PRZEGLĄD ELEKTROTECHNICZNY

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Equations of physics that have most important impact on electromechanical engineering

***Abstract**. The following equations, laws and principles have been discussed: Newton's second law of motion, Hamilton's principle, Euler-Lagrange equation, wave equation, the first and the second law of thermodynamics, Bernoulli's equation, Ohm's laws for electric and magnetic circuit, Kirchhoff's laws for electric and magnetic circuits, Maxwell's equations, Lorentz force equation, Einstein's energy-mass relation, Einstein field equations , Planck's equation, Heisenberg's uncertainty principle, Schrödinger wave equation, and Dirac equation.*

***Streszczenie.** Omówiono następujące równania, prawa i zasady: drugie prawo dynamiki Newtona, zasada Hamiltona, równanie Eulera-Lagrange'a, równanie falowe, pierwsza i druga zasada termodynamiki, równanie Bernoulliego, prawa Ohma dla obwodu elektrycznego i magnetycznego, prawa Kirchhoffa dla obwodów elektrycznych i magnetycznych, równania Maxwella, sila Lorentza, zależność energia-masa Einsteina, równania pola Einsteina, równanie Plancka, zasada nieoznaczoności Heisenberga, równanie falowe Schrödingera i równanie Diraca. (**Równania fizyczne mające największy wpływ na inżynierię elektromechaniczną**)*

Keywords: equation, law, principle, impact, physics, electromechanical engineering.
Słowa kluczowe: równanie, prawo, zasada, wpływ, fizyka, inżynieria elektromechaniczna.

Introduction

In opinion of the author, the following equations, laws and principles of physics have the greatest impact on electromechanical engineering: Newton's second law of motion, Hamilton's principle, Euler-Lagrange equation, wave equation, the first and the second law of thermodynamics, Bernoulli's equation, Ohm's laws for electric and magnetic circuit, Kirchhoff's laws for electric and magnetic circuits, Maxwell's equations, Lorentz force equation, Einstein's energy-mass relation, Einstein field equations , Planck's equation, Heisenberg's uncertainty principle, Schrödinger wave equation, and Dirac equation

Mechanical engineering first of all is based on Newton's laws of motion, Hamilton's principle, Euler-Lagrange equation, wave equation, laws of thermodynamics, Bernoulli's equation and Einstein's energy-mass relation; however, Hamilton's principle and Euler-Lagrange equation relates both to mechanical and electrical engineering. Electrical engineering is based on Ohm's laws, Kirchhoff's laws, Maxwell's equations and Lorentz force equation. To understand quantum mechanics, particle physics and universe, electromechanical engineer should also have knowledge of Einstein field equations, Planck's equation, Heisenberg's uncertainty principle, Schrödinger wave equation, and Dirac equation.

Newton's second law of motion

The Newton's[1] *second law of motion* states that the acceleration a of an object is dependent upon two variables: the net force $\mathbf{F}$ acting upon the object and the mass m of the object, i.e.:

(1) $$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}$$

where $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$ is the linear acceleration, $\mathbf{v}$ is the linear speed, t is the time and

(2) $$\mathbf{p} = m\mathbf{v}$$

is the *momentum*. The higher the force and the lower the mass, the higher the acceleration of an object. Similar equation for rotary motion is

(3) $$\mathbf{T} = J\frac{d\mathbf{\Omega}}{dt} = m\mathbf{r}\times\frac{d\mathbf{v}}{dt} = \mathbf{r}\times\frac{d\mathbf{p}}{dt} = \frac{d\mathbf{l}}{dt}$$

where $\mathbf{T}$ is the torque, $J = mr^2$ is the moment of inertia of a point mass, $\Omega = \mathbf{v}/\mathbf{r}$ is the angular velocity, $\mathbf{r}$ is the radius and

(4) $$\mathbf{l} = \mathbf{r}\times\mathbf{p}$$

is the *angular momentum*.

Hamilton's principle

Hamilton's[2] principle (1835), also called *the principle of least action*, means mathematically that for real motion the variation of an action S between two times on a real path is always equal to zero [1] (Fig. 1), i.e.,

(5) $$\delta S = \delta\int_{t_1}^{t_2}\mathfrak{L}\left(\xi,\dot{\xi},t\right)\mathrm{d}t = 0 \quad\Rightarrow\quad \delta\int_{t_1}^{t_2}\left(E_{\mathrm{k}} - E_{\mathrm{p}}\right)\mathrm{d}t = 0$$

The definite integral S from t_1 to t_2, called *action*, is expressed in one-dimensional form as follows.

(6) $$S = \int_{t_1}^{t_2}\mathfrak{L}\left(\xi,\dot{\xi},t\right)dt$$

[1] Isaac Newton (1642–1727), English mathematician and phycisist, invented the calculus in the 1660s and formulated the theory of universal gravity.

