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A hybrid solution approach to the Korteweg-de Vries and Burgers' equations

Abstract The primary purpose of this paper is to analyze the application of a new integral transform together with a homotopy perturbation method to construct approximate solutions of the initial-value problem for Korteweg-de Vries and Burgers' equations. The new integral transform homotopy perturbation method (NIHPTM) compared to other methods, offers the simple technique to handle such type partial differential equations. The 5th-order approximation results obtained in illustrative examples compared with the explicit solutions of the considered problems show the proposed approach's efficiency and validity.

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1. Introduction. The non-linear differential equations, namely the Burgers' equation and the Korteweg-de Vries (KdV) equation are the ones of the most important equations in the branch of fluid mechanics. The Burgers' equation, originally proposed as a simplified model of turbulence occurs in many problems connected with flows in viscous media as well as in description of groundwater phenomena. The KdV equation arises in modelling such phenomena as shallow water waves, ion acoustic waves in plasma, long internal waves in a density-stratified ocean, and acoustic waves on a crystal lattice. It is a challenging task to solve them through the analytical and numerical approach.

The Burgers' equation is defined as (cf.[10])

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\,\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad x \in \mathbb{R},\ t \in [0,\infty). \tag{1}$$

Eq. (1) admits the exact solution given by [20],

$$u(x,t) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}\left(x - \frac{t}{2}\right). \tag{2}$$

The Korteweg-de Vries equation is defined as being [2, 10],

$$\frac{\partial u(x,t)}{\partial t} - 6\,u(x,t)\,\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \quad x \in \mathbb{R},\ t \in [0,\infty) \tag{3}$$

and admits the exact solution given as [20],

$$u(x,t) = -\frac{k^2}{2}\text{sech}^2\frac{k}{2}\left(x - k^2 t\right), \tag{4}$$

which is equivalently written as [20],

$$u(x,t) = -2k^2\frac{e^{k\left(x-k^2t\right)}}{\left(1+e^{k(x-k^2t)}\right)^2}. \tag{5}$$

Several authors reflect various theoretical and computational approaches to solving Burgers' equation and the equation of Korteweq-de Vries. For info, please refer to [1, 2, 9, 10, 11, 15, 18, 20]. The NIHPTM approach[5, 13, 14, 17] is a combined approach for solving many types of diiferential problems. In 2013 Artion Kashuri and Akli Fundo[7] introduced the proposed methodology. Recently, the proposed approach have been successfully employed to solve engineering and science problems, such as Burgers' equation resulting from longitudinal dispersion phenomena[13, 19], discontinued nanotechnology problem[14], and sixth-order Cahn-Hillard time-fractional equation[12].
In this paper, we have demonstrate the NIHPTM technique for finding an approximate analytical solution of the equation of Korteweg-de Vries (KdV) and Burgers'. Finally, with the help of appropriate initial condition of Burgers' and KdV equation, solution obtained by the NIHPTM approach compared with the exact and HAM solution as described in [10].

2. Solution using NIHPTM In the present section we first address the few important principles of the new integral transform and then we apply the proposed method to the given problem.
A new integral transform is defined over the set of functions

$$S = \left\{ u(t) \mid |u(t)| \leq M\ e^{\dfrac{t}{k_j^2}},\ t \in [0,\infty),\ M, k_1, k_2 > 0 \right\}$$

where, k_1, k_2 and M are finite, as following [8, 16]:

$$K[u(t)] = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v^2}}u(t)dt = A(v),\ v \in (k_1, k_2),\ k_1, k_2 > 0.$$

The inverse new integral transform is given by

$$K^{-1}[A(v)] = u(t) \text{ for } t \geq 0 \tag{6}$$

Definition 2.1 A function $f(t)$ is said to be of exponential order $\frac{1}{k^2}$, where k is a positive real number, if there exist positive constants N and M such that, $|f(t)| \leq Me^{\frac{t}{k^2}}$, for all $t \geq N$.

Theorem 2.2 *If $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order $\frac{1}{k^2}$ then $A(v)$ exists for $|v|<k$.* [7]

The above theorem state the sufficient condition for existance of a new integral transform.

