminus M$. Thus the Lebesgue measure $\mu$ and the measure $\mu\varphi$ have no common extension, for $\mu(M) = 2\pi$ and $\mu\varphi(M) = \mu(M\varphi) = \mu(K \setminus M) = 0$.

The result of P. Erdös and K. Urbanik called attention to the necessity of looking for some other approches to the problem of Łoś which would provide positive answers.

The idea which turned out to be most fruitful belonged to S. Hartman. He pointed out that from the purely probabilistic point of view the existence of a common extension of two Haar measures, each defined by a compact topology in a group $G$ is of less importance than the existence for given compact topologies $\tau_1$ and $\tau_2$ in $G$ a measure algebra defined by an invariant measure in the group $G$ and containing isomorphic images (in the sense of measure algebras) of the measure algebras defined by the classes $\mathscr{B}_{\tau_1}$, $\mathscr{B}_{\tau_2}$ and the measures $\mu\tau_1, \mu\tau_2$, respectively. The question of the existence of such a measure algebra was studied by K. Urbanik. In [13] he proved the following result:

*Let $G$ be an infinite group and $T$ the family of the compact topologies in $G$. We pick a topology $\tau_0$ of the family $T$. Then for each $\tau \in T$ the measure algebra defined by the class $\mathscr{B}_{\tau_0}$ and the measure $\mu\tau_0$ contains an isomorphic image (in the sense of measure algebras) of the measure algebra defined by the class $\mathscr{B}_\tau$ and the measure $\mu_\tau$.*

Still the nature of the isomorphisms in question was not known. It was not even known whether for each compact topology in a group $G$ there exists an isomorphism of this kind induced by a one-to-one mapping of $G$ onto itself. It was not known any example of a group and two different compact topologies in it such that the corresponding Haar measures have a common extension, either.

If we confine our considerations to the Abelian groups, then the following will lead us to the formulation of the answer to all these questions.

Let $G$ be an Abelian group and $T$ the set of all compact topologies in it. If one look on a topology of $G$ from the point of view of the topological group it defines, then the set $T$ is naturally divided into classes of topologies which define the same topological group, i.e. into classes of isomorphic topologies. P. Erdös and K. Urbanik have found two topologies in the circle that belong to the same class and the Haar measures induced by them do not have common extension. We shaw that after a suitable selection of topologies each from a different class the corresponding Haar measures do have a common extension.

We formulate the above result in an equivalent form in the following

MAIN THEOREM. *Let $G$ be an Abelian group and $T$ the family of the compact topologies in it. Then for each $\tau \in T$ there exists an automorphism $\varphi_\tau$ of $G$ such that the family $\{\mu_\tau \varphi_\tau\colon \tau \in T\}$ of the Haar measures has a common invariant extension.*

In other words,

*If $G$ is an Abelian group and $T$ the set of the compact topologies in $G$, then there exists a semi-regular set of compact topologies in $G$ isomorphic to $T$.*

## 3. Preliminaries (abstract groups, Cartesian products)

From now on all groups are additively written Abelian groups. We shall use a great deal of the theory of Abelian groups. For the general references we send the reader to the books [2] and [8]. Here we want only to establish some notations and prove a simple lemma.

We denote by $C_{p^n}$ the cyclic group of order $p^n$, by $C_{p^\infty}$ the group of the roots of the unity of order $p^n$, $n = 1, 2, \ldots$, by $R$ the additive group of rationals and by $I_p$ the group of $p$-adic integers. We denote by $\sum_{t\in T} G_t$ the direct sum of groups $G_t$, $t\in T$.

LEMMA 3.1. *Let $G$ be a $p$-group and $B$ a basic subgroup of it such that $G/B$ is the direct sum of at least $\mathfrak{m}$ $C_{p^\infty}$'s, with $\mathfrak{m} \geqslant \aleph_0$. Let*

$$B = \sum_{n=1}^{\infty} \sum_{\alpha\in\Omega_n} C_{p^n}^{(\alpha)}.$$

*Then for each set $\Omega_n$ there exists a subset $N_n$ of it such that $\overline{\overline{N}}_n \leqslant \mathfrak{m}$ and for each group*

$$B' = \sum_{n=1} \sum_{\alpha\in M_n} C_{p^n}^{(\alpha)},$$

*where $M_n$ is an arbitrary set between $N_n$ and $\Omega_n$, the factor group $G/B'$ contains a divisible group $H_p$ which is the direct sum of $\mathfrak{m}$ copies of $C_{p^\infty}$.*

Lemma 3.1 is an immediate consequence of the following fact noticed by J. Łoś:

*If $G$ is a group, $N$ a subgroup of $G$ and $H$ an arbitrary subgroup of $G/N$ with $\overline{\overline{H}} \geqslant \aleph_0$, then there exists a subgroup $N_0$ with $N_0 \subset N$ and $\overline{\overline{N}}_0 \leqslant \overline{\overline{H}}$ such that for any group $N'$ such that $N_0 \subset N' \subset N$ there is a subgroup $H'$ of $G/N'$ isomorphic with $H$.*

Proof. Let

$$H = \bigcup_{\alpha\in\Omega} (a_\alpha + N),$$

where $a_\alpha$, $\alpha\in\Omega$, is a selection from the cosets of $G/N$ belonging to $H$. We put

$$N_0 = \mathrm{gp}\{a_{\alpha_1} + a_{\alpha_2} - a_{\alpha_3}:\ \alpha_1, \alpha_2, \alpha_3 \in \Omega\} \cap N.$$

If $N'$ is any group such that $N_0 \subset N' \subset N$ and

$$H' = \bigcup_{\alpha\in\Omega} (a_\alpha + N')$$

then the mapping

$$\varphi\colon H \to H'$$

defined by

$$(a_\alpha + N)\varphi = a_\alpha + N'$$

is an isomorphism of $H$ onto $H'$.

The Cartesian product of a family $\{G_\alpha: \alpha \in \Xi\}$ of groups we denote by $\sum\limits_{\alpha\in\Xi}^* G_\alpha$.

For every $\alpha \in \Xi$ we denote by $\pi_\alpha$ the projection on the group $G_\alpha$ in $\sum\limits_{\alpha\in\Xi}^* G_\alpha$.

If $\tau_\alpha$ is for each $\alpha \in \Xi$ a compact topology of the group $G_\alpha$, then the product topology consists of the unions of the sets

$$V = V_{\alpha_1} + \ldots + V_{\alpha_k} + \sum_{\alpha\in\Xi-\{\alpha_1,\ldots,\alpha_k\}}^* G_\alpha, \quad V_{\alpha_i} \in \tau_{\alpha_i}, \ \alpha_i \in \Xi,$$

and is a compact topology of the group $\sum\limits_{\alpha\in\Xi}^* G_\alpha$. We denote it by $\mathop{\mathscr{P}}\limits_{\alpha\in\Xi} \tau_\alpha$.

If $\Xi = \{1, 2\}$, we then write $\tau_1 \times \tau_2$ for $\mathop{\mathscr{P}}\limits_{\alpha\in\Xi} \tau_\alpha$.

We have

$$\tau_\alpha = (\mathop{\mathscr{P}}_{\alpha\in\Xi} \tau_\alpha) \wedge G_\alpha = (\mathop{\mathscr{P}}_{\alpha\in\Xi} \tau_\alpha)\pi_\alpha, \quad \alpha \in \Xi.$$

Let $\{X_\alpha: \alpha \in \Xi\}$ be a family of sets, for each $\alpha \in \Xi$, let $\mathscr{B}_\alpha$ be a $\sigma$-field of subsets of $X_\alpha$ and $\mu_\alpha$ a normed measure defined on $\mathscr{B}_\alpha$ in $X_\alpha$. Then the direct product measure defined in the product $\mathop{\mathscr{P}}\limits_{\alpha\in\Xi} X_\alpha$ on the $\sigma$-field

$$\mathop{\mathscr{P}}_{\alpha\in\Xi} \mathscr{B}_\alpha = [\{\mathop{\mathscr{P}}_{\alpha\in\delta} M_\alpha \times \mathop{\mathscr{P}}_{\alpha\in\Xi\setminus\delta} X_\alpha: \bar{\bar{\delta}} < \aleph_0, \ \delta \in \Xi, \ M_\alpha \in \mathscr{B}_\alpha\}]_\sigma$$

we denote by $\mathop{\mathscr{P}}\limits_{\alpha\in\Xi} \mu_\alpha$.

If $X_\alpha$ is for each $\alpha \in \Xi$ a group and $\tau_\alpha$ a compact topology in it, then

$$\mathop{\mathscr{P}}_{\alpha\in\Xi} \mathscr{B}_{\tau_\alpha} = \mathscr{B}_\tau \quad \text{and} \quad \mathop{\mathscr{P}}_{\alpha\in\Xi} \mu_{\tau_\alpha} = \mu_\tau, \quad \text{where} \quad \tau = \mathop{\mathscr{P}}_{\alpha\in\Xi} \tau_\alpha.$$

3.2. *For each $\alpha \in \Xi$ let $X_\alpha$ be a set, $\mathscr{B}_\alpha$ a $\sigma$-field of subsets of $X_\alpha$ and $\mu_\alpha$ a normed measure on $\mathscr{B}_\alpha$. If $\nu_\alpha$ denotes for each $\alpha \in \Xi$ the measure defined in $\mathop{\mathscr{P}}\limits_{\alpha\in\Xi} X_\alpha$ on the cylinders*

$$C_\alpha = M_\alpha \times \mathop{\mathscr{P}}_{\beta\in\Xi\setminus\{\alpha\}} X_\beta, \quad M_\alpha \in \mathscr{B}_\alpha,$$

*by the equality*

$$\nu_\alpha(C_\alpha) = \mu_\alpha(M_\alpha),$$

*then the measure $\mathop{\mathscr{P}}\limits_{\alpha\in\Xi} \mu_\alpha$ is a common extension of the measures $\nu_\alpha$ in which they are independent.*

Let $\Xi$ be a directed set. To each $a \in \Xi$ we correspond a set $X_a$ and a family of $\sigma$-fields $\mathscr{B}(t, a)$, $t \in T_a$, of subsets of $X_a$. Further, suppose that on each $\sigma$-field $\mathscr{B}(t, a)$ a normed measure $\mu_a^t$ is defined and that for a fixed $a$ the measures $\mu_a^t$, $t \in T_a$, have a common extension $\mu_a$ defined on the $\sigma$-field

$$\mathscr{B}_a = [\bigcup_{t \in T_a} \mathscr{B}(t, a)]_\sigma.$$

Suppose, in addition, that the measures $\mu_a^t$, $t \in T_a$, are independent in their common extension $\mu_a$.

Now, for any pair $a > \beta$ of elements of $\Xi$ we suppose that there exists a mapping $\gamma_{a,\beta}$ of $T_a$ onto $T_\beta$ and that the mappings $\gamma_{a,\beta}$ and the sets $T_a$, $a \in \Xi$, form an inverse system whose inverse limit exists. Denote it by $T$. Let $\bar{a}$ be the canonic mapping of $T$ onto $T_a$. It is plain that for each $a \in \Xi$ the measure $\bar{\mu}_a = \underset{\beta \geqslant a}{\mathscr{P}} \mu_\beta$ defined on the $\sigma$-field $\bar{\mathscr{B}}_a = \underset{\beta \geqslant a}{\mathscr{P}} \mathscr{B}_\beta$ is a common extension of the measures $\bar{\mu}_a^t = \underset{\beta \geqslant a}{\mathscr{P}} \mu_\beta^{t\bar{\beta}}$, $t \in T$, defined on the $\sigma$-fields $\bar{\mathscr{B}}(t, a) = \underset{\beta \geqslant a}{\mathscr{P}} \mathscr{B}(t\bar{\beta}, \beta)$, $t \in T$, respectively.

Lemma 3.3. *Let $F$ be a finite subset of $T$ and for some $a \in \Xi$ let $\Delta = \{\delta\}$ be a partition of $F$ such that for each $\beta \geqslant a$ and $\delta', \delta'' \in \Delta$*

$$\delta'\bar{\beta} = \delta''\bar{\beta} \quad \textit{implies} \quad \delta' = \delta''. \tag{3.3}$$

*If $M_t$ is for each $t \in F$ a set belonging to $\bar{\mathscr{B}}(t, a)$, then the sets $M_\delta = \bigcap_{t \in \delta} M_t$, $\delta \in \Delta$, are independent, that is*

$$\bar{\mu}_a(\bigcap_{\delta \in \Delta'} M_\delta) = \prod_{\delta \in \Delta'} \bar{\mu}_a(M_\delta) \quad \textit{for each } \Delta' \subset \Delta. \tag{3.4}$$

Proof. Suppose first that the sets $M_t$, $t \in F$, are cylinders of the form

$$M_t = M_t^{\beta(t)} \times P_{\beta(t)}, \quad \text{where} \quad P_{\beta(t)} = \underset{\substack{\gamma \geqslant a \\ \gamma \neq \beta(t)}}{\mathscr{P}} X_\gamma \text{ for some } \beta(t) \in \Xi, \tag{3.5}$$

$$M_t^{\beta(t)} \in \mathscr{B}(t\bar{\beta}(t), \beta(t)).$$

Now we define a family $\Psi$ of sets $\psi$ as follows:

$$\Psi = \{\psi: t, s \in \psi \text{ if and only if } t, s \in F, \text{ and } \beta(t) = \beta(s)\}.$$

For each $\psi$ the common value for $\beta(t)$ with $t \in \psi$ we denote by $\beta(\psi)$. Obviously $\Psi$ is a partition of $F$. For each $\psi \in \Psi$ the set $M_\psi = \bigcap_{t \in \psi} M_t$ is a cylinder of the form

$$M_\psi = M_\psi^{\beta(\psi)} \times P_{\beta(\psi)}, \quad \text{where} \quad M_\psi^{\beta(\psi)} \in \mathscr{B}_{\beta(\psi)}.$$

Hence, since $\bar{\mu}_a$ is the product measure of the measures $\mu_\beta$, $\beta \geqslant a$, we have

$$(3.6)\qquad \bar{\mu}_a\Big(\bigcap_{\psi\in\Psi'} M_\psi\Big) = \prod_{\psi\in\Psi'} \bar{\mu}_a(M_\psi) \quad \text{for every } \Psi' \subset \Psi.$$

For each $\psi\in\Psi$ we define a partition $\Gamma_\psi$ of $\psi$ putting

$$(3.7)\qquad \Gamma_\psi = \{\gamma\colon s, t\in\gamma \text{ if and only if } s, t\in\psi \text{ and } s\bar{\beta} = t\bar{\beta} \text{ for } \beta = \beta(\psi)\}.$$

It is plain that for different $\gamma$, $\gamma\in\Gamma_\psi$, the sets $\bigcap_{t\in\gamma} M_t^{\beta(\psi)}$ belong to different $\sigma$-fields $\mathscr{B}(t\bar{\beta}, \beta)$, $\beta = \beta(\psi)$. Then for each $\Gamma'_\psi \subset \Gamma_\psi$ we have

$$(3.8)\qquad \bar{\mu}_a\Big(\bigcap_{\gamma\in\Gamma'_\psi}\bigcap_{t\in\gamma} M_t\Big) = \bar{\mu}_a\Big[\bigcap_{\gamma\in\Gamma'_\psi}\bigcap_{t\in\gamma} (M_t^{\beta(\psi)} \times P_{\beta(\psi)})\Big]$$
$$= \mu_{\beta(\psi)}\Big(\bigcap_{\gamma\in\Gamma'_\psi}\bigcap_{t\in\gamma} M_t^{\beta(\psi)}\Big)$$
$$= \prod_{\gamma\in\Gamma'_\psi} \mu_{\beta(\psi)}(M_t^{\beta(\psi)}) = \prod_{\gamma\in\Gamma'_\psi} \bar{\mu}_a\Big(\bigcap_{t\in\gamma} M_t\Big).$$

Let $\Phi = \bigcup_{\psi\in\Psi} \Gamma_\psi$. By (3.6) and (3.8) we have

$$(3.9)\qquad \bar{\mu}_a\Big(\bigcap_{\gamma\in\Phi'}\bigcap_{t\in\gamma} M_t\Big) = \prod_{\gamma\in\Phi'} \bar{\mu}_a\Big(\bigcap_{t\in\gamma} M_t\Big) \quad \text{for every } \Phi' \subset \Phi.$$

It follows immediately from (3.7) and (3.3) that $\Phi$ is a refinement of the partition $\Delta$. Hence for each $\delta\in\Delta$ we have

$$\delta = \bigcup_{\gamma\in\Phi^\delta} \gamma, \quad \text{where} \quad \Phi^\delta = \{\gamma\colon \gamma \subset \delta, \gamma\in\Phi\}.$$

Let $\Phi_1 = \bigcup_{\delta\in\Delta'} \Phi^\delta$. Obviously $\Phi_1 \subset \Phi$. Thus applying (3.8) for $\Phi' = \Phi_1$ and $\Phi' = \Phi^\delta$ successively we get

$$(3.10)\qquad \bar{\mu}_a\Big(\bigcap_{\delta\in\Delta'} M_\delta\Big) = \bar{\mu}_a\Big(\bigcap_{\gamma\in\Phi_1}\bigcap_{t\in\gamma} M_t\Big) = \prod_{\delta\in\Phi_1} \bar{\mu}_a\Big(\bigcap_{t\in\gamma} M\Big)$$
$$= \prod_{\delta\in\Delta'}\prod_{\gamma\in\Phi^\delta} \bar{\mu}_a\Big(\bigcap_{t\in\gamma} M_t\Big) = \prod_{\delta\in\Delta'} \bar{\mu}_a\Big(\bigcap_{\gamma\in\Phi^\delta}\bigcap_{t\in\gamma} M_t\Big) = \prod_{\delta\in\Delta'} \bar{\mu}_a(M_\delta)$$

as required.

Suppose now that the sets $M_t$, $t\in F$, are arbitrary cylinders such that $M_t\in\bar{\mathscr{B}}(t, a)$. Then for each $t\in F$

$$M_t = \bigcap_{l\in L} M_t^l,$$

where $L$ is a finite set, $M_t^l$ belongs to $\bar{\mathscr{B}}(t, a)$ and is of the form

$$(3.11)\qquad M_t^l = M_t^{\beta_l} \times \mathop{\mathscr{P}}_{\substack{\beta\geqslant a\\ \beta\neq\beta_l}} X_\beta, \qquad M_t^{\beta_l} \in \mathscr{B}(t\bar{\beta}_l, \beta_l).$$

Moreover, for each $F' \subset F$

$$\bar{\mu}_a(\bigcap_{t\in F'} M_t) = \prod_{i\in L} \bar{\mu}_a(\bigcap_{t\in F'} M_t^i).$$

Thus equality (3.4) for the sets $M_t$, $t\in F$, being arbitrary cylinders belonging to $\bar{\mathscr{B}}(t, a)$ respectively, follows at once from the last equality and (3.10) applied for each $i$, $i\in L$, to the sets $M_t^i$, $t\in F$.

For an element $t_0$ belonging to $F$ and a collection of arbitrary but fixed cylinders $N_t\in\mathscr{B}(t, a)$, $t\in F\setminus\{t_0\}$, let $\boldsymbol{E}_0$ be the family of sets $M_{t_0}\in \bar{\mathscr{B}}(t_0, a)$ for which equality (3.4) holds provided $M_t = N_t$ for $t\in F\setminus\{t_0\}$. The previous reasoning shows that if $\boldsymbol{A}_0$ is the family of the cylinders of the form (3.11) with $t = t_0$, then $\boldsymbol{A}_0 \subset \boldsymbol{E}_0$. $\boldsymbol{A}_0$ is complementative and the finite intersections of sets of $\boldsymbol{A}_0$ belong to $\boldsymbol{E}_0$. From the elementary properties of measure follows that the families $\boldsymbol{A}_0$ and $\boldsymbol{E}_0$ satisfy also conditions (iii) and (iv) of Lemma 1.5. Thus $\bar{\mathscr{B}}(t_0, a) \subset [\boldsymbol{A}_0]_\sigma \subset \boldsymbol{E}_0$. Since the sets $N_t$, $t\in F\setminus\{t_0\}$, are arbitrary cylinders of the form (3.11) belonging to $\bar{\mathscr{B}}(t, a)$ respectively, we may repeat the above reasonning for any $t_1\in F\setminus\{t_0\}$ considering the family $\boldsymbol{E}_1$ of all the sets $M_{t_1}$ of $\bar{\mathscr{B}}(t_1, a)$ for which (3.4) holds provided $M_t = N_t$ for $t\in F\setminus\{t_0, t_1\}$ and $M_{t_0}$ is arbitrary set of $\bar{\mathscr{B}}(t_0, a)$. Thus $\bar{\mathscr{B}}(t_1, a) \subset \boldsymbol{E}_1$. Repeating the same reasonning successively changing every time the set $N_t$ for an arbitrary set of $\mathscr{B}(t, a)$ we get after finitely many steps the proof of Lemma 3.3.

## 4. Preliminaries (automorphisms, duality theory)

The identity automorphism of a group $G$ we denote by $e|G$. If $G = \sum_{a\in\Xi}^{*} G_a$ and $\varphi_a$ is for each $a\in\Xi$ an automorphism of the group $G_a$, then the automorphism $\varphi$ of the group $G$ defined by the equality

$$\langle x_a\rangle_{a\in\Xi}\varphi = \langle x_a\varphi_a\rangle_{a\in\Xi}, \quad x_a\in G_a,$$

we denote by $\sum_{a\in\Xi}^{*}\varphi_a$. We have $\varphi\pi_a = \pi_a\varphi_a = \pi_a\varphi$ for each $a\in\Xi$. If $\tau_a$ is for each $a\in\Xi$ a compact topology of $G_a$, then

$$\mathscr{P}_{a\in\Xi}(\tau_a\varphi_a) = \mathscr{P}_{a\in\Xi}\tau_a \quad\text{and}\quad \mathscr{P}_{a\in\Xi}\mu_{\tau_a}\varphi_a = (\mathscr{P}_{a\in\Xi}\mu_{\tau_a})\varphi.$$

If $G$ is a group, $\varphi$ an automorphism of it and $H$ a subgroup of $G$ such that $H\varphi = H$, then the automorphism induced by $\varphi$ in $H$ we denote by $\varphi \wedge H$.