[2] William Rowan Hamilton (1805 – 1865) was an Irish mathematician, astronomer, and physicist.

where ξ is general coordinate. The *lagrangian*

(7) $$\mathfrak{L} = E_k - E_p$$

is defined as the difference between the kinetic energy E_k and the potential energy E_p. The function of the state of energy, called the *lagrangian*, with the use of a set of abstract coordinates, called generalized coordinates, allows to capture all energy phenomena occurring together in the electromechanical energy conversion device.

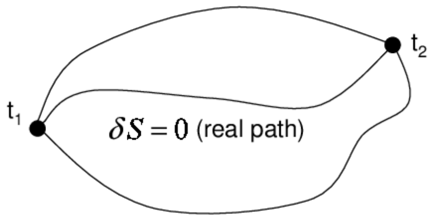

Fig.1. Variation of the action in the Hamilton principle.

Euler-Lagrange equation

The Euler[3]-Lagrange[4] *equation of motion* is a solution to the Hamilton-Larmor[5] *variational principle*. This equation is a system of *n* differential equations of the 2nd order and describes a non-conservative system with *n* degrees of freedom, in which there are dissipative (energy dissipating) elements represented by the Rayleigh[6] function *Ra* and external extortions Q_k (forces or torques) [1]:

(8) $$\left[\frac{\partial \mathfrak{L}\left(\dot{\xi},\xi,t\right)}{\partial \dot{\xi}_k}\right] - \frac{\partial \mathfrak{L}\left(\dot{\xi},\xi,t\right)}{\partial \xi_k} + \frac{\partial\, Ra\left(\dot{\xi},\xi,t\right)}{\partial \dot{\xi}_k} = Q_k$$

where ξ is the general coordinate, k = 1,2,3,…n is the number of degrees of freedom, Q_k are external forces or torques. The first term on the left-hand side represents inertia of the system, the second term represents stiffness and the third term represents damping (losses). In Newton's notation

(9) $$\dot{\xi} = \xi' = \frac{\mathrm{d}\xi}{\mathrm{d}t}$$

is the first derivative of the general coordinate ξ. The Rayleigh function

(10) $$Ra\left(\dot{\xi},\xi,t\right) = \frac{1}{2} r_k \left(\dot{\xi}\right)^2$$

expresses losses due to friction (damping) in mechanical systems and resistance in electrical circuits. In general, *r* denotes either the coefficient of friction *D* or the electric resistance *R*.

[3] Leonhard Euler (1707 – 1783) was a Swiss mathematician, physicist, astronomer, geographer, logician and engineer.

[4] Joseph-Louis Lagrange, born Giuseppe Luigi Lagrangia (1736 – 1813) was an Italian mathematician, physicist and astronomer, later naturalized French. He made major contributions to the development of physics, celestial mechanics, calculus, algebra, number theory, and group theory.

[5] Joseph Larmor (1857 – 1942) was an Irish and British physicist and mathematician who made breakthroughs in the understanding of electricity, dynamics, thermodynamics, and the electron theory of matter.

[6] Lord Rayleigh, in full John William Strutt, 3rd Baron Rayleigh (1842 – 1919), English physical scientist who made fundamental discoveries in the fields of acoustics and optics. Nobel Prize in Physics in 1904.

In electromechanical energy conversion, Eurel-Lagrange equation is a generalized form of equations describing

- linear motion

(11) $$m\frac{d^2x}{dt^2} + D\frac{dx}{dt} + K\,x = F_x$$

- rotary motion

(12) $$J\frac{d^2\vartheta}{dt^2} + D_\vartheta\frac{d\vartheta}{dt} + K_s\vartheta = T_\vartheta$$

These equations also results from *d'Alembert's principle*, (principle of virtual work), which states that the work of the the sum of external forces and inertia forces on the path being a virtual displacement, i.e., the virtual work is equal to zero [2]. Similar equation for a series electric circuit with resistance *R*, inductance *L*, and capacitance *C* has the form

(13) $$L\frac{d^2q}{d\,t^2} + R\frac{dq}{dt} + \frac{1}{C}q = u$$

where *q* is the electric chargé [C = As], d*q*/d*t* = *i* is the electric current [A] and *u* is the electric voltage [V].

Wave equation

The *wave equation* is a second-order partial differential equation of a scalar variable in terms of one or more space variable and time variable. The 3D wave equation is

(14) $$\nabla^2\psi = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial\,t^2}$$

where *v* is the velocity of wave and

(15) $$\nabla^2\psi = \frac{\partial^2\psi}{\partial\,x^2} + \frac{\partial^2\psi}{\partial\,y^2} + \frac{\partial^2\psi}{\partial\,z^2}$$

is *Laplacian* of ψ. For a plane wave (1D)

(16) $$\frac{\partial^2\psi}{\partial\,x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2\psi}{\partial\,t^2}$$

Solution to this equations is:

- for a plane *progressive* (forward-traveling) sinusoidal wave

(17) $$\psi(x,t) = A_0\cos\left(\omega t - \frac{2\pi}{\lambda}x - \varphi\right)$$

- for a plane *retrograde* (backward-traveling) sinusoidal wave

(18) $$\psi(x,t) = A_0\cos\left(\omega t + \frac{2\pi}{\lambda}x + \varphi\right)$$

where A_0 is the amplitude, λ is the wavelength, *f* is the frequency, *T* = 1/f is the period, ω is the pulsation, *v* is the velocity and φ is the initial phase of the wave.

