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Secretary problem and two almost the same consecutive applicants

Abstract The classical secretary problem involves sequentially interviewing a pool of n applicants with the aim of hiring exactly the best one in the pool; nothing less is good enough. The decision maker's strategy should maximize the probability of appropriate selection. The various modifications of the aim under the probability maximization criterion do not contain the issue of selecting the pairs of secretaries of very close absolute ranks. This paper is devoted to such a concern, which is formulated in a rigorous way. The effectiveness of the threshold rules is analyzed. It is shown that the asymptotic probability of success in this class of strategies may achieve 0.5.

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1. Introduction. The classical *secretary problem* can be formulated as follows:

- There is a set A_n of n rankable applicants.
- A company wants to hire the best applicant from the set A_n.
- The applicants come sequentially in a random order to be interviewed by the company.
- After interviewing an applicant, the company has to immediately decide if the applicant is selected or rejected.
- The rejected applicants cannot be recalled.
- The company knows only the number n and the relative ranks of the applicants being interviewed so far.

The first articles presenting a solution to the secretary problem are [3], [4], and [8]. An optimal stopping rule maximizes the probability of selecting the best applicant. As n tends to infinity, the optimal stopping rule for the classical secretary problem is as follows: Reject the first ne^{-1} applicants and then select the first applicant, who is the best among all the applicants being interviewed so far. The probability of selecting the best applicant with this asymptotic optimal stopping rule is equal to e^{-1}.

Quite many generalizations of the secretary problem have been investigated. To name just a few examples, in [17] an optimal stopping rule is shown for selecting an applicant, who is "representative" for the given set of applicants; a selection of the median object and a selection of any object from a set of middle ranks are considered. The articles [2, 14, 16, 19, 21] research an optimal stopping rule for selecting the best or the worst applicant and for selecting the second best applicant (also known as postdoc problem).

In [13, 16, 22], the selection of more than one candidate is investigated.

In [6], a version of the secretary problem is investigated with the goal to stop on the element closest to the center of the interval $(0, 1)$, where the observed elements are independent random variables from the uniform distribution on the interval $(0, 1)$.

For more variations of the secretary problem, we recommend the articles [9], [10], [11], and [12].

Some recent surveys on generalizations of the secretary problem can be found in [2], [5], and [20].

In the current article, we consider a variant of the secretary problem, that consists in selecting two consecutive applicants whose absolute ranks differ by one. Let A_n be a totally ordered set of n applicants. Given $P \subseteq A_n$ and $x \in A_n$, let $\mathbf{RelRank}(P, x) = |\{z \in P \mid z \leq x\}|$ be the *relative rank of* x *with regard to* P, and let $\mathbf{RelRank}_n(x) = \mathbf{RelRank}(A_n, x)$ be the *absolute rank* of the candidate x. We formulate the variant as follows:

- A company wants to hire two applicants $x, y \in A_n$ such that their absolute ranks differ by one; formally $\mathbf{RelRank}_n(x) - \mathbf{RelRank}_n(y) \in \{-1, 1\}$.
- The applicants come sequentially in a random order to be interviewed by the company. The manager of company observes $\xi_1, \xi_2, \ldots, \xi_n$–the sequences of candidates.
- After interviewing an applicant ξ_j, the company has to decide if the previous applicant, i.e. ξ_{j-1} is selected or rejected.
- After interviewing an applicant ξ_j, the company is allowed to immediately decide if the applicants ξ_{j-1}, ξ_j are selected or rejected.
- The rejected applicants cannot be recalled.
- The company knows only the number n and the relative ranks of the applicants being interviewed so far.

From the formulation, it follows that the two selected applicants have to be consecutive in the random sequence.

Let α be a real constant with $0 < \alpha < 1$ and let $x_1, x_2, \ldots, x_n \in A_n$ be a random sequence of distinct applicants. Suppose the following stopping rule $\tau_n(\alpha)$: select the first pair of applicants x_{j-1}, x_j such that $\mathbf{RelRank}(P_j, x_{j-1}) - \mathbf{RelRank}(P_j, x_j) \in \{-1, 1\}$, where

$$P_j = \{x_i \mid 1 \leq i \leq j\} \quad \text{and} \quad j > \lfloor \alpha n \rfloor.$$

In case that there is no such pair, the stopping rule $\tau_n(\alpha)$ stops at the last applicant.

REMARK 1.1 Note that the stopping rule $\tau_n(\alpha)$ rejects first $\lfloor \alpha n \rfloor - 1$ applicants.

Let $\mathbf{P}_{n,\tau}(\alpha)$ be the probability that $\tau_n(\alpha)$ selects x_j such that

$$\mathbf{RelRank}_n(x_{j-1}) - \mathbf{RelRank}_n(x_j) \in \{-1, 1\}.$$

The main result of the current article is the following theorem. The proof of the theorem is presented at the end of the article.

THEOREM 1.2 *If α is a real constant and $0 < \alpha < 1$ then*

$$\lim_{n\to\infty} \boldsymbol{P}_{n,\tau}(\alpha) \leq \frac{1}{2},$$

while the equality holds for $\alpha = \frac{1}{2}$.

Less formally stated, with the given stopping rule $\tau_n(\alpha)$, the optimal strategy is to reject the first half of applicants, and then to select the first consecutive pair of applicants, whose relative ranks differ by one between the applicants being interviewed so far. The probability of success with this rule tends to 2^{-1} as n tends to infinity.

REMARK 1.3 The integer sequence A002464[1] expresses "the number of permutations of length n without rising or falling successions". Let $a(n)$ denote this integer sequence A002464. It is known [1, 15], that $\lim_{n\to\infty} \frac{a(n)}{n!} = e^{-2}$.

It follows that if $x_1, x_2, \ldots, x_n \in A_n$ is a random sequence of n distinct applicants, then with probability e^{-2} there is no $j \in \{2, \ldots, n\}$ such that $\mathbf{RelRank}_n(x_{j-1}) - \mathbf{RelRank}_n(x_j) \in \{-1, 1\}$ as n tends to infinity. It gives the upper bound for the asymptotic probability of success of the optimal algorithm equal to $1 - e^{-2}$.

