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Some remarks on a mathematical model of COVID-19 pandemic with health care capacity

Abstract In this paper, a SEIR model proposed in the article "Dynamic analysis of mathematical model with health care capacity for COVID-19 pandemic" by S. Çakan (2020) is analysed. The model describes COVID-19 pandemic spread affected by healthcare capacity and is expressed by a system of delay differential equations. To prove the local stability of stationary states, S. Çakan uses linearization technique, though she does this as if the equations did not depend on the delay. Additionally, it is shown that the crucial argument used by S. Çakan to prove boundedness of the solutions is not correct, which implies that the proofs of global stability in the original article are not correct either. In this paper, improved proofs of local and global stability of the stationary states are provided. For local stability of the stationary states a standard linearization technique is used. Global stability of the stationary states is proved based on Lyapunov's functionals. Although the functionals are the same as those proposed by S. Çakan, additional properties of the solutions (in the case of disease–free stationary state) and the functional (in the case of the endemic stationary state) are proved.

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1. Introduction First cases of coronavirus disease 2019 (COVID-19) were identified in December 2019 in China. Since then, the disease has spread worldwide, leading to an ongoing pandemic and becoming one of the biggest current world's problems. In order to predict dynamics of epidemic, compartmental epidemiological SIR and SEIR models were adapted and modified to reflect some aspects of the disease (see [1], [5]).

Predictive mathematical models are crucial to describe the course of the pandemic and plan effective control strategies. Therefore, researchers made efforts to deliver such models as soon as possible. Unfortunately, the rush and shortened publication process made it easier to overlook some fundamental mistakes in the analysis. In this paper we are going to point out and correct lapses in the article by S. Çakan's [8] .

1.1. Presentation of the model

The model proposed by S. Çakan is a modification of the standard SEIR model. In the article [8] the author divides the whole population into four separable compartments: Susceptible (S), Exposed (E), Infectious (I) and Recovered (R). The last group – R – not only contains individuals that have already recovered from the disease, but also all individuals that cannot infect others (e.g. due to isolation). Let us denote by S, E, I, R the size of groups S, E, I, R, respectively. In [8] three modifications of a standard SEIR model were proposed. First – a direct transmission from Exposed to Recovered has been added to describe individuals that have been infected but cannot infect others (for various reasons, e.g. quarantine). The sketch of transmissions between compartments is presented on Fig. 1. Second – S. Çakan introduced time delay into equations describing dynamics of E and I, substituting the term "$-\gamma E(t)$" which describes transition between Exposed and Infectious compartments by the term "$-\gamma\beta S(t)I(t)\,\mathrm{e}^{-d\tau}$". She claimed that such modification describes the incubation period of COVID-19. Although we do not agree with this justification, we believe that it becomes reasonable after considering a system without Exposed and Recovered compartments (which was actually done in [8]). Moreover, this modification causes problems with non-negativity of solutions, which we discuss later.

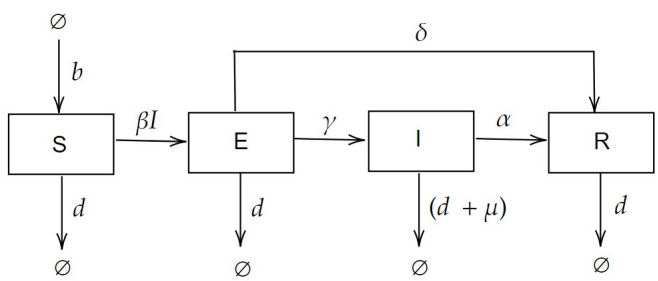

Figure 1: Compartmental transition diagram for the model from [8]

The last modification proposed by S. Çakan is the most important. Specifically, in [8] it is assumed that recovery and disease-induced death rates depend on the healthcare system capacity $c \in [0, 1]$ which may change during the epidemic. We assume that $c = 1$ means that the healthcare system is fully efficient, while $c = 0$ represents the crisis of the healthcare system, when all hospital opportunities are consumed away. Thus, recovery rate α and disease-induced death rate μ depend on the condition of the healthcare system c and we write $\alpha = \alpha_1 + \alpha_2 c$ and $\mu = \mu_1 + \mu_2(1 - c)$. Now, α_1 can be interpreted as natural recovery rate (without medical help), and $(\alpha_1 + \alpha_2)$ is the maximal recovery rate. Similarly, $\mu_1 + \mu_2$ is the natural disease-induced death rate, while μ_1 represents the minimal disease-induced death rate that can be

achieved due to efficient healthcare. In this paper we limit ourselves to the case of c being a constant. The model considered in [8] consists of four delay differential equations

$$\begin{aligned}
\dot{S}(t) &= b - \beta S(t)I(t) - dS(t),\\
\dot{E}(t) &= \beta S(t)I(t) - \gamma\beta S(t-\tau)I(t-\tau)\,\mathrm{e}^{-(d+\delta)\tau} - (d+\delta)E(t),\\
\dot{I}(t) &= -\Big(\alpha_1 + \alpha_2 c + \mu_1 + \mu_2(1-c) + d\Big)I(t) + \\
&\qquad + \gamma\beta S(t-\tau)I(t-\tau)\,\mathrm{e}^{-(d+\delta)\tau},\\
\dot{R}(t) &= \Big(\alpha_1 + \alpha_2 c\Big)I(t) + \delta E(t) - dR(t).
\end{aligned} \tag{1}$$

The other parameters used in System (1) have standard interpretations, namely:

- b – birth rate,
- d – natural death rate ($\frac{1}{d}$ can be interpreted as average lifetime),
- β – effective contact rate,
- γ – progression rate, i.e. fraction of infected individuals transitioning from group E to I,
- τ – length of the incubation period, i.e. average time that an infected individual spends in group E.

It is worth mentioning that we replaced the term "$\mathrm{e}^{-d\tau}$" from article [8] with "$\mathrm{e}^{-(d+\delta)\tau}$" in the second and third equations, as we believe that this term should correspond to the change of the size of group E, not only due to the mortality, but also to direct transition into R compartment. However, this change is only quantitative and does not influence the qualitative properties of the considered model.

To close System (1) we impose the following initial condition:

$$S(t) = \phi_1(t), \quad E(t) = \phi_2(t), \quad I(t) = \phi_3(t), \quad R(t) = \phi_4(t), \tag{2}$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are non-negative continuous functions defined on $[-\tau, 0)$.

