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APPLICATION OF THE TAYLOR DIFFERENTIAL TRANSFORMATION FOR SOLVING THE INTEGRO-DIFFERENTIAL EQUATIONS

Abstract. A method of solving the integro-differential equations is presented. The discussed equations will be solved by the Taylor differential transformation. By using appropriate properties of this transformation the integro-differential equation will be transformed to a respective recurrence equation. Unfortunately, the high degree of generality and complexity of such defined problem does not allow to obtain the solution in general form. Each equation requires a special method of solution.

1. Introduction

The Taylor differential transformation [4], due to its specific properties significantly simplifying most of the considered problems, is widely applied to various problems in mathematics, engineering and technics. Authors of the present paper have already used these properties for solving, among others, the nonlinear ordinary differential equations [5], the problems from the calculus of variations [8], systems of nonlinear ordinary differential equations [7] and the Stefan problem [6].

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Also the other scientists have applied the Taylor transformation to various problems, for example to the several types of differential equations (ordinary and partial) or to integral equations [1, 2, 9–13]. The Taylor differential transformation is known for a long time, but only recently the methods based on this transformation became useful due to the development of computers and programs enabling to execute the symbolic calculations. In this paper we use the *Mathematica* software [3, 14, 15].

In this paper we investigate the integro-differential equations of the following form

$$f(x, y(x), y'(x), \ldots, y^{(n)}(x)) + \sum_{i=1}^{m} \int_a^b g_i(x, y(x))dx + \sum_{i=1}^{k} \int_a^x h_i(t, y(t))dt = 0, \quad (1)$$

in the class of continuous functions y possessing the continuous derivatives of all n orders in the interval $\langle a, b\rangle$ and satisfying the conditions

$$y(a) = A_1,\ y'(a) = A_2, \ldots, y^{(n-1)}(a) = A_n,$$

where $a, b, A_1, \ldots, A_n \in \mathbb{R}$, f is the continuous function in set $\langle a, b\rangle \times \mathbb{R}^{n+1}$, whereas the functions g_i, $i = 1, 2, \ldots, m$, and h_j, $j = 1, 2, \ldots, k$, are continuous in set $\langle a, b\rangle \times \mathbb{R}$.

The function f can be nonlinear, whereas the integrals arisen in this equation can take the particular forms of integrals appearing in the Fredholm and Volterra integral equations, or they can have such form only with respect to the boundaries of integration and the antiderivatives $g(x)$ and $h(x)$ can be nonlinear also with respect to the sought function $y(x)$.

We will solve equation (1) by using the Taylor differential transformation and, due to its properties, we will transform equation (1) to a more simple form, usually recurrent with respect to the coefficients. In this way we will be able to find the approximate solution (and sometimes even exact solution) of the considered equation.

2. The Taylor transformation

Let us assume that we consider only such functions of the real variable x, defined in some region $X \subset \mathbb{R}$, that can be expanded into the Taylor series within some neighborhood of point $\alpha \in X$. We call such functions as the originals and denote by the small letters of Latin alphabet, for example f, y, u, v, w, and so on. Thus, if the function y is the original, then the following equality holds

$$y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(\alpha)}{k!}(x-\alpha)^k, \tag{2}$$

where $\alpha \in X$ denotes the point, in the neighborhood of which the function y is expanded into the Taylor series.

Each original corresponds to a function Y_α of nonnegative integer arguments $k = 0, 1, 2, \ldots$, according to formula

$$Y_\alpha(k) = \frac{y^{(k)}(\alpha)}{k!}, \quad k = 0, 1, 2, \ldots \tag{3}$$

The function Y_α will be called the image of the function y, the T_α–function of the function y or the transform of the function y, and the discussed transformation will be called the Taylor transformation.

The obvious fact is that, by having the T_α–function Y_α one can find, according to formulas (2) and (3), the corresponding original in the form of its expansion into the Taylor series, that is

$$y(x) = \sum_{k=0}^{\infty} Y_\alpha(k)(x-\alpha)^k, \quad x, \alpha \in X. \tag{4}$$

Transformation (3), assigning to each original its image, will be called the direct transformation. Whereas the transformation (4), assigning the corresponding original to the image, will be called the inverse transformation. Connection between these both transformations will be denoted by using the following symbols

$$Y_\alpha(k) = \mathcal{T}[y(x); k, \alpha]$$

for the direct transformation and

$$y(x) = \mathcal{T}^{-1}[Y_\alpha(k); x]$$

for the inverse transformation, where $\mathcal{T}$ and $\mathcal{T}^{-1}$ symbolize the proper transformations.