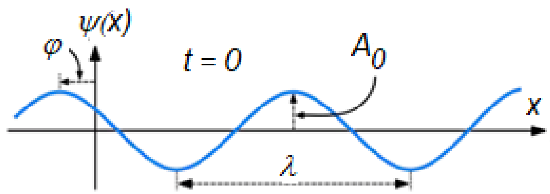

Fig.2. Amplitude A_0, wavelength λ and initial phase φ of the wave.

The pulsation ω and wavelength λ are expressed, respectively, as

(19) $$\omega = 2\pi f = \frac{2\pi}{T} \qquad\qquad \lambda = \frac{\mathrm{v}}{f} = \mathrm{v}T$$

The first law of thermodynamics

The *first law of thermodynamics* states that the change ΔU in internal energy U of a closed system equals the net heat transfer Q into the system minus the net work W done by the system, i.e.,

$$\Delta U = Q - W \tag{20}$$

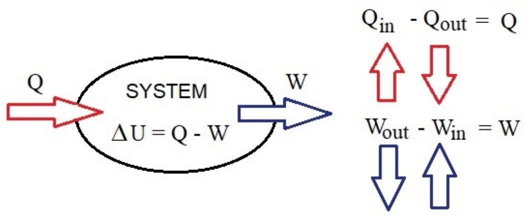

Fig. 3. The first law of thermodynamics

By convention, if Q is positive, then there is a net heat transfer into the system. If W is positive, then there is net work done by the system. So, positive Q adds energy to the system by heat, and positive W takes energy from the system by work.

The second law of thermodynamics

The *second law of thermodynamics* relies on a concept of *entropy*. Entropy is a measure of the disorder of a system. Entropy also describes how much energy is not available to do the work. The more disordered a system and higher the entropy, the less of a system's energy is available to do the work. The change in entropy is:

$$\Delta S = \frac{Q}{T} \quad \left[\frac{\mathrm{J}}{\mathrm{K}}\right] \tag{21}$$

where Q is the heat that transfers energy during a process, and T is the absolute temperature at which the process takes place.

The *second law of thermodynamics* states that heat transfers energy spontaneously from higher- to lower-temperature objects, but never spontaneously in the reverse direction (Clausius[7] statement) [3]. This is because entropy increases for heat transfer of energy from hot to cold (Fig. 4).

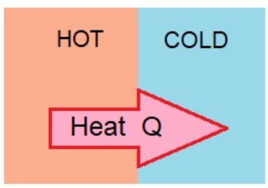

Fig. 4. Heat transfer occurs spontaneously from hot to cold, but not from cold to hot.

The total entropy ΔS_{tot} of a system either increases or remains constant in any spontaneous process: it never decreases, i.e.,

$$\Delta S_{tot} = \Delta S_{syst} + \Delta S_{env} > 0 \tag{22}$$

Based on this equation, it is seen that the entropy of the local system ΔS_{syst} can be negative as long as the entropy of the environment ΔS_{env} is positive and greater in magnitude.

In the universe, it is possible for the entropy of one part of the universe to decrease, as long as the total change ΔS_{tot} in entropy of the universe increases.

[7] Rudolf Julius Emanuel Clausius, (1822 - 1888), German mathematical physicist who formulated the second law of thermodynamics and is credited with making thermodynamics a science.

Bernoulli's equation

Bernoulli's[8] *equation* relates the pressure, velocity, and height of any two points (1 and 2) in a steady streamline flowing fluid of density ρ (Fig. 5). Bernoulli's equation is usually written as follows [2,4]:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 \tag{23}$$

The variables P_1, v_1, h_1 refer to the pressure, velocity, and height of the fluid at point 1, whereas the variables P_2, v_2 and h_2 refer to the pressure, velocity, and height of the fluid at point 2, ρ is the fluid density [kg/m^3], g = 9.81 m/s^2 is the acceleration due to gravity.

Aircraft *lift force* is produced on the basis of Bernoulli's principle (Fig. 6). On the same principle, the rotor of a wind turbine spins.

