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Optimization and decision-making for a service contract on machine maintenance

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Abstract

We investigate a novel maintenance service contract model. The service provider of the machine must determine the optimal pricing structure and staffing levels, while the client selects an appropriate plan for the warranty duration. We consider linear, quadratic, polynomial, and exponential pricing functions for three types of warranties. By building a service model with a non-cooperative game, we obtain the Nash equilibrium of the bargaining solution. Numerical analysis reveals that the optimal warranty period decreases monotonically with service provider revenue and increases monotonically with the maximum prospective service time. Additionally, the market size does not affect the ideal warranty duration when the machine's lifetime is constant.

Keywords: *service contract, reliability, queueing system, optimization, game theory*

1. Introduction

The role of warranties in maintenance service contracts is pivotal, offering peace of mind for consumers and a predictable revenue stream for providers. This mutual benefit underscores the need for a deep understanding of these contracts, traditionally built upon qualitative techniques. However, a comprehensive quantitative approach, particularly in the context of warranty management, has been less explored.

Previous studies of maintenance service contracts have been driven by qualitative techniques. Murthy and Ashgarizadeh [9] made the first contribution to this theme. The authors developed a game theory model to describe the ideal consumer and agent strategy. They used a Stakelberg game at the best choices under the premise that there is just one customer and one repairman. Ashgarizadeh and Murthy [13] expanded their setting by including numerous clients and a repairman. The authors investigated the best customer and agent tactics. Additionally, Murthy and Ashgarizadeh [10] expanded the concept to

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include many clients and service channels (or repairmen). They determined the agent's best course of action on pricing, the number of clients to serve, and the number of servers. Diamaludin et al. [2] investigated preventative maintenance procedures when products are sold with warranties and examined the models mentioned in the literature. Jackson and Pascual [6] created a model in which the failure intensity is linear with time and calculated the ideal number of preventative maintenance operations, the number of customers, and the unit's lifespan. Wang [19] examined three planned maintenance service contracts with a single agent and client under availability and reliability limitations. The author developed a model to precisely address the impact of regular inspections and repairs made during inspections on the maintenance service contract for both parties.

Since they control the post-sale service, warranties are crucial components of service management. As far as we know, there is little literature on maintenance service contracts with warranties. The papers listed below are accessible. Murthy and Padmanabhan [15] initially considered a concept in which service contracts are considered extended warranties. Murthy and Blischke established a strategic approach to warranty management [13], where decisions about warranties are made in the context of the product life cycle and from a commercial perspective that bridges technical and business concerns. Yeh and Lo [20] looked into preventive maintenance warranty practices for products that could be repaired. By reducing overall maintenance costs, Pascual and Ortega [16] established a methodology to identify the ideal life cycle and gaps between overhauls. The reader is referred to the outstanding survey papers by Murthy and Blischke [12], Murthy and Djamaludin [14], and the book by Murthy and Blischke [11] for extensive overviews of the key results and methodologies. By applying a gas turbine manufacturer that offers engine maintenance services, Hong et al. [4] offered a mechanism design method. Luo and Wu [8] studied the optimization of the warranty policy for different products produced by a single producer whose failures are statistically dependent.

Recent studies in the field of warranty and maintenance servicing contracts have made significant contributions. For instance, Iskandar et al. [5] studied maintenance service contracts and warranty when the products are repairable. The authors developed a genetic algorithm to derive the optimal price option for the consumer. Zheng and Su [21] introduced a flexible two-dimensional basic warranty policy, highlighting the complexities and nuances of warranty regions. Additionally, Zheng et al. [22] explored two-stage flexible warranty decision-making by considering downtime loss and underscoring the balance between service efficiency and customer satisfaction. Zheng and Zhou [24] provided insights into how products deteriorate with age and under varying conditions by comparing three preventive maintenance warranty policies. Zheng et al. [23] reflected the growing importance of sustainability and long-term serviceability in industrial applications by focusing on optimal post-warranty maintenance contracts for wind turbines, with an emphasis on availability. Li et al. [7] focused on designing warranty policies by modeling product lifetime through distribution functions, addressing both self-announcing and degradation failures. This involves a comprehensive approach to reliability modeling and warranty policy design. Additionally, there is research on random replacement policies to sustain post-warranty reliability, looking at warranty terms based on a limited number of random working cycles and investigating random maintenance policies from the consumer's perspective (for instance, Shang et al. [18]). These studies collectively expand the understanding of warranty management, emphasizing the need for adaptable and strategic approaches in diverse industrial contexts.