Note that the variables R and E do not appear in equations for S and I in System (1). Therefore, throughout the paper, except for Subsection 2.1, we consider the model

$$\begin{aligned}
\dot{S}(t) &= b - \beta S(t)I(t) - dS(t),\\
\dot{I}(t) &= \gamma\beta S(t-\tau)I(t-\tau)\,\mathrm{e}^{-(d+\delta)\tau} - \big(\eta + \sigma c\big)I(t)
\end{aligned} \tag{3}$$

where

$$\eta := \mu_1 + \mu_2 + \alpha_1 + d, \qquad \sigma := \alpha_2 - \mu_2, \tag{4}$$

and $c \in [0,1]$ is a constant. In the next section (see 2.1), we show that for a non-negative initial condition, the coordinate E may change the sign. It implies that there does not exist a bounded set invariant with respect to evolution governed by System (3), in contrary to the statement in [8]. In consequence, we need to prove some additional properties of Lyapunov's function in order to conclude global stability of the steady states of System (3).

To close System (3) we assume that

$$S(t) = \phi_1(t), \quad I(t) = \phi_2(t), \tag{5}$$

where ϕ_1, ϕ_2 are non-negative functions continuous on $[-\tau, 0)$.

The paper is organised as follows. In the next section we prove basic properties of the model. We discuss nonnegativity, boundedness and continuation of solutions, as well as define an invariant region for System (3). Then, in Section 3 we derive formulas for stationary states of System (3) and we examine local and global stability of these states.

2. Basic properties of the model

In this section, we prove existence and uniqueness of solutions to Systems (1) and (3) for fixed initial conditions. Moreover, we discuss nonnegativity of solutions to these systems. Then we prove that the solution to System (3) is well defined for all $t \geq 0$ for a non-negative initial condition.

THEOREM 2.1 For any non-negative initial function (2) that is continuous on $[-\tau, 0)$ there exists a unique solution to System (1). For any non-negative initial function (5) that is continuous on $[-\tau, 0)$ there exists a unique solution to System (3).

PROOF The right-hand side of (1), as well as the right-hand side of (3), is continuous and satisfies the Lipschitz's condition on each compact subset of the space of continuous functions. Thus, the assertion of Theorem follows from the standard theory of delay differential equations, see e.g. [4]. ∎

2.1. Nonnegativity of solutions

In the article [8], S. Çakan claims that the solution to System (1) remains non-negative if the initial condition is non-negative. However, the author does not prove this statement, which in fact appears to be false. Let us consider the following example.

PRZYKŁAD 2.1 Let us take the initial condition (2) for System (1) such that $\phi_1(0) = \phi_2(0) = \phi_3(0) = 0$, $\phi_1(-\tau) = \phi_3(-\tau) = 1$, ϕ_1, ϕ_2, ϕ_3 are non-negative and continuous and ϕ_4 is an arbitrary, non-negative, continuous function. Then

$$E(0) = \phi_2(0) = 0 \text{ and}$$

$$\dot{E}(0) = -\gamma\beta\phi_1(-\tau)\phi_3(-\tau)\,\mathrm{e}^{-(d+\delta)\tau} = -\gamma\beta\,\mathrm{e}^{-(d+\delta)\tau} < 0,$$

so there exists $\xi > 0$ such that $E(t) < 0$ for $t \in (0, \xi)$.

One can argue that initial condition assumed in Example 2.1 is artificial and cannot describe a realistic epidemiological situation. Nonetheless, the solution E to System (1) can be negative also for epidemiologically relevant initial conditions. In this case, we only present an example of a numerical solution, as it is not easy to prove analytically the negativity of E.

Przykład 2.2 Let us consider the epidemic that starts at time $t = 0$. We assume that $\phi_2(t) = \phi_4(t) = 0$ for all $t \in [-\tau, 0]$ and that the whole population is susceptible for $t < 0$, that is $\phi_1(t) = \frac{b}{d}$ and $\phi_3(t) = 0$ for $t < 0$. Finally, we assume that at time $t = 0$ a small population of individuals becomes infectious, which means $\phi_1(0) = \frac{b}{d} - \varepsilon$ and $\phi_3(0) = \varepsilon$ for a certain $\varepsilon > 0$. For the following set of parameters:

$$\begin{array}{llll} \beta = 1.75 \cdot 10^{-8}, & b = 4000, & \gamma = 1.2, & \alpha_1 = 0.4, \\ \alpha_2 = 0.4, & d = 0.000015, & \delta = 0.005, & \mu_1 = 0.05, \\ \mu_2 = 0.05, & c = 0.4, & \tau = 10, & \varepsilon = 1 \end{array}$$

the function E is negative for $t \in [150, 500]$, see Fig. 2.

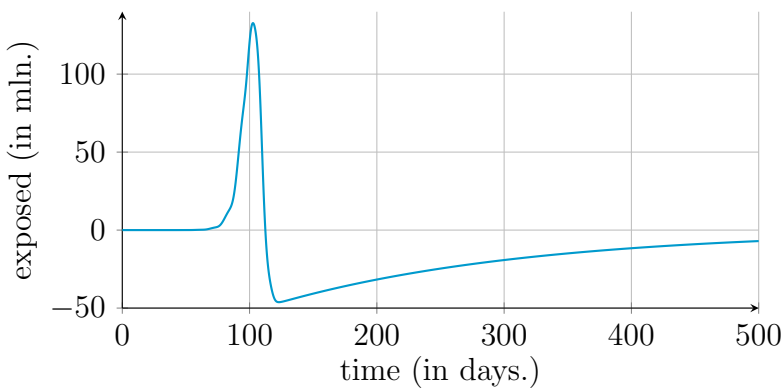

Figure 2: The graph of the coordinate $E(t)$ of the solution to System (1) for parameters and initial condition given in Example 2.2.

In the following, we restrict our analysis to System (3). First, we show that the solution to System (3) with a non-negative initial condition is non-negative. However, we cannot prove that $S(t) + I(t)$ is bounded from above.

Let us define $\mathcal{C}$ as a space of all continuous functions defined on the interval $[-\tau, 0]$ with values in $\mathbb{R}^2$.

Theorem 2.2 For any non-negative initial condition (2), the solution to System (3) is non-negative.