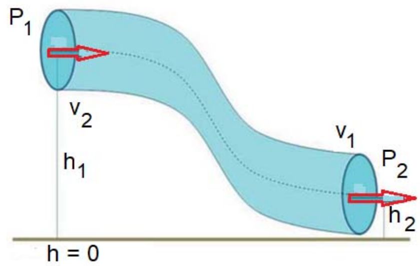

Fig. 5. Illustration of Bernoulli's equation

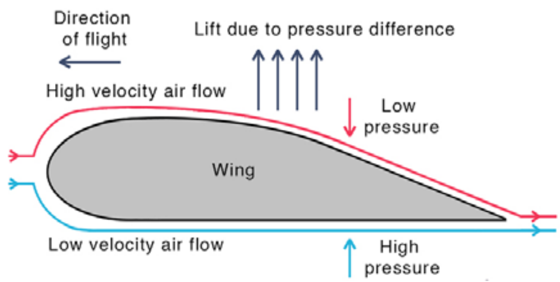

Fig. 6. Lift of an aircraft.

Ohm's law

Ohm's[9] *law* states that at a constant temperature, the electrical current I flowing through a fixed linear resistance R is directly proportional to the voltage U applied across it, and inversely proportional to the resistance R, i.e.,

$$I = \frac{U}{R} \tag{24}$$

The unit of resistance is *Ohm* (Ω). The reciprocal of resistance R is conductance G expressed in *Siemens* (S), The resistance of a conductor can be calculated on the basis the so called *second Ohm's law*, i.e.,

$$R = \frac{l}{\sigma S} \tag{25}$$

where l is the length of the conductor [m], S is the cross-section area of the conductor [m^2] and σ is the electric conductivity of the conductor [S/m]. The so called *Ohms law formula wheel* (Fig. 7) shows all the mathematical relationships between current I, voltage U, resistance R and power P.

[8] Daniel Bernoulli (1700 – 1782) was a Dutch-born member of the Swiss mathematical family. His most important work considered the basic properties of fluid flow, pressure, density and velocity, and gave the Bernoulli principle.

[9] Georg Simon Ohm (1789 – 1854) was a German physicist who discovered the electric circuit law, named after him.

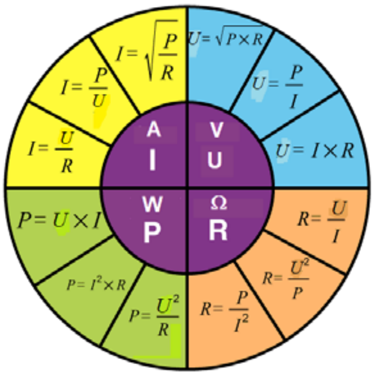

Fig. 7. Ohm's law formula wheel

Similar laws can be written for magnetic circuits. In magnetic circuits the resistance is replaced by reluctance R_μ [1/H], the current is replaced by the magnetic flux Φ [Wb = Vs], the electromotive force (EMF) E is replaced by the magnetomotive force MMF [A] and the electric voltage U is replaced by the magnetic voltage V_μ [A]. The reciprocal of reluctance R_μ is permeance Λ [H].

Kirchhoff's laws

Kirchhoff's[10] laws govern the conservation of charge and energy in electrical circuits.

Kirchhoff's current law or first Kirchhoff's law states that the algebraic sum of all currents entering and exiting a node must equal zero, i.e.,

(26) $$\sum_{i=1}^{n} I_i = 0$$

Kirchoff's voltage law or second Kirchhoff's law states that the sum of all source voltages U and voltage drops RI around a closed path in a circuit is zero, i.e.,

(27) $$\sum U - \sum RI = 0$$

Replacing the electric voltage U with the magnetic voltage V_μ, the electric current I with the magnetic flux Φ, and the electric resistance R with the reluctance R_μ, similar equations (26) and (27) can be written for magnetic circuits. Table 1 shows Ohm's laws and Kirchhoff's laws for electric and magnetic circuits.

Table 1. Fundamental laws for magnetic and electric circuits

Law	Electric circuit	Magnetic circuit
Ohm's law Prawo Ohma	$R=\frac{U}{I}$ $G=\frac{I}{U}$	$R_\mu=\frac{V_\mu}{\Phi}$ $\Lambda_\mu=\frac{\Phi}{V_\mu}$
2nd Ohm's law	$R=\frac{l}{\sigma S}$ $G=\frac{S}{\rho l}$	$R_\mu=\frac{l}{\mu S}$ $\Lambda_\mu=\frac{\mu S}{l}$
Kirchhoff's current law	$\sum I=0$	$\sum \Phi=0$
Kirchhoff's voltage law	$\sum U-\sum RI=0$	$\sum V_\mu-\sum R_\mu \Phi=0$

Maxwell's equations

Maxwell's[11] *equations* are *fundamental equations of electromagnetic fields* published in 1873 on the basis of and in making the connection between light and electromagnetic waves.

- Biot[12]-Savart[13] law (1820)

(28) $$\mathbf{H}=\frac{i}{4\pi}\int_l \frac{\mathbf{dl}\times\mathbf{1_r}}{r^2}$$

- Faraday[14] law (1831)

(29) $$e=-N\frac{\mathrm{d}\varphi(x,t)}{\mathrm{d}t}=-N\left(\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\right)$$

- and Gauss's[15] law (1840)

(30) $$\oint \mathbf{D}\cdot\mathbf{dS}=Q$$

In the above equations i is the instantaneous electric current, **dl** is the element of conductor with current, $\mathbf{1_r}$ is the unit vector, r is the distance from **dl** to to the point at which the field is being calculated, e is the instantaneous value of the induced voltage (EMF), φ is the instantaneous value of magnetic flux, N is the number of turns, **D** is the vector of the electric flux density [As/m^2], **dS** is the element of surface enclosing the electric charge Q [C].