2. Probabilistic point of view Although our proofs are purely combinatorial, we explain in this section the probabilistic model that is commonly used when dealing with the secretary problem and its variants; see, for instance, [7] and [20].

Suppose a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Elementary events are permutations of all candidates and the probability measure $\mathbf{P}$ is evenly distributed over Ω. Observations of relative ranks R_k define the sequence of σ-fields $\mathcal{F}_k = \sigma(R_1, R_2, \ldots, R_k)$, where $k \in \{1, 2, \ldots, n\}$. The random variables

[1] v. [18], https://oeis.org/A002464

R_k are independent and $\mathbf{P}\{R_k = i\} = \frac{1}{k}$, where $k \in \{1, 2, \dots, n\}$ and $i \in \{1, 2, \dots, k\}$. Let $\mathfrak{M}^n$ denote the set of all Markov moments τ relative to the sequence of σ-fields $\{\mathcal{F}_k\}_{k=1}^n$.

The classical secretary problem can be stated as follows: We are looking for $\tau^* \in \mathfrak{M}^n$ such that

$$\mathbf{P}\{Z_{\tau^*} = 1\} = \sup_{\tau \in \mathfrak{M}^n} \mathbf{P}\{Z_\tau = 1\},$$

where $\{Z_k\}_{k=1}^n$ are the absolute ranks of the observations. In the observation process they are hidden and they are recovered at the very end of the selection procedure.

The problem considered in this article can be formulated as follows: Find $\tau^* \in \mathfrak{M}^n$ such that

$$\mathbf{P}\{(Z_{\tau^*} - Z_{\tau^*-1}) \in \{-1, 1\}\} = \sup_{\tau \in \mathfrak{M}^n} \mathbf{P}\{(Z_\tau - Z_{\tau-1}) \in \{-1, 1\}\}.$$

In addition we present an open question: Is it a problem to solve the double stopping problem; it means to find $\tau, \nu \in \mathfrak{M}^n$ such that

$$\mathbf{P}\{(Z_{\tau^*} - Z_{\nu^*}) \in \{-1, 1\}\} = \sup_{\tau, \nu \in \mathfrak{M}^n} \mathbf{P}\{(Z_\tau - Z_\nu) \in \{-1, 1\}\}?$$

3. Preliminaries For the whole article, suppose that $n > 3$, where $n = |A_n|$.

Let $\mathbb{R}^+$ denote the set of all positive real numbers, let $\mathbb{Z}$ denote the set of all integers, and let $\mathbb{N}^+$ denote the set of all positive integers.

Given $P \subseteq A_n$ with $P \neq \emptyset$, let $\min\{P\} \in P$ and $\max\{P\} \in P$ be the applicants such that $\min\{P\} \leq x$ and $\max\{P\} \geq x$ for all $x \in P$. Thus $\min\{P\}$ and $\max\{P\}$ are the applicants from P with the minimal and the maximal rank, respectively.

Given $j \in \mathbb{N}^+$, let $A_n^j = \{(x_1, \dots, x_j) \mid x_i \in A_n \text{ for all } i \in \{1, 2, \dots, j\}\}$ and let $A_n^+ = \bigcup_{j \in \mathbb{N}^+} A_n^j$. The elements of A_n^+ are called *sequences of applicants* or just *sequences.*

Suppose $\vec{x}, \vec{y} \in A_n^+$, where $\vec{x} = (x_1, x_2, \dots, x_i)$ and $\vec{y} = (y_1, y_2, \dots, y_j)$. Let $\vec{x} \circ \vec{y} \in A_n^{i+j}$ denote the concatenation of $\vec{x}$ and $\vec{y}$; formally

$$\vec{x} \circ \vec{y} = (x_1, x_2, \dots, x_i, y_1, y_2 \dots, y_j) \in A_n^{i+j}.$$

Remark 3.1 We consider that $A_n = A_n^1$; it means that if $x \in A_n$ then $x = (x)$ is a sequence of length 1.

Given $\vec{x} = (x_1, x_2, \dots, x_k) \in A_n^+$, let $|\vec{x}| = k$ denote the length of $\vec{x}$, let $\vec{x}[i] = x_i \in A_n$, and let

$$\vec{x}[i, j] = (x_i, x_{i+1}, \dots, x_j) \in A_n^{j-i+1},$$

where $i, j \in \{1, 2, \ldots, k\}$ and $i \leq j$.

Given $\vec{x} \in A_n^+$, let $\text{toSet}(\vec{x}) = \{\vec{x}[i] \mid 1 \leq i \leq |\vec{x}|\}$ be the set of applicants of the sequence $\vec{x}$.

Given $k \in \mathbb{N}^+$, $k \leq n$, let

$$\Gamma_n(k) = \{\vec{x} \in A_n^k \mid \vec{x}[i] = \vec{x}[j] \text{ implies } i = j \text{ for all } i, j \in \{1, 2, \ldots, k\}\}.$$

Let $\Gamma_n = \Gamma_n(n)$. Obviously $|\Gamma_n(k)| = \frac{n!}{(n-k)!}$ and in particular $|\Gamma_n| = n!$.

Remark 3.2 The set $\Gamma_n(k)$ contains sequences of k distinct applicants. The sequences of Γ_n represent the sequences of applicants that the company is interviewing when selecting the applicants.

Given $k \in \mathbb{N}^+$, let $\Delta(k) = \{-1, 1, k-1, 1-k\} \subseteq \mathbb{Z}$. Given $P \subseteq A_n$, let

$$\begin{aligned}\text{Adj}(P) = \{\{x, y\} \mid x, y \in P \text{ and}\\ \mathbf{RelRank}(P, x) - \mathbf{RelRank}(P, y) \in \Delta(|P|)\}.\end{aligned}$$

If $\{x, y\} \in \text{Adj}(P)$ then we say that the applicants x, y are *adjacent* with regard to P.