Proof We write the right-hand side of (3) in a functional form, that is we define a function $f = [f_1, f_2]^T : \mathcal{C} \to \mathbb{R}^2$ by a formula

$$f_1(\phi) = b - \beta\phi_1(0)\phi_2(0) - d\phi_1(0)$$
$$f_2(\phi) = \gamma\beta\phi_1(-\tau)\phi_2(-\tau)\,\mathrm{e}^{-(d+\delta)\tau} - \big(\eta + \sigma c\big)\phi_2(0).$$

Now, we will examine if the right-hand side of (3) fulfills the assumptions of Theorem 3.4 from [6]. Firstly, we need to know that there exists a unique solution to System (3), which we proved in Theorem 2.1. Second, we need to check if for any $t \in [0, +\infty)$ and for any ϕ such that $\phi \geq 0$ the following implication holds:

$$\phi_i(0) = 0 \text{ for some } i \in \{1, 2\} \implies f_i(t, \phi) \geq 0.$$

This is an easy observation. By way of explanation, if $\phi_1(0) = 0$, then $f_1(\phi) = b > 0$. Whereas if $\phi_2(0) = 0$, then

$$f_2(\phi) = \gamma\beta\phi_1(-\tau)\phi_2(-\tau)\,\mathrm{e}^{-(d+\delta)\tau} \geq 0.$$

Thus, assertion of Theorem follows from [6, Th 3.4]. ∎

2.2. Boundedness of solutions

Solutions of classical SEIR model (without delays) satisfy the condition: if $S(0) + I(0) + E(0) + R(0) \leq b/d$, then $S(t) + I(t) + E(t) + R(t) \leq b/d$ for all $t \geq 0$. In the case considered here, this implication holds as well, but due to the failure of nonnegativity property for E, we cannot infer that each variable is bounded by b/d. This is a genuine issue for proving continuation of solutions as well as for using Lyapunov's theorems to prove global stability of steady states. In order to solve this problem, we provide the following lemmas and their proofs.

The first lemma states that the function I is either constant, equal to 0 on the interval $[0, \tau]$ (and consequently for all $t \geq 0$), or there exists such $\bar{t} \in [0, \tau]$ that $I(t)$ is positive for all $t > \bar{t}$.

Lemma 2.1 *The solution to System (3) with a non-negative initial condition (5) has the following properties.*

1. *If $I(0) = 0$ and $S \cdot I \equiv 0$ on $[-\tau, 0]$, then $I \equiv 0$ for all $t \geq 0$.*
2. *If $I(0) = 0$ and $S \cdot I \not\equiv 0$ on $[-\tau, 0]$, then $I \equiv 0$ on $[0, \tau - \kappa]$ and $I > 0$ for $t > \tau - \kappa$, where $\kappa = -\inf\{t \in [-\tau, 0] \colon S(t) \cdot I(t) > 0\}$.*
3. *If $I(0) > 0$, then $I > 0$ for all $t \geq 0$.*

PROOF Note that it is enough to prove the assertions of Lemma for $t \in [0, \tau]$. Then the mathematical induction allows us to extend the result for all $t \geq 0$.

Note that for $t \in [0, \tau]$ inequality

$$h(t) - \bar{\eta} I(t) \leq \dot{I}(t) \leq h(t) - \hat{\eta} I(t) \tag{6}$$

holds, where $h(t) = \gamma\beta S(t-\tau)I(t-\tau)\,\mathrm{e}^{-(d+\delta)\tau}$, $\bar{\eta} = \eta + |\sigma|$ and $\hat{\eta} = \eta - |\sigma|$. If $t \in [0, \tau]$, then $t - \tau \in [-\tau, 0]$, so values of $h(t)$ for $t \in [0, \tau]$ are defined due to the initial condition. Therefore, Gronwall's inequality implies that

$$\left(I(0) + \int_0^t h(s)\,\mathrm{e}^{-\bar{\eta}s}\,ds\right)\mathrm{e}^{-\bar{\eta}t} \leq I(t) \leq \left(I(0) + \int_0^t h(s)\,\mathrm{e}^{-\hat{\eta}s}\,ds\right)\mathrm{e}^{-\hat{\eta}t}. \tag{7}$$

Additionally, since the initial condition is non-negative, so is the function h on $[0, \tau]$.

First, assume that $I(0) = 0$ and $S \cdot I \equiv 0$ on $[-\tau, 0]$. Then $h \equiv 0$ on $[0, \tau]$ and, due to (7), $I \equiv 0$ on $[0, \tau]$. This proves the first point of the lemma.

Second, assume that $I(0) = 0$ and $S \cdot I \not\equiv 0$ on $[-\tau, 0]$. From the definition of κ it is clear that $h \equiv 0$ on $[0, \tau - \kappa]$. Thus, Inequality (7) implies that $I \equiv 0$ on $[0, \tau - \kappa]$. Moreover, $S(t) \cdot I(t) > 0$ for $t \in (-\kappa, -\kappa + \varepsilon)$ for a certain $\varepsilon > 0$, based on continuity of I. Consequently, $h(t) > 0$ for $t \in (\tau - \kappa, \tau - \kappa + \varepsilon)$. Based on the same Inequality (7), we claim the positivity of $I(t)$ for $t > \tau - \kappa$. That completes the proof of the second point of the lemma.

The third point of the lemma follows from (7) and from nonnegativity of the function h. ∎

Now, we use Lemma 2.1 to prove that the function S has an upper bound.

LEMMA 2.2 *For a non-negative initial condition* (5) *the solution to System* (3) *fulfills*

$$S(t) \leq \frac{b}{d} \text{ for } t \in [-\tau, 0] \implies S(t) \leq \frac{b}{d} \text{ for } t \geq 0.$$

PROOF Note that, according to Lemma 2.1, $I(t) \geq 0$ holds for all $t \in [-\tau, \infty)$.

Proof by the *method of steps*. We prove that $S(t) \leq \frac{b}{d}$ for $t \in [0, \tau]$. Then, by mathematical induction, we deduce that $S(t) \leq \frac{b}{d}$ for $t \in [0, \infty)$.

Suppose that the assertion of Lemma does not hold, i.e there exists $\tilde{t} \in (0, \tau]$, such that $S(\tilde{t}) > \frac{b}{d}$. The continuity of the function S implies existence of $t_1 \in [0, \tilde{t})$, satisfying the following conditions: $S(t_1) = \frac{b}{d}$; $\dot{S}(t_1) \geq 0$; $S(t) \leq \frac{b}{d}$ for $t \in [0, t_1]$. Additionally, there exists $\varepsilon > 0$ such that $\dot{S}(t) > 0$ and $S(t) > \frac{b}{d}$ for $t \in (t_1, t_1 + \varepsilon)$.

Observe that

$$\dot{S}(t_1) = b - \beta\frac{b}{d}I(t_1) - d\frac{b}{d} = -\beta\frac{b}{d}I(t_1) \leq 0. \tag{8}$$

Thus, if $I(t_1) > 0$, then $\dot{S}(t_1) < 0$, but this contradicts the assumption $\dot{S}(t_1) \geq 0$.

If $I(t_1) = 0$ then, based on Lemma 2.1, the condition $t \in [0, t_1)$ implies $I(t) = 0$. Hence, $\dot{S}(t) = b - dS(t)$ holds for $t \in [0, t_1)$. Consequently, $S \equiv S(t_1) \equiv \frac{b}{d}$ on $[0, t_1)$ (due to uniqueness of solutions of (3)).

