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Derivation of equations of motion and supplementary natural boundary conditions for a slender single column subjected to Euler's load using Hamilton's principle

Krzysztof Kuliński[1]

ABSTRACT:
In this paper the derivation process of motion equations and boundary conditions for a slender mechanical system on the basis of Hamilton's principle is presented. In order to apply the Hamilton's principle, first of all it is necessary to define appropriate general variables that describe the motion of the considered system. In the case of slender mechanical systems, natural coordinates are usually used, which are well suited to the system geometry and its motional characteristics. Based on Hamilton's principle, an appropriate action functional is constructed, which is the Lagrangian integral covering the appropriate general variables and time. The Lagrangian describes the well-known difference between the kinetic and potential energy of a system. A step by step derivation of a motion equation and supplementary natural boundary conditions in regard to an example of a slender clamped-free column subjected to Euler's load is discussed. The obtained equation with a set of geometrical and natural boundary conditions gives the possibility to thoroughly analyse both analytically or numerically system dynamics and/or static response. Despite that the discussed method is time consuming and requires advanced mathematical techniques, it makes it possible to obtain exact or approximate motion equations even for complex problems, what can be difficult or even impossible to achieve using other known methods.

KEYWORDS:
Hamilton's principle; slender system; equations of motion derivation; static and dynamic response; Euler's load

1. Introduction

Slender mechanical structures, characterized by their high aspect ratio, are ubiquitous in various fields of engineering. Whether they are tall buildings, long-span bridges, slender towers, or slender machine components, these structures are subjected to a wide range of external and internal forces that can induce instability and generate vibrations. The consequences of instability and excessive vibrations can be catastrophic, leading to structural failures, reduced performance, increased maintenance costs and compromised safety. Consequently, engineers and researchers have devoted considerable efforts to comprehending the complex dynamics of slender structures and have developed innovative solutions to enhance their stability and mitigate vibrations. The derivation of motion equations typically involves applying principles of physics, such as Newton's laws of motion, Lagrange's equations, variational methods, Hamilton's principle, numerical methods (FEM) etc. to analyse and describe the dynamic behaviour of studied system.

Pedersen and Pedersen [1] discussed a step by step derivation of motion equations for a flexible mechanical system using the principle of virtual work, Newton's second law and the law of angular momentum, where inertia is treated as a force. An example of a rigid body with a circular beam

[1] Czestochowa University of Technology, Faculty of Civil Engineering, ul. Akademicka 3, 42-218 Częstochowa, Poland, e-mail: krzysztof.kulinski@pcz.pl, orcid id: 0000-0001-9850-9144

element is investigated, where the coordinate system is located at the centre of mass and away of its centre. Vassiliou and Makris [2] investigated the dynamic response of a vertically restrained rocking column with an elastic tendon passing through its centreline. A variational formulation is employed and the direct approach of a nonlinear motion equation derivation is exhibited. Kounadis [3] demonstrated a derivation of coupled differential equations establishing the governing motion equation for a restrained Timoshenko beam subjected to uniformly distributed compressive follower force using the virtual work and Hamilton's principle. Moreover, it is stated that one gets a different set of equations of motion when a free body diagram with sides not perpendicular to the deflected axis of the beam is considered. The Hamilton's principle has been used in the formulation and derivation of equations describing free vibrations and the stability of a geometrically nonlinear slender column subjected to a follower force directed towards the positive pole in [4]. The authors verified obtained numerical results with experimental results showing good agreement. Sokół [5] derived equations of motion and solved them studying the stability, free vibrations and load capacity of a nonlinear column with a crack simulated by a rotational spring element. Lagrange multiplier formulation has been employed to find the coupled large displacement-small deformation equations of motion in [6]. A thorough derivation of motion equations and boundary conditions using various methods and principles has been widely described in [6-9]. Nowadays, the Finite Element Method (FEM) is frequently employed to formulate and numerically solve high complexity problems giving the opportunity to thoroughly analyse static/dynamic response. Exemplary FEM based analyses for studying dynamic response in various civil engineering structures can be found in [10-12].