We now list some fundamental facts of the duality theory of Pontrjagin which will play most important role in the sequel.

The *character group* (with the standard topology) of a topological group $G$ we denote by $\hat{G}$. If $G$ is an abstract group, by $\hat{G}$ we mean the character group of the topological discrete group $G$.

4.1. *If a topological group $G$ is discrete, then $\hat{G}$ is compact, and vice versa, if $G$ is compact, then $\hat{G}$ is discrete* [12], p. 242).

4.2. *If a topological group $G$ is locally compact, then $G = \hat{\hat{G}}$* ([12], p. 256).

4.3. *The character group of a reduced compact topological group is periodic (discrete) group* ([8], p. 55).

4.4. *The character group of a divisible compact (or discrete) topological group is torsion-free. The character group of a torsion-free compact (or discrete) topological group is divisible* ([8], p. 55).

4.5. *If $G_a$, $a \in \Xi$, are discrete topological groups, then*

$$\widehat{\Big(\sum_{a \in \Xi} G_a\Big)} = \sum_{a \in \Xi}^{*} \hat{G}_a$$

*and for the character topology $\tau$ in the group $\sum\limits_{a\in\Xi}^{*}\hat{G}_a$ we have $\tau = \mathop{\mathscr{P}}\limits_{a\in\Xi} \tau_a$, where $\tau_a$ is for each $a \in \Xi$ the character topology of the group $\hat{G}_a$* ([12], p. 259).

4.6. *If $G$ is a locally compact topological group and $H$ a closed subgroup of it, then $\hat{H} \simeq \hat{G}/N$, where $N$ is a closed subgroup of $\hat{G}$ and constitutes the annihilator of the subgroup $H$, i.e. it consists of the characters which map $H$ into zero. We have $N \simeq \widehat{(G/H)}$* ([12], p. 257).

4.7. *The annihilator of a pure closed subgroup of a compact (or discrete) topological group $G$ is a pure (closed) subgroup of $\hat{G}$* [4].

4.8. *If $G$ is a discrete topological group and $H$ a pure subgroup of it, then*

$$\hat{G} = \hat{H} + \widehat{(G/H)}$$

*with $\widehat{(G/H)}$ being closed in $\hat{G}$* ([11] and [4], p. 75).

4.9. *The character group of a discrete topological group isomorphic with $C_{p^\infty}$ is $I_p$ and the character topology in $I_p$ coincides with the $p$-adic topology in $I_p$. The character group of the group $C_{p^n}$ is isomorphic with $C_{p^n}$.*

4.10. *Two compact topologies $\tau_1, \tau_2$ in a group $G$ are isomorphic if and only if $\hat{G}_{\tau_1} \simeq \hat{G}_{\tau_2}$. The group of the continuous automorphisms of a compact topological group $G$ is isomorphic with the group of the automorphisms of the group $\hat{G}$* (compare [1]).

4.11. *If $G$ is a discrete (or compact) topological group which does not contain any pure cyclic subgroup of order $p^n$, then $\hat{G}$ has the same property.*

For, if $C$ is a pure cyclic subgroup of order $p^n$ of $\hat{G}$, then by 4.7 its annihilator $M$ is a pure subgroup in the character group $\hat{\hat{G}} \simeq G$. By 4.6 and 4.9 $G/M \simeq \hat{C} \simeq C$, which implies that $M$ is a direct summand of $G$ and thus $C$ is a cyclic pure subgroup of $G$, which is not possible.

4.12. *If $G$ is a compact group, then $\bar{\bar{G}} = 2^{\mathfrak{m}}$, where $\mathfrak{m} = \bar{\bar{\hat{G}}}$.*

## 5. Compact groups

As was proved in [5] and [6]

*A group $G$ admits (at least one) compact topology if and only if*

(i) $G = D+\sum_p^* D_p$, *where $p$ are prime integers;*

(ii) $D = \sum_{\iota\in\Delta} R_\iota + \sum_p \sum_{\iota\in\Delta_p} C^\iota_{p\infty}$ *with* 1. $\bar{\Delta} = 2^{\mathfrak{m}}$, $\mathfrak{m} \geqslant \aleph_0$, 2. $\bar{\Delta}_p$ *is*

(5.1) *finite or* $\bar{\Delta}_p = 2^{\mathfrak{n}_p}$, 3. $\bar{\Delta} \geqslant \bar{\Delta}_p$ *for each $p$;*

(iii) $D_p = \sum_{n=1}^{\infty *} \sum_{\iota\in\Gamma_n^p}^* C^\iota_{p^n} + \sum_{\iota\in\Gamma_0^p}^* I^\iota_p$.

5.2. *For each compact topology in the group $G$ the maximal divisible subgroup $D$ is closed in it.*

In what follows any group called $D_p$ will be understood to be of the form (5.1), (iii).

By [8], p. 55, we have

5.3. *For each compact topology $\tau$ of a group $G$ of the form*

$$G = \sum_p^* D_p$$

*we have* $\tau = \mathscr{P}_\tau \wedge D_p$.

We consider next the problem of the uniqueness of the decomposition (5.1). The question is to what extend the algebraic structure of the group $G$ defines the cardinals $\bar{\Delta}$, $\bar{\Delta}_p$, $\bar{\Gamma}_n^p$, $n = 0, 1, 2, \ldots$, $p = 2, 3, 5, \ldots$

It follows immediately from the uniqueness of the maximal divisible subgroup of the group $G$ and the uniqueness of its representation, as the sum of groups $C_{p\infty}$ and groups of rationals, that the cardinals $\bar{\Delta}, \bar{\Delta}_p$, $p = 2, 3, 5, \ldots$, are unique. Similarly, since $G/D = \sum_p^* D_p$ and for any two different primes $p$ and $q$ the group $D_p$ has no non-trivial homomorphism into the group $D_q$, the groups $D_p$ in the decomposition (5.1), (i) are defined uniquely by the group $G$.

The situation with the cardinals $\bar{\Gamma}_n^p$ and $\bar{\Gamma}_0^p$ $(n = 1, 2, \ldots$, $p = 2, 3, 5, \ldots)$ is somewhat more complicated. We first prove the following

Lemma 5.4. *If $G$ is a group which admits (at least one) compact topology, then the cardinals $p^{\bar{\Gamma}_n^p}$, $n = 1, 2, \ldots$, $p = 2, 3, 5, \ldots$, are defined uniquely by $G$. Hence, if the generalized continuum hypothesis is assumed, then the cardinals $\bar{\Gamma}_n^p$ $(n = 1, 2, 3, \ldots$, $p = 2, 3, 5, \ldots)$ themselves are unique.*

Proof. Since the groups $D_p$ in (5.1), (i) are defined uniquely by the group $G$, for each prime $p$ the set $\{x\colon px = 0,\ x \in D_p\} = P_p$ is also defined uniquely by the group $G$. For each $n = 1, 2, 3, \ldots$, by (5.1), (iii), we have

$$P_p \cap p^n D_p = \sum_{\iota \in \Gamma_n^p}^{*} C_p^{\iota} = H.$$

Hence $H$ is defined uniquely and thus $\overline{\overline{H}} = p^{\overline{\overline{\Gamma_n^p}}}$ is unique.

In general, the cardinals $\overline{\overline{\Gamma_0^p}}$, $p = 2, 3, 5, \ldots$, are far from being unique (see Corollary 6.13). However, as we shall prove later (Theorem 6.24), for each group $G$ having a compact topology there exists a decomposition of the form (5.1) such that for each prime $p$ the cardinal $p^{\overline{\overline{\Gamma_0^p}}}$ is maximal among the corresponding cardinals in various decompositions of the form (5.1) of the group $G$. Obviously the maximal cardinal $p^{\overline{\overline{\Gamma_0^p}}}$ is unique.

The following lemma gives conditions on the group $G$ under which for each prime $p$ the cardinal $p^{\overline{\overline{\Gamma_0^p}}}$ is unique:

LEMMA 5.5. *Suppose that there exists a decomposition of the type* (5.1) *of the group $G$ such that for each prime $p$ all but a finite number of the sets $\Gamma_n^p$, $n = 1, 2, \ldots$, are empty. Then for each prime $p$ the cardinal $p^{\overline{\overline{\Gamma_0^p}}}$ is defined uniquely by the group $G$. Hence, if the generalized continuum hypothesis is assumed, the cardinals $\overline{\overline{\Gamma_0^p}}$ themselves are unique.*

Proof. It follows from the assumption of the lemma that for each prime $p$ the maximal periodic subgroup $P$ of the group $D_p$ in the decomposition (5.1) of $G$ is of finite exponent ([2]). Hence $P$ is a direct summand of $D_p$ and the group

$$H = D_p/P = \sum_{\iota \in \Gamma_0^p}^{*} I_p^{\iota}$$

is uniquely defined by the group $G$, since $D_p$ is unique. Consider the group $F = H/pH$. We have

$$F = \sum_{\iota \in \Gamma_0^p}^{*} C_p^{\iota}.$$

Thus $\overline{\overline{F}} = p^{\overline{\overline{\Gamma_0^p}}}$, which completes the proof of the lemma, since $F$ is uniquely defined by the group $D_p$.

We prove another lemma on the cardinals $p^{\overline{\overline{\Gamma_n^p}}}$, with $n = 1, 2, \ldots$, $p = 2, 3, 5, \ldots$

([2]) A group is *of finite exponent* $n$ if for each element $x$ of the group $nx = 0$ and $n$ is the least integer for which $nx = 0$ for all elements $x$ of the group. A group of finite exponent is called sometimes a *group of bounded order*; comp. [8].

Lemma 5.6. *For a compact topology $\tau$ in the group $G$ let $T_p$ be the $p$-component of the maximal periodic subgroup $T$ of the group $\hat{G}_\tau$. Let*

$$B_p = \sum_{n=1}^{\infty} \sum_{\iota\in\Omega_n^p} C_{p^n}^{(\iota)}$$

*be a basic subgroup of the group $T_p$. Then $\hat{T}_p \simeq D_p$ and $p^{\overline{\overline{\Omega_n^p}}} = p^{\overline{\overline{\Gamma_n^p}}}$, $n = 1, 2, \ldots$, for each decomposition of the type* (5.1) *of the group $G$.*

Proof. Since $T$ is pure in $\hat{G}$, by 4.2 and 4.8, we have

$$G \simeq \hat{\hat{G}} = \widehat{(\hat{G}/T)} \dotplus \hat{T}. \tag{5.7}$$

By 4.4 the group $\widehat{(\hat{G}/T)}$ is the maximal divisible subgroup of $G$ and, by 4.3, $\hat{T}$ is reduced. We have

$$T = \sum_p T_p. \tag{5.8}$$

Let $B_p$ be a basic subgroup of the group $T_p$. Since $B_p$ is pure in $T_p$ and $T_p/B_p$ is the direct sum of $C_{p^\infty}$'s, by 4.8, 4.9 and 4.5, we have

$$\hat{T}_p = \sum_{n=1}^{\infty}{}^{*} \sum_{\iota\in\Omega_n^p}{}^{*} C'_{p^n} + \sum_{\iota\in U}{}^{*} I'_p. \tag{5.9}$$

Putting $\hat{T} = D'_p$ and $\widehat{(\hat{G}_\tau/T)} = D'$, from (5.7), (5.8), 4.5, 4.4 and (5.9) we obtain a decomposition of the type (5.1), with $D'$ and $D'_p$ in place of $D$ and $D_p$, respectively, of the group $G$. Hence by Lemma 5.4 we have $p^{\overline{\overline{\Omega_n^p}}} = p^{\overline{\overline{\Gamma_n^p}}}$, $n = 1, 2, \ldots$, and by the uniqueness of the group $D_p$ for every prime $p$, $D_p \simeq \hat{T}_p$.

Lemma 5.10. *If a group $G$ is of the form*

$$G = \sum_p{}^{*} D_p,$$

*where for each prime $p$ the group $D_p$ is the direct sum of finitely many cyclic $p$-groups and finitely many, say $n_p$, groups of $p$-adic integers, then $G$ has exactly one compact topology.*

Proof. The group $G$ has evidently a compact topology $\tau$ which is the product topology of the discrete topologies in the cyclic groups and the $p$-adic topologies in the groups of $p$-adic integers. We prove that this is the only compact topology in $G$. For, let $\bar{\tau}$ be a compact topology in $G$. Then, since $G$ is reduced, by 4.3, $\hat{G}_\tau$ is periodic. Hence it is the direct sum of its primary $p$-components, i.e. $\hat{G}_\tau = \sum T_p$. Hence, by 4.5, $G = \sum_p^* \hat{T}_p$.

By Lemma 5.6 $\hat{T}_p = D_p$, and by 4.5 $\bar{\tau} = \mathscr{P}_p(\bar{\tau} \wedge D_p)$. Hence it is sufficient to prove that for each prime $p$ the topology $\bar{\tau} = \bar{\tau} \wedge D_p$ is equal to the topology $\tau_p = \tau \wedge D_p$. To prove this we show first that the topologies $\bar{\tau}_p$ and $\tau_p$ are isomorphic, this means that the character groups $\hat{D}_{p,\bar{\tau}_p}$ and $\hat{D}_{p,\tau_p}$ are isomorphic (compare 4.10). The maximal periodic subgroup $T$ of the group $D_p$ is finite, so it is closed in the topology $\bar{\tau}_p$. We have

$$D_{p,\bar{\tau}_p}/T \stackrel{\text{df}}{=} I = \sum_{i=1}^{n_p} I_p^{(i)}.$$

Since $I$ is reduced and torsion-free, by 4.3 and 4.4, we get

$$\hat{I} = \sum_{i=1}^{m_p} C_{p^\infty}^i.$$

Since, by 4.5 and 4.9, $\hat{\hat{I}} = I = \sum_{i=1}^{m_p} I_p^i = \sum_{i=1}^{n_p} I_p^i$, we have $\overline{\overline{I/pI}} = p^{m_p} = p^{n_p}$ which, since $n_p$ is finite, gives $m_p = n_p$. Since $\hat{I}$ is divisible, by 4.6, 4.5 and 4.9 we get $\hat{D}_{p,\bar{\tau}_p} \simeq \hat{T} + \hat{I} = \hat{T} + \sum_{i=1}^{n_p} C_{p^\infty}^i$. Hence, by 4.5 and 4.9, $\hat{D}_{p,\tau_p} \simeq \hat{D}_{p,\bar{\tau}_p}$. But since each homomorphism of the group $I_p$ into a cyclic $p$-group or the group $I_p$ itself is continuous in the $p$-adic topology of $I_p$, i.e. in the topology $\tau_p \wedge I_p^i$, $i = 1, 2, \ldots, n_p$, each automorphism of the group $D_p$ is continuous in the topology $\tau_p$. Thus, since $\bar{\tau}_p$ and $\tau_p$ are isomorphic, $\bar{\tau}_p = \tau_p$.

## 6. Theorems on the groups $D_p$

All modules we consider here are modules over the ring of $p$-adic integers. It follows immediately from (5.1), (iii) that a group $D_p$ is a module.

I. Kaplansky showed in [5], p. 55, that

6.1. *Every group $D_p$ is a module complete in its $p$-adic topology.*

Following Kulikoff [10] we call a submodule $B$ of a module $m$ a *basic submodule* if

(i) $B$ is a direct sum of cyclic modules;

(ii) $B$ is a pure submodule of $m$;

(iii) $m/B$ is a divisible module.

6.2. *Every module has at least one basic submodule* ([10], p. 305).

6.3. *Every basic submodule $B$ of a module $m$ is generated by a set $A$, called basis of $B$, with the following properties:*

(i) *For each finite set* $\delta \subset A$ *we have* $m\{\delta\} = \sum_{x\in\delta} m\{x\}$ (3);

(ii) $m\{A\}$ *is pure in* $m$;

(iii) $A$ *is a maximal set having properties* (i) *and* (ii) ([10], p. 305).

Property (i) implies that $m\{A\} = \sum_{x\in A} m\{x\}$.

6.4. *Every two basic submodules of a module are isomorphic* ([10], p. 309).

6.5. *The set of the infinite cyclic direct summands in a direct decomposition into cyclic modules of a basic submodule* $B$ *of a module* $m$ *has cardinal equal to the number of the cyclic direct summands in a direct decomposition into cyclic modules of the module* $m/m\{T, pm\}$, *where* $T$ *is the maximal periodic submodule of the module* $m$ ([10], p. 308).

6.6. *If* $m$ *is a module complete in its* $p$*-adic topology, then* $m$ *is equal to the completion of any basic submodule of* $m$ ([8], p. 53).

It follows immediately from 6.6 that

6.7. *Let* $m_1$ *and* $m_2$ *be two modules each complete in its* $p$*-adic topology. Let* $B_1$ *and* $B_2$ *be two basic submodules of* $m_1$ *and* $m_2$ *respectively. Then every isomorphism of the modules* $B_1$ *and* $B_2$ *has the unique extension to an isomorphism of* $m_1$ *and* $m_2$.

Now we are going to define a class $K$ of groups $D_p$ which will play an important role in the following

Definition of the class $K$. *We say that a group* $D_p$ *belongs to the class* $K$ *if and only if it can be represented in the form* (5.1), (iii) *such that*

(i) $\bar{\bar{\Gamma}}{}_0^p + \bar{\bar{\Gamma}}{}_1^p + \bar{\bar{\Gamma}}{}_2^p + \ldots \geq \aleph_0$

*and*

(ii) *either* (a) $\bar{\bar{\Gamma}}{}_0^p \geq \sum_{n=1}^{\infty} \bar{\bar{\Gamma}}{}_0^p$ *or* (b) *there exists an infinite set* $N = \{n_i\colon i = 1, 2, \ldots\}$ *of positive integers such that* $\bar{\bar{\Gamma}}{}_{n_{i+1}}^p \geq \bar{\bar{\Gamma}}{}_{n_i}^p$ *and for each positive integer* $n$ *there is an element* $n_i$ *in the set* $N$ *such that* $\bar{\bar{\Gamma}}{}_{n_i}^p \geq \bar{\bar{\Gamma}}{}_n^p$.

As it will be shown soon the class $K$ consists of the groups $D_p$ which regarded as $p$-adic modules have basic submodules with a large number of infinite direct summands in every direct decomposition into cyclic modules. We shall prove also that each group $D_p$ is the direct sum of a group of finite exponent and a group of the class $K$ or it is the direct sum of finitely many groups of $p$-adic integers and finite cyclic groups.

First we prove the following

(3) We recall that $m\{A, B, \ldots\}$ denotes the $p$-adic module generated by the elements of the sets $A, B, \ldots$

THEOREM 6.8. *If a group $D_p$ belongs to the class $K$, then for any basic submodule $B$ of the module $D_p$ we have*

$$B = T + K,$$

*where $K$ is the direct sum of $\bar{\bar{D}}_p$ infinite cyclic modules.*

The proof of the theorem is based on the following

LEMMA. *Let $\{\Gamma_n\colon n = 1, 2, \ldots\}$ be a family of disjoint sets such that the cardinals $\bar{\bar{\Gamma}}_n$ satisfy condition* (ii), (b) *of the definition of the class $K$ provided $\Gamma_n = \Gamma_n^n$ for every $n = 1, 2, \ldots$ Let $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$. Then there exists a set $\Omega$ of cardinal $2^{\bar{\bar{\Gamma}}}$ consisting of sequences $\varepsilon = \langle \varepsilon_\xi \rangle_{\xi \epsilon \Gamma}$ with $\varepsilon_\xi = 0$ or $\varepsilon_\xi = 1$ and having the following property:*

(6.9) *For every two sequences $\varepsilon, \eta$ of the set $\Omega$ there exists an infinite set $\{n_i\colon i = 1, 2, \ldots\}$ of positive integers such that for each $i = 1, 2, \ldots,$ there exists an element $\xi_{n_i}$ in $\Gamma_{n_i}$ such that $\varepsilon_{\xi_{n_i}} \neq \eta_{\xi_{n_i}}$.*

Proof [4]. By means of transfinite induction we construct an ascending sequence $\langle \Omega_\alpha \rangle_{\alpha < \omega_\lambda}$ consisting of subsets $\Omega_\alpha$ of the set $2^\Gamma$ such that each of them has property (6.9) and the set

$$\bigcup_{\alpha < \omega_\lambda} \Omega_\alpha = \Omega$$

has cardinal $2^{\bar{\bar{\Gamma}}}$.

Let

$$\Omega_1 = \{\langle \varepsilon_\xi \rangle_{\xi \epsilon \Gamma} : \varepsilon_\xi = 1\}.$$

If for some $\beta < \omega_\lambda$ the sets $\Omega_\alpha$, $\alpha < \beta$, are already defined and for all $\bar{\bar{\Omega}}_\alpha < 2^{\bar{\bar{\Gamma}}}$, then for $\beta$ being a limit-number we put

$$\Omega_\beta = \bigcup_{\alpha < \beta} \Omega_\alpha.$$

For $\beta = \alpha + 1$ we consider two cases: (a) $\bar{\bar{\Omega}}_\alpha < 2^{\bar{\bar{\Gamma}}_n}$ for some integer $n$ and (b) $\bar{\bar{\Omega}}_n \geqslant 2^{\bar{\bar{\Gamma}}_n}$ for all integers $n$.

(a) By condition (ii), (b) of definition of the class $K$ there exists an infinite set of integers $n_i$, $i = k, k+1, \ldots,$ such that

$$\bar{\bar{\Omega}}_\alpha < 2^{\bar{\bar{\Gamma}}_n} \leqslant 2^{\bar{\bar{\Gamma}}_{n_i}}.$$

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[4] As has been noticed by J. Łoś, the lemma can be deduced from Tarski's theorem on the number of independent subsets in a set of a given cardinal, but a more direct simple reasonning is at hand.

Hence for each $\Gamma_{n_i}$ there exists a sequence $\langle\varepsilon_\xi^i\rangle_{\Gamma_{n_i}}$ consisting of zeros and ones such that $\langle\varepsilon_\xi^i\rangle_{\xi\in\Gamma_{n_i}}\notin\Omega_\alpha|\Gamma_{n_i}$, where by $\Omega_\alpha|A$, $A\subset\Gamma$, we mean the class of the partial sequences $\langle\eta_\xi\rangle_{\xi\in A}$ for which $\langle\eta_\xi\rangle_{\xi\in\Gamma}\in\Omega_\alpha$. We put

$$\Omega_\beta = \Omega_\alpha\cup\{\langle\varepsilon_\xi\rangle_{\xi\in\Gamma}\},$$

where $\varepsilon_\xi = \varepsilon_\xi^i$ for $\xi\in\Gamma_{n_i}$, $i = k, k+1, \ldots$ Obviously the set has property (6.9).