The research gap addressed by this paper is particularly significant in the field of maintenance service contracts, specifically in the context of warranties. Before this study, the existing literature had several limitations or areas that were not fully explored, which this paper aims to address:

Limited analysis of warranty management in service contracts. Previous studies have primarily focused on the qualitative aspects of maintenance service contracts without a comprehensive quantitative model that integrates warranty management with pricing and staffing strategies.

Lack of comprehensive game-theoretic models. Although some studies have used game theory in the context of service contracts, there is a scarcity of research employing advanced game-theoretic models, like non-cooperative games, to analyze the complex negotiation dynamics between service providers and clients.

Inadequate consideration of pricing structures. Most of the earlier research in this field did not thoroughly investigate different pricing functions (linear, quadratic, polynomial, and exponential) and their impact on warranty terms and service provider strategies.

Neglect of the interplay between market size and machine lifetime. The existing literature has not adequately addressed how the market size and the constant lifetime of machines interact to influence warranty durations, leaving a gap in strategic planning for companies.

Insufficient focus on optimal staffing levels. Previous studies often overlooked the importance of determining optimal staffing levels concerning warranty service contracts, which is crucial for operational efficiency and customer satisfaction.

Absence of detailed bargaining dynamics analysis. While some studies have touched upon negotiation aspects within service contracts, there is a lack of in-depth analysis of the bargaining process, especially using advanced game theory methods.

By addressing these gaps, this paper significantly contributes to the field by providing a more comprehensive, quantitative, and practical approach to understanding and optimizing maintenance service contracts. It enhances the understanding of how different factors like pricing structures, market size, machine lifetime, and staffing levels interplay in the context of warranty management. This not only advances academic knowledge but also offers practical insights and tools for industry professionals to optimize their strategies in warranty management and service contracts.

We investigate a novel model and utilize a game theoretic approach to characterize the design, negotiation, and optimization of maintenance service contracts. The model is examined under the presumption that both customers and repairmen are numerous. Determining the ideal warranty fee, the warranty duration, and the number of repairmen is the agent's (service provider's) primary goal. On the other hand, the major goal for customers is how to bargain the warranty fee to get more profit. Using queueing systems and game theory, we develop a Nash bargaining solution for non-cooperative games. Several numerical examples are provided to demonstrate the suggested formulation's characteristics. The paper's main point of interest is that the optimal warranty period and the number of repairmen are unaffected by warranty pricing functions. Additionally, the number of customers does not affect the ideal warranty duration when the machine's lifetime is constant.

The promise of our research is to offer nuanced insights into the modern maintenance service contract landscape, enriched with the interplay of warranties, pricing strategies, and queue dynamics. Through an array of numerical illustrations, we aim to validate our model and underscore its unique contributions.

The rest of this paper is arranged as follows. After setting the stage with our model's parameters in Section 2, we plunge into an in-depth model analysis in Section 3. Section 4 unravels the optimal warranty price, duration, and repairers count, all through a game-theoretical lens. Subsequently, Section 5, illuminated with numerical studies, brings optimal solutions across varied pricing functions to the fore. Section 6 provides readers with a comprehensive understanding of our contributions to the field by summarizing the key points of our observations.

2. Model description

Consider a system with K customers. The original equipment manufacturer (OEM), hereafter referred to as the agent, sells a machine to each customer. Each machine has a maximum service lifespan of L_0. Beyond this period, it is advisable to retire the machine due to its age-related inefficiencies.