Maxwell introduced the so-called *displacement current*, the density of which is $\partial\mathbf{D}/\partial t$. where the electric flux density **D** is called *displacement vector*. There is continuity of the displacement current and electric current $\mathbf{J} = \sigma\,\mathbf{E}$, i.e., in a circuit with capacitor (Fig.8).

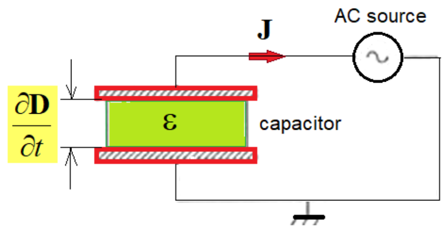

Fig. 8. The continuity of displacement current and electric current.

The differential form of the *first Maxwell's equation* is

(31) $$\nabla\times\mathbf{H}=\sigma\,\mathbf{E}+\frac{\partial\mathbf{D}}{\partial t}+\nabla\times(\mathbf{D}\times\mathbf{v})+\mathbf{v}\nabla\cdot\mathbf{D}$$

or

(32) $$curl\mathbf{H}=\sigma\,\mathbf{E}+\frac{\partial\mathbf{D}}{\partial t}+curl(\mathbf{D}\times\mathbf{v})+\mathbf{v}div\mathbf{D}$$

where **H** is the magnetic field intensity [A/m], **E** is the electric field intensity [V/m], $\mathbf{J} = \sigma\,\mathbf{E}$ the density of the electric current [A/m^2], **v** is the linear velocity (m/s), $\partial\mathbf{D}/\partial t$ is the density of the displacement current, curl(**D** × **v**) is the density of the current due to the motion of a polarized dielectric material, **v**div**D** is the density of the convection current and

(33) $$\Delta=\mathbf{1_x}\frac{\partial}{\partial x}+\mathbf{1_y}\frac{\partial}{\partial y}+\mathbf{1_x}\frac{\partial}{\partial z}$$

[10] Gustav Robert Kirchhoff (1824 – 1887) was a German physicist who contributed to the fundamental understanding of electrical circuits, spectroscopy, and the emission of black-body radiation by heated objects.

[11] James Clark Maxwell (1831-1879) born in Edinburgh, Scotland was one of the greatest scientists of the nineteenth century. He is best known for the formulation of the theory of electromagnetism and in making the connection between light and electromagnetic waves.

[12] Jean-Baptiste Biot (1774 – 1862) was a French physicist, astronomer, and mathematician who co-discovered the Biot-Savart law of mag netostatics, established the reality of meteorites, made an early balloon flight, and studied the polarization of light.

[13] Felix Savart (1791 – 1841), was a French physicist and mathematician who is primarily known for the Biot–Savart law of magnetostatic.

[14] Michael Faraday (1791 – 1867) was an English scientist who contributed to the study of electromagnetism an electrochemistry.

[15] Johann Carl Friedrich Gauss (1777 - 1855) was a German mathematician and physicist who made significant contributions to many fields in mathematics and science.

is the operator *nabla*. It is a vector operator that has no physical meaning or vector direction by itself. $\mathbf{1}_x$,$\mathbf{1}_y$ and $\mathbf{1}_z$ are unit vectors in Cartesian coordinate system.

The *second Maxwell's equation* in the differential form is

(34) $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{B} \times \mathbf{v})$$

or

(35) $$curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - curl(\mathbf{B} \times \mathbf{v})$$

where **B** is magnetic flux density vector [T = Vs/m^2].

From Gauss's law (30) for the volume charge density ρ_V [C/m^3] and through the use of Gauss's theorem

(36) $$\oint_S \mathbf{D}_S \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{D} dV$$

the *third Maxwell's equation* in differential form is

(37) $$\nabla \cdot \mathbf{D} = \rho_V$$

or

(38) $$div\mathbf{D} = \rho_V$$

The *fourth Maxwell equation* states that there are no magnetic charges, i.e., in differential form is

(39) $$\nabla \cdot \mathbf{B} = 0$$

or

(40) $$div\mathbf{B} = 0$$

However, in particle physics, a *magnetic charge* or *magnetic monopole* is a hypothetical elementary particle that is an isolated magnet with only one magnetic pole Magnetic effect is caused by moving electric charges while an electric field is caused by stationary charges. Magnetic charges are quasiparticles, represented by the quantum-mechanical Pauli matrices in Hamiltonian formulation [5].