(b) For each positive integer $n$ consider the set

$$\Omega_\alpha^n = \{\langle\varepsilon_\xi\rangle_{\xi\in\Gamma}\colon \langle\varepsilon_\xi\rangle_{\xi\in A}\in\Omega_\alpha|A,\ A = \bigcup_{j=n}^{\infty}\Gamma_j\}.$$

We have

$$\overline{\overline{\Omega_\alpha^n}} = 2^{\overline{\overline{\Gamma}}_1}\cdot\ldots\cdot 2^{\overline{\overline{\Gamma}}_{n-1}}\overline{\overline{\Omega_\alpha|\bigcup_{j=n}^{\infty}\Gamma_j}} \leqslant \overline{\overline{\Omega}}_\alpha < 2^{\overline{\overline{\Gamma}}}.$$

Hence the set $\Omega_\alpha' = \bigcup_{n=1}^{\infty}\Omega_\alpha^n$ has cardinal less than $2^{\overline{\overline{\Gamma}}}$. We take an arbitrary sequence $\varepsilon = \langle\varepsilon_\xi\rangle_{\xi\in\Gamma}$ consisting of zeros and ones which does not belong to $\Omega_\alpha'$. It follows immediately from the definitions of the sets $\Omega_\alpha^n$ and $\Omega_\alpha'$ that the set $\{\varepsilon\}\cup\Omega_\alpha = \Omega_\beta$ has property (6.9).

**Proof of Theorem 6.8.** Let $D_p\in K$. Then $D_p$ has a decomposition of the type (5.1), (iii) such that the sets $\Gamma_n^p$, $n = 0, 1, 2, \ldots$, satisfy either conditions (i) and (ii), (a) or (i) and (ii), (b) of the definition of the class $K$. We show that in both cases we have

$$\overline{\overline{D}}_p = \overline{\overline{D_p/m\{T, pD_p\}}}, \tag{6.10}$$

where $T$ is the maximal periodic submodule of the module $D_p$. This, since $D_p$ is infinite, will give us the result by 6.5.

In the first case the cardinal $\overline{\overline{\Gamma}}{}_0^p$ is infinite and $\overline{\overline{D}}_p = 2^{\overline{\overline{\Gamma}}{}_0^p}$. We have

$$D_p/m\{T, pD_p\}\supset \sum_{\iota\in\Gamma_0^p}^{*} I_p^{(\iota)}/pI_p^{(\iota)}.$$

Hence $\overline{\overline{D_p/m[T, pD_p]}} \geqslant 2^{\overline{\overline{\Gamma}}{}_0^p} = \overline{\overline{D}}_p$, as required.

In the case when $\Gamma_0^p, \Gamma_n^p$, $n = 1, 2, \ldots$, satisfy conditions (i) and (ii), (b) of the definition of the class $K$ we consider the set $\Omega$ of the sequences $\langle\varepsilon_\xi\rangle_{\xi\in\Gamma}$, with $\Gamma = \bigcup_{n=1}^{\infty}\Gamma_n^p$, defined in the lemma. For each $\xi\in\Gamma_n^p$ let $x_\xi$ be a generator of $C_{p^n}^{(\xi)}$ (compare (5.1), (iii)). For each $\varepsilon = \langle\varepsilon_\xi\rangle_{\xi\in\Gamma}\in\Omega$ we take an element

$$\bar{\varepsilon} = \langle\bar{\varepsilon}_\xi\rangle_{\xi\in\Gamma},\quad \text{where}\quad \bar{\varepsilon}_\xi = \begin{cases} 0 & \text{if}\quad \varepsilon_\xi = 0,\\ x_\xi & \text{if}\quad \varepsilon_\xi = 1.\end{cases} \tag{6.11}$$

The set $\bar{\Omega}$, consisting of the $\bar{\varepsilon}$'s defined above, is contained in $D_p$ and has cardinal $2^{\bar{\Gamma}} = \bar{\bar{D}}_p$. We assert that for any two different elements $\bar{\varepsilon}, \bar{\eta} \in \bar{\Omega}$ we have $\bar{\varepsilon} \not\equiv \bar{\eta} \pmod{m\{T, pD_p\}}$. This of course implies equality (6.10). Suppose that for some $\bar{\varepsilon}, \bar{\eta}$ we have

$$\bar{\varepsilon} - \bar{\eta} = \bar{\gamma} \in m\{T, pD_p\}.$$

Then $\bar{\gamma} = a + pb$ with $a, b \in D_p$ and $p^n a = 0$ for some integer $n$. Hence for each $\xi \in \Gamma$ we get

$$\bar{\gamma}_\xi = a_\xi + pb_\xi. \tag{6.12}$$

Since $\bar{\eta}, \bar{\varepsilon} \in \bar{\Omega}$, there exists an infinite set of positive integers $n_i$, $i = 1, 2, \ldots$, such that $\bar{\eta}_{\xi_{n_i}} \neq \bar{\varepsilon}_{\xi_{n_i}}$ for some $\xi_{n_i} \in \Gamma^p_{n_i}$ and all $i = 1, 2, \ldots$ This by (6.11) implies that for all $i = 1, 2, \ldots$ we have $\bar{\gamma}_{\xi_{n_i}} = x_{\xi_{n_i}}$. For some integer $j$ we have $n_j > n$. Then for some integer $s$, $0 \leqslant s < p^{n_j}$, $a_{\xi_{n_j}} = sp^{n_j - n} x_{\xi_{n_j}}$. Hence by (6.12) the element $\bar{\gamma}_{\xi_{n_i}} = x_{\xi_{n_i}}$ is divisible by $p$ in the group $C_{p^{n_j}}$, which is imposible.

Corollary 6.13. *If a group $D_p$ belongs to the class $K$, then*

$$D_p \simeq D_p + \sum_{\iota \in S}{}^{*} I_p^\iota, \quad \textit{where} \quad 2^{\bar{\bar{S}}} = \bar{\bar{D}}_p.$$

Indeed, the groups $D_p$ and $D_p + \sum_{\iota \in S}^{*} I_p$ are modules complete in their $p$-adic topologies. Let $B$ be a basic submodule of the module $D_p$. Since $D_p \in K$, by Theorem 6.8 $B \simeq B + \sum_{\iota \in T} I_p^\iota$ with $\bar{\bar{T}} = 2^{\bar{\bar{S}}}$. Obviously $B + \sum_{\iota \in T} I_p^\iota$ is isomorphic to a basic submodule of the module $D_p + \sum_{\iota \in S}^{*} I_p^\iota$, thus Corollary 6.13 follows at once from 6.7.

Corollary 6.14. *If a group $D_p$ belongs to the class $K$, then for each compact topology $\tau$ in it there exists a set $A_\tau^*$ satisfying the following conditions:*

(i) *$A_\tau^*$ is a maximal set of elements of infinite order of the group $D_p$ which satisfy conditions* (i) *and* (ii) *of* 6.3;

(ii) *For each set $M$ belonging to $\mathscr{B}_\tau^*$* (compare section 1) *we have* $\overline{\overline{A_\tau^* \cap M}} = \bar{\bar{D}}_p$.

Proof. We well-order the family $\mathscr{B}_\tau^{**}$ (compare section 1) of subsets $M$ of the group $D_p$ in a transfinite sequence $\langle M_\xi \rangle_{\xi < \omega_\alpha}$, where $\omega_\alpha$ is the first ordinal of cardinal $\bar{\bar{D}}_p$. For each $\xi < \omega_\alpha$ we are going to construct a set $A'_\tau(\xi) = \{x_\lambda \colon \lambda \leqslant \xi\}$ satisfying the following conditions:

(a) *for each $\lambda \leqslant \xi$ we have $x_\lambda \in M_\lambda$;*

(b) *for each $\lambda \leqslant \xi$ the element $x_\lambda$ is of infinite order;*

(c) *for each finite subset $\delta$ of $A'_\tau(\xi)$ we have $m\{\delta\} = \sum_{x \in \delta} m\{x\}$;*

(d) *the module $m\{T, A'_\tau(\xi)\}$, where $T$ is the maximal periodic submodule of $D_p$, is pure in $D_p$.*

The construction is by a transfinite induction. Let $\pi$ be the homomorphism

$$\pi\colon D_p \xrightarrow{\text{onto}} D_p/T.$$

The set $M_1\pi$ contains an element $\bar{x}_1$ which is not divisible by $p$ in the module $D_p/T$. Suppose not, then $M_1\pi \subset pD_p/T$ and hence $M_1 \subset m\{T, pD_p\}$. But since $\mu_\tau(M_1) > 0$, the module $D_p/m\{T, pD_p\}$ is finite, which contradicts to Theorem 6.8 and 6.5. If $x_1$ is an element of $M_1$ such that the element $\bar{x}_1$ is the image of $x_1$ by the homomorphism $\pi$, then for the set $A'_\tau(1) = \{x_1\}$ conditions (a)-(d) are satisfied.

Suppose now that we have already constructed the sets $A'_\tau(\xi)$ for all $\xi < \lambda$ with $\lambda < \omega_\lambda$. Let $\eta$ be the homomorphism

$$\eta\colon D_p \xrightarrow{\text{onto}} D_p/m\{T, x_1, x_2, \ldots, x_\xi, \ldots,\colon \xi < \lambda\}.$$

The set $M_\lambda\eta$ is not contained in $pD_p\eta$. Suppose $M_\lambda\eta \subset pD_p\eta$; then the homomorphism

$$\gamma\colon D_{\bar{p}}\eta \xrightarrow{\text{onto}} D_p\eta/pD_p\eta$$

maps the set $M_\lambda\eta$ into zero. Hence, since $\mu_\tau(M_\lambda) > 0$, the module $D_p\eta\gamma$ is finite. Let $\overline{\overline{D_p\eta\gamma}} = p^k$. Then

$$D_p = m\{T, pD_p, x_1, \ldots, x_\xi, \ldots, K\colon \xi < \lambda\},$$

where $K$ is a set of cardinal $p^k$ of elements of $D_p$ such that $K\eta\gamma = D_p\eta\gamma$. Hence $\overline{\overline{D_p/m\{T, pD_p\}}}$ is either equal to $p^k\cdot\bar{\lambda} < \overline{\overline{D}}_p$ or is finite, which in both cases contradicts to Theorem 6.8 and 6.5. We define $A'_\tau(\lambda)$ putting

$$A'_\tau(\lambda) = \bigcup_{\xi<\lambda} A'_\tau(\xi) \cup \{x_\lambda\},$$

where $x_\lambda$ is an element of $M_\lambda$ such that $x_\lambda\eta \notin pD_p\eta$. It is easy to check that the set $A'_\tau(\lambda)$ satisfies conditions (a)-(d). If $A_\tau$ is a set containing all the sets $A'_\tau(\xi)$, $\xi < \omega_a$, and satisfying condition (ii), then $A_\tau$ satisfies also condition (i). Indeed, for each set $M_\xi$, $\xi < \omega_a$, of the family $\mathscr{B}^{**}_\tau$ we have $x_\xi \in M_\xi \cap A'_\tau(\xi)$ and for different $M_\xi$ the points $x_\xi$ are different, moreover, each set of the family $\mathscr{B}^*_\tau$ contains $\overline{\overline{D}}_p$ different sets of the family $\mathscr{B}^{**}_\tau$; thus

$$\overline{\overline{M_\xi \cap A^*_\tau}} = \overline{\overline{D}}_p.$$

Now we are going to prove the second of the already mentioned properties of the groups belonging to the class $K$. We prove the following theorem.

THEOREM 6.15. *For each group $D_p$ we have*

$$D_p = D_p^1 + D_p^2, \tag{6.15}$$

*where $D_p^1$ is a group of finite exponent and $D_p^2$ either belongs to the class $K$ or it is the direct sum of finitely many groups of p-adic integers. Moreover, for each compact topology $\tau$ in $D_p$ there is an automorphism $\varphi_\tau$ of $D_p$ such that the groups $D_p^1$ and $D_p^2$ are both closed in the topology $\tau\varphi_\tau$.*

Proof. Suppose that there is a decomposition of the type (5.1), (iii) of the group $D_p$ such that $\sum_{n=0}^{\infty} \bar{\Gamma}_n^p < \aleph_0$. Then the group $D_p$ is the direct sum of finitely many groups of $p$-adic integers and finitely many finite $p$-groups. Since, moreover, by Lemma 5.7 such a group has exactly one compact topology, namely the product topology of the topologies of the factors in the decomposition (5.1), (iii) of $D_p$, the theorem is proved. Suppose then that $\sum_{n=0}^{\infty} \bar{\bar{\Gamma}}_n^p \geqslant \aleph_0$. If in addition $\bar{\bar{\Gamma}}_0^p \geqslant \sum_{n=1}^{\infty} \bar{\bar{\Gamma}}_n^p$ or the condition (ii), (b) of definition of the class $K$ is satisfied, then the group $D_p$ itself belongs to the class $K$ and there is nothing to prove. Thus we have to consider the remaining case, i.e.

$$\bar{\bar{\Gamma}}_0^p + \bar{\bar{\Gamma}} \geqslant \aleph_0, \quad \bar{\bar{\Gamma}}_\bullet < \bar{\bar{\Gamma}}, \quad \text{where} \quad \Gamma = \bigcup_{n=1}^{\infty} \Gamma_n^p$$

and

(6.16) *there is no infinite set of integers $n_i$, $i = 1, 2, \ldots$, such that (a) $\bar{\bar{\Gamma}}_{n_{i+1}}^p \geqslant \bar{\bar{\Gamma}}_{n_i}^p$ and (b) for each $\Gamma_n^p$ and some $i$, $i = 1, 2, \ldots$, we have $\bar{\bar{\Gamma}}_{p_i}^n \geqslant \bar{\bar{\Gamma}}_n^p$.*

Denote by $N$ the set of non-negative integers. For each $n \in N$ let

$$L_n = \{k: \bar{\bar{\Gamma}}_k^p \geqslant \bar{\bar{\Gamma}}_n^p\}.$$

It follows from (6.16) that the set

$$A = \{\bar{\bar{\Gamma}}_n^p: L_n \text{ is finite}\}$$

is non-void. Let $\bar{\bar{\Gamma}}_m^p$ be the least cardinal belonging to $A$. Then (since $L_m$ is finite) the set $N \setminus L_m$ is infinite and for each $k \in N \setminus L_m$ there exists infinitely many sets $\Gamma_n^p$ such that $\bar{\bar{\Gamma}}_n^p \geqslant \bar{\bar{\Gamma}}_k^p$. Hence by an easy induction we pick in the set $N \setminus L_m$ an infinite subset $\{n_i: i = 1, 2, \ldots\}$ such that $\bar{\bar{\Gamma}}_{n_{i+1}}^p \geqslant \bar{\bar{\Gamma}}_{n_i}^p$ and for each $n \in N \setminus L_m$ we have $\bar{\bar{\Gamma}}_{n_i}^p \geqslant \bar{\bar{\Gamma}}_n^p$ for some $i = 1, 2, \ldots$ Let $\delta = L_m \setminus \{0\}$. Put

$$\begin{aligned} D_p^1 &= \sum_{n \in \delta} \sum_{\iota \in \Gamma_n^p}^{*} C_{p^n}^{\iota}, \\ D_p^2 &= \sum_{n \in \delta'}^{*} \sum_{\iota \in \Gamma_n^p}^{*} C_{p^n}^{\iota} + \sum_{\iota \in \Gamma_0^p}^{*} I_p, \quad \text{where} \quad \delta' = N \setminus (L_m \cup \{0\}). \end{aligned} \tag{6.17}$$

Since $\delta$ is a finite set, $D_p^1$ is a group of finite exponent. The group $D_p^2$ is either the direct sum of finitely many groups of $p$-adic integers, when $\Gamma_n^p$ is void for all $n \in \delta'$ and $\Gamma_0^p$ is finite, or $D_p^2 \in K$, in the other case.

In order to complete the proof of the theorem we show that for each compact topology $\tau$ in $D_p$ there exist groups $D_p^1(\tau)$ and $D_p^2(\tau)$, both closed in the topology $\tau$, such that

(6.18) $$D_p^1(\tau) \text{ is isomorphic to } D_p^1,$$

(6.19) $$D_p^2(\tau) \text{ is isomorphic to } D_p^2,$$

(6.20) $$D_p = D_p^1(\tau) + D_p^2(\tau).$$

It is easy to see that if $\varphi_\tau = \varphi_\tau^1 + \varphi_\tau^2$, where $\varphi_\tau^1$ and $\varphi_\tau^2$ are isomorphisms of $D_p^1(\tau)$ onto $D_p^1$ and $D_p^2(\tau)$ onto $D_p^2$, respectively, then the groups $D_p^1$ and $D_p^2$ are closed in the topology $\tau\varphi_\tau$. To prove the existence of the groups $D_p^1(\tau)$ and $D_p^2(\tau)$ we consider the character group $\hat{D}_{p,\tau}$ of the topological group $D_{p,\tau}$. By 4.3 $\hat{D}_{p,\tau}$ is a torsion $p$-group. Let $B$ be a basic subgroup of it and let

$$B = \sum_{n=1}^{\infty} \sum_{\iota \in \Omega_n} C_{p^n}^{\iota}.$$

Let further

$$B_\delta = \sum_{n \in \delta} \sum_{\iota \in \Omega_n} C_{p^n}^{\iota} \quad \text{and} \quad B_{\delta'} = \sum_{n \in \delta'} \sum_{\iota \in \Omega_n} C_{p^n}^{\iota}.$$

The group $B_\delta$ is a direct summand of the group $\hat{D}_{p,\tau}$ (since it is pure and has finite exponent). Thus get

(6.21) $$\hat{D}_{p,\tau} = B_\delta + U.$$

We prove

6.22. *The group $U$ does not contain any pure subgroup of order $p^n$ with $n \in \delta$.*

To see this we note first that if $S$ is a subgroup of the group $\hat{D}_{p,\tau}$ satisfying the following conditions:

(i) $B_{\delta'} \subset S$;

(ii) $S \cap B_\delta = 0$;

(iii) $S$ is pure in $\hat{D}_{p,\tau}$;

(iv) the subgroups $S$ and $B_\delta$ generate a pure subgroup of $\hat{D}_{p,\tau}$;

(v) $S$ is a maximal subgroup satisfying conditions (i)-(iv);

then

(6.23) $$\hat{D}_{p,\tau} = B_\delta + S.$$

For, by condition (ii), all we have to prove is that $\hat{D}_{p,\tau}/S = B_\delta$. But this follows from the fact, that by (iii) and (iv) the subgroup $\mathrm{gp}\{S, B_\delta\}/S$, isomorphic to $B_\delta$, is a pure subgroup of the group $\hat{D}_{p,\tau}/S$. Thus, since $B_\delta$ is of finite exponent, we get $\hat{D}_{p,\tau}/S = B_\delta + M$. But by (v) we necessarily have $M = 0$, which completes the proof of (6.23). The group $B_{\delta'}$ is a basic subgroup of $S$. Hence $S$ does not contain any pure cyclic subgroup of order $p^n$ with $n \in \delta$. In the converse case it would contain a basic subgroup containig a direct summand being a cyclic group of order $p^n$ with $n \in \delta$, which is not possible, since every two basic subgroups of a group are isomorphic. The relations

$$U \simeq \hat{D}_{p,\tau}/B_\delta \simeq S$$

complete the proof of 6.22.

Let

$$\hat{B}_\delta = D_p^1(\tau) \quad \text{and} \quad \hat{U} = D_p^2(\tau).$$

Obviously both $D_p^1(\tau)$ and $D_p^2(\tau)$ are closed in the topology $\tau$ and (6.20) holds. It is easy to see that by 4.5 and Lemma 5.6 (6.18) holds too. We prove (6.19). By 4.11 the group $\hat{U}$ does not contain any pure cyclic subgroup of order $p^n$ with $n \in \delta$. Both $\hat{B}_\delta$ and $\hat{U}$ can be regarded as modules. Let $A'$ and $A''$ be basis (compare 6.3) of basic submodules $H'$ and $H''$ of the modules $\hat{U}$ and $D_p^2$, respectively. Let further $A_1$ and $A_2$ be basis of $\hat{B}_{\delta'}$ and $D_p^1$, respectively. The module $D_p^1 + H''$ is a basic submodule of the module $D_p^1 + D_p^2 = D_p$ and $\hat{B}_\delta + H''$ is a basic submodule of the module $\hat{B}_\delta + \hat{U} = D_p$. By 6.4 there exists an isomorphism $\varphi$ of the module $D_p$ onto $D_p$ such that the set $A_2 \cup A''$ is mapped by $\varphi$ onto $A_1 \cup A'$. Since the orders of the elements of the sets $A_1$ and $A_2$ are of the form $p^n$ with $n \in \delta$, and neither $\hat{U}$ nor $D_p^2$ contains a pure cyclic subgroup of order $p^n$ with $n \in \delta$, the isomorphism $\varphi$ maps the set $A_2$ onto $A_1$ and $A''$ onto $A'$. Hence $D_p^1 \varphi = \hat{B}_\delta$ and $H''\varphi = H'$. But since by 6.1 the modules $\hat{U}$ and $D_p^2$ are complete in their $p$-adic topologies, by 6.7 the isomorphism $\varphi$ can be extended to the isomorphism of $D_p^2$ and $\hat{U}$, which completes the proof of (6.19) and the proof of the theorem.

THEOREM 6.24. *For each group $D_p$ there exists a decomposition of the type* (5.1), (iii) *such that the cardinal $2^{\overline{\overline{\Gamma_0^p}}}$ is maximal among the corresponding cardinals in various decompositions of the type* (5.1), (iii) *of the group $D_p$.*

Proof. It follows from Theorem 6.15 and Corollary 6.13 that

$$D_p = D_p^1 + D_p^2 + \sum_{\iota \in \Gamma_0^p}^{*} I_p^\iota,$$

where $D_p^1$ has finite exponent and either $\Gamma_0^p$ is finite and $D_p^2 = 0$, or $2^{\bar{\bar{\Gamma}}_0^p} = \overline{\overline{D_p^2 + \sum_{\iota\epsilon\Gamma_0^p}^* I_p^\iota}}$. Clearly $2^{\bar{\bar{\Gamma}}_0^p}$ is the maximal cardinal among the corresponding cardinals in various decompositions of the type (5.1), (iii) of the group $D_p$.