Lemma 2.1 states that for $t > t_1$ there are two possibilities: either $I > 0$ on $(t_1, \tau]$ or $I \equiv 0$ on $(t_1, t_1 + \varepsilon_1)$, for a certain $\varepsilon_1 > 0$.

In the first case, the inequality $\dot{S}(t) < 0$ holds for $t \in (t_1, t_1 + \varepsilon)$ for a certain $\varepsilon > 0$ (due to Inequality (8) and differential continuity of S). But it contradicts the assumption that $\dot{S}(t) > 0$ for $t \in (t_1, t_1 + \varepsilon)$. The second case implies that $S \equiv \frac{b}{d}$ for $t \in (t_1, t_1 + \varepsilon_1)$, which contradicts the assumption that $S(t) > \frac{b}{d}$ for $t \in (t_1, t_1 + \varepsilon)$.

Thus, inequality $S(t) \leq \frac{b}{d}$ holds for $t \in [0, \tau]$. By induction we conclude that $S(t) \leq \frac{b}{d}$ for all $t \geq 0$. ∎

Finally, we show that if the model parameters satisfy a certain inequality, then $I(t)$ is bounded.

LEMMA 2.3 *If the inequality*

$$\frac{\gamma\beta b\, \mathrm{e}^{-(d+\delta)\tau}}{d(\eta + \sigma c)} < 1 \tag{9}$$

holds, then for any solution to System (3) with a non-negative initial condition (5) we have

$$I(t) \leq \sup_{s\in[-\tau,0]} I(s) \text{ for all } t \geq 0.$$

PROOF Note that (according to Lemma 2.1) if $\sup_{s\in[-\tau,0]} I(s) = 0$, then $I(t) = 0$ for all $t \geq -\tau$. For this reason, the assertion of Lemma holds in this case.

Now, assume that $\sup_{s\in[-\tau,0]} I(s) > 0$ and the assertion of Lemma does not hold. Then, $I(t) > \sup_{s\in[-\tau,0]} I(s) > 0$ for a certain $t > 0$. Observe that the set

$$\mathcal{A} := \left\{ t \in [0, \infty) \colon I(t) = \sup_{s\in[-\tau,0]} I(s) > 0, \quad \dot{I}(t) \geq 0 \right\}.$$

is nonempty.

Next, let us define $\tilde{t} := \inf \mathcal{A}$. Then $I(t) \leq I(\tilde{t})$ for $t \in [-\tau, \tilde{t}]$ and $I(\tilde{t}) > 0$. For $\frac{\gamma\beta b\, \mathrm{e}^{-(d+\delta)\tau}}{d(\eta+\sigma)} < 1$ the following inequalities are true.

$$\begin{aligned} \dot{I}(\tilde{t}) &\leq \frac{b\beta\gamma\, \mathrm{e}^{-(d+\delta)\tau}}{d} I(\tilde{t} - \tau) - (\eta + \sigma c) I(\tilde{t}) \leq \\ &\leq (\eta + \sigma c_0) I(\tilde{t}) \left(\frac{b\beta\gamma\, \mathrm{e}^{-(d+\delta)\tau}}{d(\eta + \sigma c)} - 1 \right) < 0. \end{aligned} \tag{10}$$

The first inequality of (10) is a direct consequence of Lemma 2.2, the second follows from the definition of $\tilde{t}$. Hence, if $\frac{\gamma\beta b\,\mathrm{e}^{-(d+\delta)\tau}}{d(\eta+\sigma)} < 1$, then $\dot{I}(\tilde{t}) < 0$, which contradicts $\dot{I}(\tilde{t}) \geq 0$. ∎

2.3. Continuation of solutions and the invariant set

In this section, we prove that, for a non-negative initial condition, the solution to System (3) is well defined for all $t \geq 0$.

PROPOSITION 2.1 The solution to System (3) with a non-negative initial condition (5) exists for all $t \in [-\tau, \infty)$.

PROOF The right-hand side of (3) is well defined on $\mathbb{R}_+$ and the solution to System (3) is non-negative, so the solution is bounded on every compact interval. According to Lemma 2.2, the function S is bounded.

Now, we prove that I is bounded on $[0, \tau]$. For $t \in [0, \tau]$ we have

$$\dot{I}(t) = h(t) - \big(\eta + c\sigma\big)I(t), \tag{11}$$

where

$$h(t) = \gamma\beta\phi_1(t-\tau)\phi_2(t-\tau)\,\mathrm{e}^{-(d+\delta)\tau}\,.$$

The function h is given on $[0, \tau]$ by the initial condition. Equation (11) is a non-autonomous linear ordinary differential equation, so its solution on $[0, \tau]$, namely the function I, is bounded. By mathematical induction, we deduce that I is bounded on any compact interval of $[0, +\infty)$. This completes the proof. ∎

Note that, due to Lemma 2.2, we can define an invariant region for System (3).

DEFINITION 2.1 The set Γ given by the formula

$$\Gamma = \left\{ \big(x, y\big) \in [0, +\infty)^2 \colon x \leq \frac{b}{d} \right\} \tag{12}$$

is the invariant region for System (3).

3. Stability of stationary states

In [8], in order to derive the formula for the basic reproduction number $\mathcal{R}_0$, the next-generation matrix method [3, 7] has been applied. The method is commonly used to calculate the basic reproduction number. However, the next generation matrix method was introduced and proved for systems without delay and we are not aware of any strict generalisation of this method to systems with time delay. Therefore, we provide the formula for $\mathcal{R}_0$ before justifying that it is a proper expression of the basic reproduction number for System (3).

Definition 3.1 Let

$$\mathcal{R}_0 = \frac{\gamma\beta b\,\mathrm{e}^{-(d+\delta)\tau}}{d(\eta+\sigma c)}. \tag{13}$$

Assume that $S(t) \approx b/d$ and $I(t) \approx 0$, I and S are constant and $I(t) > 0$ for $t \leq 0$. Then, I increases if and only if $\dot{I}(0) > 0$, which is equivalent (under these assumptions) to the inequality $\mathcal{R}_0 < 1$. This heuristically justifies the formula for $\mathcal{R}_0$. Afterwards, we prove that if $\mathcal{R}_0 < 1$, then the disease-free stationary state is globally asymptotically stable.

Coordinates of any stationary state $\bar{P} = (\bar{S}, \bar{I})$ of System (3) satisfy the following algebraic relations

$$\begin{aligned} 0 &= b - \beta\bar{S}\bar{I} - d\bar{S}, \\ 0 &= \big(\gamma\beta\bar{S}\,\mathrm{e}^{-(d+\delta)\tau} - (\eta+\sigma c)\big)\bar{I}. \end{aligned} \tag{14}$$

Solving (14) and using (13), we obtain formulas for stationary states.