In this paper, we delve into the world of slender mechanical structures, exploring the fundamental concepts of deriving equations regarding static and dynamic system response on the basis of Hamilton's principle. Challenges posed by these phenomena, the mathematical methods and techniques used to derive and solve discussed equations are emphasized. Through the study of a simple case regarding a slender clamped-free column subjected to an axial compressive load, the aim is to shed a light on the cutting-edge approach of deriving motion equation giving the possibility of reliable examination of stability and vibration frequency in slender mechanical systems.

2. Structural model

For the exemplary derivation of a motion equation using Hamilton's principle, a slender column with clamped-free boundary conditions was adopted. On the free end an axial compressive force *P* was applied, which was expected to buckle the system out of plane and shortens the column by Δ as shown in Figure 1. Before the load was applied, the column was assumed to be perfectly straight and initially unstressed – any geometrical imperfections and material nonlinearities were neglected. Moreover, it was assumed that the column cross-sectional dimensions are small compared to its length, the cross-section was uniform along the column's length, the column had experienced little in the way of deformations and the material behaved linearly within the elastic range. The described key assumptions classified the problem into Euler-Bernoulli beam theory, where rotational bending effects and shear deformations are completely neglected.

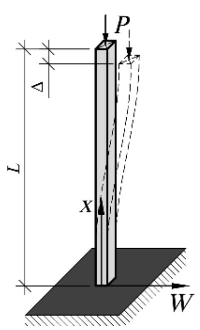

Fig. 1. The adopted clamped-free column subjected to the Euler's load for the derivation of equations of motion

3. Derivation of equations of motion

In this section the derivation of equations of motion and natural boundary conditions for the clamped-free column shown in Figure 1 is presented on the basis of Hamilton's principle, taking into account Lagrangian integration and variational calculus techniques. The Hamilton's principle is the fundamental variational method used in mechanics – it states that from all possible movements being consistent with constraints of an analysed system, only that movement is realized for which the integral is:

$$\delta\int_{t_1}^{t_2}(T-V+L_e)dt=0 \tag{1}$$

where δ stands as variational operator, T and V denotes kinetic and potential energy of the system, respectively; L_e – work of external force on the free column's end.

First of all, one needs to define all the forces involved (kinetic energy, potential energy and the work of external force) in regard to the analysed system. Taking into account that the cross--section of the analysed column is uniform along its length, the kinetic energy can be written as:

$$T=\frac{1}{2}\rho A\int_0^L\left(\frac{\partial W(x,\ t)}{\partial t}\right)^2 \tag{2}$$

where ρ denotes the material density, A is the cross-section area, W is the transversal displacement, L is the column length and t refers to time.

The total potential energy stored in the system due to the forced bending resulting from compressive force P is:

$$V=\frac{1}{2}\int_0^L M(x,t)\cdot\frac{\partial^2W(x,t)}{\partial x^2}dx=\frac{1}{2}EJ\int_0^L\left[\frac{\partial^2W(x,t)}{\partial x^2}\right]^2dx \tag{3}$$

where: $M(x,t)=EJWII(x,t)$, E denotes the column's Young's moduli and J – the cross-section moment of inertia and the Roman numerals defines the order of the derivative with respect to space x.

The work of the external force at the column's free end can be expressed as:

$$L_e=P\cdot\Delta=\frac{1}{2}P\int_0^L\left[\frac{\partial^2W(x,\ t)}{\partial x^2}\right]^2dx \tag{4}$$

where Δ denotes the displacement, being the difference in column length between the undeformed and deformed state.

