SCIENTIFIC JOURNAL OF POLISH NAVAL ACADEMY

2019 (LX) 4 (219)

sciendo

DOI: 10.2478/sjpna-2019-0024

A GENERALIZATION OF DIFFERENCE MODELS OF THE BITTNER OPERATIONAL CALCULUS

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ABSTRACT

In the paper, there has been constructed a discrete model of the non-classical Bittner operational calculus with a derivative understood as the operation $S_b\{x(k)\} := \{x(k+n) - b(k)x(k)\}$, which is a generalization of the n^{th}-order forward difference. It has also been pointed out that there is a possibility to generalize operational calculus models with backward and central differences of higher orders.

Keywords:

operational calculus, derivative, integrals, limit conditions, forward difference, backward difference, central difference.

Research article

NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* [1–4] is referred to as a system

$$CO(L^0, L^1, S, T_q, s_q, Q)^1, \tag{1}$$

where L^0 and L^1 are linear spaces (over the same field $\mathscr{F}$ of scalars) such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$), called the (abstract) *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ and $s_q \in \mathscr{L}(L^1, L^1)$ such that $ST_qf = f, f \in L^0$ and $s_qx = x - T_qSx, x \in L^1$. These operations are called *integrals* and *limit conditions*, respectively. The kernel of S, i.e. $\operatorname{Ker} S$ is called a set of *constants* for the derivative S. It easy to verify that the limit conditions $s_q, q \in Q$ are projections of L^1 onto the subspace $\operatorname{Ker} S$.

The following auxiliary theorems are used throughout the paper:

Lemma 1 (Th. 3 [4]). *An abstract differential equation*

$$Sx = f, \quad f \in L^0, x \in L^1$$

with a limit condition

$$s_qx = x_{0,q}, \quad x_{0,q} \in \operatorname{Ker} S$$

has exactly one solution

$$x = x_{0,q} + T_qf.$$

Lemma 2 (Th. 4 [4]). *With the given derivative* $S \in \mathscr{L}(L^1, L^0)$ *the projection* $s_q \in \mathscr{L}(L^1, \operatorname{Ker} S)$ *determines an integral* $T_q \in \mathscr{L}(L^0, L^1)$ *from the condition*

$$x = T_qf \quad \textit{if and only if} \quad Sx = f, s_qx = 0.$$

Moreover, the projection s_q *is a limit condition corresponding to the integral* T_q.

If we define objects (1), then we have in mind a *representation* or a *model* of the operational calculus. An example of the operational calculus (1) representation is a *discrete model*, in which the derivative S is understood as an n^{th}*-order forward difference*, i.e.

$$\Delta_n x(k) := x(k + n) - x(k),$$

where n is a specified natural number.

[1] The abbreviation *CO* originates from the French *calcul opératoire* (the operational calculus).

FORWARD DIFFERENCE MODEL

Let $\mathbb{N}_0$ and $\mathbb{C}$ mean a set of non-negative integers and a set of complexes, respectively. Moreover, let $L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$ be a linear space of complex sequences $x = \{x(k)\}_{k\in\mathbb{N}_0}$ with usual sequences addition and sequences multiplication by complex numbers.

In [9] it was proved that to the derivative

$$S\,x \equiv \Delta_n x = \{x(k+n) - x(k)\}^2, \tag{2}$$

there correspond the below integrals

$$T_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\Big[\sum_{i=0}^{k-1}\varepsilon_j^{k-i}\,x(i) - \sum_{i=0}^{k_0-1}\varepsilon_j^{k-i}\,x(i)\,\Big]\right\}^3 \tag{3}$$

and limit conditions

$$s_{k_0}x := \left\{\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=k_0}^{k_0+n-1}\varepsilon_j^{k-i}\,x(i)\right\}, \tag{4}$$

where $x = \{x(k)\} \in L^0, k_0 \equiv q \in Q := \mathbb{N}_0$ and

$$\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$$

are n^{th} roots of unity, i.e.

$$\varepsilon_j = \cos\frac{2j\pi}{n} + \mathrm{i}\,\sin\frac{2j\pi}{n}, \quad j \in \overline{0, n-1}^4,$$

whereas ‘i’ means the imaginary unit.