In the used notation, for example for the function $y(x) = \sinh x$ and $\alpha = 0$ we have

$$Y_0(k) = \mathcal{T}[\sinh x; k, 0] = \mathcal{T}\left[\frac{1}{2}\sum_{k=0}^{\infty}\frac{1-(-1)^k}{k!}x^k; k, 0\right] = \frac{1+(-1)^{k+1}}{2k!},$$

where $k = 0, 1, 2, \ldots$ Whereas in case of the inverse transformation we get for the above function

$$y(x) = \mathcal{T}^{-1}[Y_0(k); x] = \mathcal{T}^{-1}\left[\frac{1+(-1)^{k+1}}{2k!}; x\right] = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1-(-1)^k}{k!}x^k = \sinh x.$$

The Taylor transformation possesses a number of properties causing that the application of this tool, with the aid of computational platforms giving the possibility to execute the symbolic calculations, *Mathematica* for example, is quite simple.

Since we use here the specific case of the Taylor series, the Maclaurin series, hence α, equal to zero, will be henceforward omitted.

In particular, the following properties are especially useful [4]:

$$\mathcal{T}[x^n; k] = \delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases} \tag{5}$$

$$\mathcal{T}[e^{ax}; k] = \frac{a^k}{k!}, \tag{6}$$

$$\mathcal{T}[\sin(ax); k] = \frac{a^k}{k!}\sin\frac{\pi k}{2}, \tag{7}$$

$$\mathcal{T}[\cos(ax); k] = \frac{a^k}{k!}\cos\frac{\pi k}{2}, \tag{8}$$

$$\mathcal{T}[\arctan; k] = \begin{cases} 0, & k = 0, \\ \frac{1}{k}\sin\frac{\pi k}{2}, & k \geq 1, \end{cases} \tag{9}$$

$$\mathcal{T}[c \cdot u(x); k] = c \cdot U(k), \tag{10}$$

$$\mathcal{T}[u(x) \pm w(x); k] = U(k) \pm W(k), \tag{11}$$

$$\mathcal{T}[u(x) \cdot w(x); k] = \sum_{r=0}^{k} U(r)W(k-r), \tag{12}$$

$$\mathcal{T}[u'(x); k] = (k+1)U(k+1), \tag{13}$$

$$\mathcal{T}[u^{(n)}(x); k] = \frac{(k+n)!}{k!}U(k+n), \tag{14}$$

$$\mathcal{T}[u(x)\cdot w'(x);k]=\sum_{r=0}^{k}(k+1-r)U(r)W(k-r+1), \tag{15}$$

$$\mathcal{T}[u(x)\cdot w''(x);k]=\sum_{r=0}^{k}(k-r+1)(k-r+2)U(r)W(k-r+2), \tag{16}$$

$$\mathcal{T}[u'(x)\cdot w'(x);k]=\sum_{r=0}^{k}(r+1)(k-r+1)U(r+1)W(k-r+1), \tag{17}$$

$$\mathcal{T}[\int_0^x u(t)dt;k]=\begin{cases}0, & k=0,\\ \frac{U(k-1)}{k}, & k\geq 1,\end{cases} \tag{18}$$

$$\mathcal{T}[\int_0^x u(t)w(t)dt;k]=\begin{cases}0, & k=0,\\ \sum\limits_{r=0}^{k-1}\frac{U(r)W(k-r-1)}{k}. & k\geq 1,\end{cases} \tag{19}$$

where $a, c \in \mathbb{R}$, $n \in \mathbb{N}\cup\{0\}$, and $k = 0, 1, 2, \ldots$

3. Computational examples

Example 1. Let us consider the equation

$$2y(x)y'(x)+\int_0^\pi(\pi-2x)y(x)dx=\sin 2x, \tag{20}$$

for $0 \leq x \leq \pi$, with the condition

$$y(0)=0, \tag{21}$$

the exact solution of which is given by the function $y(x) = \sin x$.

Equation (20), under the assumption that $\int_0^\pi(\pi-2x)y(x)dx=\lambda\in\mathbb{R}$ and after the usage, among others, of properties (7), (10), (11) and (15), is transformed to the form

$$2\sum_{r=0}^{k}(k+1-r)Y(r)Y(k+1-r)+\lambda\delta(k-0)-\frac{2^k}{k!}\sin\frac{\pi k}{2}=0,\quad k\geq 0, \tag{22}$$

and additionally, by the initial condition (21) we have $Y(0) = 0$.