Remark 3.3 The set $\text{Adj}(P)$ contains all sets $\{x, y\}$ such that the difference of relative ranks of x, y with regard to P is from the set $\Delta(|P|)$.

The set $\Delta(k)$ contains the differences in relative ranks, that "the company is looking for"; it means $\{-1, 1\} \subseteq \Delta(k)$. In addition the set $\Delta(k)$ contains values $1-k, k-1$. In consequence the applicants $\max\{P\}$ and $\min\{P\}$ are adjacent with regard to P.

4. Consecutive applicants We present a technical lemma that we use in the proof of Proposition 4.4.

Lemma 4.1 *If $P \subseteq A_n$, $|P| \geq 3$, $x \in P$, and*

$$\upsilon_n(P, x) = |\{y \in P \mid y \neq x \text{ and } \{x, y\} \in \text{Adj}(P)\}|,$$

then $\upsilon_n(P, x) = 2$.

Proof Clearly, we have that

- If $x \in P \setminus \{\max\{P\}\}$ then there is exactly one applicant $y \in P$ such that $\mathbf{RelRank}(P, x) - \mathbf{RelRank}(P, y) = -1$.
- If $x \in P \setminus \{\min\{P\}\}$ then there is exactly one applicant $y \in P$ such that $\mathbf{RelRank}(P, x) - \mathbf{RelRank}(P, y) = 1$.
- If $x = \max\{P\}$ then there is exactly one applicant $y \in P$ such that $\mathbf{RelRank}(P, x) - \mathbf{RelRank}(P, y) = |P| - 1$.

- If $x = \min\{P\}$ then there is exactly one applicant $y \in P$ such that $\mathbf{RelRank}(P, x) - \mathbf{RelRank}(P, y) = 1 - |P|$.

Note that x satisfies exactly two of the four mentioned conditions. Each condition produces exactly one y corresponding to x. The lemma follows. ∎

Remark 4.2 Lemma 4.1 is the reason that we have defined that $\max\{P\}$ and $\min\{P\}$ are adjacent with regard to P. Otherwise the number of y's used in the definition of $\upsilon_n(P, x)$ would depend on $x \in P$. As a result the proof of the next proposition will be more simple.

Given $r, k \in \{3, 4, \ldots, n\}$ with $r < k$, let

$$\begin{aligned}\Lambda_n(r, k) = \{\vec{x} \in \Gamma_n \mid \mathbf{RelRank}_n(\vec{x}[k-1]) - \mathbf{RelRank}_n(\vec{x}[k]) \in \{-1, 1\} \\ \text{and } \{\vec{x}[i-1], \vec{x}[i]\} \notin \mathrm{Adj}(\mathrm{toSet}(\vec{x}[1, i])) \\ \text{for all } i \in \{r+1, r+2, \ldots, k-1\}\}.\end{aligned}$$

Let $\Lambda_n(r) = \bigcup_{k=r+1}^{n} \Lambda_n(r, k)$. Obviously $\Lambda_n(r, k) \cap \Lambda_n(r, \overline{k}) = \emptyset$ if $k \neq \overline{k}$.

Remark 4.3 We have that if $\vec{x} \in \Lambda_n(r)$ then there is exactly one k such that $r < k \leq n$ and $\vec{x} \in \Lambda_n(r, k)$.

Given $r, k \in \{3, 4, \ldots, n\}$ with $r < k$, and $z, \overline{z} \in A_n$, let

$$\Lambda_n(r, k, z, \overline{z}) = \{\vec{x} \in \Lambda_n(r, k) \mid \vec{x}[k-1] = z \text{ and } \vec{x}[k] = \overline{z}\}.$$

The sets $\Lambda_n(r, k, z, \overline{z})$ form a partition of the set $\Lambda_n(r, k)$. We derive a formula for the size of the sets $\Lambda_n(r, k, z, \overline{z})$.

Proposition 4.4 *If $z, \overline{z} \in A_n$, $\boldsymbol{RelRank}_n(z) - \boldsymbol{RelRank}_n(\overline{z}) \in \{-1, 1\}$, $r, k \in \{3, 4, \ldots, n\}$, and $r < k$ then*

$$|\Lambda_n(r, k, z, \overline{z})| = (n-2)! \frac{(r-1)(r-2)}{(k-2)(k-3)}.$$

Proof Given $\vec{y} \in A_n^+$, let

$$\omega(\vec{y}) = A_n \setminus (\{z, \overline{z}\} \cup \mathrm{toSet}(\vec{y})) .$$

Given $j \in \mathbb{N}^+$ and $D \subseteq A_n^+$, let

$$\mathrm{Suffix}_n(D, j) = \{\vec{y} \in A_n^j \mid \text{ there is } \vec{x} \in A_n^+ \text{ such that } \vec{x} \circ \vec{y} \in D\}.$$

Let $k, z, \overline{z}$ be given. For $j \in \{1, 2, \ldots, n\}$, we define the sets $H(j) \subseteq \mathrm{Suffix}_n(\Gamma_n, n-j+1)$ as follows. Let

$$H(n) = \begin{cases} \{\overline{z}\} & \text{if } k = n \\ \{x \mid x \in A_n \setminus \{z, \overline{z}\}\} & \text{otherwise.} \end{cases}$$

Given $j \in \{k+1, k+2, \ldots, n-1\}$, let

$$H(j) = \{(x) \circ \vec{y} \mid \vec{y} \in H(j+1) \text{ and } x \in \omega(\vec{y})\}.$$

Let

$$H(k) = \begin{cases} H(n) = \{\overline{z}\} & \text{if } k = n \\ \{(\overline{z}) \circ \vec{y} \mid \vec{y} \in H(k+1)\} & \text{otherwise.} \end{cases}$$