In [9] there was also considered a more general operational calculus model with the derivative

$$S_b\{x(k)\} := \{x(k+n) - b\,x(k)\}, \tag{5}$$

where $\{x(k)\} \in L^0, b \in \mathbb{C} \setminus \{0\}$.

[2] $\{x(k)\}$ means a symbol of the sequence x, i.e. $x = \{x(k)\}$, whereas $x(k)$ means the k^{th}-term of the sequence $\{x(k)\}$, where $k \in \mathbb{N}_0$. This notation orginates from J. Mikusiński [7].

[3] We assume that $\sum_{i=0}^{-1} x(i) := 0$.

[4] $\overline{0, n-1} := \{0, 1, \ldots, n-1\}$.

A generalization of the forward difference (5) is an operation

$$S_b\{x(k)\} := \{x(k+n) - b(k)x(k)\}^5, \tag{6}$$

which will be called an n^{th}*-order forward difference with the base* $b = \{b(k)\}$.

In this paper, we shall determine integrals T_{b,k_0} and limit conditions s_{b,k_0} corresponding to the derivative (6) with an assumption that $b = \{b(k)\}$ is a real sequence satisfying the condition

$$\bigwedge_{k\in\mathbb{N}_0} b(k) > 0.$$

In order to do that, we shall first determine a positive sequence $e = \{e(k)\}$ such that $e \in \operatorname{Ker} S_b$ and

$$e(0) = e(1) = \ldots = e(n-1) = 1.$$

Thus, we have

$$\ln(e(k+n)) - \ln(e(k)) = \ln(b(k)), \quad k \in \mathbb{N}_0,$$

so

$$h(k+n) - h(k) = v(k), \quad k \in \mathbb{N}_0 \tag{7}$$

and

$$h(0) = h(1) = \ldots = h(n-1) = 0, \tag{8}$$

where

$$h = \{h(k)\} := \{\ln(e(k))\}, \quad v = \{v(k)\} := \{\ln(b(k))\}.$$

The initial value problem (IVP) (7), (8) can be shown as

$$S\,h = v, \quad s_0 h = 0,$$

where S is the forward difference (2), while s_0 is the limit condition (4) for $k_0 := 0$.

On the basis of Lemma 1, a solution to this problem takes the form of

$$h = T_0 v,$$

where T_0 is the integral (3) for $k_0 := 0$.

Eventually, the sequence

$$e = \exp(T_0 v) \tag{9}$$

is the requested constant for the derivative (6), i.e.

$$e(k+n) = b(k)e(k), \quad k \in \mathbb{N}_0.$$

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5 The sequences multiplication in (6) means usual coordinate-wise (Hadamard) multiplication.

Example 1. When b is a constant sequence and $n := 3$, then from (9) we obtain[6]

$$e(k) = b^{\frac{1}{9}\left(3k-3+3\cos\left(\frac{2k\pi}{3}\right)+\sqrt{3}\sin\left(\frac{2k\pi}{3}\right)\right)}, \quad k \in \mathbb{N}_0, \tag{10}$$

that is

$$\{e(k)\} = \{1, 1, 1, b, b, b, b^2, b^2, b^2, b^3, b^3, b^3, b^4, b^4, b^4, \ldots\}. \tag{11}$$

For $n := 4$, we get in turn

$$e(k) = b^{\frac{1}{8}\left((-1)^k-3+2\left(k+\cos\left(\frac{k\pi}{2}\right)+\sin\left(\frac{k\pi}{2}\right)\right)\right)}, \quad k \in \mathbb{N}_0,$$

so

$$\{e(k)\} = \{1, 1, 1, 1, b, b, b, b, b^2, b^2, b^2, b^2, b^3, b^3, b^3, b^3, b^4, b^4, b^4, b^4, \ldots\}.$$