By taking $k = 0$ in equation (22) we get

$$2Y(0)Y(1) + \lambda = 0 \Rightarrow \lambda = 0.$$

For $k = 1$ we have

$$-2 + 2((Y(1))^2 + 2Y(0)Y(2)) = 0 \Rightarrow Y(1) = \pm 1.$$

Thus we obtain two families of solutions generated by the values $Y(1) = 1$ and $Y(1) = -1$. By taking $Y(1) = 1$, for $k = 2$ we obtain

$$6(Y(1)Y(2) + Y(0)Y(3)) \Rightarrow Y(2) = 0,$$

for $k = 3$ we have

$$\frac{4}{3} + 4((Y(2))^2 + 2Y(1)Y(3) + 2Y(0)Y(4)) = 0 \Rightarrow Y(3) = -\frac{1}{6},$$

and for $k = 4$ we get

$$10(Y(2)Y(3) + Y(1)Y(4) + Y(0)Y(5)) = 0 \Rightarrow Y(4) = 0.$$

As a result of calculations with *Mathematica* we get the values given in Table 1.

Table 1
Values of $Y(i)$ for $0 \le i \le 9$

i	$Y(i)$	
0	0	
1	1	-1
2	0	0
3	$-\frac{1}{6} = \frac{1}{3!}$	$\frac{1}{6} = \frac{1}{3!}$
4	0	0
5	$\frac{1}{120} = \frac{1}{5!}$	$-\frac{1}{120} = -\frac{1}{5!}$
6	0	0
7	$-\frac{1}{5040} = -\frac{1}{7!}$	$\frac{1}{5040} = \frac{1}{7!}$
8	0	0
9	$\frac{1}{362880} = \frac{1}{9!}$	$-\frac{1}{362880} = -\frac{1}{9!}$

It is easy to notice that we obtain two solutions

$$y_1(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} \quad \text{and} \quad y_2(x) = -\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!},$$

leading to the well known functions

$$y_1(x) = \sin x \quad \text{and} \quad y_2(x) = -\sin x,$$

being the exact solutions of the investigated problem, which is easy to check.

Example 2. Let us consider the equation

$$y''(x) - 3y'(x) + 3y(x) + \int_0^x (e^{-t} - 1)y(t)dt = \frac{x^3}{3} - 4x - 2, \tag{23}$$

for $0 \leq x \leq 2$, with conditions

$$y(0) = y'(0) = -4, \tag{24}$$

the exact solution of which is defined by the function $y(x) = (x^2 - 4)e^x$.

Equation (23), in result of applying, among others, the properties (5), (6), (10), (11), (13), (14) and (19), is transformed to the form

$$\begin{aligned} &(k+1)(k+2)Y(k+2) - 3(k+1)Y(k+1) + 3Y(k) + 2\delta(k-0) + \\ &+ \begin{cases} \sum_{r=0}^{k-1} \dfrac{\left(\frac{(-1)^r}{r!} - \delta(r-0)\right) Y(k-r-1)}{k}, & k \geq 1 \\ 0, & k = 0 \end{cases} + 4\delta(k-1) = \frac{1}{3}\delta(k-3) \end{aligned} \tag{25}$$

and additionally, by the initial conditions (24) we have $Y(0) = Y(1) = -4$.

By taking $k = 0$ in equation (25) we receive

$$2 + 3Y(0) - 3Y(1) + 2Y(2) = 0 \Rightarrow Y(2) = -1.$$

For $k = 1$ we have

$$4 + 3Y(1) - 6Y(2) + 6Y(3) = 0 \Rightarrow Y(3) = \frac{1}{3}.$$

Further calculations, by using the *Mathematica* software, lead us to the results collected in Table 2.