Let us summarize our assumptions and given data:

- Each machine's purchase price is C_p.
- A machine yields a revenue of R dollars per unit of time when operational, and none otherwise.
- All machines have similar reliability statistics since they are identical.
- Machines operate independently.
- The lifetime of each machine is exponentially distributed with parameter λ.
- Upon malfunction, machines are immediately sent to the agent for repairs.
- The repair cost for a single machine after it is damaged is C_r. Within the warranty, the agent pays the whole cost.
- There are N repair personnel, such that $1 \leq N \leq K$. Their service charge is C_n dollars.
- The repair time, denoted by χ, follows an exponential distribution with a repair rate of μ. The machines become as good as new after repair.
- Machines are serviced on a first-come, first-served basis.
- Post-repair, machines revert to a pristine operational state.
- Each machine is covered under a warranty for L time units where $0 \leq L \leq L_0$. If a machine malfunctions during this period, the agent handles all repairs with free of charge after paying a fixed price $P(L)$. Here we assume that $P(L)$ is strictly increasing, concave and twice differentiable in the interval $[0, L_0]$ which is reasonable in reality because of the economics of scale.
- If the agent does not repair a malfunctioning machine within time T, a delay penalty of $\delta(B - T)$ is incurred where B is the time taken to repair the malfunctioned machine.
- Post-warranty, a portion of repair costs, C_r, are borne by the customer ($C_r < C_s$).
- The agent services K clients, all of whom share the same risk propensity.
- We also assume that L_0 is large enough and $\lambda E[B] \ll 1$, meaning the expected time for machine repair is considerably shorter than the expected malfunction time.

In the ensuing sections, we will employ queueing and game theories to determine the optimal values for warranty duration (L), cost ($P(L)$), and number of repair personnel (N).

3. Model analysis

This section provides a comprehensive examination of the proposed model. For any customer j, let random variables X_j and Y_j denote the number of failures in intervals $[0, L]$ and $(L, L_0]$, respectively. Let B_{ji} denote the time taken to restore system functionality after i failures. The distribution of B_{ji} aligns with that of B, representing the cumulative downtime (waiting plus repair time) for machine j after its ith failure. The revenue accrued by the jth customer is represented by U_j. This yields the equation:

$$U_j = R\left(L - \sum_{i=1}^{X_j} B_{ji}\right) + \delta\left(\sum_{i=1}^{X_j} \max(0, B_{ji} - T\right) + R\left(L_0 - L - \sum_{i=1}^{Y_j} B_{ji}\right) - C_p - P(L) - C_s Y_j \quad (1)$$

Further, let us denote the agent's profit as V. This gives us:

$$V = \sum_{j=1}^{K}\left(C_p + P(L) - C_r X_j - \delta\left(\sum_{i=1}^{X_j} \max(0, B_{ji} - T\right) + (C_s - C_r)Y_j\right) - NC_n \quad (2)$$

Let $M(t)$ be the number of failure machines under repairing or waiting for repairing at time t, $M(t) = 0, 1, \ldots, K$. Because both the lifetime and the repairing time the machines are exponential distributed, it is easy to see that $\{M(t), t \geq 0\}$ is a finite-state Markov process which can also mirror an $M/M/N$ queueing system with a limited source pool of size K. In this queueing system, then the number of customers (machines) in the servers or the waiting line is k, the customer arrival rate (the failure rate) is $\lambda_k = (K-k)\lambda$. Recalling that N is the number of repairmen, hence the service rate $\mu_k = \min(k, N)\mu$, $k = 0, 1, \ldots, K$.

The probability is described as: $p_k = \lim_{t\to\infty} P\{M(t) = k\}$, for all $k = 0, 1, \ldots, K$. Drawing from [3], we infer the subsequent results in traditional queueing theory:

Proposition 1.

$$p_j = \begin{cases} \dfrac{K!}{j!(K-j)!}\rho^j p_0, & j = 0, 1, \ldots, N-1 \\ \dfrac{K!}{(K-j)!N!N^{j-N}}\rho^j p_0, & j = N, \ldots, K \end{cases}$$

where

$$\rho = \frac{\lambda}{\mu}, \quad p_0 = \left(\sum_{i=0}^{N-1} \frac{K!}{i!(K-i)!}\rho^i + \sum_{i=N}^{K} \frac{K!}{(K-i)!N!N^{i-N}}\rho^i\right)^{-1}$$