Proposition 3.1 If the inequality $\mathcal{R}_0 > 1$ holds, then System (3) has two stationary states P_0 (disease-free) and P_* (endemic):

$$\begin{aligned} P_0 &= (S_0, I_0) = \left(\frac{b}{d}, 0\right), \\ P_* &= (S_*, I_*) = \left(\frac{b}{d\mathcal{R}_0}, \frac{d}{\beta}\big(\mathcal{R}_0 - 1\big)\right). \end{aligned}$$

Otherwise, System (3) has only disease-free stationary state P_0.

3.1. Local stability

To begin with, we prove that the disease-free stationary state is locally asymptotically stable for $\mathcal{R}_0 < 1$, whereas the endemic stationary state is locally asymptotically stable for $\mathcal{R}_0 > 1$. Next, we construct Lyapunov's functionals and we prove that for any non-negative initial condition (5), the solution to (3) converges to one of the steady states (disease-free or endemic depending on $\mathcal{R}_0$).

Let us start the analysis of local stability by expressing the characteristic function W for System (3) and for a stationary state $\bar{P} = (\bar{S}, \bar{I})$ of (3). The formula for W is as follows

$$W(\lambda) = \det \begin{bmatrix} -\beta\bar{I} - d - \lambda & -\beta\bar{S} \\ \gamma\beta\bar{I}\,\mathrm{e}^{-(d+\delta)\tau}\,\mathrm{e}^{-\lambda\tau} & \gamma\beta\bar{S}\,\mathrm{e}^{-(d+\delta)\tau}\,\mathrm{e}^{-\lambda\tau} - (\eta+\sigma c) - \lambda \end{bmatrix}, \tag{15}$$

which is equivalent to

$$\begin{aligned} W(\lambda) = (-\beta\bar{I} - d - \lambda)\big(\gamma\beta\bar{S}\,\mathrm{e}^{-(d+\delta)\tau}\,\mathrm{e}^{-\lambda\tau} - (\eta+\sigma c) - \lambda\big) + \\ + \gamma\beta^2\bar{S}\bar{I}\,\mathrm{e}^{-(d+\delta)\tau}\,\mathrm{e}^{-\lambda\tau}. \end{aligned} \tag{16}$$

We would like to point out that we have obtained the same conditions for local stability as those presented in [8]. Nonetheless, in [8] the formulas for stationary states have been obtained in a nonrigorous manner, as the dependence of characteristic functions (polynomials in [8]) on time delay was overlooked.

3.1.1. Local stability of disease–free stationary state

PROPOSITION 3.2 The disease-free stationary state P_0 is locally asymptotically stable, if $\mathcal{R}_0 < 1$. Otherwise, if $\mathcal{R}_0 > 1$ it is unstable.

PROOF For stationary state P_0, we obtain

$$W(\lambda) = (-\lambda - d)U(\lambda),$$

where

$$U(\lambda) := (\eta + \sigma c)\mathcal{R}_0 \, \mathrm{e}^{-\lambda\tau} - (\eta + \sigma c) - \lambda. \tag{17}$$

Because every root of W is either a root of U or it is equal to $-d < 0$, it is enough to prove that all roots of U have negative real parts. Note that λ is a root of U if and only if it is a solution to

$$\lambda = A + B \, \mathrm{e}^{-\lambda\tau}, \tag{18}$$

where

$$A := -(\eta + \sigma c), \quad B := (\eta + \sigma c)\mathcal{R}_0. \tag{19}$$

Let us make an easy observation that $A < 0$ and $B = |A|\mathcal{R}_0 > 0$. Therefore, if $\mathcal{R}_0 < 1$, then $A + B < 0$ (and $A < B$ obviously). In this case it is known that all solutions to (18) have negative real parts (see e.g. Th 4.7, [6]). Similarly, if $\mathcal{R}_0 > 1$, then $A + B < 0$ and consequently, there exits a solution to (18) with a positive real part. ∎

3.1.2. Local stability of endemic stationary state Let us begin by expressing the formula of the characteristic function W for the stationary state P_* as

$$W(\lambda) = p(\lambda) + q(\lambda) \, \mathrm{e}^{-\lambda\tau}, \tag{20}$$

where

$$\begin{aligned} p(\lambda) &= \lambda^2 + \lambda\big(d\mathcal{R}_0 + (\eta + \sigma c)\big) + d(\eta + \sigma c)\mathcal{R}_0, \\ q(\lambda) &= -\lambda(\eta + \sigma c) - d(\eta + \sigma c). \end{aligned} \tag{21}$$

It is important not to ignore the dependence of $\mathcal{R}_0$ on the delay τ and therefore the dependence of the function p on τ. Below, we present a theorem that allows us to prove local stability of the endemic stationary state.

THEOREM 3.1 (CF. [2, 6]) Let W be in the form of (20), where $p, q : \mathbb{C} \to \mathbb{C}$ are analytic functions and $\lambda > 0$. Let us suppose that the following conditions hold:

(a) $p(\lambda) \neq 0$ for all $z \in \mathcal{C}$ such that $\mathrm{Re}(\lambda) \geq 0$,

(b) $|q(iy)| < |p(iy)|$ for all $0 \leq y < \infty$,

(c) $\lim\limits_{\lambda \to \infty, \mathrm{Re}(\lambda) \geq 0} \left| \dfrac{q(\lambda)}{p(\lambda)} \right| = 0.$

Then for every root λ of the function W the inequality $\mathrm{Re}(\lambda) < 0$ for $\tau \geq 0$ is true.

The proof of Theorem 3.1 is based on a contour integration for fixed parameters (and fixed τ), see [2]. Thus, the dependence of p on τ does not change the assertion of Theorem 3.1 and we can use it in the case considered in this paper.

PROPOSITION 3.3 If $\mathcal{R}_0 > 1$, then the endemic stationary state P_* is locally asymptotically stable.

PROOF We prove that the assumptions of Theorem 3.1 hold.

Note that p is a quadratic polynomial of λ and it can be rewritten as

$$p(\lambda) = \Big(\lambda + d\mathcal{R}_0\Big)\Big(\lambda + (\eta + \sigma c)\Big),$$

then it is easy to note that p has real, negative roots. For this reason the condition (a) of Theorem 3.1 holds.