Theorem 2.3 (Uniqueness Theorem [7]) *Suppose f and g are piecewise continuous on $[0,\infty)$ and exponential type $\frac{1}{k^2}$ and if $F(v)$ and $G(v)$ are a new integral transforms of $f(t)$ and $g(t)$ respectively then*

$$K[f(t)] = K[g(t)] \Rightarrow \ f(t) = g(t)$$

Theorem 2.4 [8, 16] *Let $\frac{\partial^i f(x,t)}{\partial t^i}$, $i=0,1,\dots,n$ be continuous with respect to t on $[0,\infty)$ and of exponential order $\frac{1}{k^2}$. Let $A(x,v)$ be a new integral transform of $f(x,t)$, then*

$$K\left[\frac{\partial^n f(x,t)}{\partial t^n}\right] = \frac{A(x,v)}{v^{2n}} - \sum_{i=0}^{n-1} \frac{1}{v^{2(n-i)-1}} \frac{\partial^i}{\partial t^i} f(x,0), \ \textit{for } n \geq 1.$$

To illustrate the main idea of NIHPTM, we consider:

$$\mathcal{D}u + \mathcal{R}u + \mathcal{N}u = q(x,t), \quad x \in \mathbb{R},\, t \in [0,\infty) \tag{7}$$

with initial conditions

$$u(x,0) = m(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}\,, \tag{8}$$

where $\mathcal{D} = \frac{\partial^2}{\partial t^2}$ represents second order linear differential operator, $\mathcal{R}$ is the linear differential operator with respect to x, and $\mathcal{N}$ represents the general nonlinear differential operator with respect to variable x. Taking a new integral transform on Eq. (7), we get

$$K\left[\mathcal{D}u\right] + K\left[\mathcal{R}u\right] + K\left[\mathcal{N}u\right] = K\left[q(x,t)\right]. \tag{9}$$

Using properties of the new integral transform and the given conditions (8) we get

$$K\left[u\right] = v^4 K\left[q(x,t)\right] + v\left[m(x) + v^2 g(x)\right] - v^4 K\left[\mathcal{R}u + \mathcal{N}u\right]. \tag{10}$$

Now, employing the inverse new integral transform on Eq. (10), we get

$$u(x,t) = H(x,t) - K^{-1}\{v^4 K\left[\mathcal{R}u + \mathcal{N}u\right]\}, \tag{11}$$

where $H(x,t)$ represents combined source term and the initial conditions prescribed. According to the homotopy perturbation method [5] we consider the family of perturbed equations of the form

$$u(x,t) = H(x,t) - pK^{-1}\{v^4 K\left[\mathcal{R}u + \mathcal{N}u\right]\}, \tag{12}$$

where $p \in [0,1]$ represents the imbedding parameter [4, 6]. Now, we employ the homotopy perturbation method [5] and put

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \tag{13}$$

Then the nonlinear term can be decomposed as

$$\mathcal{N}u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u_0, u_1, \ldots, u_n), \tag{14}$$

where $H_n(u_0, u_1, \ldots, u_n)$ is the He's polynomial (see [3, 5]) given by

$$H_n\left(u_0, u_1, \cdots, u_n\right) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left\{\mathcal{N}\left[\sum_{i=0}^{\infty} p^i u_i\right]\right\}_{p=0}, n \geq 0. \tag{15}$$

Substituting Eq.(13) and Eq.(14) into Eq. (11), we have

$$\sum_{n=0}^{\infty} p^n u_n(x,t) =$$
$$= H(x,t) - p\left\{K^{-1}\left[v^4 K\left[\mathcal{R}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + \sum_{n=0}^{\infty} p^n H_n(u_0, u_1, \ldots, u_n)\right]\right]\right\}. \tag{16}$$

The following approximations are obtained when comparing the coefficient of like powers of p.

$$\begin{aligned}
p^0 &: u_0(x,t) = H(x,t),\\
p^1 &: u_1(x,t) = -K^{-1}\{v^4 K\left[\mathcal{R}u_0(x,t) + H_0(u_0)\right]\},\\
p^2 &: u_2(x,t) = -K^{-1}\{v^4 K\left[\mathcal{R}u_1(x,t) + H_1(u_0, u_1)\right]\},\\
&\ldots
\end{aligned}$$

Thus in the series form, we have:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \tag{17}$$

Eq. (17) gives the approximate NIHPTM solution.