Proposition 2. The mean number of failed machines is

$$E[H] = \sum_{i=0}^{K} i p_i \quad (3)$$

The mean number of failed machines waiting for repair is

$$E[H_q] = \sum_{i=N}^{K} (i-N)p_i \tag{4}$$

Proposition 3. In the steady state, the distribution function for the waiting time (B_q) of an arriving failed machine can be obtained by

$$B_q(t) = 1 - \sum_{j=N}^{K-1} \sum_{i=N}^{j} \frac{(N\mu t)^{i-N}}{(i-N)!} \times \frac{K-j}{K-E[H]} p_j \mathrm{e}^{-N\mu t},\ t \geq 0. \tag{5}$$

Notice that the sojourn time (B) equals the waiting time (B_q) plus the repair time (χ). We can then derive the steady-state distribution function of the sojourn time in the representative of an incoming failing machine

$$\begin{aligned} B(t) = P\{B_q + \chi \leq t\} &= \int_0^t P\{B_q \leq t-x\} dP\{\chi \leq x\} \\ &= \int_0^t \left(1 - \sum_{j=N}^{K-1} \sum_{i=N}^{j} \frac{[N\mu(t-x)]^{i-N}}{(i-N)!} \times \frac{K-j}{K-E[H]} p_j \mathrm{e}^{-N\mu(t-x)} \right) \mu \mathrm{e}^{-\mu x} dx \\ &= 1 - \mathrm{e}^{-\mu t} - \mathrm{e}^{-\mu t} \sum_{j=N}^{K-1} \sum_{i=N}^{j} p_j \frac{K-j}{K-E[H]} \frac{N^{i-N}}{(N-1)^{i-N+1}} \\ &\quad + \mu \mathrm{e}^{-N\mu t} \sum_{j=N}^{K-1} \sum_{i=N}^{j} \sum_{k=1}^{i-N+1} p_j \frac{K-j}{K-E[H]} \frac{(N\mu)^{i-N} t^{i-N-k+1}}{[(N-1)\mu]^k (i-N-k+1)!} \end{aligned} \tag{6}$$

We can display the equation that was employed to arrive at the expression provided by (6):

$$\int_0^t u^n \mathrm{e}^{-mu} du = \frac{n!}{m^{n+1}} - \mathrm{e}^{-mt} \sum_{k=1}^{n+1} \frac{n!}{m^k (n-k+1)!} t^{n-k+1} \tag{7}$$

Proposition 4. Mean time needed to get a malfunctioning machine recovered and running is given by

$$E[B] = E[B_{ji}] = \int_0^\infty t dB(t) = \frac{E[H_q]}{\lambda(K-E[H])} + \frac{1}{\mu} \tag{8}$$

Proposition 5. The expected penalty cost can be obtained by

$$\begin{aligned} Q(N) =& E\left(\max\{0, B_{ji} - T\}\right) = \int_T^\infty (t-T) dB(t) \\ &= \frac{1}{\mu} \mathrm{e}^{-\mu T} \left(1 + \sum_{j=N}^{K-1} \sum_{i=N}^{j} p_j \frac{K-j}{K-E[H]} \frac{N^{i-N}}{(N-1)^{i-N+1}} \right) \end{aligned} \tag{9}$$

$$- N\mu^2 \mathrm{e}^{-N\mu T} \sum_{j=N}^{K-1} \sum_{i=N}^{j} \sum_{k=1}^{i-N+1} p_j \frac{K-j}{K-E[H]} \frac{(N\mu)^{i-N}}{(N-1)\mu)^k}$$

$$\times \left(\sum_{l=0}^{i-N-k+1} \frac{lT^{i-N-k-l+2}}{(N\mu)^{l+1}(i-N-k-l+2)!} + \frac{i-N-k+2}{(N\mu)^{i-N-k+3}} \right)$$

$$+ \mu \mathrm{e}^{-N\mu T} \sum_{j=N+1}^{K-1} \sum_{i=N+1}^{j} \sum_{k=1}^{i-N} p_j \frac{K-j}{K-E[H]} \frac{(N\mu)^{i-N}}{((N-1)\mu)^k}$$

$$\times \left(\sum_{l=0}^{i-N-k} \frac{lT^{i-N-k-l+1}}{(N\mu)^{l+1}(i-N-k-l+1)!} + \frac{i-N-k+1}{(N\mu)^{i-N-k+2}} \right)$$