It is easy to notice that also for any $n \in \mathbb{N}$ we have (cf. fig. 1)

$$e(k) = b^{\lfloor k/n \rfloor}, \quad k \in \mathbb{N}_0{}^7. \tag{12}$$

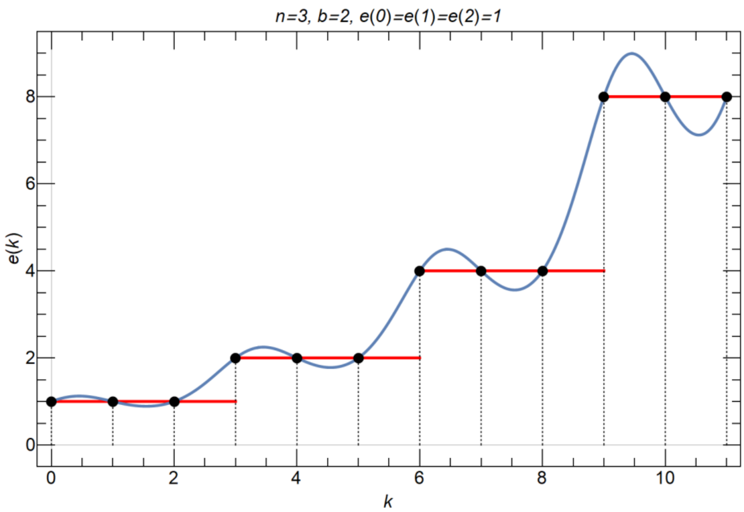

Fig. 1. Graphs of the sequence (11) for $k \in \overline{0, 11}$
and of the functions (10) and (12) for $k \in [0, 11]$

Let us consider a difference equation

$$S_b\{x(k)\} = \{f(k)\},$$

[6] All symbolic and numerical calculations and graphs in this paper have been done using *Mathematica*®.

[7] $\lfloor r \rfloor$ means the floor (the integer part) of a real number r.

i.e.

$$x(k+n) - b(k)x(k) = f(k), \quad k \in \mathbb{N}_0. \tag{13}$$

Hence, we have

$$\frac{x(k+n)}{e(k+n)} - \frac{x(k)}{e(k)} = \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{N}_0,$$

that is

$$y(k+n) - y(k) = g(k), \quad k \in \mathbb{N}_0, \tag{14}$$

where

$$y(k) := \frac{x(k)}{e(k)}, \quad g(k) := \frac{f(k)}{e(k+n)}, \quad k \in \mathbb{N}_0. \tag{15}$$

The equation (14) can be presented in the form of

$$S\{y(k)\} = \{g(k)\}, \tag{16}$$

where $S \equiv \Delta_n$ is the operation (2).

From Lemma 1 it follows that the sequence

$$\{y(k)\} = s_{k_0}\{y(k)\} + T_{k_0}\{g(k)\},$$

where T_{k_0} and s_{k_0} are operations (3) and (4), is a solution to the equation (16).

From (15) we get $x(k) = e(k)\, y(k), k \in \mathbb{N}_0$. Eventually,

$$\{x(k)\} = \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\} + \{e(k)\}T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\} \tag{17}$$

is a solution to the equation (13).