2.1. The Burgers' Equation We solve Burgers' equation (1) together with an initial condition $u_0 = u(x,0) = \dfrac{1}{2} - \dfrac{1}{2}\tanh\dfrac{1}{4}(x)$ using NIHPTM as follows.
We apply new integral transform on Eq. (1), with $\mathcal{D}[u(x,t)] = \dfrac{\partial}{\partial t}u(x,t)$, $\mathcal{R}[u(x,t)] = \dfrac{\partial^2}{\partial x^2}u(x,t)$ and $\mathcal{N}[u(x,t)] = u(x,t)\dfrac{\partial}{\partial x}u(x,t)$ and get

$$K\left[u\right] = v\left(\frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x)\right) + v^2 K\left[u_{xx} - uu_x\right]. \tag{18}$$

Taking inverse transform on Eq. (18), we get

$$u(x,t) = \left(\frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x)\right) + K^{-1}\left\{v^2 K\left[u_{xx} - uu_x\right]\right\}. \tag{19}$$

Using homotopy perturbation method to Eq. (19) and proceeding as described above we obtain

$$\sum_{n=0}^{\infty} p^n u_n(x,t) == \left(\frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x)\right)$$
$$+ pK^{-1}\left\{v^2 K\left[\sum_{n=0}^{\infty} p^n u_{nxx}(x,t) - \sum_{n=0}^{\infty} p^n H_n(u_0(x,t),\dots,u_n(x,t))\right]\right\}, \tag{20}$$

where

$$H_n(u_0,u_1,\cdots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[\left(\sum_{i=0}^{\infty} p^i u_i\right)\left(\sum_{i=0}^{\infty} p^i \frac{\partial}{\partial x}u_i\right)\right]_{p=0}.$$

We represent some of the terms of $H_n(u_0,\dots,u_n)$ as follows:

$$\begin{aligned}
H_0(u_0(x,t)) &= u_0(x,t)u_{0x}(x,t),\\
H_1(u_0(x,t),u_1(x,t)) &= u_0(x,t)u_{1x}(x,t) + u_1(x,t)u_{0x}(x,t),\\
H_2(u_0(x,t),u_1(x,t),u_2(x,t)) &= u_0(x,t)u_{2x}(x,t) + u_1(x,t)u_{1x}(x,t)\\
&\quad + u_2(x,t)u_{0x}(x,t),\\
H_3(u_0(x,t),u_1(x,t),u_2(x,t),u_3(x,t)) &= u_0(x,t)u_{3x}(x,t) + u_1(x,t)u_{2x}(x,t)\\
+u_2(x,t)u_{1x}(x,t) + u_3(x,t)u_{0x}(x,t),&\\
&\cdots
\end{aligned}$$

Comparing the coefficient of same power of p in Eq. (20), we get

$$\begin{aligned}
p^0 &: u_0 = u(x,0) = \frac{1}{2} - \frac{1}{2}\tanh\frac{1}{4}(x)\\
p^1 &: u_1(x,t) = K^{-1}\left\{v^2 K\left[u_{0xx}(x,t) - H_0(u_0(x,t))\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16} t \,\mathrm{sech}\left[\frac{x}{4}\right]^2 \\
p^2 : u_2(x,t) &= K^{-1}\left\{v^2 K\left[u_{1xx}(x,t) - H_1(u_0(x,t), u_1(x,t))\right]\right\} \\
&= \frac{1}{128} t^2 \,\mathrm{sech}\left[\frac{x}{4}\right]^2 \tanh\left[\frac{x}{4}\right] \\
p^3 : u_3(x,t) &= K^{-1}\left\{v^2 K\left[u_{2xx}(x,t) - H_2(u_0(x,t), u_1(x,t), u_2(x,t))\right]\right\} \\
&= \frac{1}{6144} t^3 \left(-2 + \cosh\left[\frac{x}{2}\right]\right) \mathrm{sech}\left[\frac{x}{4}\right]^4 \\
&\dots
\end{aligned}
$$

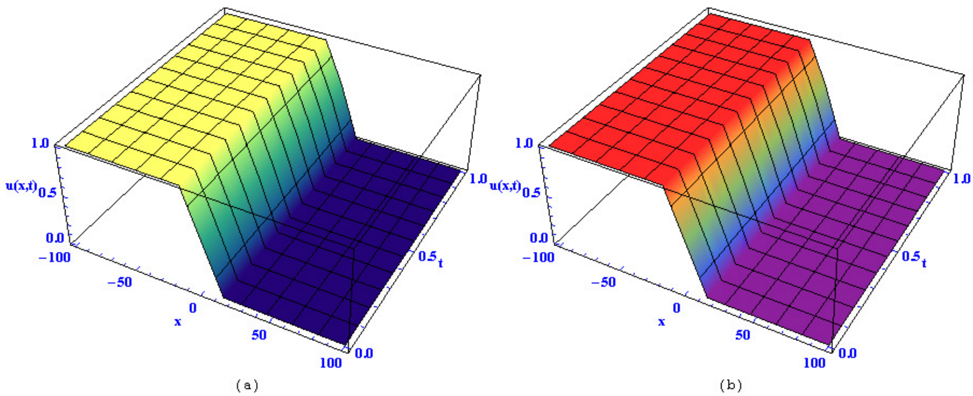

Figure 1: Comparison between exact solution and approximate solution of Burgers' equation (1)