Let $H(k-1) = \{(z) \circ \vec{y} \mid \vec{y} \in H(k)\}$. Given $j \in \{r, r+1, \ldots, k-2\}$, let

$$H(j) = \big\{(x) \circ \vec{y} \mid \vec{y} \in H(j+1) \text{ and } \\ x \in \omega(\vec{y}) \text{ and } \{x, \vec{y}[1]\} \notin \mathrm{Adj}(\omega(\vec{y}) \cup \{\vec{y}[1]\})\big\}.$$

Given $j \in \{1, 2, \ldots, r-1\}$, let

$$H(j) = \{(x) \circ \vec{y} \mid \vec{y} \in H(j+1) \text{ and } x \in \omega(\vec{y})\}.$$

We have defined sets $H(j)$ for $j \in \{1, 2, \ldots, n\}$. It is straightforward to verify that $\Lambda(r, k, z, \overline{z}) = H(1)$. We derive the formulas for the size of $H(j)$. Note that if $\vec{y} \in H(j+1) \subseteq \mathrm{Suffix}_n(\Gamma_n, n-j)$, then $|\vec{y}| = n - j$. From the definition of $H(j)$ it follows that

- $$|H(n)| = \begin{cases} 1 & \text{if } k = n \\ n-2 & \text{otherwise.} \end{cases}$$
- If $j \in \{k+1, k+2, \ldots, n-1\}$ then $|H(j)| = (j-2)|H(j+1)|$. Realize that if $\vec{y} \in H(j+1)$ then $|\vec{y}| = n - j$, $\mathrm{toSet}(\vec{y}) \cap \{z, \overline{z}\} = \emptyset$, and $|\{z, \overline{z}\}| = 2$. It follows that $|\omega(\vec{y})| = (n - (n-j) - 2) = j - 2$.
- $$|H(k)| = \begin{cases} |\{\overline{z}\}| = 1 & \text{if } k = n \\ |H(k+1)| & \text{otherwise.} \end{cases}$$
- $|H(k-1)| = |H(k)|$.
- If $j \in \{r, r+1, \ldots, k-2\}$ then $|H(j)| = (j-2)|H(j+1)|$. Realize that if $\vec{y} \in H(j+1)$ then $|\vec{y}| = n - j$ and $\{z, \overline{z}\} \subseteq \mathrm{toSet}(\vec{y})$. It follows that $|\omega(\vec{y})| = (n-(n-j)) = j$. Moreover, since $|\omega(\vec{y}) \cup \{\vec{y}[1]\}| \geq 3$, Lemma 4.1 implies that there are exactly two distinct applicants $x, \overline{x} \in \omega(\vec{y})$ such that

  $$\{x, \vec{y}[1]\}, \{\overline{x}, \vec{y}[1]\} \in \mathrm{Adj}(\omega(\vec{y}) \cup \{\vec{y}[1]\}).$$
- If $j \in \{1, 2, \ldots, r-1\}$ then $|H(j)| = (n-(n-j))|H(j+1)| = j|H(j+1)|$. Realize that if $\vec{y} \in H(j+1)$ then $|\vec{y}| = n - j$ and $\{z, \overline{z}\} \subseteq \mathrm{toSet}(\vec{y})$. It follows that $|\omega(\vec{y})| = (n - (n-j)) = j$.

j	n	n-1	...	k+1	k	k-1	k-2	...	r+1	r	r-1	...
g(j)	n-2	n-3	...	k-1	1	1	k-4	...	r-1	r-2	r-1	...

Table 1: The values of $g(j)$.

Let $g(j) = \frac{|H(j)|}{|H(j+1)|}$ for $j \in \{1, 2, \ldots, n-1\}$ and let $g(n) = n - 2$. The value of $g(n)$ represents the size of $H(n)$, when $k \neq n$. Table 1 shows the values of $g(j)$; the case with $k = n$ is represented by the table shrunk to the columns from $j = k$ to $j = 1$. From the table we can derive that

$$|H(1)| = (n-2)! \frac{(r-1)(r-2)}{(k-2)(k-3)}.$$

Since $\Lambda(r, k, z, \overline{z}) = H(1)$ this completes the proof. ∎

Using Proposition 4.4, we can present a formula for the size of $\Lambda_n(r, k)$.

Lemma 4.5 *If* $r, k \in \{3, 4, \ldots, n\}$ *and* $r < k$ *then*

$$|\Lambda_n(r, k)| = 2(n-1)! \frac{(r-1)(r-2)}{(k-2)(k-3)}.$$

Proof It is clear that

$$\Lambda_n(r, k) = \bigcup_{(z, \overline{z}) \in A_n^2} \Lambda_n(r, k, z, \overline{z}).$$

If $z_1, z_2, z_3, z_4 \in A_n$ and $(z_1, z_2) \neq (z_3, z_4)$ then

$$\Lambda_n(r, k, z_1, z_2) \cap \Lambda_n(r, k, z_3, z_4) = \emptyset.$$

If $z, \overline{z} \in A_n$ and $\mathbf{RelRank}_n(z) - \mathbf{RelRank}_n(\overline{z}) \notin \{-1, 1\}$ then from the definition of $\Lambda_n(r, k)$ and $\Lambda_n(r, k, z, \overline{z})$ we have that

$$\Lambda_n(r, k, z, \overline{z}) = \emptyset.$$

Let $T = \{(z, \overline{z}) \in A_n^2 \mid \mathbf{RelRank}_n(z) - \mathbf{RelRank}_n(\overline{z}) \in \{-1, 1\}\}$. It follows then that

$$|\Lambda_n(r, k)| = \sum_{(z, \overline{z}) \in T} |\Lambda_n(r, k, z, \overline{z})|. \tag{1}$$

Obviously we have that

$$|T| = 2(n-1). \tag{2}$$

The lemma follows from (1), (2), and Proposition 4.4. This completes the proof. ∎

Given a sequence $\vec{x} \in \Gamma_n$ and $r \in \{3, 4, \dots, n-1\}$, the next theorem shows the probability that $\vec{x} \in \Lambda_n(r)$.