If we assume that

$$\{\widetilde{c}(k)\} := s_{k_0}\left\{\frac{x(k)}{e(k)}\right\},$$

then the sequence $\{\widetilde{c}(k)\} \in \operatorname{Ker} S$ is n-periodic , i.e.

$$\widetilde{c}(k+n) = \widetilde{c}(k), \quad k \in \mathbb{N}_0.$$

Let

$$s_{b,k_0}\{x(k)\} := \{e(k)\}s_{k_0}\left\{\frac{x(k)}{e(k)}\right\}, \quad k_0 \in Q := \mathbb{N}_0, \{x(k)\} \in L^1. \tag{18}$$

Thus, for each $k \in \mathbb{N}_0$ we obtain

$$\begin{aligned} S_b s_{b,k_0} x(k) &= e(k+n)\widetilde{c}(k+n) - b(k)e(k)\widetilde{c}(k) \\ &= e(k+n)(\widetilde{c}(k+n) - \widetilde{c}(k)) = e(k+n) \cdot 0 = 0, \end{aligned}$$

so $s_{b,k_0} \in \mathscr{L}(L^1, \operatorname{Ker} S_b)$. Moreover, for each $k \in \mathbb{N}_0$ we have

$$\begin{aligned} s_{b,k_0}^2 x(k) &= s_{b,k_0}[e(k)\widetilde{c}(k)] = e(k) s_{k_0}\left[\frac{e(k)\widetilde{c}(k)}{e(k)}\right] \\ &= e(k) s_{k_0}\widetilde{c}(k) = e(k)\widetilde{c}(k) = s_{b,k_0} x(k), \end{aligned}$$

because $s_{k_0}\{\widetilde{c}(k)\} = \{\widetilde{c}(k)\}$. Finally, s_{b,k_0} is a projection of L^1 onto $\operatorname{Ker} S_b$ for each $k \in \mathbb{N}_0$.

From Lemma 2 it follows that the projection s_{b,k_0} determines an *integral* T_{b,k_0} from the formula (17). Namely,

$$T_{b,k_0}\{f(k)\} := \{e(k)\} T_{k_0}\left\{\frac{f(k)}{e(k+n)}\right\}, \quad k_0 \in Q, \{f(k)\} \in L^0. \tag{19}$$

What is more, s_{b,k_0} is a *limit condition* corresponding to the integral (19).

Therefore, the system (6), (18), (19) forms a discrete Bittner operational calculus model.

Example 2. A solution to the IVP

$$x(k+3) - 4^k x(k) = 0, \quad k \in \mathbb{N}_0 \tag{20}$$

$$x(5) = 0, x(6) = 1, x(7) = 2 \tag{21}$$

obtained by using the RSolve command in *Mathematica®*, has a form of

$$x(k) = \frac{1}{3} \cdot 4^{\frac{1}{6}(k-7)(k+4)} \left(4^{5/3} + 2 + 2\left(4^{5/3} - 1\right)\cos\left(\frac{2k\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{2k\pi}{3}\right)\right), \quad k \in \mathbb{N}_0, \tag{22}$$

from which we get

$$\{x(k)\} = \{4^{-3}, 2\cdot 4^{-5}, 0, 4^{-3}, 2\cdot 4^{-4}, 0, 1, 2, 0, 4^6, 2\cdot 4^7, 0, 4^{15}, 2\cdot 4^{17}, 0, 4^{27}, 2\cdot 4^{30}, \ldots\}. \tag{23}$$

The homogeneous equation (20) can be presented as $S_b x = 0$, where S_b is the forward difference (6) and $n := 3, b := \{4^k\}$. From (17) we obtain a solution to (20) expressed by means of the limit condition (18), i.e.

$$\{x(k)\} = s_{\{4^k\},5}\{x(k)\},$$

which is equivalent to the initial conditions (21). Hence, we have

$$x(k)=\frac{1}{3}\cdot 4^{\frac{1}{18}\left(3(k-3)k-4\cos\left(\frac{2k\pi}{3}\right)-86\right)}\left(4\cos\left(\frac{2(k-1)\pi}{3}\right)+32\cos\left(\frac{2k\pi}{3}\right)+18\right),\quad k\in\mathbb{N}_0. \quad (24)$$