Here, we use the formula that may be obtained via mathematical induction on k to derive the equation given by (9):

$$\int_T^{\infty} (t-T)t^k \mathrm{e}^{-\mu t} dt = k! \mathrm{e}^{-\mu T} \left(\sum_{l=0}^{k} \frac{T^{k+1-l}}{\mu^{l+1}(k-l)!} \frac{l}{k+1-l} + \frac{k+1}{\mu^{k+2}} \right) \tag{10}$$

We discover that the expectancies of the profit functions (1) and (2) are, respectively, given by the following result using Proposition 3 and 4.

Proposition 6. The expected customer profit and the agent profit can be given respectively by:

$$E[U] = E[U_j] = RL_0 \left[1 - \lambda E[B]\right] - C_p - P(L) - C_s\lambda(L_0 - L) + \delta\lambda LQ(N) \tag{11}$$

and

$$E[V] = K[C_p + P(L) - C_r\lambda L_0 + \lambda(L_0 - L)C_s - \delta\lambda LQ(N)] - NC_n \tag{12}$$

The subsequent propositions provide further insights into the system, covering aspects like the mean number of malfunctions, the expected downtime, and the aggregate agent profit.

4. Agent's optimal strategy

In this section, we present the agent's optimal strategy. The foundation of this strategy is based on the Nash equilibrium for non-cooperative games, the details of which are discussed after formalizing the game. The Nash equilibrium condition requires that the total profits be equally divided between the agent and the customers, from which the bargaining solution for $P(L)$ is derived. This is consistent with the insights from [17] and [1]. Consequently, the equilibrium condition leads to:

$$E[V] = KE[U] \tag{13}$$

from which we obtain:

$$P(L) = [C_s + \delta Q(N)]\lambda L + \frac{\Delta(N)}{2}$$
$$\Delta(N) = RL_0\,(1 - \lambda E[W]) - \lambda L_0(2C_s - C_r) - 2C_p + \frac{N}{K}C_n \tag{14}$$

To maximize the agent's revenue, the following two steps are undertaken:

1. Identify the optimal warranty period, $\hat{L}(N)$.
2. Based on Proposition 6, ascertain the number of repairmen N^* that maximizes the revenue. Since the number of repairmen is finite, it is sufficient to consider N such that $1 \le N \le K$.

Considering equation (14), we derive the following proposition:

Proposition 7. For a given N, equation (14) has a unique root $\hat{L}(N)$ in the interval $(0, L_0)$ if and only if

$$\Delta(N)\left(P(L_0) - (C_s + \delta Q(N))\,\lambda L_0 - \frac{\Delta(N)}{2}\right) > 0$$

Proof. Consider the function:

$$f(L) = 2P(L) - 2\,(C_s + \delta Q(N))\,\lambda L - \Delta(N)$$

It is clear that:

1. $f(0) = -\Delta(N)$.
2. Because we assume that the price $P(L)$ is a concave function of the warranty duration L, hence $f''(L) = P''(L) \le 0$. Therefore, for any given N, $f(L)$ is concave.

Invoking the properties of concave functions and the zero point theorem, if $f(0)f(L_0) < 0$, then $f(L) = 0$ has a unique root $\hat{L}(N)$ in $(0, L_0)$. □

From the agent's perspective, the expected profit, given N, is:

$$E[V; N] = \frac{KL_0}{2}\,(R\,(1 - \lambda E[W]) - \lambda C_r) - \frac{N}{2}C_n \tag{15}$$

Due to the complexity of $E[W]$, a direct differentiation approach to find the optimal number of repairmen is challenging. Instead, an exhaustive method yields:

$$N^* = \text{argmax}\,(E[V; 1], E[V; 2], \ldots, E[V; K]) \tag{16}$$

Thus, the optimal warranty price and duration are $P^* = P(L^*)$ and $L^* = \hat{L}(N^*)$, respectively.