Table 2

Values of $Y(i)$ for $0 \le i \le 9$ in Example 2

i	0	1	2	3	4	5	6	7	8	9
$Y(i)$	-4	-4	-1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{15}$	$\frac{13}{360}$	$\frac{19}{2520}$	$\frac{13}{10080}$	$\frac{17}{90720}$

By taking 7 successive values $Y(i)$ we receive the approximate solution $y_7(x) = \sum\limits_{i=0}^{6} Y(i)x^i$:

$$y_7(x) = -4 - 4x - x^2 + \frac{x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \frac{13x^6}{360}$$

and for 15 terms we have the approximate solution $y_{15}(x)$ in the form

$$\begin{aligned} y_{15}(x) = & -4 - 4x - x^2 + \frac{x^3}{3} + \frac{x^4}{3} + \frac{2x^5}{15} + \frac{13x^6}{360} + \frac{19x^7}{2520} + \frac{13x^8}{10080} + \frac{17x^9}{90720} + \\ & + \frac{43x^{10}}{1814400} + \frac{53x^{11}}{19958400} + \frac{x^{12}}{3742200} + \frac{19x^{13}}{3742200} + \frac{89x^{14}}{43589145600}. \end{aligned}$$

The plots of the above obtained approximate solutions, together with the plots of their absolute errors $\Delta_n(x)$ defined by equation

$$\Delta_n(x) = |y(x) - y_n(x)|,$$

are presented in Figures 1 and 2, where the solid line denotes the exact solution (in the left figure) and the error Δ (in the right figure), whereas the dashed line represents the approximate solution (in the left figure).

We can observe the general form of elements $Y(i)$ presented in Table 3.

Table 3

Values $Y(i)$ and their general form for $0 \le i \le 9$ in Example 2

i	0	1	2	3	4	5	...	k
$Y(i)$	-4	-4	-1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{15}$	...	
$Y(i)$	$\frac{0\cdot(-1)-4}{0!}$	$\frac{1\cdot 0-4}{1!}$	$\frac{2\cdot 1-4}{2!}$	$\frac{3\cdot 2-4}{3!}$	$\frac{4\cdot 3-4}{4!}$	$\frac{5\cdot 4-4}{5!}$	...	$\frac{k\cdot(k-1)-4}{k!}$

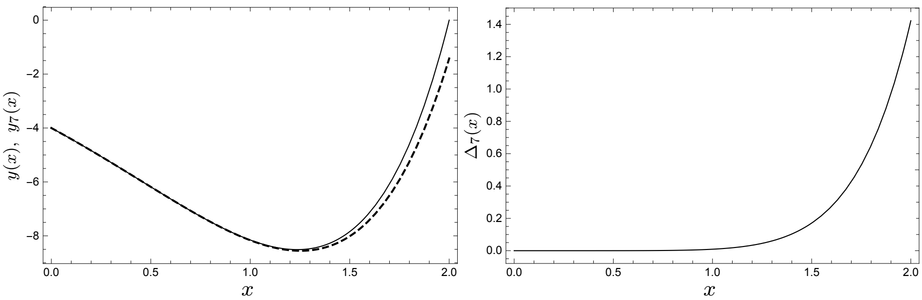

Fig. 1. Exact solutions $y(x)$ and approximate solution $y_7(x)$ together with the absolute error of the approximate solution

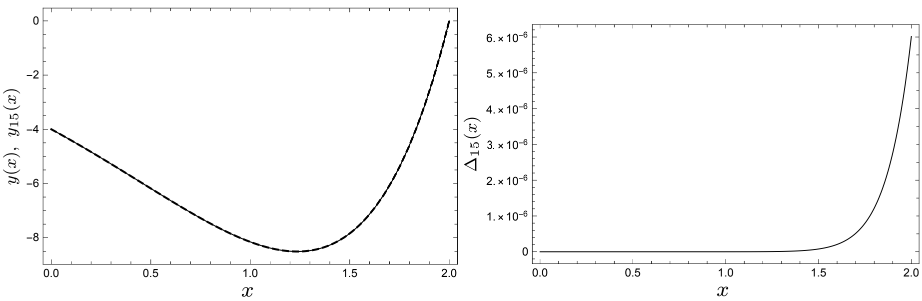

Fig. 2. Exact solutions $y(x)$ and approximate solution $y_{15}(x)$ together with the absolute error of the approximate solution

Thus we get $y(x) = \sum_{i=0}^{\infty} \frac{i \cdot (i-1) - 4}{i!} x^i = (x^2 - 4)e^x$, which is the exact solution.

Example 3. We investigate the equation

$$3(x^3 - 2x)y(x) + (1 + x^2)^2 y''(x) + \frac{8}{4\ln 2 - 3} \int_0^1 (3xy(x) - 1)y(x)dx + \\ + 12 \int_0^x (1 - t^2)y(t)dt = x^3 + 9(\arctan x - x), \tag{26}$$

for $0 \leq x \leq 1$, with conditions

$$y(0) = y'(0) = 0, \tag{27}$$

the exact solution of which is given by function $y(x) = x \arctan x$.