As a consecutive step, according to Eq. (1) one has to perform the operation of variation and integration by parts of the given forces (2)-(4). By taking advantage of integration commutativity with respect to the time variable t and space coordinate x and the commutativity in calculating variation, the step by step solution to Eq. (2) is:

$$\delta T=\frac{1}{2}\rho A\int_{t_1}^{t_2}\frac{\partial W(x,\ t)}{\partial t}\cdot 2\delta\frac{\partial W(x,\ t)}{\partial t}dt\Rightarrow\rho A\int_{t_1}^{t_2}\frac{\partial W(x,\ t)}{\partial t}\cdot\frac{\partial[\delta W(x,\ t)]}{\partial t}dt\Rightarrow \tag{5a}$$

$$\rho A\left\{\frac{\partial W(x,\ t)}{\partial t}\cdot\delta W(x,\ t)\bigg|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{\partial^2W(x,\ t)}{\partial t^2}\cdot\delta W(x,\ t)dt\right\} \tag{5b}$$

In the Eq. (5_b) the first member is equal to zero, since $\delta W(x,t)$ for $t=t_1$ and $t=t_2$ is equal zero. Hence, the expression (5_b), keeping in mind mentioned integration commutativity, may be rewritten as:

$$\delta T = -\rho A \int_0^L \frac{\partial^2 W(x,\ t)}{\partial t^2} \cdot \delta W(x,\ t) dx \tag{5c}$$

The step by step solution to get the variation of potential energy from Eq. (2) is as follows:

$$\delta V = \frac{1}{2} EJ \int_0^L \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot 2\delta \frac{\partial^2 W(x,\ t)}{\partial x^2} dx \Rightarrow EJ \int_0^L \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta \frac{\partial^2 W(x,\ t)}{\partial x^2} dx \Rightarrow \tag{6a}$$

$$EJ \left\{ \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L - \int_0^L \frac{\partial^3 W(x,\ t)}{\partial x^2} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} dx \right\} \Rightarrow \tag{6b}$$

$$\begin{gathered} EJ \left\{ \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L - \left[\frac{\partial^3 W(x,\ t)}{\partial x^3} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L + \right.\right. \\ \left.\left. - \int_0^L \frac{\partial^4 W(x,\ t)}{\partial x^4} \cdot \delta W(x,\ t)\, dx \right] \right\} \Rightarrow \end{gathered} \tag{6c}$$

$$\begin{gathered} EJ \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L - EJ \frac{\partial^3 W(x,\ t)}{\partial x^3} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L + \\ + EJ \int_0^L \frac{\partial^4 W(x,\ t)}{\partial x^4} \cdot \delta W(x,\ t)\, dx \end{gathered} \tag{6d}$$

The variation of external force work at the free column's end from Eq. (4) is:

$$\delta L_e = \frac{1}{2} P \int_0^L \frac{\partial W(x,\ t)}{\partial x} \cdot 2\delta \frac{\partial W(x,\ t)}{\partial x} dx \Rightarrow P \int_0^L \frac{\partial W(x,\ t)}{\partial x} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} dx \Rightarrow \tag{7a}$$

$$\delta L_e = P \frac{\partial W(x,\ t)}{\partial x} \cdot \delta W(x,\ t) \Big|_0^L - P \int_0^L \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta W(x,\ t)\, dx \tag{7b}$$

After substituting the variations of energy terms given by Eqs. (5_c), (6_d) and (7_b) into Eq. (1) and performing minor simplifications, the following expression is obtained:

$$\begin{gathered} \delta \int_{t_1}^{t_2} \left\{ -\rho A \int_0^L \frac{\partial^2 W(x,\ t)}{\partial t^2} \cdot \delta W(x,\ t) dx - EJ \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L + \right. \\ + EJ \frac{\partial^3 W(x,\ t)}{\partial x^3} \cdot \delta \frac{\partial W(x,\ t)}{\partial x} \Big|_0^L - EJ \int_0^L \frac{\partial^4 W(x,\ t)}{\partial x^4} \cdot \delta W(x,\ t)\, dx + \\ \left. + P \frac{\partial W(x,\ t)}{\partial x} \cdot \delta W(x,\ t) \Big|_0^L - P \int_0^L \frac{\partial^2 W(x,\ t)}{\partial x^2} \cdot \delta W(x,\ t)\, dx \right\} dt = 0 \end{gathered} \tag{8}$$