Now, let us check the condition (b) of Theorem 3.1. For an arbitrary $y \in [0, +\infty)$, we have

$$\begin{aligned} |q(iy)|^2 &= (\eta + \sigma c)^2 (y^2 + d^2), \\ |p(iy)|^2 &= (-y^2 + d(\eta + \sigma c)\mathcal{R}_0)^2 + y^2 (d\mathcal{R}_0 + (\eta + \sigma c))^2. \end{aligned}$$

Therefore, inequality $|q(iy)| < |p(iy)|$ is equivalent to

$$d^2(\eta + \sigma c)^2 < y^4 + y^2 (d\mathcal{R}_0)^2 + (d\mathcal{R}_0(\eta + \sigma c))^2. \tag{22}$$

Inequality (22) is true for all $y \in [0, \infty)$ if and only if

$$d^2(\eta + \sigma c)^2 < (d\mathcal{R}_0(\eta + \sigma c))^2 \iff \mathcal{R}_0 > 1.$$

Therefore, the condition (b) holds if and only if $\mathcal{R}_0 > 1$.

Finally, the condition (c) of Theorem 3.1 holds as the polynomial p has a higher degree than the polynomial q. To conclude, for $\mathcal{R}_0 > 1$ the assertion of Proposition 3.3 follows from Theorem 3.1. ∎

3.2. Global stability

Finally, we examine global stability of stationary states P_0 and P_* using Lyapunov's functions. System (3) generates dynamical system in $\mathcal{C}$. Knowing that $S(t) \leq b/d$, we prove global stability in the set $\chi \subset \mathcal{C}$ which is invariant with respect to the evolution of System (3). Let us define

$$\chi = \left\{ \phi \in \mathcal{C} \colon \phi(t) \in \Gamma \quad \forall t \in [-\tau, 0] \right\} \subset \mathcal{C}, \tag{23}$$

where Γ is defined by (12).

The results provided here are the same as stated in [8]. However, the proofs from [8] needed some improvement. In contrary to S. Çakan we claim that invariant set χ is not bounded. Even though we make use of Lyapunov's functionals proposed in [8], in order to do so, we need to make sure that the solutions to System of (3) have some additional properties, that have not been considered in [8].

THEOREM 3.2 The disease-free stationary state P_0 is globally asymptotically stable in χ if $\mathcal{R}_0 < 1$.

PROOF Let us define $L \colon \mathcal{C} \to \mathbb{R}$ by

$$L(\phi) = \phi_2(0) + \gamma\beta\, \mathrm{e}^{-(d+\delta)\tau} \int_{-\tau}^{0} \phi_1(a)\phi_2(a)da, \quad \text{where } \phi = [\phi_1, \phi_2]^T. \tag{24}$$

Below, we show that L is a Lyapunov's functional on the set χ and satisfies the assumptions of Theorem 3.1 [4, Chapter 5]. However, we cannot use this theorem without proving that the solution to System (3) remains in χ (which is already done) and is bounded (we need to prove it).

Let us denote the solution to System (3) by $x = [S, I]^T$. Then (24) can be expressed as

$$L(x_t) = I(t) + \gamma\beta\, \mathrm{e}^{-(d+\delta)\tau} \int_{t-\tau}^{t} S(a)I(a)da.$$

It seems obvious that L is continuous on χ. In order to examine if L is actually a Lyapunov's functional, we calculate the derivative of L along the trajectory of x_t:

$$\dot{L}(x_t) = \frac{\mathrm{d}}{\mathrm{d}t}\big(L(x_t)\big) = I(t)\Big(\gamma\beta\, \mathrm{e}^{-(d+\delta)\tau}\, S(t) - (\eta + \sigma c)\Big). \tag{25}$$

Note that $x_t = [S_t, I_t]^T \in \chi$, so $S(t) \leq \frac{b}{d}$. It implies that

$$\dot{L}(x_t) \leq I(t)\Big(\frac{\gamma\beta b}{d}\, \mathrm{e}^{-(d+\delta)\tau} - (\eta + \sigma c)\Big) = I(t)(\eta + \sigma c)(\mathcal{R}_0 - 1). \tag{26}$$

We obtained the last equality using Formula (13) for $\mathcal{R}_0$. Finally, for $\mathcal{R}_0 < 1$, the inequality $\dot{L}(x_t) < 0$ holds. Thus, L is a Lyapunov's functional.

Next, we find the explicit formula for the set $\Sigma := \{\phi \in \chi \colon \dot{L}(\phi) = 0\}$. We have $\dot{L}(\phi) = 0$ if and only if

$$\phi_2(0) = 0 \ \vee \ \phi_1(0) = \frac{d}{b\mathcal{R}_0}. \tag{27}$$

However, for $\phi \in \chi$ we have $\phi_1(0) \leq \frac{b}{d} < \frac{d}{b\mathcal{R}_0}$ as $\mathcal{R}_0 < 1$. Thus, the second alternative of (27) does not hold. Therefore, $\phi_2(0) = 0$ which in turn implies that

$$\Sigma = \left\{\phi = (\phi_1, \phi_2)^T \in \chi : \phi_2(0) = 0\right\}.$$

Note that the only subset of Σ, invariant to System (3) is

$$G := \left\{\phi = (\phi_1, \phi_2)^T \in \mathcal{C} : \phi_1 \equiv \frac{b}{d}, \phi_2 \equiv 0\right\} \subset \Sigma.$$

Indeed, if $\phi_2(0) = 0$ then, by Lemma 2.1, $\phi_2 \equiv 0$ holds on $[-\tau, 0]$. The uniqueness of solutions to System (3) implies that $\phi_1 \equiv \frac{b}{d}$ on $[-\tau, 0]$.

In order to complete the proof we show that the solution to System (3) with the initial condition $\phi \in \chi$ is bounded in Γ. From the definition of Γ we have $S(t) \in \left[0, \frac{b}{d}\right]$. The boundedness of $I(t)$ for $\mathcal{R}_0 < 1$ follows from Lemma 2.3. ∎

Note, that stability of the disease-free stationary state in the case $\mathcal{R}_0 = 1$ is not decided. Although we suspect that the the disease-free stationary state is stable in this case, we cannot prove that. In the case of local stability, the linearization theorem does not work in the case $\mathcal{R}_0 = 1$. In the case of global stability, for any solution such that $S(t) \equiv 0$, $I(t) \geq 0$ we have $\dot{L}(x_t) = 0$ and thus chosen Lyapunov functional does not allow us to deduce that disease-free stationary state is attractive.

THEOREM 3.3 The endemic stationary state P_* is globally asymptotically stable in the set

$$\chi_* := \chi \setminus \left\{\phi = (\phi_1, \phi_2)^T \in \mathcal{C} \colon \phi_2(t) = 0 \text{ for } t \in [-\tau, 0]\right\} \tag{28}$$

if $\mathcal{R}_0 > 1$.