Theorem 4.6 *If* $r \in \{3, 4, \dots, n-1\}$ *then*

$$\frac{|\Lambda_n(r)|}{|\Gamma_n|} = 2\frac{r-1}{n} - 2\frac{(r-1)(r-2)}{n(n-2)}.$$

Proof Since $\Lambda_n(r) = \bigcup_{k=r+1}^{n} \Lambda_n(r,k)$ and $\Lambda_n(r,k) \cap \Lambda_n(r,\overline{k}) = \emptyset$ if $k \neq \overline{k}$, we have that

$$|\Lambda_n(r)| = \sum_{k=r+1}^{n} |\Lambda_n(r,k)|. \tag{3}$$

Recall that $|\Gamma_n| = n!$ and note that $\frac{1}{(k-2)(k-3)} = \frac{1}{k-3} - \frac{1}{k-2}$. Then from Lemma 4.5 and (3) it follows that

$$\frac{|\Lambda_n(r)|}{|\Gamma_n|} = \sum_{k=r+1}^{n} \frac{|\Lambda_n(r,k)|}{n!} = \sum_{k=r+1}^{n} 2\frac{(n-1)!}{n!}\frac{(r-1)(r-2)}{(k-2)(k-3)} =$$

$$2\frac{(r-1)(r-2)}{n}\sum_{k=r+1}^{n}\frac{1}{(k-2)(k-3)} = 2\frac{(r-1)(r-2)}{n}\left(\frac{1}{r-2} - \frac{1}{n-2}\right) =$$

$$2\frac{r-1}{n} - 2\frac{(r-1)(r-2)}{n(n-2)}.$$

This completes the proof. ∎

Given a sequence $\vec{x} \in \Gamma_n$, $\alpha \in \mathbb{R}^+$, and $\alpha < 1$, the next lemma shows the probability that $\vec{x} \in \Lambda_n(\lfloor \alpha n \rfloor)$ as n tends to infinity.

Lemma 4.7 *If* $\alpha \in \mathbb{R}^+$, $\alpha < 1$ *then*

$$\lim_{n\to\infty} \frac{|\Lambda_n(\lfloor \alpha n \rfloor)|}{|\Gamma_n|} = 2\alpha - 2\alpha^2 \leq \frac{1}{2},$$

while the equality holds for $\alpha = \frac{1}{2}$.

Proof From Theorem 4.6 we get that

$$\begin{aligned}\lim_{n\to\infty} \frac{|\Lambda_n(\lfloor \alpha n \rfloor)|}{|\Gamma_n|} &= \lim_{n\to\infty}\left(2\frac{\lfloor \alpha n \rfloor - 1}{n} - 2\frac{(\lfloor \alpha n \rfloor - 1)(\lfloor \alpha n \rfloor - 2)}{n(n-2)}\right)\\ &= \lim_{n\to\infty}\left(2\frac{\alpha n - 1}{n} - 2\frac{(\alpha n - 1)(\alpha n - 2)}{n(n-2)}\right)\\ &= 2\alpha - 2\alpha^2.\end{aligned}$$

It is easy to verify that the function $f(\alpha) = 2\alpha - 2\alpha^2$ has a maximum for $\alpha = 2^{-1}$ and that $f(2^{-1}) = 2^{-1}$. This ends the proof. ∎

5. Bijections on sequences of applicants Let $\rho : A_n \to A_n$ be a function defined as follows. Given $x \in A_n$, let

$$\rho(x) = \begin{cases} y \in A_n \text{ such that} & \\ \quad \mathbf{RelRank}_n(y) - \mathbf{RelRank}_n(x) = 1 & \text{if } x \neq \max\{A_n\} \\ \min\{A_n\} & \text{if } x = \max\{A_n\}. \end{cases}$$

Given $x \in A_n$, we define that $\rho^1(x) = \rho(x)$ and $\rho^{i+1}(x) = \rho(\rho^i(x))$, where $i \in \{1, 2, \dots, n-1\}$. We define a function $\rho : \Gamma_n \to \Gamma_n$ as follows:

$$\rho(\vec{x}) = (\rho(\vec{x}[1]), \rho(\vec{x}[2]), \dots, \rho(\vec{x}[n])) \in \Gamma_n, \text{ where } \vec{x} \in \Gamma_n.$$

Given $\vec{x} \in \Gamma_n$, let $\rho^1(\vec{x}) = \rho(\vec{x})$ and $\rho^{i+1}(\vec{x}) = \rho(\rho^i(\vec{x}))$, where $i \in \{1, 2, \dots, n-1\}$. Let $R_n(\vec{x}) = \{\rho^i(\vec{x}) \mid i \in \{1, 2, \dots, n\}\}$. It is clear that $\rho(x)$, $\rho(\vec{x})$ are bijections, $\rho^n(x) = x$, $\rho^n(\vec{x}) = \vec{x}$, and

$$|R_n(\vec{x})| = n. \tag{4}$$

Example 5.1 Let $A_n = \{1, 2, \dots, 9\}$ and let $\vec{x} = (3, 5, 4, 6, 2, 7, 8, 9, 1) \in \Gamma_n$. Then we have that:
$\rho^1(\vec{x}) = (4, 6, 5, 7, 3, 8, 9, 1, 2)$,
$\rho^2(\vec{x}) = (5, 7, 6, 8, 4, 9, 1, 2, 3)$,
$\rho^3(\vec{x}) = (6, 8, 7, 9, 5, 1, 2, 3, 4)$.

We present a simple lemma that we will need. We omit the proof. The lemma shows that the adjacent applicants on positions $i-1, i$ in a sequence $\vec{x} \in \Gamma_n$ remain adjacent in the sequence $\rho(\vec{x})$.