Equation (26), under the assumption that $\frac{8}{4\ln 2-3}\int_0^1 (3xy(x)-1)y(x)dx = \lambda \in \mathbb{R}$ and after applying, among others, the properties (5), (9), (10)–(12), (16) and (19), is transformed to the form

$$\begin{aligned}
&3\sum_{r=0}^{k}(\delta(r-3)-2\delta(r-1)Y(k-r))+\lambda\delta(k-0)+\\
&+\sum_{r=0}^{k}(k-r+1)(k-r+2)(\delta(r-0)+2\delta(r-2)+\delta(r-4))Y(k-r+2)+\\
&\quad+12\begin{cases}\displaystyle\sum_{r=0}^{k-1}\frac{(\delta(r-0)-\delta(r-2))\,Y(k-r-1)}{k}, & k\geq 1\\ 0, & k=0\end{cases}+\\
&\quad+9\delta(k-1)-\delta(k-3)-9\begin{cases}\dfrac{1}{k}\sin\dfrac{\pi k}{2}, & k\geq 1\\ 0, & k=0\end{cases}=0,
\end{aligned}\tag{28}$$

and additionally, the initial conditions (27) yield $Y(0)=Y(1)=0$.

By taking $k=0$ in equation (28) we get

$$\lambda+2Y(2)=0\Rightarrow Y(2)=-\frac{\lambda}{2}.$$

For $k=1$ we have

$$6(Y(0)+Y(3))=0\Rightarrow Y(3)=0,$$

for $k=2$ we get

$$4(Y(2)+3Y(4))=0\Rightarrow Y(4)=-\frac{Y(2)}{3}=\frac{\lambda}{6},$$

and for $k=3$ we obtain

$$2-Y(0)-2Y(2)+12Y(3)+20Y(5)=0\Rightarrow Y(5)=-\frac{\lambda+2}{20}.$$

By using the *Mathematica* software we get the successive elements $Y(i)$ dependent on λ collected in Table 4.

To obtain the solution independent on λ we need to determine this value. For this purpose we will use the following fact

$$\frac{8}{4\ln 2-3}\int_0^1 (3ty(t)-1)y(t)dx=\lambda.$$

Table 4

Values $Y(i)$ dependent on λ for $0 \leq i \leq 9$ in Example 3

i	0	1	2	3	4	5	6	7	8	9
$Y(i)$	0	0	$-\frac{\lambda}{2}$	0	$\frac{\lambda}{6}$	$-\frac{\lambda+2}{20}$	$-\frac{\lambda}{10}$	$\frac{29(\lambda+2)}{420}$	$\frac{19\lambda-2}{280}$	$-\frac{127(\lambda+2)}{1680}$

Substituting the function $y_n(t) = \sum\limits_{i=0}^{n-1} Y(i)t^i$ instead of the function $y(t)$ in the above relation we get the equation, the real root of which makes the function y_n unique and this function exactly will be taken as an approximate solution of the considered equation.

So, for $n = 7$ the equation possesses two real roots: $\lambda = -1.8734$ and $\lambda = -0.0933$. In result of verification, consisted in substituting function $y_n(x)$ in place of function $y(x)$ in equation (26), it turned out that $\lambda = -0.0933$ is unacceptable. For $\lambda = -1.8734$ (let us notice that the exact value of λ is equal to -2) we get the solution presented in Figure 3 (description of figure is the same as explained before).

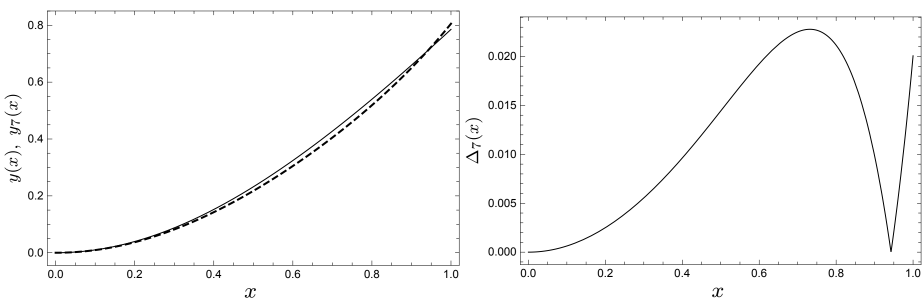

Fig. 3. Exact solutions $y(x)$ and approximate solution $y_7(x)$ together with the absolute error of the approximate solution

In case of $n = 31$ we have the similar situation – we also get here two values: $\lambda = -1.9903$ and $\lambda = -0.0466$, but for the latter value of λ the equation (26) is not satisfied. For $\lambda = -1.9903$ we obtain the solution presented in Figure 4.