Using *Mathematica*® we can easily verify the equality of functions (22) and (24) in the domain $\mathbb{N}_0$. Namely, after running the code

```
x1=...;
x2=...;
FullSimplify[x1==x2, Assumptions->Element[k,Integers] && k>=0]
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where ... denote right-hand sides of (22) and (24), respectively, we obtain the boolean value True (cf. fig. 2).

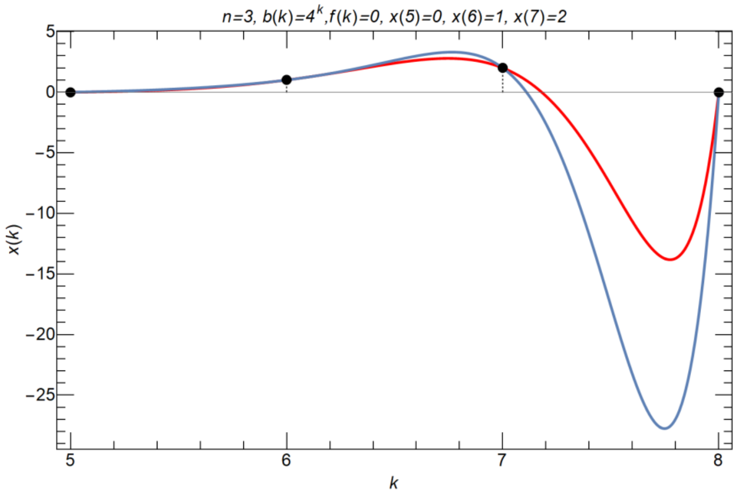

Fig. 2. Graphs of the sequence (23) for $k \in \overline{5,8}$
and of the functions (22) and (24) for $k \in [5,8]$

Example 3. A non-linear equation

$$u(k+n)[u(k)]^{-b(k)} = g(k), \quad k \in \mathbb{N}_0{}^8 \quad (25)$$

with the below initial conditions

$$u(k_0) = u_0, u(k_0+1) = u_1, \ldots, u(k_0+n-1) = u_{n-1} \quad (26)$$

can be converted to the linear equation (13), where

$$x(k) := \ln(u(k)), \quad f(k) := \ln(g(k)), \quad k \in \mathbb{N}_0,$$

[8] We assume that the sequences in (25) are positive.

to which there correspond the initial conditions

$$x(k_0+i)=\ln(u_i),\quad i\in\overline{0,n-1}.$$

Thus, the sequence

$$u(k)=\exp(x(k)),\quad k\in\mathbb{N}_0,$$

where $x(k)$ has been determined from (17), is a solution to the problem (25), (26). For instance, for the problem

$$u(k+1)[u(k)]^{-(k+1)}=a^k,\quad u(k_0)=u_0,\quad (a,u_0>0)$$

we get the sequence

$$u(k)=\frac{(au_0)^{k!/k_0!}}{a},\quad k\in\mathbb{N}_0.$$

Example 4. A non-linear difference equation

$$\alpha(k)u(k+n)u(k)+\beta(k)u(k+n)+\gamma(k)u(k)=0,\quad k\in\mathbb{N}_0 \tag{27}$$

is called the n^{th}*-order Riccati equation* (cf. [5,6]).

Let us assume that $u(k)\neq 0$ and $\beta(k)\gamma(k)<0$ for each $k\in\mathbb{N}_0$. Then, the equation (27) can also be converted to the equation (13) if we take

$$x(k):=\frac{1}{u(k)},\ b(k):=-\frac{\beta(k)}{\gamma(k)},\ f(k):=-\frac{\alpha(k)}{\gamma(k)},\quad k\in\mathbb{N}_0.$$

Then, to the initial conditions

$$u(k_0)=u_0,u(k_0+1)=u_1,\ldots,u(k_0+n-1)=u_{n-1} \tag{28}$$

there correspond the following conditions

$$x(k_0+i)=\frac{1}{u_i},\quad i\in\overline{0,n-1}.$$

After determining the sequence $\{x(k)\}$ from (17), we obtain a solution to the problem (27),(28):

$$u(k)=\frac{1}{x(k)},\quad k\in\mathbb{N}_0.$$

For instance, we convert the non-linear Cauchy problem[9]

$$(1-2k-k^2)u(k+2)u(k)+(k+2)u(k+2)-u(k)=0,\quad k\in\mathbb{N}_0 \tag{29}$$

[9] *Mathematica*® failed to solve this problem, while *Maple*™ gave the solution (33).

$$u(0) = 1, \quad u(1) = \frac{1}{2} \tag{30}$$