Upon substituting the values of N^*, L^*, and P^* into equations (11) and (12), the expected profits for the agent and each customer are:

$$E[U^*] = \frac{L_0}{2}\,(R\,(1 - \lambda E[W]) - \lambda C_r) - \frac{N^*}{2K}C_n \tag{17}$$

$$E[V^*] = \frac{KL_0}{2}\,(R\,(1 - \lambda E[W]) - \lambda C_r) - \frac{N^*}{2}C_n \tag{18}$$

5. Numerical analysis

We will delve into the numerical analysis across three distinct scenarios, each defined by the price functions associated with the service contract. Specifically, we consider linear, quadratic, polynomial, and exponential categorizations.

For a practical understanding of how parameter variations influence our model's optimal solutions, we provide several numerical examples computed using MATLAB software. It's pertinent to note that for the examples below, the parametric values are chosen such that the condition

$$\Delta(N)\left(P(L_0)-\lambda\left(C_s+\delta Q(N)\right)L_0-\frac{1}{2}\Delta(N)\right)>0$$

is satisfied. Our base assumptions are as follows:

$$\begin{array}{lll} C_p = 3.8\times 10^5\$, & C_r = 5\times 10^2\$, & C_s = 5.2\times 10^3\$ \\ C_n = 1\times 10^4\$, & \delta = 7\times 10^2\$, & T = 2\text{h} \\ \lambda = 0.06\text{h}, & \mu = 0.5\text{h} & \end{array}$$

The results are presented in Tables 1–3 and Figures 1, 2.

Consider a scenario where the warranty cost is a linear function of the warranty duration, represented as $P(L) = aL$, with $a > 0$. Substituting this into (14) and solving for L, we obtain:

$$\hat{L}_1 = \frac{\Delta(N)}{2a - 2\lambda[C_s + \delta Q(N)]}. \tag{19}$$

From this, the optimal warranty duration and its cost can be expressed as:

$$L_1^* = \frac{\Delta(N^*)}{2a - 2\lambda[C_s + \delta Q(N^*)]}, \tag{20}$$

$$P_1^* = \frac{a\Delta(N^*)}{2a - 2\lambda[C_s + \delta Q(N^*)]}. \tag{21}$$

Table 1 provides a detailed numeric breakdown for this scenario.

Table 1. Results for $P(L) = 20L$ with $R = 4$ (10^2\$/h) and $L_0 = 5$ (10^3 h)

K	N^*	L^*	P^*	$E[V^*]$
2	2	3035	60700	1600000
4	3	3038.8	60777	3204800
6	3	3042.5	60850	4812200
8	4	3042.7	60854	6419100
10	4	3044.2	60884	8026200
12	5	3044	60880	9633700
14	5	3044.9	60898	11241000
16	6	3044.7	60893	12849000
18	6	3045.3	60906	14457000
20	7	3045.1	60901	16064000

Next, assuming that the warranty price follows a quadratic polynomial function of its duration, i.e., $P(L) = bL^2 + cL$ with $b > 0$ and $c \geq 0$. Solving the following quadratic equation

$$bL^2 + (c - \lambda(C_s + \delta Q(N)))\, L = \frac{1}{2}\Delta(N) \tag{22}$$

for L, we get

$$\hat{L}_2 = \frac{1}{2b}\left(\lambda\,(C_s + \delta Q(N)) - c + \sqrt{(c - \lambda(C_s + \delta Q(N)))^2 + 2b\Delta(N)}\right) \tag{23}$$

Furthermore, we obtain the optimal solutions:

$$L_2^* = \frac{1}{2b}\left(\lambda\,(C_s + \delta Q(N^*)) - c + \sqrt{(c - \lambda(C_s + \delta Q(N^*)))^2 + 2b\Delta(N^*)}\right) \tag{24}$$

and

$$P_2^* = b(L_2^*)^2 + cL_2^* \tag{25}$$

Table 2 displays the best options for this scenario.