We conclude this section by defining another class containig groups $D_p$.

DEFINITION OF THE CLASS $\boldsymbol{B}$. *We say that a group $D_p$ belongs to the class $\boldsymbol{B}$ if it is of the form*

$$D_p = \sum_{\iota\epsilon\Gamma_n^p}^* C_{p^n}^\iota, \quad \textit{where} \quad \bar{\bar{\Gamma}}_n^p \geqslant \aleph_0. \tag{6.25}$$

THEOREM 6.26. *If a group $D_p$ is of finite exponent, then*

$$D_p = \sum_{n\epsilon\delta} D_p^{(n)},$$

*where $\delta$ is finite and for each $n\epsilon\delta$ we have $D_p^n \epsilon \boldsymbol{B}$ or $D_p^n$ is finite. Moreover, for each compact topology $\tau$ in the group $D_p$ there exists an automorphism $\varphi_\tau$ of $D_p$ such that the groups $D_p^n$, $n\epsilon\delta$, are closed in the topology $\tau\varphi_\tau$.*

Proof. Let

$$D_p = \sum_{n\epsilon\delta} \sum_{\iota\epsilon\Gamma_n^p}^* C_{p^n}^{(\iota)} \quad \text{with} \quad \bar{\bar{\delta}} < \aleph_0.$$

Put

$$D_p^{(n)} = \sum_{\iota\epsilon\Gamma_n^p}^* C_{p^n}^\iota, \qquad n\epsilon\delta.$$

If $\tau$ is a compact topology of $D_p$, then the character group $\hat{D}_{p,\tau}$ is of the form $\hat{D}_{p,\tau} = \sum_{n\epsilon\delta} D(n)$, where for each $n\epsilon\delta$ there exists an isomorphism

$$\varphi_\tau^n\colon\ \hat{D}(n) \xrightarrow{\text{onto}} D_p^{(n)}.$$

By 4.2 and 4.5 we have $D_p = \sum_{n\epsilon\delta} \hat{D}(n)$. Since the groups $\hat{D}(n)$ are closed in the topology $\tau$, the groups $D_p^n$ are closed in the topology $\tau\varphi_\tau$, where $\varphi_\tau = \sum_{n\epsilon\delta} \varphi_\tau^n$.

## 7. A decomposition of compact groups

In this section we prove only one lemma, which concerns groups admitting compact topologies. It is rather doubtful whether the lemma is of any interest by itself, however it is indespensable in the proof of the Main Theorem.

LEMMA 7.1. *Let $G$ be a group admitting compact topologies. Then*

$$G = D + A + B,$$

*where $D$ is the maximal divisible subgroup of $G$ and $\bar{\bar{A}} \leqslant \bar{\bar{D}}$. Moreover, for each compact topology $\tau$ in $G$ there exists an automorphism $\varphi_\tau$ of the group $G$ such that the group $B$ is closed in the topology $\tau\varphi_\tau$.*

Proof. If $D$ is the trivial subgroup of $G$, then there is nothing to prove. Suppose $D$ is non-trivial. Then, by (5.1), (ii), 1. we have $\bar{\bar{D}} \geqslant 2^{\aleph_0}$. Now we choose the decomposition (5.1) of the group $G$ in the way that each of the cardinals $2^{\bar{\bar{\Gamma}}_0^p}$, $p = 2, 3, 5, \ldots$, is maximal among the corresponding cardinals in various decompositions of the group $G$. (The possibility of the choice follows from Theorem 6.24.) By (6.17) for each prime $p$ the groups $D_p^1$ and $D_p^2$ in the decomposition (6.15) can be chosen in such a way that the given decomposition of the type (5.1), (iii) of the group $D_p$ is a refinement of the decomposition (6.15) of the group $D_p$. If we choose the decompositions (5.1) and (6.15) in the way described above, then for each prime $p$ and a finite set $\delta$ of integers we have

$$D_p^1 = \sum_{n\in\delta} \sum_{\iota\in\Gamma_n^p}^{*} C_{p^n}^{\iota} \tag{7.2}$$

and

$$\bar{\bar{D}}_p^2 = 2^{\bar{\bar{\Gamma}}_0^p},$$

where $\Gamma_n^p$, $n = 0, 1, 2, \ldots$, are the corresponding sets in the chosen decomposition of the type (5.1), (iii) of $D_p$.

Put

$$A = \sum_p^{*} D_p', \quad B = \sum_p^{*} D_p'',$$

where for each prime $p$ the groups $D_p'$ and $D_p''$ are defined as follows:

(7.3)

$$D_p' = \sum_{n=1}^{\infty} \sum_{\iota\in S_n^p}^{*} C_{p^n}^{\iota} + \sum_{\iota\in S_0^p}^{*} I_p^{\iota},$$

where $S_n^p = \Gamma_n^p$ if $2^{\bar{\bar{\Gamma}}_n^p} \leqslant \bar{\bar{D}}$, $S_n^p \subset \Gamma_n^p$ and $2^{\bar{\bar{S}}_n^p} = \bar{\bar{D}}$ if $2^{\bar{\bar{\Gamma}}_0^p} > \bar{\bar{D}}$, $n = 0, 1, 2, \ldots$;

$$D_p'' = \sum_{n=1}^{\infty}{}^{*} \sum_{\iota\in\Psi_n^p}^{*} C_{p^n}^{\iota} + \sum_{\iota\in\Psi_0^p}^{*} I_p^{\iota},$$

where $\Psi_n^p = \Gamma_n^p \setminus S_n^p$, $n = 0, 1, 2, \ldots$

It follows immediately from (7.2) and (7.3) that

(7.4) $$\text{if } \bar{\bar{D}}_p^2 \leqslant \bar{\bar{D}}, \text{ then } D_p^2 \subset D_p'.$$

It is plain that

(7.5) $$D+A+B = G \quad \text{and} \quad \bar{\bar{A}} \leqslant \bar{\bar{D}}.$$

In order to prove the second part of the lemma we show that for each compact topology $\tau$ in the group $G$ there exists a decomposition

(7.6) $$G = X+Y$$

of the group $G$ such that $Y$ is closed in the topology $\tau$, $X$ is isomorphic with $D+A$ and $Y$ is isomorphic to $B$. Hence the automorphism $\varphi_\tau$ is obtained by putting $\varphi_\tau = \varphi_\tau^1+\varphi_\tau^2$, where $\varphi_\tau^1, \varphi_\tau^2$ are isomorphisms of $X$ onto $D+A$ and of $Y$ onto $B$, respectively.

Consider the character group $\hat{G}_\tau$ of the topological group $G_\tau$. Let $T$ be its maximal periodic subgroup and $K$ a subgroup of $\hat{G}_\tau$ such that

(7.7) $$\bar{\bar{K}} = \overline{\overline{\hat{G}_\tau/T}} \quad \text{and} \quad \operatorname{gp}\{T, K\} = \hat{G}_\tau.$$

By (5.7) and 4.4 $\widehat{(\hat{G}_\tau/T)} \simeq D$; hence by 4.12 we have

(7.8) $$2^K = \bar{\bar{D}}.$$

Obviously $\hat{G}_\tau/K$ is a periodic group. Let $T_p$ be the $p$-component of the group $T$. We have

(7.9) $T_p = T_p^1+T_p^2$,

*where $T_p^1$ has finite exponent, $T_p^2$ contains a basic subgroup $B_p^2$ such that $\overline{\overline{T_p^2/B_p^2}} = \bar{\bar{T}}_p^2$.*

For the proof of the existence of the decomposition (7.9) see [2], p. 106. Let

(7.10) $$T_p^1 = \sum_{n\in\gamma}\sum_{\iota\in\Omega_n^p} C^\iota_{p^n}.$$

We put $B_p = T_p^1+B_p^2$. The group $B_p$ is a basic subgroup of the group $T_p$. We have

(7.11) $$B_p = \sum_{n=1}^{\infty}\sum_{\iota\in\Omega_n^p} C^\iota_{p^n} \quad \text{and (by Lemma 5.6)} \quad 2^{\bar{\bar{\Omega}}_n^p} = 2^{\bar{\bar{\Gamma}}_n^p}.$$

We define a group $B_p'$ being a direct summand of the group $B_p$ in the following way:

If $2^{\bar{\bar{T}}_p^2} \leqslant \bar{\bar{D}}$, then we put

$$(7.12) \qquad B'_p = \sum_{n=1}^{\infty} \sum_{\iota \in M_n^p} C'_{pn}, \quad \text{where } M_n^p = \Omega_n^p, \text{ for } 2^{\bar{\bar{\Omega}}_n^p} \leqslant \bar{\bar{D}}, \; M_n^p \subset \Omega_n^p$$

$$\text{and } 2^{\bar{\bar{M}}_n^p} = \bar{\bar{D}} \text{ for } 2^{\bar{\bar{\Omega}}_n^p} > \bar{\bar{D}}.$$

If $2^{\bar{\bar{T}}_p^2} > \bar{\bar{D}}$, then, since $\bar{\bar{D}} \geqslant \mathfrak{c}$, we have $\bar{\bar{T}}_p^2 > \aleph_0$. Hence by (7.9) the group $T_p/B_p = T_p^2/B_p^2$ is the direct sum of $\bar{\bar{T}}_p^2$ $C_{p\infty}$'s. Let $\mathfrak{m}$ be a cardinal such that $2^{\mathfrak{m}} = \bar{\bar{D}}$ (compare 4.12). By Lemma 3.1 for each set $\Omega_n^p$ of (7.11) there exists a subset $N_n^p$ of it such that $\bar{\bar{N}}_n^p \leqslant \mathfrak{m}$ and for each set $M_n^p$ between $N_n^p$ and $\Omega_n^p$ for the group

$$(7.13) \qquad B'_p = \sum_{n=1}^{\infty} \sum_{\iota \in M_n^p} C'_{pn}$$

the group $T_p/B'_p$ contains a subgroup $H_p$ being the direct sum of $\mathfrak{m}$ $C_{p\infty}$'s. We use (7.13) for the definition of the group $B'_p$ in the case $2^{\bar{\bar{T}}_p^2} > \bar{\bar{D}}$ selecting the sets $M_n^p$ in such a manner that

$$(7.14) \qquad M_n^p = \Omega_n^p, \qquad \text{if} \quad 2^{\bar{\bar{\Omega}}_n^p} \leqslant \bar{\bar{D}},$$

$$M_n^p \subset \Omega_n^p, \quad N_n^p \subset M_n^p \quad \text{and} \quad 2^{\bar{\bar{M}}_n^p} = \bar{\bar{D}}, \quad \text{if} \quad 2^{\bar{\bar{\Omega}}_n^p} > \bar{\bar{D}}.$$

Let $H'_p$ be a subgroup of $T_p$ such that $\bar{\bar{H}}_p = \bar{\bar{H}}'_p$ and $\mathrm{gp}\{H'_p, B'_p\}/B'_p = H_p$. For each prime $p$ we define a group $U'_p$ putting

$$(7.15) \qquad U'_p = \begin{cases} \mathrm{gp}\{T_p^2, B'_p\}, & \text{if} \quad 2^{\bar{\bar{T}}_p^2} \leqslant \bar{\bar{D}}, \\ \mathrm{gp}\{H'_p, B'_p\}, & \text{if} \quad 2^{\bar{\bar{T}}_p^2} > \bar{\bar{D}}. \end{cases}$$

Let $U_p$ be a pure subgroup of $T_p$ such that $U'_p \subset U_p$ and $\bar{\bar{U}}'_p = \bar{\bar{U}}_p$. Clearly

$$(7.16) \qquad 2^{\bar{\bar{U}}_p} \leqslant \bar{\bar{D}}.$$

Let $X' = \mathrm{gp}\{K, \sum_p U_p\}$. We define $X''$ to be a pure subgroup of $\hat{G}_\tau$ containing $X'$ and such that $\bar{\bar{X}}' = \bar{\bar{X}}''$ such a subgroup exists, since $X'$ is infinite. By (7.8) and (7.16) we have

$$(7.17) \qquad 2^{\bar{\bar{X}}'} \leqslant \bar{\bar{D}}.$$

Let $Y' = \hat{G}_\tau/X''$. Since $X''$ is pure in $\hat{G}_\tau$, by 4.8 we have $G_\tau = \hat{X}''+\hat{Y}'$, and $\hat{Y}'$ is closed in $G_\tau$. We put $X = \hat{X}''$ and $Y = \hat{Y}'$. We are going to prove

$$X \simeq A+D \quad \text{and} \quad Y \simeq B. \tag{7.18}$$

Since $K \subset X''$, the group $\hat{G}_\tau/X''$ is periodic. Hence, by 4.3, $Y$ is reduced and thus $D \subset X$. Let $X = D+M$. $M$ is a reduced group admitting compact topologies, for, if $\eta$: $G \xrightarrow{\text{onto}} G/Y = X$ and $\pi$: $X \xrightarrow{\text{onto}} X/D = M$, then the topology $\tau\eta\pi$ is a compact topology in $M$. Hence by (5.1), (ii) and (5.1), (iii) we get

$$M = \sum_p{}^* \Big(\sum_{n=1}^{\infty}{}^* \sum_{\iota P_n^p}{}^* C'_{p^n} + \sum_{\iota P_0^p}{}^* I'_p\Big), \tag{7.19}$$

where for each prime $p$ and non-negative integer $n$ the set $P_n^p$ corresponds to the set $\Gamma_n^p$ in (5.1), (iii) and the set $P_0^p$ is such that $2^{\bar{P}_0^p}$ is the maximal cardinal among the corresponding cardinals in various decompositions of the type (5.1) of the group $M$. We prove that $M \simeq A$. We show that for each prime $p$ and non-negative integer $n$ we have

$$2^{\bar{P}_n^p} = 2^{\bar{S}_n^p}. \tag{7.20}$$

This will give us the first of the isomorphisms (7.18).

For each prime $p$ let

$$B''_p = \sum_{n=1}^{\infty} \sum_{\iota K_n^p} C'_{p^n}$$

be a basic subgroup of the $p$-component $R_p$ of the maximal periodic subgroup of $X''$. Since $X''$ is pure in $\hat{G}_\tau$, $R_p$ is pure in $T_p$ and hence the group $B''_p$ can be extended to a basic subgroup $B^*_p$ of $T_p$. We have

$$B^*_p = \sum_{n=1}^{\infty} \sum_{\iota {}^*\Omega_n^p} C'_{p^n},$$

where ${}^*\Omega_n^p = \Omega_n^p$, and thus $\bar{\bar{K}}_n^p \leqslant \bar{\bar{\Omega}}_n^p$.

By Lemma 5.6 we have

$$2^{\bar{K}_n^p} = 2^{\bar{P}_n^p}.$$

In order to prove the equality $2^{\bar{K}_n^p} = 2^{\bar{S}_n^p}$ we consider first the case $n \geqslant 1$. If, moreover, $2^{\bar{S}_n^p} < \bar{\bar{D}}$, then by (7.3) and (7.11)

$$2^{\bar{S}_n^p} = 2^{\bar{\Gamma}_n^p} = 2^{\bar{\Omega}_n^p} = 2^{{}^*\bar{\Omega}_n^p}.$$

Using (7.12) and (7.13) we see that the inequality $2^{\bar{\bar{\Omega}}_n^p} < \bar{\bar{D}}$ implies $\Omega_n^p = M_n^p$. Since $B_p' \subset X''$ and $B_p'$ is pure in $\hat{G}_\tau$ (and hence in $X''$), $B_p'$ can be extended to a basic subgroup of $R_p$. Thus $\bar{\bar{M}}_n^p \leqslant \bar{\bar{K}}_n^p$, which together with the previous inequalities gives $\bar{\bar{\Omega}}_n^p = \bar{\bar{M}}_n^p \leqslant \bar{\bar{K}}_n^p \leqslant \bar{\bar{\Omega}}_n^p$ and completes the proof of (7.20) for $n \geqslant 1$ and $2^{\bar{\bar{S}}_n^p} < \bar{\bar{D}}$. If $2^{\bar{\bar{S}}_n^p} = D$, then by (7.3) $2^{\bar{\bar{\Gamma}}_n^p} \geqslant \bar{\bar{D}}$. Hence, by (7.11), $2^{\bar{\bar{\Omega}}_n^p} \geqslant \bar{\bar{D}}$. By (7.12), (7.13) and (7.14) we get $2^{\bar{\bar{M}}_n^p} = \bar{\bar{D}}$. Thus, since $B_p'$ can be extended to a basic subgroup of $X''$, we have $2^{\bar{\bar{K}}_n^p} \geqslant \bar{\bar{D}}$, which by (7.17) implies $2^{\bar{\bar{K}}_n^p} = \bar{\bar{D}}$. We have then $2^{\bar{\bar{F}}_n^p} = \bar{\bar{D}} = 2^{\bar{\bar{S}}_n^p}$.

For the case $n = 0$ we note first that by (7.4) we have either $D_p^2 \subset D_p'$, then by (7.3) $S_0^p = \Gamma_0^p$ and the group $D_p''$ has finite exponent, or $\bar{\bar{D}}_p^2 > \bar{\bar{D}}$. In the first case the group $D_p$ cannot contain a torsion-free direct summand of cardinal greater than $\bar{\bar{D}}$. Thus, by 4.8, 4.5, 4.9, and Lemma 5.6 the group $T_p$ has no pure subgroup $E$ that the factor group $T_p/E$ is the direct sum of $\mathfrak{n}$ $C_{p^\infty}$'s with $2^{\mathfrak{n}} > \bar{\bar{D}}$. Hence by (7.9) $2^{\bar{\bar{T}}_p^2} \leqslant \bar{\bar{D}}$ and by (7.15) $T_p^2 \subset U_p$. By 4.5 and (7.9) we get $D_p \simeq \hat{T}_p = \hat{U}_p \dot{+} S$, where $S$ has finite exponent. But hence the maximal value for $2^{\bar{\bar{T}}_0^p}$ in the decomposition (5.1), (iii) of $D_p$ is the same as the maximal value for the corresponding cardinal in the decomposition (5.1), (iii) of $\hat{U}_p$. Since $U_p$ is a pure subgroup in $T_p$, by 4.8, we get $\hat{U}_p \subset \hat{X}'' \frown D_p$. Hence the equality $2^{\bar{\bar{T}}_0^p} = 2^{\bar{\bar{F}}_0^p}$ is an immediate consequence of the choice of the decomposition (7.19). In the other case, i.e. when $\bar{\bar{D}}_p^2 > \bar{\bar{D}}$, by (7.3) we have $2^{\bar{\bar{S}}_0^p} = \bar{\bar{D}}$. But $\bar{\bar{X}} \leqslant \bar{\bar{D}}$. Then the only thing which is left to show in this case is, that $X$ contains a direct summand being the complete direct sum of $\mathfrak{m}$ $I_p$'s with $2^{\mathfrak{m}} = \bar{\bar{D}}$. It follows from (7.9), 4.5 and Lemma 5.6 that $2^{\bar{\bar{T}}_p^2} \geqslant \bar{\bar{D}}$. Hence by (7.13) and (7.15) $U_p$ contains the pure subgroup $B_p'$ such that $U_p/B_p'$ contains a direct summand, which is the direct sum of $\mathfrak{m}$ $C_{p^\infty}$'s. This by 4.8, 4.5, 4.9 gives the result.

Now we are going to prove the second of the isomorphisms (7.18). Since $Y$ is reduced and admits compact topologies, then by (5.1), (ii) and (5.1), (iii) it is of the form

$$Y = \sum_p{}^* \Big[\sum_{n=1}^{\infty}{}^* \sum_{\iota \in \Phi_n^p}{}^* C_{p^n}^{(\iota)} + \sum_{\iota \in \Phi_0^p}{}^* I_p^{(\iota)}\Big]. \tag{7.21}$$

We suppose that the decomposition (7.21) is chosen in such a way that the cardinals $2^{\bar{\bar{\Phi}}_0^p}$, $p = 2, 3, 5, \ldots$, are maximal among the corresponding cardinals in various decompositions of the type (5.1) of the group $Y$.

As above we prove the second of the isomorphisms (7.18) showing that for each prime $p$ and non-negative integer $n$ we have

$$2^{\overline{\overline{\Psi}}_n^p} = 2^{\overline{\Psi}_n^p}. \tag{7.22}$$

We consider the case $n \geqslant 1$ and $n = 0$ separately.

Let $n \geqslant 1$. If $2^{\overline{\overline{\Gamma}}_n^p} \leqslant \overline{\overline{D}}$, then by (7.3) $\Psi_n^p$ is empty and $S_n^p = \Gamma_n^p$. Hence, by Lemma 5.6, $2^{\overline{\overline{\Omega}}_n^p} = 2^{\overline{\overline{\Gamma}}_n^p} \leqslant \overline{\overline{D}}$, which by (7.12) and (7.14) gives $M_n^p = \Omega_n^p$. Thus by (7.11), (7.12) and (7.13) there is no pure cyclic subgroup of order $p^n$ belonging to $\hat{G}_\tau$ and disjoint with $B'_p$. Hence, since $X''$ is pure, $\hat{G}_\tau/X'' = Y'$ does not contain any pure cyclic subgroup of order $p^n$. This by 4.11 implies that also $\hat{Y}' = Y$ does not contain any pure cyclic subgroup of order $p^n$. Thus $\Phi_n^p$ is empty. On the other hand, if $2^{\overline{\overline{\Gamma}}_p^n} > \overline{\overline{D}}$, then by (7.3) $\overline{\overline{\Psi}}_p^n = \overline{\overline{\Gamma}}_p^n$ and since $\overline{\overline{X}} \leqslant \overline{\overline{D}}$, the equality $2^{\overline{\overline{\Psi}}_n^p} = 2^{\overline{\overline{\Phi}}_n^p}$ follows immediately from Lemma 5.4.