Lorentz force equation

The *Lorentz*[16] *force* (or electromagnetic force) [6] is the combination of electric and magnetic force on a point charge q due to electromagnetic fields, i.e.,

(41) $$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where q **E** is the electric force, q(**v**×**B**) is the magnetic force. The magnetic flux density can be determined by the Lorentz force, as follows,

(42) $$\mathbf{B} = \frac{\mathbf{F}}{q\,\mathbf{v}} \qquad \left[\frac{\mathrm{N}}{\mathrm{C}\frac{\mathrm{m}}{\mathrm{s}}} = \frac{\mathrm{Vs}}{\mathrm{m}^2} = \mathrm{T} \right]$$

One Tesla (T) is equal to the Lorentz force of 1 N that acts on a 1 C charge moving at 1 m/s.

Einstein's energy-mass relation

Einstein's[17] *theory of special relativity* expresses that mass m and energy E are the same physical entity and can be changed into each other [7]. Mass–energy equivalence is the relationship between mass m and energy E in a system's rest frame, where the two quantities differ only by a multiplicative constant and the units of measurement, i.e.,

(43) $$E = mc^2$$

where $c = 2.997930\times10^8$ m/s is the velocity of light. The theory of relativity assumes that the velocity of light is always constant

(44) $$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = const$$

where ε_0 = (1/36π)×10^{-9} F/m is the electric permittivity and μ_0 = 0.4π×10^{-6} H/m is the magnetic permeability of free space. On 20 May 2019 a revision to the SI system has gone into effect, making the vacuum permeability no longer a constant but rather a value that needs to be determined experimentally. The following conclusions arise from eqn (43):

- Even masses at rest have an energy inherent to them.
- Mass can be converted into pure energy. Eqn (43) tells exactly how much energy can be obtained from converting mass.
- Mass can be make out of pure energy $m = E/c^2$.

Using eqn (43), the energy of the electron with rest mass m_e = 9.10938356×10^{-31} kg moving at the velocity of 0.8 velocity of light is

$$E_e = m_e(0.8c)^2 = 9.10938356\times10^{-31}\times(0.8\times2.997930\times10^8)^2 = 5.24\times10^{-14}\ \mathrm{J}$$

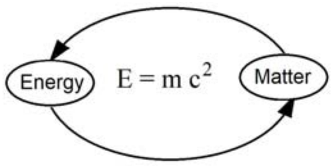

Fig. 8. The mass–energy equivalence: mass m and energy E are the same physical entity and can be changed into each other.

The variation of mass m with velocity v is given by the Lorentz transformations, i.e.,

(45) $$m' = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m is the rest mass of the particle. For example, the mass of electron at 0.8c is

$$m_e' = \frac{m}{\sqrt{1 - \frac{(0.8c)^2}{c^2}}} = \frac{m}{\sqrt{1 - 0.64}} = 1.667m$$

Theoretically, if $v = c$, the mass $m' \to \infty$. No matter how much energy is put into accelerating a mass, its velocity can only approach, not reach, the speed of light.

Similarly, the time t' measured by astronaut's clock moving with velocity v >> 0 is longer than the time t indicated by the clock on the Earth at v = 0, i.e.,

(46) $$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

For example, at velocity v = 0.999 c, the time t' is 223.6 times longer than time t at v = 0 (Fig.9).

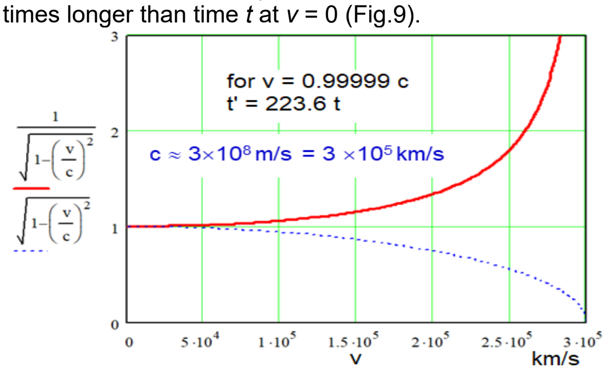

Fig. 9. Variation of time with the velocity [8].

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[16] Hendrik Antoon Lorentz (1853-1928) was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect.

[17] Albert Einstein (1879-1955), German-born physicist, winner of the 1921 Nobel Prize in Physics. Best known for his explanation of photoelectric effect and formulation of General Theory of Relativity.

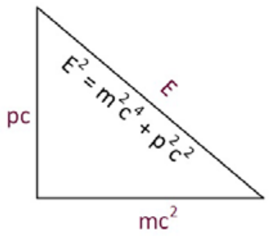

Fig. 10. Full and general relationship for any moving object.

Newton's gravity has no way to account for this, but in Einstein's General Theory of Relativity, the curvature of space means that falling into a gravitational field makes the gain of energy, and climbing out of a gravitational field makes the loss of energy. The full and general relationship for any moving object, isn't just $E = mc^2$, but (Fig. 10)

(47) $$E = \sqrt{m^2c^4 + p^2c^2}$$

where p is momentum – eqn (2). The mass of a photon m = 0 (photons do not have mass), so that the momentum of a photon

(48) $$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

where h = 6.62607004×10^{-34} kgm^2/s is *Planck's constant*.