Let us also point our attention in this example to the convergence region of the Maclaurin series of the function $y(x) = x \arctan x = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2n-1}$, which is the interval $(-1, 1)$. For $x = 1$ the approximate solution $y_n(x)$ behaves like it was discussed in the example, which results from the (only) conditional convergence of the number series $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ (moreover, its convergence speed is not high – to

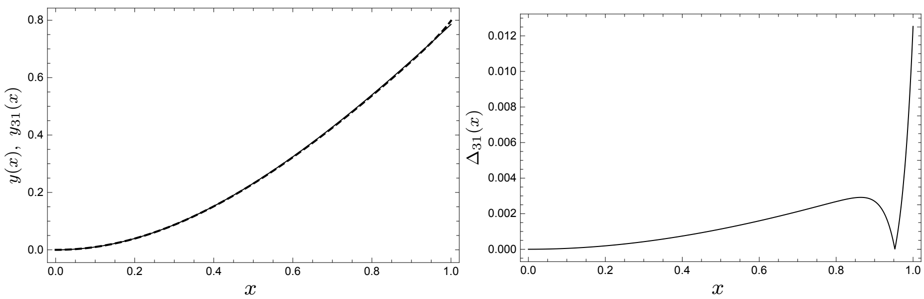

Fig. 4. Exact solutions $y(x)$ and approximate solution $y_{31}(x)$ together with the absolute error of the approximate solution

obtain the precision rate of the sum equal to $\frac{1}{m}$ we need to take the partial sum with $\frac{1}{2}(m+1)$ terms).

Example 4. Now we investigate the equation

$$\begin{aligned}\frac{4}{\pi}(y(x)-y^{(4)})+\frac{1}{\pi}\int_0^{\pi}(t-x-y(t))^2dt+\\ -2\int_0^x t(1+t+3\cos t-\sin t-y(t))dt=5,\end{aligned} \tag{29}$$

for $0\leq x\leq\pi$, with conditions

$$y(0)=3,\quad y'(0)=0,\quad y''(0)=-3,\quad y'''(0)=1, \tag{30}$$

the exact solution of which is defined by function $y(x)=x-\sin x+3\cos x$.

In the process of solution we will use the following relations

$$\begin{aligned}\frac{1}{\pi}\int_0^{\pi}(t-x-y(t))^2dt=\frac{1}{\pi}\int_0^{\pi}(t^2-2tx+x^2)dt-\frac{2}{\pi}\int_0^{\pi}ty(t)dt+\\ +\frac{2x}{\pi}\int_0^{\pi}y(t)dt+\frac{1}{\pi}\int_0^{\pi}y^2(t)dt=h_1(x)-\alpha+\beta x+\gamma,\end{aligned}$$

where $h_1(x)=\frac{1}{\pi}\int_0^{\pi}(t^2-2tx+x^2)dt$, $\alpha=\frac{2}{\pi}\int\limits_0^{\pi}ty(t)dt$, $\beta=\frac{2}{\pi}\int\limits_0^{\pi}y(t)dt$ and

$\gamma = \dfrac{1}{\pi}\displaystyle\int\limits_0^\pi y^2(t)dt$ and additionally

$$-2\int_0^x t(1+t+3\cos t-\sin t-y(t))dt = -2\int_0^x t(1+t+3\cos t-\sin t)dt+ \\ +2\int_0^x ty(t)dt = h_2(x) + 2\int_0^x ty(t)dt,$$

where $h_2(x) = -2\displaystyle\int_0^x t(1+t+3\cos t-\sin t)dt$.