2.2. The Korteweg-de Vries (KdV) equation Applying same procedure as in Subsection 2.1 to Eq. (3) with initial condition $u_0 = u(x,0) = -2\dfrac{e^x}{(1+e^x)^2}$ and with $\mathcal{D}[u(x,t)] = \dfrac{\partial}{\partial t}u(x,t)$, $\mathcal{R}[u(x,t)] = \dfrac{\partial^3}{\partial x^3}u(x,t)$ and $\mathcal{N}[u(x,t)] = 6u(x,t)\dfrac{\partial}{\partial x}u(x,t)$, we have

$$
\begin{aligned}
u(x,t) = \frac{1}{17280\,(1+e^x)^{11}} e^x \big(&-34560\,(1+e^x)^9 - 34560\,(-1+e^x)\,(1+e^x)^8\, t \\
&- 17280\,(1+e^x)^7\left(1 - 4e^x + e^{2x}\right) t^2 \\
&- 2880\,(1+e^x)^6\left(-1 + 11e^x - 11e^{2x} + e^{3x}\right) t^3 \\
&- 120\,(1+e^x)^3\left(1 + 48e^x - 681e^{2x} + 1256e^{3x} - 441e^{4x} + 24e^{5x} + e^{6x}\right) t^4 \\
&- \big(-1 - 1795e^x + 44752e^{2x} - 249736e^{3x} + 487318e^{4x} - 444694e^{5x} \\
&+ 202840e^{6x} - 36064e^{7x} + 1603e^{8x} + e^{9x}\big) t^5\big) + \cdots
\end{aligned}
$$

3. Result and discussion. Approximate solution of the equation of Burgers' and the equation of Korteweg-de Vries has been successfully obtained

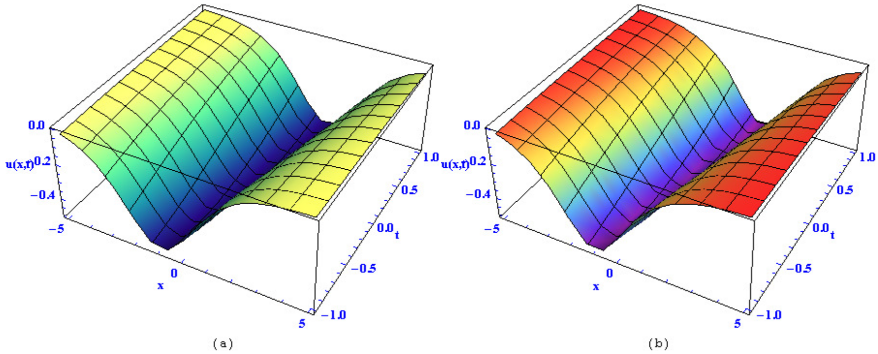

Figure 2: Comparison between exact solution and approximate solution of KdV equation (3)

through a hybrid approach, namely NIHPTM. The comparison between the method proposed and the method of homotopy analysis with the exact solution of the equation of Burgers and the equation of KdV has been shown in Table 1 & 2 as a tabular value. Figs 2.1 & 2.2 shows its graphical relation with the same solution. From this analysis, we can see that, with the exact solution, the result observed by the proposed approach is very similar. For the solution of a strongly non-linear partial differential equation, the proposed method is straightforward to handle and trustworthy. An essential advantage of the proposed method over the process of analyzing homotopy is that we don't need a convergence control parameter or a suitable convergence area to match the exact solution of established problems best.

4. Conclusions. In this paper, we successfully employed the NIHPTM to obtain the approximate solution of the initial-value problem for Burgers' and Korteweg-de Vries equations. The result obtained through the proposed process is quite satisfactory and matches the exact solution of the equations discussed. Thus, this analysis shows how reliable the proposed method is. The proposed method can be applied in the future in engineering and science for the largest possible nonlinear problems.

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Hybrydowe podejście do rozwiązania równań Kortewega-de Vriesa i Burgersa.