Table 2. Results for $P(L) = 0.002L^2 + 0.3L$ with $R = 4$ (10^2\$/h) and $L_0 = 5$ (10^3 h)

K	N^*	L^*	P^*	$E[V^*]$
2	2	2909.9	17808	1600000
4	3	2913.7	17853	3204800
6	3	2917.3	17897	4812200
8	4	2917.4	17898	6419100
10	4	2918.9	17916	8026200
12	5	2918.7	17913	9633700
14	5	2919.6	17924	11241000
16	6	2919.4	17921	12849000
18	6	2920	17929	14457000
20	7	2919.7	17926	16064000

Table 3. Results for $P(L) = 0.2e^{0.003L} - 0.2$ with $R = 4$ (10^2\$/h) and $L_0 = 5$ (10^3 h)

K	N^*	L^*	P^*	$E[V^*]$
2	2	2861.1	1068.1	1600000
4	3	2864.7	1079.9	3204800
6	3	2868.3	1091.5	4812200
8	4	2868.4	1091.9	6419100
10	4	2869.9	1096.7	8026200
12	5	2869.7	1096	9633700
14	5	2870.6	1098.9	11241000
16	6	2870.3	1098	12849000
18	6	2870.9	1100.1	14457000
20	7	2870.7	1099.2	16064000

For the scenario where $P(L)$ is of an exponential form, specifically $P(L) = \alpha e^{\beta L} - \alpha$ with constants $a > 0$ and $\beta > 0$, we present the optimal solutions in Table 3.

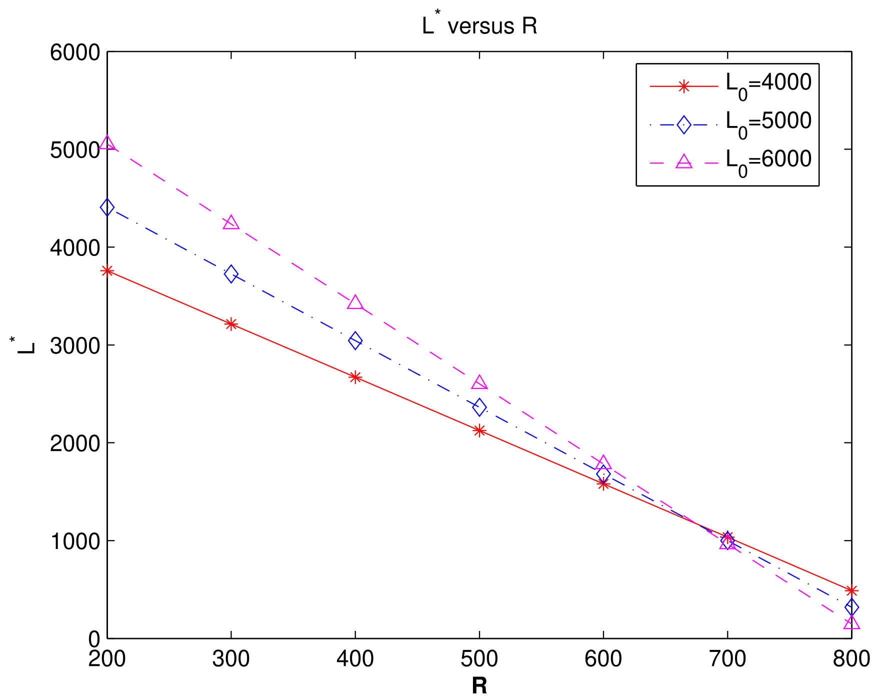

Figure 1. The optimal warranty length L^* versus R for different L_0 ($P(L) = 20L, K = 10$)