Einstein field equations

The Einstein field equations (EFE) can be written in the following form [9]:

(49) $$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress–energy tensor, Λ is the *cosmological constant* and κ is the *Einstein gravitational constant*. Einstein originally introduced the constant Λ in 1917 to counterbalance the effect of gravity and achieve a static universe. Einstein's cosmological constant Λ was abandoned after Edwin Hubble's[18] confirmation that the universe was expanding. The Einstein tensor is defined as:

(50) $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, and R is the scalar curvature. This is a symmetric second-degree tensor that depends on only the metric tensor and its first and second derivatives. The *Einstein gravitational constant* is defined as

(51) $$\kappa = \frac{8\pi G}{c^4} \approx 2.076647442844 \times 10^{-43} \quad \left[\mathrm{N}^{-1}\right]$$

where G = 6.67430(15)×10^{-11} m^3·kg^{-1}·s^{-2} is the *Newtonian constant of gravitation*, i.e.,

(52) $$F = G\frac{m_1 \times m_2}{r^2}$$

and c is the speed of light in vacuum. Putting (50) and (51) into (49), the EFE can be written as

(53) $$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

In standard units, each term on the left has units of 1/length^2. The expression on the left represents the curvature of spacetime as determined by the metric. The expression on the right represents the stress–energy–momentum content of spacetime. The EFE can then be interpreted as a set of equations dictating how stress–energy–momentum determines the curvature of spacetime.

In the EFE the term containing the cosmological constant Λ was absent from the version in which he originally published them. Einstein then included the term with the cosmological constant to allow for a universe that is not expanding or contracting. This effort was unsuccessful because:

- any desired steady state solution described by this equation is unstable, and
- observations by Edwin Hubble showed that our universe is expanding.

Einstein then abandoned Λ. For many years the cosmological constant was almost universally assumed to be zero. More recent astronomical observations have shown an accelerating expansion of the universe, and to explain this a positive value of Λ is needed. The cosmological constant is negligible at the scale of a galaxy or smaller.

Einstein accepted the modern cosmological view that the universe is expanding (Fig. 11) long after many of his contemporaries. Until 1931, Einstein believed that the universe was static. He changed his mind when American astronomer Edwin Hubble showed Einstein his observations of redshift in the light emitted by far away nebulae -- today known as galaxies (Fig. 12).

E. Hubble and his colleague at Mt. Wilson Observatory, Milton Humason, estimated the expansion rate of the universe to be 500 km/s/Mpc (kilometer per second per megaparsec). A parsec (symbol pc) is a distance equal to approximately 3.26 light-years or 30.9 trillion km.

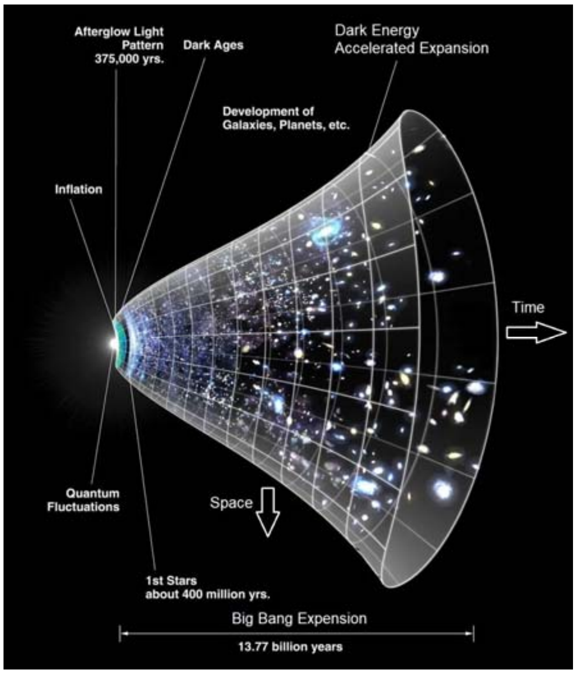

Fig. 11. The universe's expansion over time. The curve changes noticeably about 7.5 billion years ago, when objects in the universe began flying apart as a faster rate. Astronomers theorize that the faster expansion rate is due to a mysterious, dark force that is pulling galaxies apart. Credit: NASA/STSci.

The universe is full of matter and the attractive force of gravity pulls all matter together. The Hubble Space Telescope (launched into low Earth orbit in 1990) observations of very distant supernovae (powerful and luminous explosion of stars) showed that, a long time ago, the universe was actually expanding more slowly than it is

[18] Edwin Powell Hubble (1889-1953), one of the leading astronomers of the 20th century, His doscovery in the 1920s that countless galaxies exist beyond our own Milky Way galaxy revolutionized our understanding of the universe.

today. Now, the universe is expanding at an accelerating rate. Maybe it was a result of a long-discarded version of Einstein's theory of gravity that contained a cosmological constant.