By determining the functions $h_1(x)$ and $h_2(x)$ and by applying, among others, the properties (5), (7), (8), (10)–(12), (14) and (19), we get the relation written below

$$\begin{aligned} &\frac{4}{\pi}\left(Y(k) - \frac{(k+4)!}{k!}Y(k+4)\right) + \delta(k-0)\left(\gamma - \alpha + 1 + \frac{\pi^2}{3}\right) + \\ &+ \delta(k-1)(\beta-\pi) - \frac{2}{3}\delta(k-3) - \frac{6}{k!}\cos\frac{\pi k}{2} + \frac{2}{k!}\sin\frac{\pi k}{2} + \\ &- 2\sum_{r=0}^{k}\frac{\delta(k-r-1)}{r!}\cos\frac{\pi r}{2} - 6\sum_{r=0}^{k}\frac{\delta(k-r-1)}{r!}\sin\frac{\pi r}{2} + \\ &+ \begin{cases} \displaystyle\sum_{r=0}^{k-1}\frac{\delta(r-1)Y(k-r-1)}{k}, & k \ge 1 \\ 0, & k = 0 \end{cases} = 0, \quad \text{for} \quad k \ge 0, \end{aligned} \tag{31}$$

and moreover, from the initial conditions (30) we get $Y(0) = 3$, $Y(1) = 0$, $Y(2) = -\frac{3}{2}$, $Y(3) = \frac{1}{6}$.

By taking in equation (31) the successive values $k \ge 0$ due to the *Mathematica* software we are able to calculate as follows

$Y(4) = \frac{1}{288}\left(-3\pi\alpha + 3\pi\gamma + \pi^3 - 15\pi + 36\right)$, $Y(5) = -\frac{1}{480}\pi(\pi-\beta)$, $Y(6) = -\frac{1}{240}$, $Y(7) = \frac{1}{5040}$, $Y(8) = \frac{-3\pi\alpha+3\pi\gamma+\pi^3-15\pi+36}{483840}$, $Y(9) = -\frac{\pi(\pi-\beta)}{1451520}, \ldots$

One can observe that the approximate solution $y_n(x)$ depends on parameters α, β and γ. To make the solution unique, we need to determine values of these parameters on the way of solving the system of equations

$$\begin{cases} \frac{2}{\pi}\int\limits_0^\pi t y_n(t)dt = \alpha, \\ \frac{2}{\pi}\int\limits_0^\pi y_n(t)dt = \beta, \\ \frac{1}{\pi}\int\limits_0^\pi (y_n)^2(t)dt = \gamma. \end{cases} \tag{32}$$

So, for $n = 7$ we obtain the couple of solutions: $\alpha = 0.5828$, $\beta = 1.8214$, $\gamma = 2.4989$ and $\alpha = 12.0135$, $\beta = 6.1059$, $\gamma = 15.0464$, however the verification shows that the latter one is incorrect. For the former solution we receive the result presented in Figure 5 (description of figure is the same as in the previous examples).

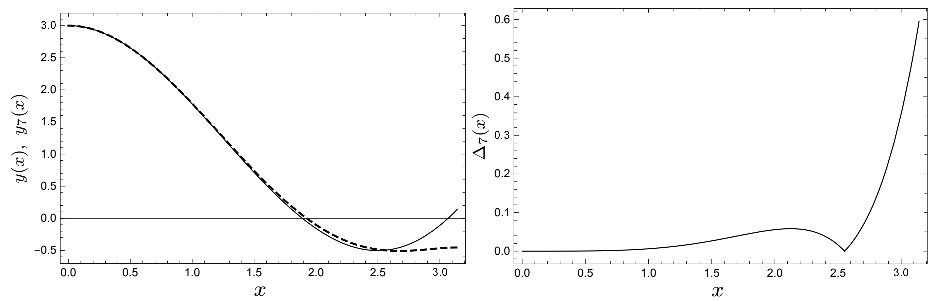

Fig. 5. Exact solutions $y(x)$ and approximate solution $y_7(x)$ together with the absolute error of the approximate solution

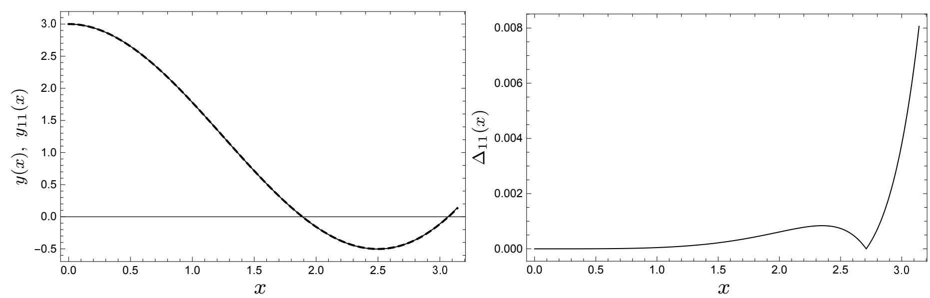

Fig. 6. Exact solutions $y(x)$ and approximate solution $y_{11}(x)$ together with the absolute error of the approximate solution