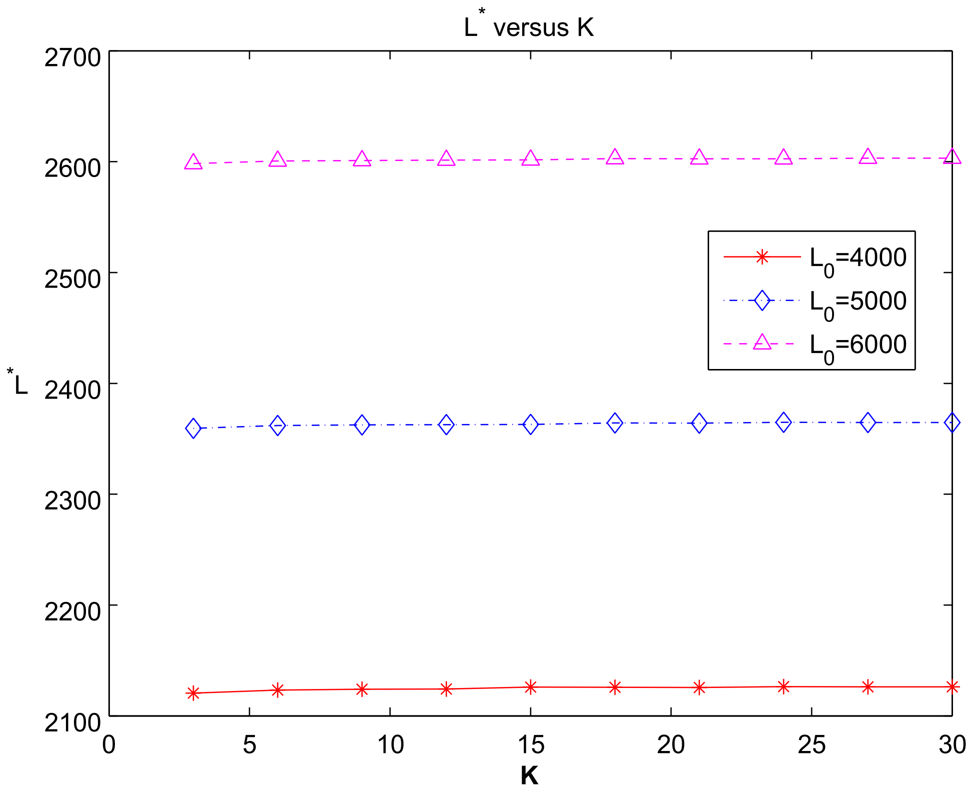

Figure 2. The optimal warranty length L^* versus K for different L_0 ($P(L) = 20L, R = 500$)

Insights drawn from Tables 1–3 underscore that the nature of the warranty pricing functions doesn't dictate the optimal number of repairmen, N^*. Intuitively, with an increase in the clientele (K), both the optimal expected profit for the agent and the ideal number of repairmen (N^*) showcase a direct relationship. Figure 1 traces the influence of per unit time revenue (R) of a machine on the optimal warranty duration (L^*) for different values of L_0. There's a discernible decline in L^* as R ascends. Moreover, the longer the lifespan L_0, the more pronounced the decrement in the optimal warranty duration L^*. As per Figure 2, L^* remains relatively uninfluenced by the number of clients, K. Predictably, the best warranty duration, L^*, steadily rises with an increase in the lifespan value L_0.

6. Concluding remarks

In this study, we developed a comprehensive model to evaluate maintenance service contracts for machines with uncertain lifetimes. By framing a non-cooperative game between clients and service providers, we explored the dynamics of cost-minimization and revenue-maximization. Our analysis led to the determination of the Nash equilibrium, providing optimal solutions for both parties.

Key findings include the inverse relationship between optimal warranty duration and service provider revenue. This relationship intensifies with higher baseline warranty durations. Interestingly, the optimal warranty price remains unaffected by the number of repairmen, indicating a separation between service capacity and pricing strategy.

The study also highlights a direct correlation between the increase in clientele and both the optimal expected profit for service providers and the number of repairmen required. Notably, the type of warranty pricing function does not influence the optimal number of repairmen.

Looking forward, this research opens avenues for exploring various contract forms, deeper analysis of non-linear pricing models, and the potential to extend this model to different types of machines. This work contributes significantly to the understanding of maintenance service contracts, offering valuable insights for both academic research and practical applications in service management.

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