Lemma 5.2 *If $\vec{x} \in \Gamma_n$, $i \in \{2, 3, \dots, n\}$, $\{\vec{x}[i-1], \vec{x}[i]\} \in \mathrm{Adj}(\mathrm{toSet}(\vec{x}[1, i]))$, and $\vec{y} \in R_n(\vec{x})$ then $\{\vec{y}[i-1], \vec{y}[i]\} \in \mathrm{Adj}(\mathrm{toSet}(\vec{y}[1, i]))$.*

Given $j, d \in \mathbb{N}^+$ with $j > 3$ and $\vec{x} \in \Gamma_n$, let

$$\begin{aligned} R_n(\vec{x}, j, d) = \{\vec{y} \in R_n(\vec{x}) \mid \\ \mathbf{RelRank}(P_j, \vec{y}[j-1]) - \mathbf{RelRank}(P_j, \vec{y}[j]) \in \{-d, d\}, \\ \text{where } P_j = \mathrm{toSet}(\vec{y}[1, j])\}. \end{aligned}$$

Recall that if applicants $\vec{x}[j-1], \vec{x}[j]$ are adjacent with regard to a set $P_j = \mathrm{toSet}(\vec{x}[1, j])$, then $\mathbf{RelRank}(P_j, \vec{x}[j-1]) - \mathbf{RelRank}(P_j, \vec{x}[j]) \in \{-d, d\}$, where d is either 1 or $j-1$. The next lemma offers an insight on how many sequences from $R_n(\vec{x})$ have the values 1 and $j-1$, respectively.

Lemma 5.3 *If $\vec{x} \in \Gamma_n$, $j \in \{3, 4, \dots, n\}$, $P_j = \mathrm{toSet}(\vec{x}[1, j])$,*

$$\boldsymbol{RelRank}(P_j, \vec{x}[j-1]) - \boldsymbol{RelRank}(P_j, \vec{x}[j]) \in \{1-j, j-1\},$$

and $h = \boldsymbol{RelRank}_n(\max\{P_j\}) - \boldsymbol{RelRank}_n(\min\{P_j\})$ then

- $|R_n(\vec{x}, j, 1)| = h$, *and*
- $|R_n(\vec{x}, j, j-1)| = n - h$.

PROOF Note that $\{\vec{x}[j-1], \vec{x}[j]\} \in \mathrm{Adj}(P_j)$. Hence Lemma 5.2 and (4) imply that

$$|R_n(\vec{x}, j, 1) \cup R_n(\vec{x}, j, j-1)| = n. \tag{5}$$

The condition $\mathbf{RelRank}(P_j, \vec{x}[j-1]) - \mathbf{RelRank}(P_j, \vec{x}[j]) \in \{1-j, j-1\}$ implies that

$$\{\vec{x}[j-1], \vec{x}[j]\} = \{\min\{P_j\}, \max\{P_j\}\}. \tag{6}$$

Let $i \in \{j-1, j\}$ be such that $\vec{x}[i] = \min\{P_j\}$ and let

$$Y = \{\vec{y} \in R_n(\vec{x}) \mid \mathbf{RelRank}_n(\vec{y}[i]) \leq n - h\}.$$

From (6) it is easy to see that $\vec{y} \in Y$ if and only if $\mathbf{RelRank}(P_j, \vec{y}[j-1]) - \mathbf{RelRank}(P_j, \vec{y}[j]) \in \{1-j, j-1\}$. Since obviously $|Y| = n - h$, the lemma follows then from (5). This completes the proof. ■

Given $\vec{x} \in \Gamma_n$, let

$$\Upsilon_\alpha(\vec{x}) = \{j \in \{\lfloor \alpha n \rfloor + 1, \lfloor \alpha n \rfloor + 2, \dots, n\} \mid \{\vec{x}[j-1], \vec{x}[j]\} \in \mathrm{Adj}(P_j), \\ \text{where } P_j = \mathrm{toSet}(\vec{x}[1, j])\}.$$

Given $r, k \in \{3, 4, \dots, n\}$ with $r < k$, let

$$\Phi_n(r, k) = \{\vec{x} \in \Gamma_n \mid \mathbf{RelRank}_n(\vec{x}[k-1]) - \mathbf{RelRank}_n(\vec{x}[k]) \in \{-1, 1\} \\ \text{and } \mathbf{RelRank}(P_j, \vec{x}[j-1]) - \mathbf{RelRank}(P_j, \vec{x}[j]) \notin \{-1, 1\}, \text{ where} \\ P_j = \mathrm{toSet}(\vec{x}[1, j]) \text{ and } j \in \{r+1, r+2, \dots, k-1\}\}$$

and let $\Phi_n(r) = \bigcup_{k=r+1}^{n} \Phi_n(r, k)$. It is easy to see that $\Phi_n(\lfloor \alpha n \rfloor)$ is the set of sequences of Γ_n on which the stopping rule $\tau_n(\alpha)$ "wins". It follows that

$$\mathbf{P}_{n,\tau}(\alpha) = \frac{|\Phi_n(\lfloor \alpha n \rfloor)|}{|\Gamma_n|}. \tag{7}$$

EXAMPLE 5.4 The following example illuminates the difference between the sets $\Lambda_n(r, k)$ and $\Phi_n(r, k)$.

Let $\vec{x} = (3, 5, 4, 6, 2, 7, 8, 9, 1) \in \Gamma_n$ with $r = 3$. The stopping rule $\tau_n(\alpha)$ selects a pair $(7, 8)$ with $k = 7$. However note that $\vec{x} \notin \Lambda_n(3, 7)$ because $\{6, 2\} \in \mathrm{Adj}(\mathrm{toSet}(\vec{x}[1, 5]))$. On the other hand, we have that $\vec{x} \in \Phi_9(3, 7)$.

Consider a sequence $\vec{x} \in \Gamma_n$ such that the stopping rule $\tau_n(\alpha)$ wins on $\vec{x}$ but $\vec{x} \notin \Lambda_n(r)$. We show some properties of sequences from $R_n(\vec{x})$ that we will apply in the proof of Theorem 6.1.