In case of $n = 11$ the situation is similar – the system of equations (32) returns two sets of solutions: $\alpha = 0.7586$, $\beta = 1.868$, $\gamma = 2.4702$ and $\alpha = 11.3925$, $\beta = 5.8436$, $\gamma = 14.0483$, but for the latter triple the equation (29) is significantly not fulfilled. For the former triple of α, β and γ we obtain the solution presented in Figure 6.

4. Conclusion

In this paper we have presented the possibility of applying the Taylor differential transformation to solve the selected types of the integro-differential equations. The discussed examples indicate that the proposed method is efficient in solving the problems of considered kind and as the additional advantage of this method one can take into account the simplicity of its application.

Moreover, we would like to notice that the fact of taking value one by variables m and k, denoting the upper bounds of summation in equation (1), should not be questionable. Example 4 shows that although the summation is seemingly limited to one term, the method of solution increases in fact this number. Additionally, Example 4 shows that the discussed method can be used for solving also a wider class of problems – the antiderivative h is the function of two variables x and t, which makes the integral on this function more complex as it is presented in equation (1).

References

1. Biazar J., Eslami M., Islam M.R.: *Differential transform method for special systems of integral equations.* J. King Saud Univ. – Sci. **24** (2012), 211–214.
2. Doğan N., Ertürk V.S., Momani S., Akın Ö., Yıldırım A.: *Differential transform method for solving singularly perturbed Volterra integral equations.* J. King Saud Univ. – Sci. **23** (2011), 223–228.
3. Gliński H., Grzymkowski R., Kapusta A., Słota D.: *Mathematica 8.* Jacek Skalmierski Computer Studio, Gliwice 2012 (in Polish).
4. Grzymkowski R.: *Taylor transformation and its applications.* Jacek Skalmierski Computer Studio, Gliwice 2015 (in Polish).
5. Grzymkowski R., Hetmaniok E., Pleszczyński M.: *A novel algorithm for solving the ordinary differential equations.* In: *Selected Problems on Experimental*

Mathematics, E. Hetmaniok, D. Słota, T. Trawiński, R. Wituła (eds.), Silesian University of Technology Press, Gliwice 2017, 103–112.

6. Grzymkowski R., Hetmaniok E., Pleszczyński M.: *The Taylor transformation hybrid method applied for solving the Stefan problem.* Silesian J. Pure Appl. Math. **6** (2016), 111–123.
7. Grzymkowski R., Pleszczyński M.: *Application of the Taylor transformation to the systems of ordinary differential equations.* The 24th International Conference on Information and Software Technologies (ICIST 2018), Lithuania (2018), in press.
8. Grzymkowski R., Pleszczyński M., Hetmaniok E.: *Application of the Taylor differential transformation in the calculus of variations.* Silesian J. Pure Appl. Math. **7** (2017), 65–82.
9. Odibat Z.M.: *Differential transform method for solving Volterra integral equation with separable kernels.* Math. Comput. Model. **48** (2008), 1144–1149.
10. Ravi Kanth A.S.V., Aruna K.: *Differential transform method for solving the linear and nonlinear Klein-Gordon equation.* Compu. Phys. Comm. **185** (2009), 708–711.
11. Ravi Kanth A.S.V., Aruna K., Chaurasia R.K.: *Reduced differential transform method to solve two and three dimensional second order hyperbolic telegraph equations.* J. King Saud Univ. – Eng. Sci. **29** (2017), 166–171.
12. Srivastava V.K., Awasthi M.K.: *Differential transform method for solving linear and non-linear systems of partial differential equations.* Phys. Lett. A **372** (2008), 6896–6898.
13. Tari A., Shahmorad S.: *Differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations.* Comput. Math. Appl. **61** (2011), 2621–2629.
14. Wolfram S.: *An Elementary Introduction to the Wolfram Language, Second Edition.* Wolfram Media, Champaign 2017.
15. Wolfram S.: *The Mathematica Book. Fifth Edition.* Wolfram Media, Champaign 2003.