Consider the case $n = 0$. If $\overline{\overline{D}}_p^2 \leqslant \overline{\overline{D}}$, then by (7.4) $D_p^2 \subset D'_p$ and $\Psi_0^p$ is empty. We have then $2^{\overline{\overline{\Gamma}}_0^p} \leqslant \overline{\overline{D}}$. But $2^{\overline{\overline{\Gamma}}_0^p}$ is maximal. Thus $2^{\overline{\overline{T}}_p^2} \leqslant \overline{\overline{D}}$. For, by (7.9), 4.5, 4.7, 4.9, the converse inequality would imply that $T_p$ contains a direct summand being the complete direct sum of $2^{\overline{\overline{T}}_p^2}$ groups $I_p$, which is imposible since, by Lemma 5.6, $\hat{T}_p \simeq D_p$ and $2^{\overline{\overline{\Gamma}}_0^p}$ is maximal. Thus by (7.15) $T_p^2 \subset U_p \subset X''$. Hence, since $X''$ is pure in $\hat{G}_\tau$, the homomorphism $\hat{G}_\tau \to \hat{G}_\tau/X''$ maps the group $T_p$ into a group of finite exponent and therefore the $p$-component of the group $Y' \simeq \hat{G}_\tau/X''$ has finite exponent, which implies that, since $\hat{Y} \simeq \hat{Y}'$, $\Phi_0^p = \emptyset$. In the case $\overline{\overline{D}}_p^2 > \overline{\overline{D}}$ by (7.3) we get $2^{\overline{\overline{\Gamma}}_0^p} > \overline{\overline{D}}$ and hence $\overline{\overline{\Gamma}}_0^p = \overline{\overline{\Psi}}_0^p$. Since $\overline{\overline{X}} \leqslant \overline{\overline{D}}$, the equalities

$$2^{\overline{\overline{\Psi}}_0^p} = 2^{\overline{\overline{\Gamma}}_0^p} = 2^{\overline{\overline{\Phi}}_0^p}$$

are immediate consequences of the uniqueness of the maximal cardinal $2^{\overline{\overline{\Gamma}}_0^p}$.

## 8. Groups in which all compact topologies are isomorphic

In this section we shall investigate the class $N$ of the groups admitting compact topologies and such that every two compact topologies in a group of this class are isomorphic. Our special interest in this class is caused by the fact that this is the class of groups for which the Main Theorem becomes trivial. We show that the class $N$ is rather narrow subclass of the class of the groups admitting compact topologies. We present a full description of the groups of the class $N$ by means of the cardinals $\overline{\overline{\Delta}}, \overline{\overline{\Delta}}_p, \overline{\overline{\Gamma}}_n^p$, $p = 2, 3, 5, \ldots$, $n = 0, 1, 2, \ldots$, of the decomposition (5.1).

The results of this section might have some interest from the purely algebraic point of view. By Pontrjagin's duality theory, any compact group $G$ is the character group of a (discrete) group $H$. Moreover, the topology in $G$ is defined up to an isomorphism by the group $H$ and *vice versa* the group $H$ is uniquely defined by the algebraic structure of $G$ and its topology, for $\hat{G} \simeq H$. Thus the problem of characterizing the groups whose compact topologies are isomorphic is the same as that of finding all the groups $G$ such that for each of them there is precisely one group $H$ with

$$G \simeq \mathrm{Hom}(H, K), \tag{8.1}$$

where $K$ is the group of rotations of the circle, i.e. the group $\sum_p^* C_{p^\infty}$. In general, the isomorphism (8.1) is not defined uniquely by the group $G$ and $H$. This means that the group $G$ can "act" on the group $H$ as $\mathrm{Hom}(H, K)$ differently. The groups $G$ for which not only the group $H$ but also the isomorphism (8.1) is defined uniquely by the algebraic structure of $G$ are the groups which admit precisely one compact topology, namely the one which is carried into $G$ by the isomorphism (8.1).

Using the generalized continuum hypothesis ([5]) we prove the following

THEOREM 8.2. *Let $G$ be a group admitting compact topologies. Then all the compact topologies in $G$ are isomorphic if and only if $G$ has a decomposition of the type* (5.1) *such that*

(i) $\Delta_p = \emptyset$ *for all primes* $p$

*and either*

(ii) $\Delta = \emptyset$ *and, for each prime* $p$, $\Gamma_n^p = \emptyset$ *for all but a finite number of non-negative integers* $n$,

*or*

(ii)′ $\Delta > \mathfrak{c}$ *and* $\Gamma_n^p = \emptyset$ *for all but a finite number of primes* $p$ *and positive integers* $n$.

Proof. We prove first the *sufficiency* of either of the pairs of conditions (i)-(ii) and (i)-(ii)′. We show that if (i)-(ii) or (i)-(ii)′ are satisfied, then the character group $\hat{G}_\tau$, where $\tau$ is an arbitrary compact topology in $G$, is defined uniquely by the algebraic structure of $G$. We consider conditions (i)-(ii) and (i)-(ii)′ separately.

Conditions (i)-(ii). In this case the group $G$ is reduced and hence by 4.3 the character group $\hat{G}_\tau$ is periodic. Let

$$\hat{G}_\tau = \sum_p G_p \tag{8.3}$$

([5]) It is rather doubtful whether the theorem can be proved without the generalized continuum hypothesis. For, it can be deduced from it that for every two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ the equality $2^{\mathfrak{m}} = 2^{\mathfrak{n}}$ implies $\mathfrak{m} = \mathfrak{n}$. It is not known whether this is weaker than the generalized continuum hypothesis.

be the decomposition of $\hat{G}_\tau$ into its $p$-components. For each prime $p$ let $B_p$ denote the basic subgroup of $G_p$. By condition (ii), for each prime $p$ there exists an integer $m$ such that $G$ contains no pure cyclic subgroup of order $p^n$ with $n > m$. Hence by 4.11 the same is true for $\hat{G}_\tau$, which means that the group $B_p$ has finite exponent. Thus, since $B_p$ is pure in $G_p$,

$$G_p = B_p + H, \tag{8.4}$$

where $H$ is a divisible $p$-group. Let

$$B_p = \sum_{n=1}^{\infty} \sum_{\iota \in T_n^p} C_{p^n}^{(\iota)}, \quad H = \sum_{\iota \in T_0^p} C_{p^\infty}^{(\iota)}. \tag{8.5}$$

Hence by (8.3), (8.4), 4.5 and 4.9 we get

$$G = \sum_p{}^{*} \Big[ \sum_{n=1}^{\infty}{}^{*} \sum_{\iota \in T_n^p}{}^{*} C_{p^n}^{(\iota)} + \sum_{\iota \in T_0^p}{}^{*} I_p^{(\iota)} \Big].$$

But by Lemmas 5.4 and 5.5 we have $\bar{\bar{T}}_n^p = \bar{\bar{\Gamma}}_n^p$ for all $n = 0, 1, 2, \ldots$ and $p = 2, 3, 5, \ldots$ Thus by (8.5), (8.4) and (8.3) we have expressed the structure of $\hat{G}_\tau$ in terms of the cardinals $\bar{\bar{\Gamma}}_n^p$, $n = 0, 1, 2, \ldots, p = 2, 3, 5, \ldots$ which in this case are uniquely defined by the group $G$ (compare Lemmas 5.4 and 5.5).

Conditions (i)-(ii)'. It follows immediately from condition (ii)' that the maximal periodic subgroup $T$ of the group $G$ is finite and hence closed in the topology $\tau$. We note that there exists a direct summand $S$ of the group $G$ complementary to $T$ and also closed in $\tau$. For, since $T$ is pure in $G$, the annihilator $N$ of $T$ in the character group $\hat{G}_\tau$ is by 4.7 a pure subgroup of $\hat{G}_\tau$ and $\hat{G}_\tau / N \simeq \hat{T} \simeq T$. Hence, since $T$ is the direct sum of cyclic groups, $N$ is a direct summand of $\hat{G}_\tau$. Thus $\hat{G}_\tau = \hat{T} + N$. Hence, by 4.5, $G = T + S$, where $S = \hat{N}$, and $S$ is closed in $\tau$. Since $S$ is torsion-free, the group $\hat{S}_{\tau'} \simeq N$, $\tau' = \tau \wedge S$, is divisible. Thus

$$N = \widehat{\sum_{\iota \in T_0}} R^{(\iota)} + \sum_p \sum_{\iota \in T_p} C_{p^\infty}^{(\iota)}.$$

Since $\hat{T} \simeq T$, it sufficies to prove that the cardinals $\bar{\bar{T}}_0$ and $\bar{\bar{T}}_p$, $p = 2, 3, 5, \ldots$, are uniquely defined by the group $G$. By 4.5 we have

$$G = T + S = T + \sum_{\iota \in T_0} R^{(\iota)} + \sum_p{}^{*} \sum_{\iota \in T_p}{}^{*} I_p^{(\iota)}.$$

Since $T$ is finite, the conditions of Lemma 5.5 are satisfied and the equalities $\bar{\bar{T}}^p = \bar{\bar{\Gamma}}_0^p$, $p = 2, 3, 5, \ldots$, follow at once. By 4.4 the group $D = \widehat{\sum_{\iota \in T_0} R^{(\iota)}}$ is the maximal divisible subgroup of $G$. Hence $D = \sum_{\iota \in \Delta} R^{(\iota)}$ and, by (5.1), (i) $\bar{\bar{\Delta}}\aleph_0 = \overline{\overline{\sum_{\iota \in \Delta} R^{(\iota)}}} = 2^{\mathfrak{n}}$, where, by 4.12, $\mathfrak{n} = \overline{\overline{\sum_{\iota \in T_0} R}} = \bar{\bar{T}}_0 \cdot \aleph_0$. Hence, since $\bar{\bar{\Delta}} > \mathfrak{c}$, we have $\mathfrak{n} > \aleph_0$ and $\bar{\bar{T}}_0 \aleph_0 = \bar{\bar{T}}_0$. Thus $\bar{\bar{\Delta}} = \bar{\bar{\Delta}}\aleph_0 = 2^{\bar{\bar{T}}_0}$ and, since the generalized continuum hypothesis is assumed, $\bar{\bar{T}}_0$ is uniquely defined by $\bar{\bar{\Delta}}$ and hence by $G$, as required.

Necessity. In order to prove the necessity of either of the pair of conditions (i)-(ii) and (i)-(ii)′ we show that if a group $G$ does not satisfy (i) or both (ii) and (ii)′, then it admits at least two compact topologies which are not isomorphic.

Condition (i). Suppose that $G$ has a decomposition of the type (5.1) such that $\Delta p \neq 0$ for some prime $p$. Then, by (5.1), (i), the maximal divisible subgroup $D$ of $G$ can be decomposed as follows:

$$D = \sum_{\iota \in T_0}^{*} H^{(\iota)} + \sum_{p}^{*} \sum_{\iota \in U_p}^{*} F_p^{(\iota)},$$

where

$$H^{(\iota)} = \sum_{\xi \in S_\iota} R^{(\xi)}, \quad F_p^{(\iota)} = \sum_{\xi \in W_\iota} R^{(\xi)} + C_{p^\infty}^{(\iota)}$$

with

$$\bar{\bar{S}}_\iota = \bar{\bar{W}}_\iota = \mathfrak{c}, \quad 2^{\bar{\bar{U}}_p} = \bar{\bar{\Delta}}_p, \quad 2^{\bar{\bar{T}}_0} = \bar{\bar{\Delta}}.$$

Every group $H^{(\iota)}$ has a compact topology $\tau_\iota$ such that $\hat{H}_{\tau_\iota}^{(\iota)} \simeq R$. Every group $F_p^{(\iota)}$ has a compact topology $\tau_\iota$ such that $\hat{F}_{p,\tau_\iota}^{(\iota)}$ is of rank 1 (see [5]) and also a compact topology $\bar{\tau}_\iota$ such that $\hat{F}_{p,\bar{\tau}_\iota}^{(\iota)}$ is an indecomposable group of rank 2 (see [5], example). Let $\tau'$ be a compact topology in the reduced part of $G$. We put

$$\tau_1 = \tau' \times \mathop{\mathscr{P}}_{\iota \in A} \tau_\iota, \quad \text{where} \quad A = T_0 \cup \bigcup_p U_p, \quad \tau_2 = \tau' \times \mathop{\mathscr{P}}_{\iota \in T_0} \tau_\iota \times \mathop{\mathscr{P}}_{p} \mathop{\mathscr{P}}_{\iota \in U_p} \bar{\tau}_\iota.$$

Then by 4.5 the group $\hat{G}_{\tau_1}$ is the direct sum of groups of rank 1 and a periodic group. The group $\hat{G}_{\tau_2}$ is the direct sum of groups of rank 1, a periodic group and indecomposable groups of rank 2. The groups $\hat{G}_{\tau_1}$ and $\hat{G}_{\tau_2}$ are not isomorphic, since by [9] each direct summand of the group $\hat{G}_{\tau_2}/P$, where $P$ is the maximal periodic subgroup of $\hat{G}_{\tau_1}$, is the direct sum of groups of rank 1. Thus $\tau_1$ and $\tau_2$ are not isomorphic.

Conditions (i)-(ii). Suppose now that the group $G$ satisfies condition (i) and condition (ii) does not hold. Then either

(a) *for each prime $p$ there exists an increasing sequence of integers $n_i$, $i = 1, 2, \ldots$, such that $\Gamma_{n_i}^p \neq \emptyset$*,

or

(b) $\Delta \neq \emptyset$.

We consider first case (a). Let

$$H = \sum_{i=1}^{\infty}{}^{*} C_{p^{n_i}},$$

where for each $i$ the group $C_{p^{n_i}}$ is equal to $C^{(\iota)}_{p^{n_i}}$ for some $\iota \in \Gamma^{p}_{n_i}$. Let $A$ be a direct summand of the reduced part of $G$ complementary to $H$. Let $D$ be the maximal divisible subgroup of $G$. The group $A$ is the complete direct sum of finite cyclic groups and groups of $p$-adic integers (might be with various $p$). We define a topology $\tau_1$ in $G$ putting

$$\tau_1 = \tau' \times \tau'' \times \tau''',$$

where $\tau'$ is an arbitrary compact topology in $D$, $\tau''$ the compact topology in $A$ being the product topology of discrete topologies in the finite cyclic direct summands and $p$-adic topologies in the groups of $p$-adic integers in the decomposition of $A$ into the complete direct sum, $\tau'''$ is the product topology $\mathscr{P}_{i=1}^{\infty} \tau_i'''$, where $\tau_i'''$ is the discrete topology in the group $C_{p^{n_i}}$, $i = 1, 2, \ldots$ By 4.5 and 4.9 the maximal periodic subgroup of the group $\hat{G}_\tau$ is the direct sum of finite cyclic groups and $C_{p^\infty}$'s. We are going to construct a topology $\tau^{\mathrm{IV}}$ in the group $H$ such that $\hat{H}_{\tau^{\mathrm{IV}}}$ is reduced and contains elements of infinite height. Then the topology

$$\tau_2 = \tau' \times \tau'' \times \tau^{\mathrm{IV}}$$

is not isomorphic to $\tau_1$; for, $\hat{G}_{\tau_2}$ is not isomorphic to $\hat{G}_{\tau_1}$, since the reduced part of the maximal periodic subgroup of the group $\hat{G}_{\tau_2}$ contains elements of infinite height, which is not true as far as the maximal periodic subgroup of the group $\hat{G}_{\tau_1}$ is concerned. In order to construct the topology $\tau^{\mathrm{IV}}$ we define a reduced group $F$ with elements of infinite height such that $\hat{F} \simeq H$. Then we put $\tau^{\mathrm{IV}}$ to be the topology transfered into $H$ from $\hat{F}$ by the isomorphism $\hat{F} \simeq H$. Let $F$ be the group generated by the symbols

$$b, a_1, a_2, \ldots$$

and the relations

$$pb = 0, \quad b = p^{n_1}a_1 = \ldots = p^{n_i}a_i = \ldots \tag{8.6}$$

It follows immediately from (8.6) that the element $b$ has infinite height in $F$ and that $F$ is reduced. We prove that $\hat{F} \simeq H$. We show first that the set

$$\{c_i : c_i = p^{n_{i+1}-n_i}a_{i+1} - a_i,\ i = 1, 2, \ldots\}$$

generates a pure subgroup $\hat{F}'$ in $F$ and that

$$F' = \sum_{i=1}^{\infty} \mathrm{gp}\{c_i\} = \sum_{i=1}^{\infty} C_{p^{n_i}}.$$

To do this we note first that

$$(8.7) \qquad p^{n_i} c_i = 0, \quad i = 1, 2, \ldots$$

It follows from (8.6) that

$$(8.8) \qquad \text{if } r_1 a_1 + \ldots + r_k a_k = \begin{cases} 0 \\ pa, \ a \in F, \end{cases} \text{ then } p^{n_i} | r_i \text{ or } p | r_i, \text{ respectively,}$$

$i = 1, 2, \ldots, k$.

We have

$$(8.9) \qquad m_1 c_1 + \ldots + m_k c_k = \sum_{i=1}^{k} (m_i p^{n_{i+1} - n_i} a_{i+1} - a_i)$$

$$= m_1 a_1 + \sum_{i=1}^{k-1} (m_i p^{n_{i+1} - n_i} - m_{i+1}) a_{i+1} + m_k p^{n_{k+1} - n_k} a_{k+1}.$$

Suppose that for some integers $m_1, \ldots, m_k$ we have

$$m_1 c_1 + \ldots + m_k c_k = 0.$$

Then by (8.8) and (8.9) we get $p^{n_i} | m_i$, which by (8.7) gives $m_i c_i = 0$, $i = 1, 2, \ldots, k$. Similarly we prove that $F'$ is pure in $F$: if $p$ divides $m_1 c_1 + \ldots + m_k c_k$ in $F$, then by (8.9) and (8.8) $p | m_i$, $i = 1, 2, \ldots, k$. It follows immediately from (8.6) and the definition of the $c_i$'s that $F/F' \simeq C_{p^\infty}$. Since $F'$ is pure in $F$, by 4.8, we get $\hat{F} = \hat{F}' + \widehat{(F/F')}$. Hence by 4.5 and 4.9 we obtain

$$F \simeq \sum_{i=1}^{\infty} C_p n_i + I_p.$$

It is plain that the group $H$ belongs to the class $K$ (compare section 6). Then, by Corollary 6.13, $H + I_p \simeq H$ and thus $\hat{F} \simeq H$.

(b) Suppose $\Delta \neq \emptyset$. Then, by (5.1), (i), we have $\bar{\bar{\Delta}} \geqslant \mathfrak{c}$. We consider first the case $\bar{\bar{\Delta}} = \mathfrak{c}$. Let $S$ be the reduced part of the group $G$ and let $\tau'$ be a compact topology in it. The maximal divisible subgroup $D$ of $G$ can be decomposed into the direct sum $D = D^1 + D^2$ such that $\bar{\bar{D}}^1 = \bar{\bar{D}}^2 = \mathfrak{c}$. If $\tau''$, $\tau'''$ are compact topologies in $D^1$ and $D^2$ respectively, such that $\hat{D}^1_{\tau''} \simeq \hat{D}^2_{\tau'''} \simeq R$, and $\tau^{\mathrm{IV}}$ is a compact topology in $D$ such that $\hat{D}_{\tau^{\mathrm{IV}}} \simeq R$, then the topologies

$$\tau_1 = \tau' \times \tau'' \times \tau''' \quad \text{and} \quad \tau_2 = \tau' \times \tau'' \times \tau^{\mathrm{IV}}$$

are not isomorphic since by 4.5, 4.3 $\hat{G}_{\tau_1}/N_1 \simeq R + R$ and $\hat{G}_{\tau_2}/N_2 \simeq R$, where $N_1 \simeq \hat{S}_{\tau_2 \wedge S}$, $N_2 \simeq \hat{S}_{\tau_2 \wedge S}$ are the maximal periodic subgroups of $\hat{G}_{\tau_1}$ and $\hat{G}_{\tau_2}$, respectively.

In the case $\bar{\bar{\Delta}} > \mathfrak{c}$ the group $G$ either satisfies condition (ii)' and then all the compact topologies in $G$ are isomorphic, or it does not satisfy (ii)'. Thus the only thing which is left to show for the proof of the theorem is the necessity of

Conditions (i)-(ii)'. Let then $\Delta_p = \emptyset$, $\bar{\bar{\Lambda}} > \mathfrak{c}$ and there exists an infinite set of primes $p_1, p_2, \ldots$ such that $\Gamma_{n_i}^{p_i} \neq \emptyset$ for some suitably chosen positive integers $n_i$. We start with two groups

$$H = \sum_{i=1}^{\infty}{}^{*} C_{p_i^{n_i}} \quad \text{and} \quad M = \sum_{\iota\in\Lambda} R^{\iota} \quad \text{with} \quad \bar{\bar{\Lambda}} = 2^{\mathfrak{c}}.$$

We rearrange the direct summands in the decomposition (5.1) of the group $G$ in such a manner that

$$G = S+K+M+H,$$

where $S = \sum_{\iota\in T} R^{(\iota)}$, $\bar{\bar{T}} = \bar{\bar{\Lambda}}$ and $K$ is the complete direct sum of finite cyclic groups and groups of $p$-adic integers. If $\tau_1 = \tau' \times \tau'' \times \tau''' \times \tau^{\mathrm{iv}}$, where $\tau', \tau'', \tau''', \tau^{\mathrm{iv}}$ are arbitrary compact topologies in the groups $S, K, M, H$, respectively, then by 4.5, 4.3, 4.4 the maximal periodic subgroup $\hat{K}_{\tau''}+\hat{H}_{\tau^{\mathrm{iv}}}$ of the group $\hat{G}_{\tau_1}$ is a direct summand of it. We put $\tau_2 = \tau' \times \tau''' \times \bar{\tau}$, where $\bar{\tau}$ is the compact topology transferred into $M+H$ by the isomorphism $\hat{F} \simeq M+H$ and $F$ is a discrete group isomorphic with $\sum_{i=1}^{\infty} C_{p_i^{n_i}}$. To see that in fact $\hat{F}$ is isomorphic with $M+H$ we note that the direct product $\sum_{i=1} C_{p_i^{n_i}} = F'$ is a pure subgroup of $F$ and that $F/F'$ is a divisible torsion-free group of cardinal $\mathfrak{c}$. Hence by 4.8 we have $\hat{F} = \hat{F}'+(\widehat{F/F'})$ and, by 4.4 and 4.9, $(\widehat{F/F'}) \simeq M$ and $\hat{F}' \simeq H$. The topology $\tau_2$ is not isomorphic with the topology $\tau_1$, since the maximal periodic subgroup $F'+\hat{K}_{\tau''}$ is not a direct summand of the group $\hat{G}_{\tau_2}$.