PROPOSITION 5.5 *If* $\vec{x} \in \Phi_n(\lfloor \alpha n \rfloor)$, $\vec{x} \notin \Lambda_n(\lfloor \alpha n \rfloor)$, $j = \min \Upsilon_\alpha(\vec{x})$, $P = \text{toSet}(\vec{x}[1,j])$, $\lfloor \alpha n \rfloor \geq 3$, *and* $h = \textbf{\textit{RelRank}}_n(\max\{P\}) - \textbf{\textit{RelRank}}_n(\min\{P\})$ *then*

- *If* $h = n-1$ *then* $|R_n(\vec{x}) \cap \Lambda_n(\lfloor \alpha n \rfloor)| = n-1$.
- *If* $h < n-1$ *then* $|R_n(\vec{x}) \cap (\Gamma_n \setminus \Phi_n(\lfloor \alpha n \rfloor))| \geq h$.

PROOF Note that $\vec{x} \in \Phi_n(\lfloor \alpha n \rfloor)$ and $\vec{x} \notin \Lambda_n(\lfloor \alpha n \rfloor)$ assert that $\Upsilon_\alpha(\vec{x}) \neq \emptyset$. Thus j is well defined.

Since j is defined as the minimal element from the set Υ_α, we have that:

$$\text{There is no } \lfloor \alpha n \rfloor < i < j \text{ such that } \{\vec{x}[i-1], \vec{x}[i]\} \in \text{Adj}(\text{toSet}(\vec{x}[1,i])). \quad (8)$$

Lemma 5.3 implies that

$$|R_n(\vec{x}, j, 1)| = h. \quad (9)$$

We distinguish two cases.

- If $h = n-1$ then for every $\vec{y} \in R_n(\vec{x}, j, 1)$ we have that $\textbf{RelRank}_n(\vec{y}[j-1]) - \textbf{RelRank}_n(\vec{y}[j]) \in \{-1, 1\}$. Thus from (8) it follows that

  $$R_n(\vec{x}, j, 1) \subseteq \Lambda_n(\lfloor \alpha n \rfloor, j) \subseteq \Lambda_n(\lfloor \alpha n \rfloor).$$

  In consequence (9) implies that $|R_n(\vec{x}) \cap \Lambda_n(\lfloor \alpha n \rfloor))| = h = n-1$.
- If $h < n-1$ then for every $\vec{y} \in R_n(\vec{x}, j, 1)$ we have that $\textbf{RelRank}_n(\vec{y}[j-1]) - \textbf{RelRank}_n(\vec{y}[j]) \notin \{-1, 1\}$. Thus from (8) it follows that

  $$R_n(\vec{x}, j, 1) \cap \Phi_n(\lfloor \alpha n \rfloor) = \emptyset.$$

  In consequence (9) implies that $|R_n(\vec{x}) \cap (\Gamma_n \setminus \Phi_n(\lfloor \alpha n \rfloor))| \geq h$.

This completes the proof. ■

6. Probability that the stopping rule $\tau_n(\alpha)$ wins Given a real positive constant $\beta < 1$, let

$$K_n(\alpha, \beta) = \{\vec{x} \in \Gamma_n \mid \textbf{RelRank}_n(\max\{P\}) - \textbf{RelRank}_n(\min\{P\}) \leq \beta n, \\ \text{where } P = \text{toSet}(\vec{x}[1, \lfloor \alpha n \rfloor])\}.$$

It is straightforward to show that

$$\lim_{n \to \infty} \frac{|K_n(\alpha, \beta)|}{|\Gamma_n|} = 0. \quad (10)$$

We show that the sizes of sets $\Phi_n(\lfloor \alpha n \rfloor)$ and $\Lambda_n(\lfloor \alpha n \rfloor)$ are "equal" as n tends to infinity.

THEOREM 6.1 *We have that*

$$\lim_{n\to\infty}\frac{|\Phi_n(\lfloor\alpha n\rfloor)|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}=1.$$

PROOF From the definitions of $\Phi_n(\lfloor\alpha n\rfloor)$ and $\Lambda_n(\lfloor\alpha n\rfloor)$ we have that

$$\Lambda_n(\lfloor\alpha n\rfloor)\subseteq\Phi_n(\lfloor\alpha n\rfloor). \tag{11}$$

Let $\beta<1$ be a real positive constant. We define a partition of the set $\Phi_n(\lfloor\alpha n\rfloor)\setminus\Lambda_n(\lfloor\alpha n\rfloor)$ into sets M_1,M_2,M_3 as follows. If $\vec{x}\in\Phi_n(\lfloor\alpha n\rfloor)\setminus\Lambda_n(\lfloor\alpha n\rfloor)$, $j=\min\{\Upsilon_\alpha(\vec{x})\}$, $P_j=\mathrm{toSet}(\vec{x}[1,j])$, and

$$h=\mathbf{RelRank}_n(\max\{P_j\})-\mathbf{RelRank}_n(\min\{P_j\})$$

then

- If $\vec{x}\in K_n(\alpha,\beta)$ then $\vec{x}\in M_1$.
- If $\vec{x}\notin K_n(\alpha,\beta)$ and $h=n-1$ then $\vec{x}\in M_2$.
- If $\vec{x}\notin K_n(\alpha,\beta)$ and $h<n-1$ then $\vec{x}\in M_3$.