PROOF First, we construct a Lyapunov's functional. We use the same functional as in [8]. Moreover, the proof that the derivative of L along trajectories of System (3) is negative is exactly the same as in [8]. Nevertheless, we need to prove the boundedness of the solution of System (3) in order to use Theorem 3.2 from [4, Chapter 5]. Let us define a function $\varphi \colon (0, \infty) \to \mathbb{R}$ as

$$\varphi(y) = y - 1 - \ln(y). \tag{29}$$

The function φ is smooth,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}y}(y) = 1 - \frac{1}{y}, \tag{30}$$

$\varphi(1) = 0$ and $\varphi(y) > 0$ for $y \neq 1$.

Let us construct $L\colon \mathcal{C} \to \mathbb{R}$. For any $\phi \in \mathcal{C}$ we define

$$\begin{aligned} L(\phi) &= L_1(\phi) + L_2(\phi) + L_3(\phi), \quad \text{where} \\ L_1(\phi) &= S_*\varphi\left(\frac{\phi_1(-\tau)}{S_*}\right), \\ L_2(\phi) &= \frac{\mathrm{e}^{(d+\delta)\tau}}{\gamma} I_*\varphi\left(\frac{\phi_2(0)}{I_*}\right), \\ L_3(\phi) &= \beta S_* I_* \int_{t-\tau}^{t} \varphi\left(\frac{\phi_2(s)}{I_*}\mathrm{d}s\right). \end{aligned} \tag{31}$$

Similarly to the proof of Theorem (3.2), let $x = [S, I]^T$ be the solution of (3). Aiming to verify if L is a Lyapunov's functional, we calculate $\dot{L}(x_t)$. Because calculation is analogous to this in [8], here we present only the final result. A series of algebraic transformations and the use of formula for the endemic stationary state lead us to the following equality:

$$\begin{aligned} \dot{L}(x_t) = {} & dS_*\left(2 - \frac{S(t-\tau)}{S_*} - \frac{S_*}{S(t-\tau)}\right) + \\ & + \beta S_* I_*\left(1 - \frac{S_*}{S(t-\tau)} + \ln\frac{S_*}{S(t-\tau)}\right) + \\ & + \beta S_* I_*\left(1 - \frac{S(t-\tau)I(t-\tau)}{I(t)S_*} + \ln\frac{S(t-\tau)I(t-\tau)}{I(t)S_*}\right). \end{aligned} \tag{32}$$

Due to properties of the logarithm and the fact that the function $y + \frac{1}{y}$ has its minimum equal to 2 at $y = 1$, each factor in separate parenthesis in (32) is non-positive. Thus, $\dot{L}(x_t) \leq 0$ and we conclude that L is a Lyapunov's functional.

Now, we need to find two subsets: $\Sigma = \{\phi \in \mathcal{C}\colon \dot{L}(\phi) = 0\} \subset \chi_*$ and $G \subset \Sigma$ – the largest subset of Σ invariant with respect to System (3). Let $\Sigma = \left\{\phi \in \chi_*\colon \dot{L}(\phi) = 0\right\}$. The following set of equations is equivalent to $\dot{L}(\phi) = 0$ and follows from the properties of logarithm:

$$\begin{aligned} 2 - \frac{\phi_1(-\tau)}{S_*} - \frac{S_*}{\phi_1(-\tau)} &= 0, \\ 1 - \frac{S_*}{\phi_1(-\tau)} + \ln\frac{S_*}{\phi_1(-\tau)} &= 0, \\ 1 - \frac{\phi_1(-\tau)\phi_2(-\tau)}{\phi_2(0)S_*} + \ln\frac{\phi_1(-\tau)\phi_2(-\tau)}{\phi_2(0)S_*} &= 0. \end{aligned} \tag{33}$$

The first and the second equation of (33) hold if and only if

$$\phi_1(-\tau) = S_*, \tag{34}$$

whereas the third equation of (33) holds if and only if

$$\phi_1(-\tau)\phi_2(-\tau) = \phi_2(0)S_*. \tag{35}$$

Because of (34), Equation (35) reduces to $\phi_2(-\tau) = \phi_2(0)$, and the set Σ can be expressed as

$$\Sigma = \left\{\phi = (\phi_1, \phi_2)^T \in \chi_* \colon \phi_1(-\tau) = S_*, \phi_2(-\tau) = \phi_2(0)\right\}. \tag{36}$$

Next, we compute the set G — the subset of Σ invariant with respect to System (3). The following implications result directly from the definition of G:

$$\phi = [\phi_1, \phi_2]^T \in G \implies \phi_1 \equiv S_*, \tag{37}$$

$$\phi = [\phi_1, \phi_2]^T \in G \implies \phi_2 \equiv I_*. \tag{38}$$

Indeed, if the initial condition $\phi \in G$, then the solution $x = (S, I)^T$ to System (3) remains in G for $t \geq 0$, that is $x_t \in G$. This, based on Equation (34), means that for all $t > 0$ we have $S(t - \tau) = S_*$ which implies $S(t) \equiv S_*$. The constraint $\phi(-\tau) = \phi(0)$ for Σ in (36) implies that for all $t \geq 0$ we have $I(t - \tau) = I(t)$. Moreover, since $S(t) \equiv S_*$, we have

$$0 = \dot{S} = b - \beta S_* I(t - \tau) - dS_* \Longrightarrow I(t - \tau) = \frac{b - dS_*}{\beta S_*} = I_*$$

for all $t \geq 0$. In consequence $I(t) \equiv I_*$, and

$$G = \left\{\phi = [\phi_1, \phi_2]^T \in \chi_* \colon \phi_1 \equiv S_*, \phi_2 \equiv I_*\right\}.$$

As we do not know if x is bounded, to complete the proof we need to examine $V(x_t)$ along the trajectories of (3) in χ_*. Below, we show that for any $\ell > 0$ there exits a constant $K = K(\ell)$ such that $|\phi(0)| < K(\ell)$ for $\phi \in \chi_*$ and $L(\phi) < \ell$. This statement is actually more complicated than it seems to be. Note that L_1 depends only on $\phi_1(-\tau)$, L_2 depends only on $\phi_2(0)$ and L_3 depends only on the integral of ϕ_2, so there is no direct dependence on $|\phi(0)|$.