to the IVP

$$x(k+2) - (k+2)x(k) = 1 - 2k - k^2, \quad k \in \mathbb{N}_0 \tag{31}$$

$$x(0) = 1, \quad x(1) = 2. \tag{32}$$

Then, on the basis of (17), the sequence (18), that is

$$s_{\{k+2\},0}\{x(k)\} = \left\{ \frac{2^{k/2}\left(2^{3/2}\left((-1)^{k+1}+1\right) + \sqrt{\pi}\left((-1)^k+1\right)\right)\Gamma\left(\frac{k+2}{2}\right)}{2\sqrt{\pi}} \right\},$$

i.e.

$$s_{\{k+2\},0}\{x(k)\} = \{1, 2, 2, 6, 8, 30, 48, 210, 384, 1890, 3840, 20790, \ldots\},$$

is the solution to the homogeneous equation

$$x(k+2) - (k+2)x(k) = 0, \quad k \in \mathbb{N}_0$$

with the conditions (32). The sequence (19), that is

$$T_{\{k+2\},0}\{1 - 2k - k^2\} = \left\{ k + 1 - \frac{2^{k/2}\left(2^{3/2}\left((-1)^{k+1}+1\right) + \sqrt{\pi}\left((-1)^k+1\right)\right)\Gamma\left(\frac{k+2}{2}\right)}{2\sqrt{\pi}} \right\},$$

i.e.

$$T_{\{k+2\},0}\{1 - 2k - k^2\} = \{0, 0, 1, -2, -3, -24, -41, -202, -375, -1880, -3829, -20778, \ldots\},$$

is in turn the solution to the non-homogeneous equation (31) with the conditions $x(0) = x(1) = 0$.

Finally, the sequences

$$x(k) = k + 1, \quad k \in \mathbb{N}_0 \quad \text{and} \quad u(k) = \frac{1}{k+1}, \quad k \in \mathbb{N}_0 \tag{33}$$

are solutions to the problems (31),(32) and (29),(30), respectively.

OTHER DISCRETE MODELS

The idea of generalizing the model with the forward difference (2) can be applied to constructing other discrete representations of the Bittner operational calculus. Namely, considering models with the *backward difference* and the *central difference*, described in [8] and [10], similarly as above we can determine integrals T_{b,k_0} and limit conditions s_{b,k_0} that correspond to the below difference operations:

$$S_b\{x(k)\} := \{x(k) - b(k)x(k-n)\}$$
$$S_b\{x(k)\} := \{x(k+n) - b(k)x(k-n)\}.$$

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UOGÓLNIENIE RÓŻNICOWYCH MODELI RACHUNKU OPERATORÓW BITTNERA

STRESZCZENIE

W artykule skonstruowano model dyskretny nieklasycznego rachunku operatorów Bittnera z pochodną rozumianą jako operacja $S_b\{x(k)\} = \{x(k+n) - b(k)x(k)\}$, która jest uogólnieniem różnicy progresywnej rzędu n. Wskazano również na możliwość uogólnienia modeli rachunku operatorów z różnicą wsteczną i różnicą centralną wyższych rzędów.

Słowa kluczowe:

rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica progresywna, różnica wsteczna, różnica centralna.

Article history

Received: 18.10.2019
Reviewed: 10.12.2019
Revised: 17.12.2019
Accepted: 18.12.2019