It is clear that $M_1\cup M_2\cup M_3=\Phi_n(\lfloor\alpha n\rfloor)\setminus\Lambda_n(\lfloor\alpha n\rfloor)$ and that M_1, M_2, and M_3 are pairwise disjoint. Then we have that

$$\begin{gathered}\lim_{n\to\infty}\frac{|\Phi_n(\lfloor\alpha n\rfloor)\setminus\Lambda_n(\lfloor\alpha n\rfloor)|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}=\\ \lim_{n\to\infty}\left(\frac{|M_1|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}+\frac{|M_2|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}+\frac{|M_3|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}\right).\end{gathered} \tag{12}$$

Note that $2\alpha-2\alpha^2>0$. Then from Lemma 4.7 and (10) we have that

$$\lim_{n\to\infty}\frac{|M_1|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}=0. \tag{13}$$

If $\vec{x}\in M_2$ then Proposition 5.5 implies that $|R_n(\vec{x})\cap\Lambda_n(\lfloor\alpha n\rfloor)|=n-1$. It follows that

$$\lim_{n\to\infty}\frac{|M_2|}{|\Lambda_n(\lfloor\alpha n\rfloor)|}\leq\lim_{n\to\infty}\frac{1}{n-1}=0. \tag{14}$$

If $\vec{x}\in M_3$ then Proposition 5.5 implies that

$$|R_n(\vec{x})\cap(\Gamma_n\setminus\Phi_n(\lfloor\alpha n\rfloor))|\geq h. \tag{15}$$

Suppose that

$$\lim_{n\to\infty}\frac{|M_3|}{|\Phi_n(\lfloor\alpha n\rfloor)|}=\delta>0. \tag{16}$$

From (4) and (15) it follows that if $\vec{x}\in M_3$ then

$$\lim_{n\to\infty}\frac{|R_n(\vec{x})\cap M_3|}{|R_n(\vec{x})\cap(\Gamma_n\setminus\Phi_n(\lfloor\alpha n\rfloor))|}\leq\frac{n-h}{h}. \tag{17}$$

Realize that if $\vec{x}, \vec{y} \in M_3$ then either $R_n(\vec{x}) \cap R_n(\vec{y}) = \emptyset$ or $R_n(\vec{x}) = R_n(\vec{y})$. Then (17) implies that

$$\lim_{n\to\infty} \frac{|M_3|}{|\Gamma_n \setminus \Phi_n(\lfloor \alpha n \rfloor)|} \leq \frac{n-h}{h}. \tag{18}$$

From (11), (16), and (18) it follows that

$$\lim_{n\to\infty} \frac{|\Gamma_n \setminus \Phi_n(\lfloor \alpha n \rfloor)|}{|\Lambda_n(\lfloor \alpha n \rfloor)|} \geq \lim_{n\to\infty} \frac{|\Gamma_n \setminus \Phi_n(\lfloor \alpha n \rfloor)|}{|\Phi_n(\lfloor \alpha n \rfloor)|} \geq \delta \frac{h}{n-h}.$$

Let $\gamma = 2\alpha - 2\alpha^2$. Note that $\gamma > 0$. Then Lemma 4.7 implies that

$$\lim_{n\to\infty} \frac{|\Gamma_n \setminus \Phi_n(\lfloor \alpha n \rfloor)|}{|\Lambda_n(\lfloor \alpha n \rfloor)|} \leq \frac{1}{\gamma}. \tag{19}$$

Realize that if $\vec{x} \notin K_n(\alpha, \beta)$ then $h \geq \beta n$. Obviously there is $\beta_0 < 1$ such that for every $\beta > \beta_0$ we have that $\delta \frac{h}{n-h} > \frac{1}{\gamma}$; this would be a contradiction to (19). Since β can be chosen arbitrarily we conclude that $\delta = 0$. Hence from (16) we have that

$$\lim_{n\to\infty} \frac{|M_3|}{|\Phi_n(\lfloor \alpha n \rfloor)|} = 0. \tag{20}$$

Lemma 4.7, (11), and $2\alpha - 2\alpha^2 > 0$ imply that $\lim_{n\to\infty} \frac{|\Phi_n(\lfloor \alpha n \rfloor)|}{|\Lambda_n(\lfloor \alpha n \rfloor)|} < \mu$, where μ is some positive real constant. Thus from (20) it follows that

$$\lim_{n\to\infty} \frac{|M_3|}{|\Lambda_n(\lfloor \alpha n \rfloor)|} = 0. \tag{21}$$

Then from (12), (13), (14), and (21) we have that

$$\lim_{n\to\infty} \frac{|\Phi_n(\lfloor \alpha n \rfloor) \setminus \Lambda_n(\lfloor \alpha n \rfloor)|}{|\Lambda_n(\lfloor \alpha n \rfloor)|} = 0. \tag{22}$$

The theorem follows from (11) and (22). This completes the proof. ■

We express the probability $\mathbf{P}_{n,\tau}(\alpha)$ by means of sizes of the sets $\Lambda_n(r)$.

LEMMA 6.2 *We have that*

$$\lim_{n\to\infty} \boldsymbol{P}_{n,\tau}(\alpha) = \lim_{n\to\infty} \frac{|\Lambda_n(\lfloor \alpha n \rfloor)|}{|\Gamma_n|}.$$

PROOF The lemma follows from (7) and Theorem 6.1. This completes the proof. ■

PROOF (PROOF OF THEOREM 1.2) Theorem 1.2 follows immediately from Lemma 6.2 and Lemma 4.7. ■

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Problem sekretarki z wyborem dwóch kandydatek o bliskich absolutnych rangach.

Josef Rukavicka

Streszczenie Klasyczny problem sekretarki to sekwencyjny problem decyzyjny, w którym celem jest wybór najlepszej kandydatki w postępowaniu rekrutacyjnym, gdy w chwili decyzji statystyk ma niepełne dane o rzeczywistej wartości akceptowanej kandydatki. Wybór kończy się niepowodzeniem, gdy wyselekcjonowana kandydatka nie jest najlepszą wśród wszystkich n, które zgłosiły się na konkurs lub żadna nie zostanie wybrana. Rekruter posługuje się strategią maksymalizującą szanse powodzenia. Zadanie rozpatrzone w tej pracy jest modyfikacją, w której celem rekrutera jest wybór dwóch bliskich co do globalnej rangi kandydatów zatrzymując się na kandydacie, którego poprzednik jest potencjalnie bliski w przyjętym sensie. Autor wyznacza strategię, która maksymalizuje prawdopodobieństwo sukcesu w tym zadaniu. Pokazano, że asymptotyczne prawdopodobieństwo sukcesu w tej klasie strategii może osiągnąć 0.5.

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