Finally the equation of motion for the system is obtained. If only all the members containing the negative integral from 0 to L from the variation of transverse displacements $W(x,t)$ are selected:

$$-\int_0^L \delta W(x,t)\, dx: \quad EJ \frac{\partial^4 W(x,\ t)}{\partial x^4} + P \frac{\partial^2 W(x,\ t)}{\partial x^2} + \rho A \frac{\partial^2 W(x,\ t)}{\partial t^2} = 0 \tag{9}$$

The next step is to pick all the members standing at the rest respective variations, which leads to obtaining the natural boundary conditions:

$$\delta \left.\frac{\partial W(x,\ t)}{\partial x}\right|_{x=0} : \quad EJ \left.\frac{\partial^2 W(x,\ t)}{\partial x^2}\right|_{x=0} = 0 \tag{10}$$

$$\delta \left.\frac{\partial W(x,\ t)}{\partial x}\right|_{x=L} : \quad EJ \left.\frac{\partial^2 W(x,\ t)}{\partial x^2}\right|_{x=L} = 0 \tag{11}$$

$$\delta W(x,\ t)|_{x=0}: \quad EJ \left.\frac{\partial^3 W(x,\ t)}{\partial x^3}\right|_{x=0} + P \left.\frac{\partial W(x,\ t)}{\partial x}\right|_{x=0} = 0 \tag{12}$$

$$\delta W(x,\ t)|_{x=L}: \quad EJ \left.\frac{\partial^3 W(x,\ t)}{\partial x^3}\right|_{x=L} + P \left.\frac{\partial W(x,\ t)}{\partial x}\right|_{x=L} = 0 \tag{13}$$

In fact, the derived boundary conditions in Eqs. (10)-(13) correspond to the arbitrary column having both ends free. At this point it should be remembered that geometrical boundary conditions originating from adopted support or supports in the system cancels or extends certain derived natural boundary conditions. Having in mind that the column from the Figure 1 has clamped support at $x = 0$, for which the transversal displacements and angle of rotation is equal to zero:

$$W(x,\ t)|_{x=0} = \left.\frac{\partial W(x,\ t)}{\partial x}\right|_{x=0} = 0 \tag{14$_{a,b}$}$$

and introducing the conditions from Eqs. (14$_{a,b}$) into Eq. (8) one gets that the boundary condition from Eq. (10) and (12) are canceled and must be replaced with the geometrical boundary condition Eq. (14$_{a,b}$) resulting from adopted support. Finally, boundary conditions for the column presented in Figure 1 are described with Eqs. (11), (13) and (14$_{a,b}$).

In order to solve the problem of stability and vibrations, the space x and time t variable needs to be separated from the transversal displacements $W(x,t)$ enclosed in equation of motion (9) and boundary conditions (11), (13) and (14$_{a,b}$) according to the following formula:

$$W(x,\ t) = y(x) \cdot T(t) \tag{15}$$

where $y(x)$ is the displacement function dependent only on the space x variable and the $T(t)$ in the case of harmonic vibrations have to satisfy the equation:

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \tag{16}$$

The solution to the boundary problem is independent of initial conditions, thus the form of the $T(t)$ function can be arbitrarily adopted as:

$$T(t) = sin(\omega t) \ \text{ or } \ T(t) = cos(\omega t) \tag{17$_{a,b}$}$$

After the separation of space and time variables in both: equation of motion and boundary conditions, it becomes possible to analytically or numerically solve the problem of stability and vibrations of a slender mechanical system.

4. Conclusion

In this paper a step by step derivation of an equation of motion and natural boundary conditions for a slender column using Hamilton's principle has been presented. It is shown that at the beginning of the derivation process, the kinetic and potential energy of the system as well as work of external forces should have been defined. Afterwards, it is necessary to perform variational operations and integration by parts in regard to defined energies. By introducing these variations into the governing Hamilton's principle equation and extracting the members standing at the appropriate variations, one gets the equations of motion of the system and set of supplementary natural boundary conditions. Despite that the presented solution of the deriving motion equations can be time consuming and requires advanced mathematical techniques, it does make it possible to obtain equations of motion even for high complexity mechanical systems, which at the same time may be difficult or even impossible using different known methods.