THEOREM 8.10. *A group $G$ admits exactly one compact topology if and only if it is of the form*

$$G = \sum_{p}{}^{*} D_p,$$

*where for each prime $p$ the group $D_p$ is the direct sum of finitely many cyclic $p$-groups and groups of $p$-adic integers.*

Proof. The sufficiency of the conditions of the theorem was proved in Lemma 5.10. In order to prove the necessity we consider the set of automorphisms of the group $G$. We show that if $G$ contains a non-trivial divisible subgroup, or for some $p$ the group $D_p$ is the complete direct sum of infinitely many cyclic $p$-groups and groups of $p$-adic integers, then for each compact topology $\tau$ in the group $G$ there exists an automorphism $\varphi$ of the group, which is not continuous in the topology $\tau$. Thus we get two different topologies $\tau$ and $\tau\varphi$ of the group $G$. Since the maximal divisible subgroup $D$ of $G$ is a direct summand of it, then every automorphism, which is not continuous in $\tau \wedge D$ defines an automorphism of $G$, which is not continuous in $\tau$. Similar reasoning leads us to the conclusion that if $G = \sum_p^* D_p$, then every automorphism of the group $D_p$ (for some prime $p$) which is not continuous in the topology $\tau \wedge D_p$ defines an automorphism of the group $G$ which is not continuous in $\tau$. Thus the theorem follows from the following simple two lemmas.

LEMMA 8.11. *If $D$ is a divisible group and $\tau$ a compact topology in it, then there exists an automorphism of $D$, which is not continuous in $\tau$.*

Proof. By 4.10 the group of the automorphisms of $D$ which are continuous in $\tau$ is isomorphic with the group of the automorphisms of $\hat{D}_\tau$. Hence it has cardinal not greater than $2^{\overline{\overline{\hat{D}_\tau}}} = \overline{\overline{D}}$. But, since $D$ is divisible, and $\overline{\overline{D}} > \aleph_0$, the group of the automorphisms of $D$ has cardinal $2^{\overline{\overline{D}}} > \overline{\overline{D}}$.

LEMMA 8.12. *If a group $D_p$ is the complete direct sum of infinitely many finite $p$-groups and groups of $p$-adic integers, then for each compact topology $\tau$ in $D_p$ there exists an automorphism which is not continuous in $\tau$.*

Proof. Similarly as in the proof of Lemma 8.11 we show that the group of the automorphisms of $D_p$ which are continuous in $\tau$ has cardinal $\leqslant D_p$. To see that the group of all the automorphisms of $D_p$ has cardinal $2^{\overline{\overline{D}}_p}$ we note that by Theorems 6.15 and 6.26

$$D_p = D_p^1(1) + \ldots + D_p^1(k) + D_p^2,$$

where either for some $i$, $i = 1, \ldots, k$, $D_p^1(i) \in \boldsymbol{B}$ and $\overline{\overline{D}}{}_p^1(i) = \overline{\overline{D}}_p$, or $D_p^2 \in \boldsymbol{K}$ and $\overline{\overline{D}}{}_p^2 = \overline{\overline{D}}_p$. It is easy to verify that in the first case the group $D_p^1(i) = \sum_{\iota \in A} C_{p^{n_\iota}}^{(\iota)}$, $\overline{\overline{A}} = \overline{\overline{D}}{}_p^1(i)$, has $2^{\overline{\overline{D}}{}_p^1(i)} = 2^{\overline{\overline{D}}_p}$ automorphisms. In the second case by Corollary 6.14 the group $D_p^2$ contains a set $A$ such that $\overline{\overline{A}} = \overline{\overline{D}}{}_p^2$ and each permutation of $A$ defines an automorphism of a basic submodule of $D_p^2$. Hence by 6.7 it defines an automorphism of $D_p^2$. Thus the group of the automorphisms of $D_p$ has again cardinal $2^{\overline{\overline{D}}{}_p^2} = 2^{\overline{\overline{D}}_p}$.

## 9. The class $M$

DEFINITION OF THE CLASS $\boldsymbol{M}$. *We say that a group $G$ belongs to the class $\boldsymbol{M}$ if for each compact topology $\tau$ in $G$ there exists a set $A_\tau$ contained in $G$ such that*

(i) *if $M$ is a Borel subset of positive Haar measure of the topological group $G_\tau$, then $\overline{\overline{M \cap A_\tau}} = \overline{\overline{G}}$,*

(ii) *for any compact topology $\tau$ in the group $G$ there exists a group $\bar{A}_\tau$ containing $A_\tau$ such that $G = \bar{A}_\tau + S_\tau$ and such that if $\tau'$, $\tau''$ are any two com-*

*pact topologies in $G$ and $\varphi_{\tau'\tau''}$ a one-to-one mapping of $A_{\tau'}$ onto $A_{\tau''}$ then $\varphi_{\tau'\tau''}$ has the unique extension to an isomorphism of the groups $\tilde{A}_{\tau'}$ and $\tilde{A}_{\tau''}$ and, moreover, $S_{\tau'} \simeq S_{\tau''}$.*

The class $M$, though defined somewhat artificially, is the class of groups for which we prove the Main Theorem at first. This will enable us to deduce it in the general case. The class $M$ is large enough to contain all the divisible groups admitting compact topologies as well as the groups of the class $K$ and $B$.

LEMMA 9.1. *Any divisible group admitting compact topology belongs to the class $M$.*

Proof. Let $G$ be a divisible group and $\tau$ a compact topology in it. We well-order the class $\mathscr{B}_{\tau}^{**}$ (compare section 1) of subsets of $G$ in the sequence $\langle M_{\xi}\rangle_{\xi<\omega_a}$, where $\omega_a$ is the first ordinal of cardinal $\bar{\bar{G}}$. By Lemma 1.4 we have $\bar{\bar{M}}_{\xi} = \bar{\bar{G}}$ for each $\xi < \omega_a$. For each $\xi < \omega_a$ we pick an element $x_{\xi}$ belonging to $M_{\xi}$ and linearly independent from the set $\{x_{\eta}\colon \eta < \xi\}$. If $A_{\tau}$ is a maximal set of linearly independent elements of $G_{\tau}$ containing the set $\{x_{\xi}\colon \xi < \omega_a\}$, $A_{\tau}$ satisfies all the requirements for the set $A_{\tau}$ in the definition of the class $M$. (In (ii) the group $\tilde{A}_{\tau}$ is the least divisible subgroup containing $A_{\tau}$ and $S_{\tau}$ is the maximal periodic subgroup of $G_{\tau}$.)

LEMMA 9.2. *The class $K$ is contained in the class $M$.*

Proof. Let $G \in K$ and let $\tau$ be a compact topology in $G$. Let $A_{\tau}$ be the set the existence of which is asserted by Corollary 6.14. Plainly, $A_{\tau}$ satisfies (i). The module

$$B(\tau) = m\{A_{\tau}\} + T(\tau),$$

where $T(\tau)$ is a suitable periodic submodule of a basic submodule of $G_{\tau}$. If $\overline{m}$ denotes the completion in the $p$-adic topology in the module $m$, then by 6.6 we have

$$G_{\tau} = \overline{B}(\tau) = \overline{m\{A_{\tau}\}} + \overline{T(\tau)}.$$

If $\tau'$ and $\tau''$ are two compact topologies of the group $G$, then every one-to-one mapping of $A_{\tau'}$ onto $A_{\tau''}$ defines uniquely an isomorphism of $m\{A_{\tau'}\}$ onto $m\{A_{\tau''}\}$. By 6.7 this isomorphism has the unique extension to the isomorphism of the modules $\overline{m\{A_{\tau'}\}} = \tilde{A}_{\tau'}$ and $\overline{m\{A_{\tau''}\}} = \tilde{A}_{\tau''}$. Since by 6.4 the modules $B(\tau')$ and $B(\tau'')$ are isomorphic and $T(\tau')$, $T(\tau'')$ are the maximal periodic submodules of $B(\tau')$ and $B(\tau'')$ respectively, we have $T(\tau') \simeq T(\tau'')$ and putting $S_{\tau'} = \overline{T}(\tau')$, $S_{\tau''} = \overline{T(\tau'')}$ we get $S_{\tau'} \simeq S_{\tau''}$, which completes the proof of Lemma 9.2.

LEMMA 9.3. *The class $B$ is contained in the class $K$.*

Proof. Let $G = \sum_{\iota\in A} C'_{p^n} \in \boldsymbol{B}$, and let $\tau$ be a compact topology in $G$. As in Lemma 9.1, we well-order the class $\mathscr{B}_\tau^{**}$ of subsets of $G$ in the sequence $\langle M_\xi\rangle_{\xi<\omega_\alpha}$, where $\omega_\alpha$ is the first ordinal of the cardinal $\overline{\overline{G}}$. In each set $M_\xi$, $\xi < \omega_\alpha$, we pick an element $x_\xi$ such that the sequence $\langle x_\eta\rangle_{n<\omega_\alpha}$ consists of elements independent over the ring $C_{p^n}$ and such that the group $\mathrm{gp}\{x_\eta\colon \eta \leqslant \xi\}$ is pure in $G$. The existence of this selection we prove as follows. Suppose that for some $\xi$, $\xi < \omega_\alpha$, there is no element $x_\xi$ in $M_\xi$ such that the set $\{x_\eta\colon \eta \leqslant \xi\}$ satisfies the above conditions. Then $M_\xi \subset \mathrm{gp}\big\{\{x_\eta\colon \eta < \xi\}, pG\big\} = N$. Hence, since $M_\xi$ has positive Haar measure, the group $G_\tau/N = S$ is finite and $G_\tau/pG_\tau = \mathrm{gp}\{x_\eta\colon \eta < \xi\} + S$. But hence $\overline{\overline{G_\tau/pG_\tau}} = \bar{\xi}p + \bar{S} < \overline{\overline{G}}$, which is impossible, since $G \in \boldsymbol{B}$. A maximal set $A_\tau$ of elements independent over the ring $C_{p^n}$ generating a pure subgroup of $G_\tau$ and containing the set $\{x_\xi\colon \xi < \omega_\alpha\}$ satisfies the requirements for the set $A_\tau$ of the definition of the class $\boldsymbol{M}$. Condition (ii) is satisfied automatically, since $\mathrm{gp}\{A_\tau\} = G_\tau$.

We conclude this section with the following

LEMMA 9.4. *If a group $G$ belongs to the class $\boldsymbol{M}$ and if $\tau$ is a compact topology in $G$, then the set $A_\tau$ can be decomposed into $\overline{\overline{G}}$ disjoint sets $A_\tau^\lambda$, $\lambda \in A$, each of full outer Haar measure and of cardinal $\overline{\overline{G}}$.*

Proof. We well-order the familly $\mathscr{B}_\tau^{**}$ in the sequence $\langle M_\xi\rangle_{\xi<\omega_\alpha}$, where $\omega_\alpha$ is the first ordinal of cardinal $\overline{\overline{G}}$. Since for each $\xi$, $\xi < \omega_\alpha$, $\overline{\overline{M_\xi \cap A_\tau}} = \overline{\overline{G}}$, we select by an easy transfinite induction a sequence $\langle x_\lambda^{(\xi)}\rangle_{\lambda<\xi}$ in such a manner that the elements $x_\beta^\eta$, $\beta < \eta \leqslant \xi$, are different and belong to $A_\tau$. Since every set $M \in \mathscr{B}_\tau^*$ contains $\overline{\overline{G}}$ different sets of the family $\mathscr{B}_\tau^{**}$, the sets

$$C_\tau^\lambda = \{x_\lambda^{(\xi)}\colon\ \xi < \omega_\alpha\}, \quad \lambda < \omega_\alpha,$$

are disjoint, of full outer Haar measure and of cardinal $\overline{\overline{G}}$. We put

$$A_\tau^1 = C_\tau^1 \cup A_\tau \setminus \bigcup_{\lambda<\omega_\alpha} C_\tau^\lambda \quad \text{and} \quad A_\tau^\lambda = C_\tau^\lambda \quad \text{for} \quad 1 < \lambda < \omega_\alpha.$$

## 10. Proof of the Main Theorem (groups of the class $M$)

Now we are in a position to prove the Main Theorem for the groups of the class $\boldsymbol{M}$. We prove it in a slightly stronger form.

THEOREM 10.1. *Let $G$ be a group belonging to the class $\boldsymbol{M}$ and let $(T\Omega)$ be a system of compact topologies of $G$* (compare section 1) *such that $\overline{\overline{\Omega}} = 2^{\mathfrak{m}}$, where $\mathfrak{m} = \overline{\overline{G}}$. Then there exists a regular system $(S\Omega)$ isomorphic to the system $(T\Omega)$.*

Proof. Let $G \in M$, $\overline{\overline{G}} = \mathfrak{m}$ and $(F\Omega)$ be a system of compact topologies of $G$ such that $\overline{\overline{\Omega}} = 2^{\mathfrak{m}}$. For each $\iota \in \Omega$ we put $\tau_\iota = \iota\ (T\Omega)$ and by $A_\iota$ we denote the subset $A_{\tau_\iota}$ of $G_{\tau_\iota}$ the existence of which is postulated in definition of the class $M$. Let further $\mathscr{B}_\iota$ be the family of the Baire sets of the group $G_{\tau_\iota}$, $\nu_\iota$ its Haar measure and $\mathscr{B}_\iota^*$ the family of the Baire sets of positive Haar measure.

Consider the group

$$\sum_{\iota \in \Omega}^{*} G_{\tau_\iota} = H.$$

For each set $E \subset \Omega$ we denote by $\mathscr{B}_E$, $\nu_E$, $\mathscr{B}_E^*$ the class of Baire sets, the Haar measure, and the class of the Baire sets of positive Haar measure of the subgroup $H_E = \sum_{\iota \in E}^{*} G_{\tau_\iota}$, respectively.

We shall use the following simple fact:

10.2. *If $M$ belongs to $\mathscr{B}_\Omega$ (or to $\mathscr{B}_\Omega^*$), then it is of the form*

$$M = M_E + H_{\Omega \setminus E}, \tag{10.2}$$

*where $\overline{\overline{E}} = \aleph_0$ and $M_E \in \mathscr{B}_E$, (or $M_E \in \mathscr{B}_E^*$ respectively).*

To see this we note first that the sets of the form (10.2) form a $\sigma$-field of subsets. Thus it sufficies to prove 10.2 for every compact set $C$ which is the intersection of countably many open sets. Let then

$$C = \bigcap_{i=1}^{\infty} \bigcup_{a} V_a^{(i)}, \quad \text{where} \quad V_a^{(i)} = V_{a_i} + H_{\Omega \setminus E_i}, \tag{10.3}$$

$E_i$ is a finite set and $V_{a_i}$ an open set in the group $H_{E_i}$. Since $C$ is compact, by (10.3),

$$C = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{n_i} V_{a_j}^{(i)}.$$

Hence, putting $E = \bigcup_{i,j} E_{ij}$ we obtain (10.2).

Since $\overline{\overline{\Omega}} = 2^{\mathfrak{m}}$, the set $\Omega$ can be regarded as the product space with the Tychonoff topology of $\mathfrak{m}$ two-points discrete spaces. Let $\mathfrak{W}$ be a base of cardinal $\mathfrak{m}$ of open sets in $\Omega$ and let $\mathfrak{W}_\sigma$ be the least $\sigma$-field containing $\mathfrak{W}$. Plainly $\overline{\overline{\mathfrak{W}}}_\sigma = \mathfrak{m}^{\aleph_0} = \mathfrak{m}$, since, by 4.12, $\mathfrak{m} = \overline{\overline{G}} = 2^{\aleph}$.

Let $\mathfrak{P}$ be the family of all countable sequences

$$\mathfrak{p} = \{W_1, W_2, \ldots\} \tag{10.4}$$

of mutually disjoint sets $W_i$, $i = 1, 2, \ldots$, belonging to $\mathfrak{W}_\sigma$ and covering together the set $\Omega$. We have $\overline{\overline{\mathfrak{P}}} = \mathfrak{m}^{\aleph_0} = \mathfrak{m}$. It is easy to see that for each countable set $E = \{\iota_1, \iota_2, \ldots\}$, $E \subset \Omega$, there exists a sequence of the form (10.4) such that $\iota_i \in W_i$ for each $i = 1, 2, \ldots$ For each $\mathfrak{p}$ we select a fixed set

$$E_\mathfrak{p} = \{\iota_1(\mathfrak{p}), \iota_2(\mathfrak{p}), \ldots\}$$

of elements of $\Omega$, such that $\iota_i(\mathfrak{p}) \in W_i$ for each $i = 1, 2, \ldots$ For each set $M_{E_\mathfrak{p}} \in \mathscr{B}^*_{E_\mathfrak{p}}$ we denote by $M_\mathfrak{p}$ the pair $(M_{E_\mathfrak{p}}, \mathfrak{p})$ and by $\mathscr{B}^*_\mathfrak{p}$ the set $\{M_\mathfrak{p}\colon M_{E_\mathfrak{p}} \in \mathscr{B}^*_{E_\mathfrak{p}}\}$. We have $\overline{\overline{\mathscr{B}^*_\mathfrak{p}}} = \mathfrak{m}$ and hence, if

$$\mathscr{B}^* = \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathscr{B}^*_\mathfrak{p},$$

then also

(10.5) $$\overline{\overline{\mathscr{B}^*}} = \mathfrak{m}.$$

For each $M_\mathfrak{p} \in \mathscr{B}^*$ and every $\iota \in \Omega$ we select a set $A_\iota(M_\mathfrak{p})$ contained in $A_\iota$ such that

(10.6) $$\overline{\overline{A_\iota(M_\mathfrak{p})}} = \mathfrak{m},$$

(10.7) $$\mu_\iota^*[A_\iota(M_\mathfrak{p})] = 1,$$

(10.8) $$\bigcup_{M_\mathfrak{p} \in \mathscr{B}^*} A_\iota(M_\mathfrak{p}) = A_\iota \text{ and } M'_{\mathfrak{p}'} \neq M''_{\mathfrak{p}''} \text{ implies } A_\iota(M'_{\mathfrak{p}'}) \cap A_\iota(M''_{\mathfrak{p}''}) = \emptyset.$$

The existence of such selection follows immediately from (10.5) and Lemma 9.4. It follows from (10.7) and Fubini theorem that for each set $E$ contained in $\Omega$ we have

(10.9) $$\nu_E^*[\mathop{\mathscr{P}}_{\iota \in E} A_\iota(M_\mathfrak{p})] = 1.$$

Equalities (10.6) and (10.8) imply that for every pair $\iota', \iota''$ of elements of $\Omega$ we can find a one-to-one mapping $s(\iota', \iota'')$ of $A_{\iota'}$ onto $A_{\iota''}$ such that for each $M_\mathfrak{p} \in \mathscr{B}^*$

$$A_{\iota'}(M_\mathfrak{p}) S(\iota', \iota'') = A_{\iota''}(M_\mathfrak{p})$$

and for every three elements $\iota', \iota'', \iota'''$ of the set $\Omega$ we have

$$s(\iota', \iota'') s(\iota'', \iota''') = s(\iota', \iota''').$$

For each $M_\mathfrak{p}$ we define an element

$$x(M_\mathfrak{p}) = \langle x_\iota(M_\mathfrak{p})\rangle_{\iota \in \Omega} = \langle x_\iota \rangle_{\iota \in \Omega} \in H$$

as follows:

For $\iota = \iota_i(\mathfrak{p})$, $i = 1, 2, \ldots$, we choose $x_\iota$ from $A_\iota(M_\mathfrak{p})$ in such a way that $\langle x_{\iota_i}(\mathfrak{p})\rangle_{i=1,2,\ldots} \in M_{E_\mathfrak{p}}$ with $M_\mathfrak{p} = (M_{E_\mathfrak{p}}, \mathfrak{p})$. (The possibility of the choice follows from (10.9) and the fact that $M_{E_\mathfrak{p}} \in \mathscr{B}^*_{E_\mathfrak{p}}$.)

For the rest of the $\iota$'s, $\iota \in \Omega$, we put

(10.10) $$x_\iota = x_{\iota_i}(\mathfrak{p}) s(\iota_i(\mathfrak{p}), \iota),$$

where $\iota_i(\mathfrak{p})$ is chosen in the way, that $\iota$ and $\iota_i(\mathfrak{p})$ belong to the same set $W_i$ of $\mathfrak{p} = \{W_1, W_2, \ldots\}$. Let

$$A = \{x(M_{\mathfrak{p}}):\ M_{\mathfrak{p}} \in \mathscr{B}^*\}.$$

We are going to prove that the set $A$ has the following properties:

(10.11)
(i) $\nu_\Omega^*(A) = 1$;
(ii) *for each* $\iota$, $\iota \in \Omega$, *the projection* $\pi_\iota\colon H \to G_\iota$ *is a one-to-one mapping of* $A$ *into* $A_\iota$;
(iii) $\overline{\overline{A_\iota \setminus A\pi_\iota}} = \mathfrak{m}$.

To prove (10.11), (i) we show that for each $M \in \mathscr{B}_\Omega^*$ the sets $A$ and $M$ have non-void intersection. By (10.2) there exists a countable set $E = \{\iota_1, \iota_2, \ldots\}$ contained in $\Omega$ and a set $M_E$ belonging to $\mathscr{B}_E^*$ such that $M$ is of the form (10.2). Let $\mathfrak{p} = \{W_1, W_2, \ldots\}$ be an element of $\mathfrak{P}$ such that for each $i = 1, 2, \ldots$ we have $\iota_i \in W_i$. If $s$ is the mapping of $H_E$ onto $H_{E_\mathfrak{p}}$ defined by the equality

$$\langle x_{\iota_i}\rangle_{i=1,2,\ldots} s = \langle x_{\iota_i} s(\iota_i, \iota_i(\mathfrak{p}))\rangle_{i=1,2,\ldots}$$

with $x_{\iota_i} \in G_{\pi_{\iota_i}}$, then $M_E s \in \mathscr{B}_{E_\mathfrak{p}}^*$. Let $M_E s = M_{E_\mathfrak{p}}$. We verify that if $M_\mathfrak{p} = (M_{E_\mathfrak{p}}, \mathfrak{p})$, then $\langle x_\iota\rangle_{\iota\in\Omega} = x(M_\mathfrak{p}) \in M$. Indeed, we have $\langle x_{\iota_i}(\mathfrak{p})\rangle_{i=1,2,\ldots} \in M_{E_\mathfrak{p}}$ and hence

$$\langle x_{\iota_i}(\mathfrak{p})\, s(\iota_i(\mathfrak{p}), \iota_i)\rangle_{i=1,2,\ldots} = \langle x_{\iota_i}(\mathfrak{p})\rangle_{i=1,2,\ldots} s^{-1} \in M_E.$$

But since for each $i = 1, 2, \ldots$ the elements $\iota_i(\mathfrak{p})$ and $\iota_i$ belong to the same set $W_i$, by (10.10) we have $x_{\iota_i}(\mathfrak{p})s(\iota_i(\mathfrak{p}), \iota_i) = x_{\iota_i}$ for all $i = 1, 2, \ldots$ and thus $\langle x_{\iota_i}\rangle_{i=1,2,\ldots} \in M_E$, which by (10.2) gives $\langle x_\iota\rangle_{\iota\in\Omega} \in M \cap A$.