Let us assume that $L(\phi) < \ell$. Nonnegativity of L_1, L_2, L_3 implies that $L_1, L_2, L_3 < \ell$. From the inequality $L_2 < \ell$ we obtain

$$L_2(\phi) = \frac{\mathrm{e}^{(d+\delta)\tau}}{\gamma} I_* \varphi\left(\frac{\phi_2(0)}{I_*}\right) < \ell \implies \varphi\left(\frac{\phi_2(0)}{I_*}\right) < \frac{\gamma\ell}{\mathrm{e}^{(d+\delta)\tau} I_*}. \tag{39}$$

By the definition of χ_*, we have $|\phi_1(0)| \leq \frac{b}{d}$. Now, let us focus on proving the boundedness of $|\phi_2(0)|$. Note that φ has the following property

$$\varphi(y) = y - 1 - \ln(y) > \frac{y}{2} \text{ for } y \geq 6. \tag{40}$$

Let us consider two cases: $\phi_2(0) < 6 \cdot I_*$ or $\phi_2(0) \geq 6 \cdot I_*$. In the first case, $\phi_2(0)$ is bounded. In the second case, Inequality (40) implies

$$\frac{\phi_2(0)}{2I_*} < \varphi\left(\frac{\phi_2(0)}{I_*}\right) < \frac{\gamma\ell}{\mathrm{e}^{(d+\delta)\tau}\, I_*} \implies \phi_2(0) < \frac{\gamma\ell}{2\,\mathrm{e}^{(d+\delta)\tau}}. \tag{41}$$

Hence, $|\phi_2(0)|$ is bounded in both cases. It implies the existence of the constant $K(\ell)$ such that $|\phi(0)| < K(\ell)$.

Thus, the assertion of the theorem follows from Theorem 3.2 from [4, Chapter 5]. ∎

4. Discussion. The main goal of this paper was to improve and correct lapses in the analysis of the SEIR model of COVID-19 spread introduced in [8] by S. Çakan.

First of all, solutions to System (1) may have negative values for the non-negative initial condition, which was presented in Examples 2.1 and 2.2. In consequence, solutions to System (3), considered throughout [8], are not necessarily bounded. Therefore, the bounded invariant set for System (3) does not necessarily exist. Thus, to prove global stability of stationary states, we used the same Lyapunov's functionals as in [8], but we made sure that following conditions were met. In the case of the disease–free stationary state we proved that solutions are bounded, while for the endemic stationary state we proved boundedness of Liapunouv's functional along trajectories.

Moreover, in [8] local stability of stationary states was analysed without taking into account the time delay. We corrected this issue.

Additionally, the function c describing healthcare capacity introduced in [8] depends on time. However, the author analysed the system as if c was a constant, while for the purpose of numerical simulation it was assumed that the function c depends on the size of the group I.

In this paper we considered the case of c being a constant parameter and corrected the analysis of the system based on theory of delay differential equations. The interesting case when c depends on the number of infectious individuals will be considered elsewhere.

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References

[1] H. Adekola, I. Adekunle, H. Egberongbe, S. Onitilo, and I. Abdullahi. Mathematical modeling for infectious viral disease: The COVID-19 perspective. *Journal of Public Affairs*, 20, 2020. doi: 10.1002/pa.2306. PMC7461001; PMID: 32904838. Cited on p. 23.

[2] F. Brauer. Absolute stability in delay equations. *Journal of Differential Equations*, 69(2):185–191, Sept. 1987. ISSN 0022-0396. doi: 10.1016/0022-0396(87)90116-1. Cited on p. 34.

[3] O. Diekmann, J. Heesterbeek, and M. Roberts. The construction of next-generation matrices for compartmental epidemic models. *J. R. Soc. Interface*, 7:873–885, 2010. doi: 10.1098/rsif.2009.0386. Cited on p. 31.

[4] J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional-differential equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993. ISBN 0-387-94076-6. doi: 10.1007/978-1-4612-4342-7. MR 1243878. Cited on pp. 26, 35, 36, and 39.

[5] M. Kochańczyk, F. Grabowski, and T.Lipniacki. Dynamics of COVID-19 pandemic at constant and time-dependent contact rates. *Mathematical Modelling of Natural Phenomena*, 15:28, 2020. doi: 10.1051/mmnp/2020011. Cited on p. 23.

[6] H. Smith. *An Introduction to Delay Differential Equations with Applications to the Life Sciences*. Springer New York, 2011. ISBN http://id.crossref.org/isbn/978-1-4419-7646-8. doi: 10.1007/978-1-4419-7646-8. Cited on pp. 28, 33, and 34.

[7] P. van den Driessche and J. Watmough. Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180(1):29–48, 2002. ISSN 0025-5564. doi: https://doi.org/10.1016/S0025-5564(02)00108-6. PMID: 12387915. Cited on p. 31.

[8] S. Çakan. Dynamic analysis of a mathematical model with health care capacity for COVID-19 pandemic. *Chaos, Solitons & Fractals*, 139:110033, 2020. ISSN 0960-0779. doi: https://doi.org/10.1016/j.chaos.2020.110033. PMC7305938; PMID: 32834594. Cited on pp. 23, 24, 25, 26, 31, 33, 35, 36, 37, and 39.

Kilka uwag o modelu pandemii COVID-19 z uwzględnieniem wydolności służby zdrowia

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Streszczenie W artykule rozważono model epidemiologiczny typu SEIR opisujący dynamikę rozprzestrzeniania się pandemii COVID-19 z uwzględnieniem wydolności służby zdrowia, zaproponowany przez S. Çakan w "Dynamic analysis of mathematical model with health care capacity for COVID-19 pandemic" (2020). Model jest opisany za pomocą układu równań różniczkowych z opóźnieniem. Nieznacznie zmodyfikowaliśmy układ zaproponowany przez S. Çakan i przeprowadziliśmy jego pogłębioną analizę. Dowody lokalnej stabilności przedstawione w oryginalnym artykule były oparte na linearyzacji, jednak pomijały zależność układu równań od opóźnienia. Dodatkowo wykazaliśmy, że kluczowy argument używany przez S. Çakan w dowodzie ograniczoności rozwiązań jest niepoprawny, co oznacza, że przedstawione przez nią dowody globalnej stabilności stanów stacjonarnych również nie są poprawne. Z tego powodu przedstawiamy tutaj poprawione dowody lokalnej i globalnej stabilności stanów stacjonarnych. W dowodzie lokalnej stabilności skorzystaliśmy z twierdzenia o linearyzacji dla układów równań różniczkowych z opóźnieniem, natomiast globalną stabilność wykazaliśmy korzystając z funkcjonałów Lapunowa. Przyjęliśmy funkcjonały Lapunowa zaproponowane przez S. Çakan, jednak po wcześniejszym udowodnieniu dodatkowych własności rozwiązań (dla stanu stacjonarnego wolnego od infekcji) oraz funkcjonału (w przypadku endemicznego stanu stacjonarnego). Dzięki temu uzyskaliśmy poprawne dowody globalnej i lokalnej stabilności stanów stacjonarnych.

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