Kunjan Shah i Himanshu Patel

Streszczenie Głównym celem tego artykułu jest analiza zastosowania nowej transformacji całkowej i metody homotopijnej perturbacji do konstrukcji przybliżonych rozwiązań zagadnienia początkowego dla równań Kortewega-de Vriesa i Burgersa. Nowa metoda homotopijnej perturbacji z transformacją całkową (NIHPTM) w porównaniu z innymi metodami oferuje prostą technikę do zastosowania w tego typu równaniach różniczkowych cząstkowych. Uzyskane aproksymacje piątego rzędu dla przykładów ilustracyjnych porównane z istniejącymi jawnymi rozwiązaniami rozważanych zagadnień pokazują skuteczność i trafność proponowanego podejścia.

Klasyfikacja tematyczna AMS (2010): 62J05; 92D20.

Słowa kluczowe: Nowa transformata całkowa, metoda zaburzeń homotopii, równanie KdV, równanie Burgersa..

A. Numerical comparison of exact and approximate solutions. The appendix contains a tabular value of the comparison between the method proposed and the method of homotopy analysis with the exact solution of the equation of Burgers and the equation of KdV has been shown in Table 1 & 2, respectively.

t	x	Exact Solution	HAM Solution	NIHPTM Solution	Absolute Error
0	$-\infty$	1.	1.	1.	0.
	⋮	⋮	⋮	⋮	⋮
	-100	1.	1.	1.	0.
	-50	0.99999999998611	0.9999999999	0.999999999986112	0.
	-20	0.99995460229829	0.9999546021	0.9999546021312975	0.
	0	0.5	0.5	0.5	0
	20	0.00004539770170	0.0000453978	0.0000453979	0.
	50	0.	0.	0.	0.
	100	0.	0.	0.	0.
	⋮	⋮	⋮	⋮	⋮
	∞	0.	0.	0.	0.
0.25	$-\infty$	1.	1.	1.	0.
	⋮	⋮	⋮	⋮	⋮
	-100	1.	1.	1.	0.
	-50	0.9999999999999	0.99999999998	0.9999999999869532	0.
	-20	0.9999573525318144	0.99995735253	0.999957352	0.
	0	0.5078118671525	0.51561991572	0.51562245026832	0.00781058
	20	0.0000468387124	0.00004832563	0.0000483247	0.
	50	0.	0.	0.	0.
	100	0.	0.	0.	0.
	⋮	⋮	⋮	⋮	⋮
	∞	0.	0.	0.	0.
0.5	$-\infty$	1.	1.	1.	0.
	⋮	⋮	⋮	⋮	⋮
	-100	1.	1.	1.	0.
	-50	0.999999999986	0.99999999998	0.99999999999	0.
	-20	0.999957352689	0.99995993630	0.999959929333	0.
	0	0.515619921465	0.53120937337	0.5312295497	0.0156096
	20	0.000048325461	0.00005144221	0.0000514344	0.
	50	0.	0.	0.	0.
	100	0.	0.	0.	0.
	⋮	⋮	⋮	⋮	⋮
	∞	0.	0.	0.	0.
0.75	$-\infty$	1.	1.	1.	0.
	⋮	⋮	⋮	⋮	⋮
	-100	1.	1.	1.	0.
	-50	0.999999999987	0.99999999998	0.99999999998848	0.
	-20	0.99995866475	0.99996236355	0.99996234068	0.
	0	0.523420357539	0.54673815197	0.54680580457	0.0233854
	20	0.000049859400	0.00005475976	0.00005473261	0.
	50	0.	1.67520×10^{-11}	1.67438×10^{-11}	1.49104×10^{-12}
	100	5.551115×10^{-17}	3.97766×10^{-23}	0.	5.55112×10^{-17}
	200	0.	7.67192×10^{-45}	0.	0.
	300	0.	1.47972×10^{-66}	0.	0.
1.0	$-\infty$	1.	1.	1.	0.
	⋮	⋮	⋮	⋮	⋮
	-100	1.	1.	1.	0.
	-50	0.999999999987744	0.9999999999	0.999999999989168	1.42397×10^{-12}
	-20	0.99995993646	0.9999646437	0.99996459107	4.65461×10^{-6}
	0	0.531209385	0.5621765008	0.5623355677	0.0311262
	20	0.0000514420269	0.0000582912	0.0000582250201	6.78299×10^{-6}
	50	0.	1.7832×10^{-11}	1.78122×10^{-11}	2.07517×10^{-12}
	100	0.	5.4781×10^{-23}	0.	0.
	200	0.	1.0565×10^{-44}	0.	0.
	300	0.	2.0379×10^{-66}	0.	0.