In order to verify (10.11), (ii) we note that for each $x(M_\mathfrak{p}) = \langle x_\iota(M_\mathfrak{p})\rangle_{\iota\in\Omega}$ the element $x_\iota(M)$ belongs to $A_\iota(M)$, $i \in \Omega$, and the set $A_\iota(M_\mathfrak{p})$ is disjoint with all the sets $A_\iota(M'_{\mathfrak{p}'})$ for $M'_{\mathfrak{p}'}$ different from $M_\mathfrak{p}$.

For the proof of (10.10), (iii) we note simply that for each $\iota$, $\iota \in \Omega$, and $M_\mathfrak{p} \in \mathscr{B}^*$ the element $x(M_\mathfrak{p})\pi_\iota$ is the only one among the elements $x(M'_{\mathfrak{p}'})\pi_\iota, M'_{\mathfrak{p}'} \in \mathscr{B}^*$, which belong to the set $A(M_\mathfrak{p})$. Thus (10.11), (iii), follows immediately from the equality $\overline{\overline{A_\iota(M_\mathfrak{p})}} = \mathfrak{m} \geqslant \aleph_0$.

For every two elements $\iota', \iota''$ of $\Omega$ let $\varphi'_{\iota',\iota''}$ be the mapping

$$\varphi'_{\iota',\iota''}\colon\ x(M_\mathfrak{p})\pi_{\iota'} \to x(M_\mathfrak{p}) \to x(M_\mathfrak{p})\pi_{\iota''}.$$

By (10.11), (iii) the mapping $\varphi'_{\iota',\iota''}$ is well-defined and one-to-one. For every three elements $\iota', \iota'', \iota'''$ of $\Omega$ we have

$$\varphi'_{\iota',\iota'''} = \varphi'_{\iota',\iota''}\varphi'_{\iota'',\iota'''}. \tag{10.12}$$

By (10.11), (iii) the function $\varphi'_{\iota',\iota''}$ can be extended to a one-to-one mapping of $A_{\iota'}$ onto $A_{\iota''}$ such that condition (10.12) is satisfied for the extended mappings. By condition (ii) of the definition of the class $M$ this mapping can be extended uniquely to an isomorphism $\bar{\varphi}_{\iota',\iota''}$ of the groups $\tilde{A}_{\iota'}$ and $\tilde{A}_{\iota''}$. Since the extension is unique, condition (10.11) is still satisfied for every three isomorphisms $\bar{\varphi}_{\iota',\iota''}, \bar{\varphi}_{\iota'',\iota'''}, \bar{\varphi}_{\iota',\iota'''}$. By condition (ii) of the definition of the class $M$ for each $\iota$, $\iota\in\Omega$, we have

$$G_{\tau_\iota} = \tilde{A}_\iota + S_{\tau_\iota}$$

and for every two indices $\iota'$, $\iota''$ of the set $\Omega$ we have $S_{\tau_{\iota'}} \simeq S_{\tau_{\iota''}}$. Thus the isomorphism $\bar{\varphi}_{\iota',\iota''}$ can be extended again to an isomorphism of the groups $G_{\tau_{\iota'}}$ and $G_{\tau_{\iota''}}$ in such a manner that condition (10.12) is satisfied for every three isomorphisms $\varphi_{\iota',\iota''}, \varphi_{\iota'',\iota'''}, \varphi_{\iota',\iota'''}$.

Consider the subgroup $G'$ of the group $H$ defined as follows:

$$G' = \{\langle x_\iota\rangle_{\iota\in\Omega}\colon x_{\iota'}\varphi_{\iota',\iota''} = x_{\iota''},\ x_\iota\in G_{\tau_\iota},\ \iota,\ \iota',\ \iota''\in\Omega\}.$$

The group $G'$ has the following properties:

1. *For every $\iota$, $\iota\in\Omega$, the projection $\pi_\iota$ of $G'$ onto $G_{\tau_\iota}$ is an isomorphism.*
2. $A\subset G'$ *and hence* $\nu^*_\Omega(G') = 1$.

Let $\bar{\mathscr{B}}$ be the $\sigma$-field of subsets $M'$ of the group $G'$ of the form $M' = G'\cap M$, where $M\in\mathscr{B}_\Omega$. Let $\mu'$ be the measure on the $\sigma$-field $\bar{\mathscr{B}}$ induced by the measure $\nu_\Omega$ and defined by the equality

$$\mu'(M') = \nu^*_\Omega(M\cap G') = \nu_\Omega(M).$$

For each $\iota\in\Omega$ we denote by $\eta_\iota$ the isomorphism of $G_{\tau_\iota}$ onto $G'$ such that $\eta_\iota\pi_\iota$ is the identity map of $G_{\tau_\iota}$. We prove that the measure $\mu'$ is common extension of the measures $\mu_\iota\eta_\iota$, $\iota\in\Omega$. Let $M\in\mathscr{B}_\iota$. Then

$$M\eta_\iota = M\pi_\iota^{-1}\cap G'\in\bar{\mathscr{B}}.$$

Hence

$$\mu'(M\eta_\iota) = \mu'(M\pi_\iota^{-1}\cap G') = \nu_\Omega(M\pi_\iota^{-1}) = \mu_\iota(M) = \mu_\iota\eta_\iota(M\eta_\iota).$$

From the last statement Theorem 10.1 follows at once. For, let $\lambda$ be an isomorphism mapping $G'$ onto $G$. Then the measure $\mu'\lambda = \mu$ is a common extension of the measures $\mu_\iota\eta_\iota\lambda$, $\iota\in\Omega$. Obviously $\eta_\iota\lambda = \zeta_\iota$ is an automorphism of $G$. Thus the system $(S\Omega)$ defined by the equality

$$\iota(S\Omega) = \iota(T\Omega)\zeta_\iota, \quad \iota\in\Omega,$$

is semi-regular and, clearly, it is isomorphic to the system $(T\Omega)$. To see that $(S\Omega)$ is regular we notice that by 3.2 and elementary properties of induced measure, the measures $\mu_\iota\eta_\iota$, $\iota\in\Omega$, are all different and independent in their common extension $\mu'$. Hence the measures $\mu_\iota\zeta_\iota = \mu_\iota\eta_\iota\lambda$, $\iota\in\Omega$, are independent in their common extension $\mu = \mu'\lambda$.

## 11. Proof of the Main Theorem (reduced groups)

If $G$ is a reduced group admitting compact topologies and $T$ is the set of compact topologies of it, then by (5.1), (ii)

$$G = \sum_p^* D_p \quad \text{and, if } \tau \in T, \text{ then} \quad \tau = \mathscr{P}_p(\tau \wedge D_p).$$

By Theorems 6.15, 6.26 and Lemmas 9.2 and 9.3 for each prime $p$ we have

$$D_p = D_p(1) + \ldots + D_p(k_p),$$

where for each $i = 1, 2, \ldots, k_p$ the group $D_p(i)$ either belongs to the class $M$ (compare section 9) or has exactly one compact topology. Moreover, for each $\tau \in T$ and each prime $p$ there is an automorphism $\varphi_\tau^p$ of the group $D_p$ such that the groups $D_p(i)$, $i = 1, 2, \ldots, k_p$, are closed in the topology $\tau\varphi_\tau^p$. Hence if

$$\varphi_\tau = \sum_p^* \varphi_\tau^p,$$

then for each prime $p$ and $i = 1, 2, \ldots, k_p$ the group $D_p(i)$ is closed in the topology $\tau\varphi_\tau$ in the group $G$ and

$$\tau\varphi_\tau = \mathscr{P}_p \mathscr{P}_{i=1}^{k_p} \big(\tau\varphi_\tau \wedge D_p(i)\big).$$

Thus we have proved the following

THEOREM 11.1. *For each reduced group $G$ admitting compact topologies we have*

$$G = \sum_{a \in \Xi}^* G_a, \tag{11.1}$$

*where $\Xi$ is a countable set and either $G_a \in M$ or it has exactly one compact topology. Moreover, if $T$ is the set of compact topologies of $G$, then there exists a set $S$ isomorphic with $T$ and such that for each $\tau \in S$ and $a \in \Xi$ the group $G$ is closed in the topology $\tau$ and*

$$\tau = \mathscr{P}_{a \in \Xi}(\tau \wedge G_a).$$

The Main Theorem for reduced groups follows easily from Theorems 11.1 and 10.1. We have

THEOREM 11.2. *If $G$ is a reduced group and $T$ the set of its compact topologies, then there exists a semi-regular set $W$ of compact topologies of $G$ isomorphic with $T$.*

Proof. Suppose $T$ is non-void. Then $G$ admits the decomposition (11.1). Let $S$ be the set of compact topologies constructed in Theorem 11.1. By Theorem 10.1 for each $\tau \in S$ there exists an automorphism $\varphi_\tau^a$ of the group $G_a$ such that the set

$$\{\mu\tau_a\varphi_\tau^a\colon \tau_a = \tau \wedge G_a, \tau \in S\}$$

has a common invariant extension $\mu_a$. (If $G_a$ has for some $a \in \Xi$ exactly one compact topology, then we put $\varphi_\tau^a = e|G_a$.) Hence, since $\tau = \mathscr{P}_{a \in \Xi} \tau_a$ for each $\tau \in S$, the measure

$$\mu = \mathscr{P}_{a \in \Xi} \mu_a$$

is an invariant common extension of the set $\{\mu_\tau\colon \tau \in W\}$, where $W = \{\tau\varphi_\tau\colon \tau \in S;\ \varphi_\tau = \sum_{a \in \Xi}^{*} \varphi_\tau^a\}$.

## 12. Proof of the Main Theorem (conclusion)

Now at last we are in a position to prove the Main Theorem in its whole generality. We prove

THEOREM 12.1. *Let $G$ be a group and let $T$ be the set of its compact topologies. Then there exists a semi-regular set $S$ of compact topologies of $G$ isomorphic with $T$.*

Proof. Suppose $T$ is non-void. Then by Lemma 7.1

$$G = D + A + B,$$

where $D$ is the maximal divisible subgroup of $G$ and $\bar{A} \leqslant \bar{\bar{D}}$. Moreover, for each compact topology $\tau$ in $G$ there exists an automorphism $\eta_\tau$ of $G$ such that the subgroup $B$ is closed in the topology $\tau\eta_\tau$. Let

$$T' = \{\tau\eta_\tau\colon \tau \in T\}. \tag{12.2}$$

The proof of the theorem splits into two cases: 1. $\bar{\bar{D}} = \bar{\bar{G}}$, 2. $\bar{\bar{D}} < \bar{\bar{G}} = \bar{\bar{B}}$.

Case 1. Since $D$ is closed in each compact topology in $G$ the set $T_0 = \{\tau \wedge D\colon \tau \in T'\}$ is a set of compact topologies in $D$. We define a system $(T_0T')$ putting for each $\tau \in T'$

$$\tau(T_0T') = \tau \wedge D.$$

Since $\bar{\bar{T'}} \leqslant 2^{\bar{\bar{G}}} = 2^{\bar{\bar{D}}}$ and, by Lemma 9.1, $D \in M$, by Theorem 10.1 for each $\tau \in T'$ there exists an automorphism $\hat{\varphi}_\tau$ of $D$ such that the system $(\hat{S}T')$ defined by the equality

$$\tau(\hat{S}T') = \tau(T_0T')\hat{\varphi}_\tau$$

is regular. Let $\pi$ denote the homomorphism $\pi\colon G \to G/D = C$. For each $\tau \epsilon T'$ the topology $\tau\pi = \bar{\tau}$ is a compact topology in $C$. Since $C$ is reduced, by Theorem 11.2 for each $\tau \epsilon T'$ there exists an automorphism $\bar{\varphi}_\tau$ of $C$ such that the set $\bar{S} = \{\bar{\tau}\bar{\varphi}_\tau\colon \tau \epsilon T'\}$ is semi-regular. For each $\tau \epsilon T'$ we put

$$\varphi_\tau = \bar{\varphi}_\tau + \hat{\varphi}_\tau.$$

Then $\varphi_\tau$ is an automorphism of $G$ and the set

$$S = \{\tau\varphi_\tau\colon \tau \epsilon T'\}$$

is isomorphic with $T'$ and hence with $T$. The set $S$ has the following properties:

(i) $\hat{S} = \{\tau \wedge D\colon \tau \epsilon S\}$. *Hence if* $\tau', \tau'' \epsilon S$ *and* $\tau' \neq \tau''$, *then* $\tau' \wedge D \neq \tau'' \wedge D$ *and the set* $\{\mu_\tau\colon \hat{\tau} \epsilon \hat{S}\}$ *has a common invariant extension* $\hat{\mu}$ *in which the measures* $\mu_\tau$, $\hat{\tau} \epsilon S$, *are independent.*

(ii) $\bar{S} = \{\tau\pi\colon \tau \epsilon S\}$. *Hence the set* $\{\mu_{\bar{\tau}}\colon \tau \epsilon \bar{S}\}$ *has a common invariant extension* $\bar{\mu}$.

We are going to prove that $S$ is semi-regular.

Consider the $\sigma$-field $\mathscr{B} = [\bigcup_{\tau \epsilon S} \mathscr{B}_\tau]_\sigma$. If $M \epsilon \mathscr{B}$, then for each $x \epsilon G$ we have $M - x \epsilon \mathscr{B}$ and hence $(M-x) \cap D \epsilon \hat{\mathscr{B}} = [\bigcup_{\hat{\tau} \epsilon \hat{S}} \mathscr{B}_\tau]_\sigma$. Thus the measure $\hat{\mu}$ is defined on the set $(M-x)\cap D$. Consider the function

$$g_M(x) = \hat{\mu}[(M-x)\cap D].$$

Since $\hat{\mu}$ is invariant, the function $g_M(x)$ is constant on the cosets $x+D$, $x \epsilon G$, and thus it defines the function

$$\bar{g}_M(\bar{x}) = g_M(x\pi^{-1}), \qquad \bar{x} \epsilon C,$$

on the group $C$. We are going to prove that $\bar{g}_M(\bar{x})$ is a measurable function with respect to the $\sigma$-field $\bar{\mathscr{B}} = [\bigcup_{\bar{\tau} \epsilon \bar{S}} \mathscr{B}_{\bar{\tau}}]_\sigma$. If $M \epsilon \mathscr{B}_\tau$ for some $\tau \epsilon S$, then, as is well known (see e. g. [3], p. 281), the function $\bar{g}_M(\bar{x})$ is measurable with respect to the $\sigma$-field $\mathscr{B}_{\bar{\tau}}$, where $\bar{\tau} = \tau\pi$. Suppose $M = \bigcap_{i=1}^{k} M_i$, where $M_i \epsilon \mathscr{B}_{\tau_i}$ and $\tau_i \neq \tau_j$ for $i \neq j$, $i, j = 1, 2, \ldots, k$. Then

$$(M-x)\cap D = \bigcap_{i=1}[(M_i - x)\cap D]$$

and further, since $(M_i - x)\cap D \epsilon \mathscr{B}_{\hat{\tau}_i}$, where $\hat{\tau}_i = \tau_i \wedge D$, $\hat{\tau}_i \neq \hat{\tau}_j$ for $i \neq j$ and the measures $\mu\hat{\tau}_i$, $i = 1, 2, \ldots, k$, are independent in their common extension $\hat{\mu}$, we have

$$\hat{\mu}\Big[\bigcap_{i=1}^{k}(M_i - x)\cap D\Big] = \prod_{i=1}^{k} \mu_{\hat{\tau}_i}[(M_i - x)\cap D\Big].$$

Hence
$$\bar{g}_M(\bar{x}) = \prod_{i=1}^{k} \bar{g}_{M_i}(\bar{x}).$$

Since for each $i = 1, 2, \ldots, k$ the function $\bar{g}_{M_i}(\bar{x})$ is measurable with respect to $\mathscr{B}_{\bar{\tau}_i}$, $\bar{\tau}_i = \tau_i \pi$, their product $\bar{g}_M(\bar{x})$ is measurable with respect to $\bar{\mathscr{B}}$. Let $\boldsymbol{E}$ be the class of sets $M$ for which the functions $\bar{g}_M(\bar{x})$ are defined and measurable with respect to $\bar{\mathscr{B}}$. If $\boldsymbol{A} = \bigcup_{\tau \in S} \mathscr{B}_\tau$, then $\boldsymbol{A}$ is a complementative class and $\boldsymbol{A} \subset \boldsymbol{E}$. Moreover, as we have just proved, the class of the finite intersections of sets of $\boldsymbol{A}$ is contained in $\boldsymbol{E}$. From the elementary properties of measure it follows that classes $\boldsymbol{A}$ and $\boldsymbol{E}$ satisfy also conditions (iii) and (iv) of Lemma 1.5. Thus by Lemma 1.5 we get $\mathscr{B} = [\boldsymbol{A}]_\delta \subset \boldsymbol{E}$, which proves the measurability of $g_M(x)$ for all the sets $M$ of $\mathscr{B}$.

For each set $M \in \mathscr{B}$ we put
$$\mu(M) = \int_G \bar{g}_M(\bar{x})\bar{\mu}(d\bar{x}).$$

Obviously $\mu$ is an invariant measure in $G$. To see that $\mu$ is an extension of the measures $\mu_\tau$, $\tau \in S$; it is sufficient to recall the well known fact (see e. g. [3], p. 282) that if $M \in \mathscr{B}_\tau$ for some $\tau \in S$, then
$$\mu_\tau(M) = \int_G \bar{g}_M(\bar{x})\mu_{\bar{\tau}}(d\bar{x}) \quad \text{with} \quad \bar{\tau} = \tau\pi.$$

Case 2. Suppose now that $\bar{\bar{D}} < \bar{\bar{G}} = \bar{\bar{B}}$. If $D = 0$, then the group $G$ is reduced and Theorem 12.1 follows immediately from Theorem 11.2. Let then $D \neq 0$. By (5.1), (i) we have $\bar{\bar{D}} \geqslant \mathfrak{c}$. Since the group $B$ is reduced (and admits compact topologies), then by Theorem 11.1
$$B = \sum_{a \in \Xi}^{*} B_a, \quad \text{where} \quad \bar{\bar{\Xi}} = \aleph_0$$

and for each $a \in \Xi$ either $B_\tau \in \boldsymbol{M}$ or it has exactly one compact topology. Following Theorem 11.1 for each $\tau \in T'$ we find an automorphism $\varphi'_\tau$ of $B$ such that if $\varphi_\tau = \varphi'_\tau + e|D + e|A$, then for any $a \in \Xi$ the group $B_a$ is closed in the topology $\tau\varphi_\tau$ and
$$\tau\varphi_\tau = \mathscr{P}_{a \in \Xi}(\tau\varphi_\tau \wedge B_a).$$

Let

(12.3) $$T'' = \{\tau\varphi_\tau\colon \tau \in T'\}.$$

Without lost of generality we may assume that the set $\Xi$ is the set of ordinals $< \omega_\zeta$ with $\bar{\bar{\omega}} = \bar{\bar{\Xi}} = \aleph_0$, such that
$$a < \beta < \omega_\zeta \text{ implies } \bar{\bar{B}}_a \leqslant \bar{\bar{B}}_\beta.$$

We introduce some notation. For each $\alpha < \omega_\zeta$ we write

$$B^\alpha = \sum_{\alpha<\beta<\omega_\zeta}^{*} B_\beta, \quad \bar{B}^\alpha = \sum_{\alpha\leqslant\beta<\omega_\zeta}^{*} B_\beta,$$

$$\pi^\alpha\colon G \to G/B^\alpha = D+A+\sum_{\beta\leqslant\alpha}^{*} B_\beta = G_\alpha,$$

$$\bar{\pi}^\alpha\colon G \to G/\bar{B}^\alpha = D+A+\sum_{\beta<\alpha}^{*} B_\beta = \bar{G}_\alpha.$$

Obviously the groups $B^\alpha$ and $\bar{B}^\alpha$ are closed in each of the topologies of $T''$. Let $\alpha_0$ be the first ordinal $< \omega_\zeta$ for which $\bar{\bar{B}}_{\alpha_0} > \bar{\bar{D}}$. The existence of $\alpha_0$ follows at once from the fact that the converse inequalities, i.e. $\bar{\bar{B}}_\alpha \leqslant \bar{\bar{D}}$, $\alpha < \omega_\zeta$, together with the equalities $\bar{\bar{D}} = 2^{\mathfrak{m}}$ and $\bar{\omega}_\zeta = \aleph_0$, imply that

$$\bar{\bar{B}} = \overline{\overline{\sum_{\alpha<\omega_\zeta}^{*} B_\alpha}} \leqslant \bar{\bar{D}}^{\aleph_0} = \bar{\bar{D}}.$$

A similar argument shows that

(12.4) $$\overline{\overline{\sum_{\alpha<\alpha_0}^{*} B_\alpha}} \leqslant \bar{\bar{D}} \quad \text{and hence} \quad \bar{\bar{\bar{G}}}_{\alpha_0} = \bar{\bar{D}}.$$

We have

(12.5) *If* $\alpha \geqslant \alpha_0$, *then* $\bar{\bar{\bar{G}}}_\alpha = \bar{\bar{B}}_\alpha$.