References

[1] Pedersen N.L., Pedersen M.L., A direct derivation of the equations of motion for 3D-flexible mechanical systems, International Journal for Numerical Methods in Engineering 1998, 41(4), 697-719.
[2] Vassiliou M.F., Makris N., Dynamics of the vertically restrained rocking column, Journal of Engineering Mechanics 2015, 141(12), 04015049.
[3] Kounadis A.N., On the derivation of equations of motion for a vibrating Timoshenko column, Journal of Sound and Vibration 1980, 73(2), 177-184.
[4] Tomski L., Uzny S., Free vibrations and the stability of a geometrically non-linear column loaded by a follower force directed towards the positive pole, International Journal of Solids and Structures 2008, 45(1), 87-112.
[5] Sokół K., An influence of the parameters of the loading heads on stability and free vibrations of a damaged column subjected to a specific load, Journal of Vibroengineering 2018, 20(3), 1299-1310.
[6] Yoo W.S., Haug E.J., Dynamics of articulated structures – Part I, Journal of Structural Mechanics 1986, 14(1), 105-126.
[7] Rao S.S., Vibration of Continuous Systems, John Wiley & Sons, Hoboken 2019.
[8] Leissa A.W., Qatu M.S., Vibrations of Continuous Systems, McGraw-Hill Education, New York 2011.
[9] Shabana A.A., Ling F.F., Vibration of Discrete and Continuous Systems, Springer, New York 1997.
[10] Kuliński K., On Innovative Concrete-Rubber Composite Blocks Reducing Effects of Dynamic Mechanical Impact: The Review of Structural Solutions, IOP Conference Series: Materials Science and Engineering, 2019, 603(4), 042018.
[11] Major M., Major I., Kucharova D., Kuliński K., Reduction of dynamic impacts in block made of concrete-rubber composites, Civil and Environmental Engineering 2018, 14(1), 61-68.
[12] Major M., Kuliński K., Major I., Dynamic analysis of an impact load applied to the composite wall structure, MATEC Web of Conferences 2017, 107, 00055.

Wykorzystanie zasady Hamiltona przy wyprowadzaniu równań ruchu oraz naturalnych warunków brzegowych w odniesieniu do smukłego jednogałęziowego słupa poddanego obciążeniu Eulera

STRESZCZENIE:
Przedstawiono proces wyprowadzania równań ruchu i naturalnych warunków brzegowych smukłego układu mechanicznego na podstawie zasady Hamiltona. Aby zastosować zasadę Hamiltona, należy przede wszystkim zdefiniować odpowiednie zmienne ogólne opisujące ruch rozpatrywanego układu. W przypadku smukłych układów mechanicznych stosuje się zwykle tzw. współrzędne naturalne, które są dobrze dopasowane do geometrii układu i jego charakterystyk ruchu. Wykorzystując zasadę Hamiltona, konstruuje się odpowiedni funkcjonał ruchu, którym jest całka Lagrange'a smukłego układu po odpowiednich zmiennych ogólnych i czasie. Lagranżjan opisuje dobrze znaną różnicę między energią kinetyczną i potencjalną układu. W pracy przedstawiono proces krok po kroku wyprowadzania równania ruchu i uzupełniających naturalnych warunków brzegowych w odniesieniu do smukłego słupa o jednym końcu utwierdzonym, a drugim wolnym, który to poddany jest obciążeniu Eulera. Otrzymane równanie wraz ze zbiorem geometrycznych i naturalnych warunków brzegowych daje możliwość dokładnej analizy analitycznej lub numerycznej odpowiedzi dynamicznej i/lub statycznej układu. Pomimo tego, że omawiana metoda jest czasochłonna i wymaga zaawansowanych technik matematycznych, pozwala ona na uzyskanie dokładnych lub przybliżonych równań ruchu nawet dla złożonych problemów, co przy innych znanych metodach może być trudne lub wręcz niemożliwe do osiągnięcia.

SŁOWA KLUCZOWE:
zasada Hamiltona; układy smukłe; wyprowadzanie równań ruchu; statyczna oraz dynamiczna odpowiedź konstrukcji; obciążenie Eulera