Tablica 1: On comparison of the NIHPTM solution with exact and HAM solutions of Eq. (1)

t	x	Exact Solution	HAM Solution	NIHPTM Solution	Absolute Error
0.05	-6	-0.004693564364865844	-0.004693564364865363 5	-0.004693612920648676	$4.8555782831971595 \times 10^{-8}$
	-3	-0.08634570715015544	-0.08634570715015137	-0.08634610232686854	3.95177×10^{-7}
	2	-0.21807963830331248	-0.21807963830330718	-0.2180801150334363	4.7673×10^{-7}
	5	-0.01396823161345761	-0.013968231613456493	-0.013968103373692229	1.2824×10^{-7}
0.25	-6	-0.0038460447136713 89	-0.0038460447137292153	-0.003851608667938083	5.56395×10^{-6}
	-3	-0.07186718166243183	-0.07186718165065627	-0.07191865077980115	0.0000514691
	2	-0.2522584503864486	-0.25225845037331035	-0.25234114233420907	0.000082692
	5	-0.017007824315900595	-0.017007824315379085	-0.016990496743281683	0.0000173276
0.5	-6	-0.0029978573890808136	-0.002997857417138099	-0.0030377441194037466	0.0000398867
	-3	-0.056906053378639285	-0.05690604775947111	-0.05733121106066425	0.000425163
	2	-0.29829291297695726	-0.2982929041406657	-0.29920535615153804	0.000912452
	5	-0.02173245972250237	-0.02173245944445047	-0.021579929259085594	0.00015253
0.75	-6	-0.002336284005515009	-0.0023362850215908897	-0.002456754077164696	0.000120469
	-3	-0.044899022133116494	-0.04489882076408693	-0.04635771318097057	0.00145889
	2	-0.3462100376919465	-0.3462095739849941	-0.3501901790458882	0.00398061
	5	-0.027731694033315973	-0.02773168287526794	-0.027166401784998746	0.000565281
1	-6	-0.0018204295917628886	-0.001820442360243653	-0.0020755704601597264	0.000255128
	-3	-0.03532791434845481	-0.03532541242658223	-0.038802636480708357	0.00347722
	2	-0.3932319186044759	-0.3932238664829637	-0.40484378655568054	0.0116199
	5	-0.03532556752567461	-0.03532541242658223	-0.033857570997076446	0.00146784

Tablica 2: On comparison of the NIHPTM solution with exact and HAM solutions of Eq. (3)

Kunjan Shah worked as an Assistant Professor in the Department of Mathematics, Centre of Education at Indian Institute of Teacher Education (IITE), Gandhinagar, Gujarat, India. He received his Ph.D. in Applied Mathematics from S.V National Institute of Technology, Surat, India under the guidance of Dr. T. R. Singh. Dr. Shah's research interests mainly include Fluid Flow Through Porous Media, Integral Transform, Homotopy Perturbation Method, Modified Homotopy Analysis Method, and Partial Differential Equation. He is awarded by the best paper award in "International Scientific Conference on Applied Sciences and Engineering", Kuala Lumpur, Malaysia December 20-21, 2014 and also chaired a session in ISCASE, Kuala Lumpur, Malaysia. References to his research papers are listed in Dr. Kunjan Shah, in Scopus under authorId=57155756900.

Himanshu Patel is a Registrar at Indian Institute of Teacher Education, Gandhinagar, Gujarat India. Dr. Patel have worked on numerical modelling of some problems related to lubrication. The study of the effect of surface roughness incorporated in my PhD work suggests to carry out the effect of roughness at the nano scale on the performance of various types of bearings (particularly, circular plates, circular step, annular plates etc.). It will be interesting to analyse the effect of longitudinal surface roughness on the bearing systems encountered in my work. It will be highly appealing to investigate the effect of longitudinal roughness on the performance of the polymer coated bearings working with magnetic fluid lubricant. The analyses involved in the modelling of roughness and the related information indicate that the above mentioned problems may be considered in almost all forms of the bearings. This field of investigation is an area in which research frontiers may be developed in various directions. I am interested to meet the people who may have same interest. References to his research papers in Indian Institute of Teacher Education, Gandhinagar Vidwan-ID : 210258

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