To prove this we note that since for all $\alpha < \omega_\zeta$ $\bar{\bar{\bar{G}}}_\alpha \geqslant \bar{\bar{B}}_\alpha$, it is sufficient to prove the inequality $\bar{\bar{\bar{G}}}_\alpha \leqslant \bar{\bar{B}}_\alpha$ for $\alpha \geqslant \alpha_0$. But since $\bar{\bar{D}} < \bar{\bar{B}}_{\alpha_0}$ for $\alpha \geqslant \alpha_0$, the result follows from the inequality

(12.6) $$\overline{\overline{\sum_{\beta<\alpha}^{*} B_\beta}} \leqslant \bar{\bar{B}}_\alpha \quad \text{for} \quad \alpha \geqslant \alpha_0.$$

To see (12.6) we note that for each $\beta < \omega_\zeta$ we have $\bar{\bar{B}}_\beta = 2^{\mathfrak{m}_\beta}$ and, since for $\gamma > \beta$ we have $\bar{\bar{B}}_\beta \leqslant \bar{\bar{B}}_\gamma$, the cardinals $\mathfrak{m}_\beta$ can be chosen in the way that for $\gamma > \beta$ we have also $\mathfrak{m}_\gamma \geqslant \mathfrak{m}_\beta$. Hence

$$\overline{\overline{\sum_{\beta<\alpha}^{*} B_\beta}} = 2^{\mathfrak{n}} \quad \text{where} \quad \mathfrak{n} = \sum_{\beta<\alpha} \mathfrak{m}_\beta.$$

But since $\alpha$ is countable and $\mathfrak{m}_\beta \leqslant \mathfrak{m}_\alpha$ for every $\beta < \alpha$, we have $\mathfrak{n} \leqslant \mathfrak{m}_\alpha$ which proves (12.6) and hence (12.5).

According to the assumption $\bar{\bar{D}} \geqslant \mathfrak{c}$. Thus among the groups $B_\alpha$, $\alpha \geqslant \alpha_0$, there is no group having exactly one compact topology, because, as one can see at the first glance, by Theorem 8.10 each group of that kind has cardinal $\leqslant \mathfrak{c}$. Thus we get

(12.7) $B_\alpha \in M$ *for each* $\alpha \geqslant \alpha_0$.

For every $a \geqslant a_0$ consider the set

$$T''(a) = \{\tau^a\colon \tau^a = \tau\pi^a,\ \tau \in T''\}.$$

Since $T''(a)$ is a set of compact topologies of $G_a$, by Lemma 1.2 we get

$$T''(a) \leqslant 2^{\bar{\bar{G}}_a} = 2^{\bar{\bar{B}}_a}.$$

Accordingly,

$$T''(0) = \{\tau_0\colon \tau_0 = \tau\bar{\pi}^{a_0},\ \tau \in T''\}.$$

Since the group $\bar{G}_{a_0}$ satisfies the conditions of case 1, for each $\tau^0 \in T''(0)$ there exists an automorphism $\varphi_{\tau 0}$ of $\bar{G}_{a_0}$ such that the set

$$S_0 = \{\tau^0\varphi_{\tau 0}\colon \tau^0 \in T''(0)\}$$

is semi-regular. For each $a \geqslant a_0$ consider the system $\big(T_a''T''(a)\big)$ defined in $B_a$ by the equality

$$\tau^a\big(T_a''T''(a)\big) = \tau^a \wedge B_a, \quad \tau^a \in T''(a).$$

By (12.7) and Theorem 10.1 for each $a \geqslant a_0$ and $\tau^a \in T''(a)$ there exists an automorphism $\varphi_{\tau a}$ of $B_a$ such that the system $\big(S_aT''(a)\big)$ defined in $B_a$ by the equality

$$\tau^a\big(S_aT''(a)\big) = \tau^a\big(T_a''T''(a)\big)\varphi_{\tau a}, \quad \tau^a \in T''(a),$$

is regular. Put

$$\varphi_\tau = \varphi_{\tau\bar{\pi}^{a_0}} + \sum_{a_0 \leqslant a < \omega_\zeta}^{*} \varphi_{\tau\pi^a}. \tag{12.8}$$

Hence the set

$$S = \{\varphi_\tau\colon \tau \in T''\} \tag{12.9}$$

has the following properties:

(i) *$S$ is isomorphic with $T$;*

(ii) *$S_0 = \{\tau\bar{\pi}^{a_0}\colon \tau \in S\}$ is semi-regular;*

(iii) *$S_a = \{\tau \wedge B_a\colon \tau \in S\}$ is regular;*

(iv) *if $\tau_1\pi^a \neq \tau_2\pi^a$ for some $a \geqslant a_0$, then $\tau_1 \wedge B_a \neq \tau_2 \wedge B_a$;*

(v) *for every two different compact topologies $\tau_1, \tau_2$ of $S$ there exists $a < \omega_\zeta$ such that $\tau_1\pi^a \neq \tau_2\pi^a$;*

(vi) *if $\tau_1, \tau_2 \in S$ and for some $a \geqslant a_0$ we have $\tau_1\bar{\pi}^a \neq \tau_2\bar{\pi}^a$, then there exists $\beta < a$ such that $\tau_1\pi^\beta \neq \tau_2\pi^\beta$.*

Property (i) follows at once from (12.2), (12.3) and (12.9).

To verify (ii) and (iii) it is sufficient to note that

$$\tau\bar{\pi}^{a_0}\varphi_{\tau\bar{\pi}^{a_0}} = \tau\varphi_\tau\bar{\pi}^{a_0} \quad \text{and} \quad (\tau \wedge B_a)\varphi_{\tau\wedge B_a} = \tau\varphi_\tau \wedge B_a.$$

In order to prove (iv) we show first that the inequality $\tau_1\pi^a \neq \tau_2\pi^a$ implies the inequality $t_1\pi^a \neq t_2\pi^a$, where $\tau_1 = t_1\varphi_{\tau_1}$, $\tau_2 = t_2\varphi_{\tau_2}$, $t_1, t_2 \in T''$. Suppose $t_1\pi^a = t_2\pi^a$. Then for all $\beta$, $a_0 \leqslant \beta < a$, we have $t_1\pi^\beta = t_2\pi^\beta$ and $t_1\bar{\pi}^{a_0} = t_2\bar{\pi}^{a_0}$. Hence, putting

$$\varphi(t_1) = \varphi_{t_1\bar{\pi}^{a_0}} + \sum_{a_0 \leqslant \beta \leqslant a}^{*} \varphi_{t_1\pi^\beta} \quad \text{and} \quad \varphi(t_2) = \varphi_{t_2\bar{\pi}^{a_0}} + \sum_{a_0 \leqslant \beta \leqslant a}^{*} \varphi_{t_2\pi^\beta}$$

we get $\varphi(t_1) = \varphi(t_2)$. Thus $t_1\pi^a\varphi(t_1) = t_2\pi^a\varphi(t_2)$ and hence by (12.8) $\tau_{1\pi^a} = t_1\varphi_{t_1}\pi^a = t_1\varphi(t_1)\pi^a = t_2\varphi(t_2)\pi^a = t_2\varphi_{t_2}\pi^a = \tau_2\pi^a$. The inequality $t_1\pi^a \neq t_2\pi^a$, by the regularity of the system $(S_aT''(a))$, implies that the topologies $(t_1 \wedge B_a)\varphi_{\tau_1\pi^a} = \tau_1 \wedge B_a$ and $(t_2 \wedge B_a)\varphi_{\tau_2\pi^a} = \tau_2 \wedge B_a$ are different.

We verify (v) by the following argument. Suppose that for some $\tau_1, \tau_2 \in S$ we have $\tau_1\pi^a = \tau_2\pi^a$ for all $a < \omega_\xi$. Let $V \in \tau_1$. Since $\bigcap_{a<\omega_\xi} B^a = 0$ and the groups $B^a$ are closed in $\tau_1$, there exists an ordinal $a < \omega_\xi$ and a set $W$ in $\tau_1$ such that $W\pi^a(\pi^a)^{-1} \subset V$. But, according to the assumption, $W\pi^a \in \tau_2\pi^a$ and hence, since $B^a$ is closed in $\tau_2$, we have $W\pi^a(\pi^a)^{-1} \in \tau_2$, which, by the symmetry, gives $\tau_1 = \tau_2$.

We prove (vi) by the same reasonning replacing $\omega_\xi$ by $a$.

The properties (v) and (vi) imply the following property of the set $S$:

(12.10). *If $\tau_1, \tau_2$ are two different topologies of the set $S$ such that $\tau_1\bar{\pi}^{a_0} = \tau_2\bar{\pi}^{a_0}$, then there exists the unique ordinal $a$ such that $\tau_1 \wedge B_\beta \neq \tau_2 \wedge B_\beta$ for all $\beta \geqslant a$ and $\tau_1\bar{\pi}^a = \tau_2\bar{\pi}^a$. We denote it by $a(\tau_1, \tau_2)$. If $\tau_1\bar{\pi}^a \neq \tau_2\bar{\pi}^a$ for all $a$, then we put $a(\tau_1, \tau_2) = a_0$.*

Let $F$ be an arbitrary but fixed finite subset of $S$. Let

$$a_0 < a_1 < \ldots < a_n$$

be the sequence of ordinals defined as follows: if $a_0, a_1, \ldots, a_k$ are already defined, then $a_{k+1}$ is the first ordinal such that $a_k < a_{k+1}$ and $a_{k+1} = a(\tau', \tau'')$ for some $\tau', \tau'' \in F$.

For each topology $\tau$ in $F$ and integer $k$, $-1 \leqslant k \leqslant n$, we define a set $\delta(\tau, k)$ putting

$$\delta(\tau, k) = \begin{cases} \{t\colon\ t \in F \text{ and } \tau\bar{\pi}^{a_k} = t\bar{\pi}^{a_k}\}, & \text{if} \quad k \geqslant 0, \\ F, & \text{if} \quad k = -1. \end{cases}$$

It follows from (12.10) that

(12.11) *if $s, t \in \delta(\tau, k-1)$, then $s\bar{\pi}^{a_k} = t\bar{\pi}^{a_k}$, $k = 0, 1, \ldots, n$.*

We have also

(12.12) $\delta(\tau, n) = \{\tau\}$ *for each* $\tau \in F$.

Let

$$\Delta(\tau, k) = \{\delta:\ \delta = \delta(t, k+1),\ t\in\delta(\tau, k)\}, \qquad k = -1, 0, 1, \ldots, n.$$

We have

$$(12.13) \qquad \delta(\tau, k) = \bigcup_{\delta\in\Delta(\tau,k)} \delta.$$

If $\delta(\tau', k)\cap\delta(\tau'', k)\neq\emptyset$ for some $\tau', \tau''\in F$, then $\delta(\tau', k) = \delta(\tau'', k)$. Thus for each fixed number $k$, $k = -1, \ldots, n$, the family

$$\Delta_k = \{\delta(\tau, k):\ \tau\in F\}$$

is a partition of the set $F$ and, by (12.13), $\Delta_{k-1}$ is a refinement of $\Delta_k$. Thus if $\delta'$, $\delta''$ are two different sets of $\Delta(\tau, k-1)$, then for $\tau'\in\delta'$, $\tau''\in\delta''$ we have $\tau'\pi^{a_k}\neq\tau''\pi^{a_k}$ and by the definition of $a_k$ and (12.10) $\tau'\wedge B_a\neq\tau''\wedge B_a$ for all $a\geqslant a_k$. In other words, if $\bar{a}$ is the mapping of $S$ onto $S_a$ defined by

$$\bar{a}:\ \tau\to\tau\wedge B_a,$$

then

$$(12.14) \qquad \delta'\bar{a}\cap\delta''\bar{a} = \emptyset \quad \text{for all } a\geqslant a_k.$$

According to property (ii) the set $S_0$ is semi-regular, i.e. the measures $\mu_{\tau_0}$, $\tau_0\in S_0$, have a common invariant extension $\mu_0$. By property (iii) the set $S_a$ is regular, i.e. the measures $\mu_{\tau^a}$, $\tau^a\in S_a$, have a common invariant extension $\mu_a$ in which they are independent.

We are going to prove that the set $S$ is semi-regular.

Write

$$\mu^a = \mathscr{P}_{a\leqslant\beta<\omega_\zeta}\mu_\beta, \qquad \mu^{a,\beta} = \mathscr{P}_{a\leqslant\gamma<\beta}\mu_\gamma.$$

For each set $M\in\mathscr{B} = [\bigcup_{\tau\in S}\mathscr{B}_\tau]_\sigma$ we have $M\cap\bar{B}^a\in\mathscr{B}^a = [\bigcup_{\tau\in S}\mathscr{B}_\tau\wedge\bar{B}^a]_\sigma$ for every $a\geqslant a_0$. Thus the function

$$g_M(x) = \mu^{a_0}[(M-x)\cap\bar{B}^{a_0}]$$

is defined for each $x\in G$. Since $\mu^{a_0}$ is invariant in $\bar{B}^{a_0}$, $g_M(x)$ defines the unique function $\bar{g}_M(\bar{x}) = g_M(\bar{x}\bar{\pi}^{a_0-1})$ on $\bar{G}^{a_0}$. We prove that the function $\bar{g}_M(\bar{x})$ is measurable with respect to the $\sigma$-field $\mathscr{B}_0 = [\bigcup_{\tau\in S}\mathscr{B}_{\tau\bar{\pi}^{a_0}}]_\sigma$. To do this we consider first the sets $M\in\mathscr{B}$ of the form

$$(12.15) \qquad M = \bigcap_{\tau\in F} M_\tau, \quad \text{where} \quad M_\tau\in\mathscr{B}_\tau.$$

We have

$$M\cap\bar{B}^{a_0} = \bigcup_{\tau\in F}(M_\tau - x)\cap\bar{B}^{a_0} = \bigcap_{\delta\in\Delta_0} M_\delta\cap\bar{B}^{a_0}, \quad \text{where} \quad M_\delta = \bigcap_{\tau\in\delta}(M_\tau - x).$$

Since for each $a$ the measures $\mu_{\tau^a}$, $\tau^a \in S_a$, are independent for different $\tau^a \in S_a$ and (12.14), with $k = 0$, implies (3.3), the conditions of Lemma 3.3 are satisfied. Hence

$$\mu^{a_0}(M \cap \bar{B}^{a_0}) = \prod_{\delta \in \Delta_0} \mu^{a_0}(M_\delta \cap \bar{B}^{a_0}).$$

Thus in order to prove the measurability of $\bar{g}_M(\bar{x})$ with respect to $\mathscr{B}_0$ it is sufficient to prove that each of the functions $\bar{f}^1_\delta(\bar{x}) = f^1_\delta(\bar{x}(\bar{\pi}^{a_0})^{-1})$, where

$$f^1_\delta(x) = \mu^{a_0}[\bigcap_{t \in \delta}(M_t - x) \cap \bar{B}^{a_0}], \qquad \delta = \delta(\tau, 1),$$

is measurable with respect to $\mathscr{B}_{\tau\bar{\pi}^{a_0}}$. To do this we prove

(12.16) *For each* $\tau \in F$ *the function* $f^k_\delta(x)$ *defined on* $\bar{G}_{a_k}$ *by* $\bar{f}^k_\delta(\bar{x}) = f^k_\delta(\bar{x}(\bar{\pi}^{a_0})^{-1})$, *where*

$$f^k_\delta(x) = \mu^{a_k}[\bigcap_{\tau \in \delta}(M_t - x) \cap \bar{B}^{a_k}], \qquad \delta = \delta(\tau, k),$$

*is measurable with respect to* $\mathscr{B}_{\tau\bar{\pi}^{a_k}}$.

We prove (12.16) by induction on $k$. For $k = n$, by (12.12), we have

$$\bigcap_{t \in \delta(\tau, n)} (M_t - x) \cap \bar{B}^{a_n} = (M_\tau - x) \cap \bar{B}^{a_n}.$$

Hence, since $\bar{B}^{a_n}$ is a subgroup of $G$ closed in $\tau$, the measurability with respect to $\mathscr{B}_{\tau\bar{\pi}^{a_n}}$ of the function $\bar{f}^n_{\{\tau\}}(\bar{x})$ follows from the well known theorem (see e.g. [3], p. 281). Suppose that (12.16) is true for $n, n-1, \ldots, k > 0$. We prove that it is true for $k-1$. We have

$$\mu^{a_{k-1}} = \mu^{a_{k-1}, a_k} \times \mu^{a_k}.$$

Since $\mu^{a_k}$ is an invariant measure in $\bar{B}^{a_k}$, for each set $X \in [\bigcup_{t \in S} \mathscr{B}_\tau \wedge \bar{B}^{a_{k-1}}]_a$ we have

$$\mu^{a_{k-1}}(X) = \int_{\bar{B}^{a_{k-1}}\bar{\pi}^{a_k}} \bar{h}_X(\bar{y})\,\mu^{a_{k-1}a_k}(d\bar{y}), \tag{12.17}$$

where $\bar{h}_X(\bar{y})$ is the function on $\bar{B}^{a_{k-1}}\bar{\pi}^{a_k}$ defined by $h_X(\bar{y}) = h_X(\bar{y}(\bar{\pi}^{a_k})^{-1})$ and $h_X(y) = \mu^{a_k}[(X - y) \cap \bar{B}^{a_k}]$, $y \in G$. Put

$$X = \bigcap_{t \in \delta(\tau, k-1)} M_t \cap \bar{B}^{a_{k-1}}. \tag{12.18}$$

Then, by (12.13),

$$(X - y) \cap \bar{B}^{a_k} = \bigcap_{t \in \delta(\tau, k-1)} (M_t - y) \cap \bar{B}^{a_k} = \bigcap_{\delta \in \Delta(\tau, k-1)} \bigcap_{t \in \delta} (M_t - y) \cap \bar{B}^{a_k}.$$

Since the measures $\pi_{\tau^a}$, $\tau^a \in S_a$, are independent for different $\tau^a \in S_a$, by (12.14), the conditions of Lemma 3.3 are satisfied (for $\mu^t_a = \mu_{\tau_a}$).

Thus

$$\mu^{a_k}[(X-y)\cap\bar{B}^{a_k}] = \bigcap_{\delta\in\Delta(\tau,k-1)} \mu^{a_k}[\bigcap_{t\in\delta}(M_t-y)\cap\bar{B}^{a_k}],$$

and hence

$$\bar{h}_X(\bar{y}) = \prod_{\delta\in\Delta(\tau,k-1)} f_\delta^k(\bar{y}), \quad \bar{y}\in\bar{B}^{a_{k-1}}\bar{\pi}^{a_k}. \tag{12.19}$$

But, according to the inductive hypothesis, for each $\delta\in\Delta(\tau,k-1)$ the function $f_\delta^k(\bar{y})$ is a function on $\bar{G}_{a_k}$ measurable with respect to $\mathscr{B}_{t\bar{\pi}^{a_k}}$, where $\delta=\delta(t,k)\in\Delta(\tau,k-1)$. But by (12.10) and the definition of $a_k$, $t\bar{\pi}^{a_k}=\tau\bar{\pi}^{a_k}$. Hence the product

$$\prod_{\delta\in\Delta(\tau,k-1)} f_\delta^k(\bar{y}) = k_X(\bar{y}), \quad \bar{y}\in\bar{G}_{a_k},$$

is a measurable function with respect to $\mathscr{B}_{\tau\bar{\pi}^{a_k}}$. Since $\mathscr{B}_{\tau\bar{\pi}^{a_k}}$ is the $\sigma$-field of Baire sets defined by a compact topology $\tau\bar{\pi}^{a_k}$ in $\bar{G}_{a_k}$, the function $k_X(\bar{x}+\bar{y})$, $\bar{x},\bar{y}\in G$, is a function of two variables measurable with respect to $\mathscr{B}_{\tau\bar{\pi}^{a_k}}$. Thus, since $k_X(\bar{y})=h_X(\bar{y})$ for $\bar{y}\in\bar{B}^{a_{k-1}}\bar{\pi}^{a_k}$, we have

$$\mu^{a_k}(X-\bar{x}) = \int_{\bar{B}^{a_{k-1}}\cdot\bar{\pi}^{a_k}} k_X(\bar{x}+\bar{y})\mu^{a_{k-1},a_k}(d\bar{y}), \quad \bar{x}\in\bar{G}_{a_k}.$$

The function $\mu^{a_k}(X-\bar{x})$ is constant on each coset $\bar{x}+\bar{B}^{a_{k-1}}\cdot\bar{\pi}^{a_k}$, $x\in\bar{G}_{a_k}$, and by [3] (p. 279) it defines a function $P(\bar{x})$ on $\bar{G}_{a_k}/\bar{B}^{a_{k-1}}\bar{\pi}^{a_k}=\bar{G}_{a_k}$ measurable with respect to $\mathscr{B}_{\tau\bar{\pi}^{a_{k-1}}}$. It follows immediately from (12.18) and the definition of the function $f_\delta^{k-1}(x)$, $\delta=\delta(\tau,k-1)$, that $P(\bar{x}) = f_\delta^{k-1}(\bar{x})$. This completes the proof of (12.16).

So far we have proved that if $M$ is the finite intersection of sets $M_\tau$, $M_\tau\in\mathscr{B}_\tau$, then the function $\bar{g}_M(\bar{x})$ is measurable with respect to $\mathscr{B}_0$. To prove that $\bar{g}_M(\bar{x})$ has this property with arbitrary $M\in\mathscr{B}$, we copy the reasonning used in the proof in case 1. We consider the class $\boldsymbol{E}$ of sets $M$ for which the function $\bar{g}_M(\bar{x})$ is measurable with respect to $\mathscr{B}_0$. Then if $\boldsymbol{A}=[\bigcup_{\tau\in S}\mathscr{B}_\tau]_\sigma$ we know that $\boldsymbol{A}\subset\boldsymbol{E}$ and, moreover, the class of the finite intersections of sets of $\boldsymbol{A}$ is contained in $\boldsymbol{E}$. Conditions (iii)-(iv) of Lemma 1.5 follow easily from the elementary properties of measure. Thus we get $\mathscr{B}=[\boldsymbol{A}]_\sigma\subset\boldsymbol{E}$, which completes the proof of measurability of the function $\bar{g}_M(\bar{x})$.

We define an invariant measure on $\mathscr{B}$ putting for each $M$

$$\mu(M) = \int_{\bar{G}_{a_0}} \bar{g}_M(\bar{x})\mu_0(d\bar{x}).$$

The proof that the measure $\mu$ is an invariant extension of the measures $\mu_\tau$, $\tau\in S$, is precisely the same as the corresponding one in case 1.

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