Astronomers and physicists still do not know what the correct explanation is, but they have given the solution a name *dark energy*. The expansion of the universe has been something like a tug-of-war between gravity (dominated by unseen *dark matter*) and the repulsive force known as *dark energy*. More is unknown than is known. It turns out that roughly 68% of the universe is dark energy. Dark matter makes up about 27%. The rest, i.e., everything on Earth, everything ever observed, all normal matter adds up to less than 5% of the universe [10].

Dark energy, which forms almost three-quarters of the universe, is the most mysterious stuff known to man.

Fig. 12. Albert Einstein with Edwin Hubble, in 1931, at the Mount Wilson Observatory, San Gabriel Mountains of the Angeles National Forest, in Southern California, looking through the lens of the 100-inch telescope through which Hubble discovered the expansion of the universe in 1929.

Planck's equation

Max Planck[19] postulated in 1900, that energy was quantized and could be emitted or absorbed only in integral multiples of a small unit, which he called *energy quantum*:

(54) $$E = hf$$

where *h* is *Planck's constant*. Planck's constant is a fundamental universal constant that defines the quantum nature of energy and relates the energy of a photon *E* to its frequency *f*. Einstein extended Planck's idea in 1905 when he introduced the concept of light *quantum*, the particle of light, or *photon*. Thus, the electromagnetic radiation wasn't continuous like a wave but isolated in the packets of light, Einstein proposed [9].

Smallest particles exhibit dual nature of particle and wave. De Broglie[20] (1924) assumed that for particles the same relations are valid as for the photon. The wavelength is

(55) $$\lambda = \frac{h}{m\mathrm{v}} = \frac{h}{p}$$

See also eqn (19). The momentum of particle is given by eqn (2) and the momentum of photon is given by eqn (48).

Heisenberg's uncertainty principle

The *uncertainty principle* says that it is not possible to measure the position *x* [m] and the momentum *p* [kg m/s] of a particle with absolute precision. Heisenberg[21] formulated the uncertainty principle as [9]

(56) $$\Delta x\,\Delta p \geq \frac{h}{4\pi} = \frac{\hbar}{2}$$

where Δ is error; *x* is position, Δx is uncertainty in position, Δp is uncertainty in momentum, *p* is momentum (48); *h* is Planck's constant and the *h*-bar constant is

(57) $$\hbar = \frac{h}{2\pi}$$

The more accurately the position is known, i.e., the smaller Δx is, the less accurately we know the momentum, i.e., the larger Δp is; and vice versa.

Schrödinger wave equation

Schrodinger[22] wave equation is a mathematical expression describing the energy and position of the electron in space and time, taking into account the matter wave nature of the electron inside an atom. Schrödinger equation has two forms: the time-dependent and the time-independent equation [11]. The time-dependent *Schrödinger wave equation* is

(58) $$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(x,t) + E_p(x)\psi(x,t)$$

(59) $$i\hbar\frac{\partial\psi(x,t)}{\partial t} = E\psi$$

where $i = \sqrt{-1}$ is the imaginary unit, *h* is Planck's constant, $E_p(x)$ is the potential energy. The *h*-bar constant is given by eqn (57). It can be derived on the basis of eqns (17), (48), *hamiltonian* energy operator

(60) $$\hat{H} = E_k + E_p$$

and the wave number

(61) $$k = \frac{2\pi}{\lambda}$$

The wave function (17) for a particle of rest mass *m* is:

(62) $$\psi(x,t) = A_0 e^{i(kx-\omega t)}$$

and A_0 is the amplitude of wave (Fig.2). The wavelength λ is given by eqns (19) and (55). The total energy of a particle

(63) $$E = \frac{m\mathrm{v}^2}{2} + E_p(x) = \frac{p^2}{2m} + E_p(x)$$

where $E_k = m\mathrm{v}^2/2$ is the kinetic energy, $E_p(x)$ is the potential energy and the momentum *p* of a particle is given by eqn (48).

Planck's quantum theory states the energy of waves are quantized such that

(64) $$E = h\nu = 2\pi\hbar\nu$$

The *time-independent Schrödinger equation* in compressed form can be expressed as:

(65) $$\hat{H}\Psi = E\Psi$$

[19] Max Karl Ernst Ludwig Planck (1858-1947) was a German theoretical physicist whose discovery of energy quanta won him the Nobel Prize in Physics in 1918.

[20] Louis Victor Pierre Raymond, 7th Duc de Broglie (1892 – 1987) was a French physicist and aristocrat who made groundbreaking contributions to quantum theory. He won the Nobel Prize in Physics in 1929.

[21] Werner Heisenberg (1901-1976), German physicist, winner of the 1932 Nobel Prize in Physics.

[22] Erwin Rudolf Josef Alexander Schrödinger (1887 – 1961) was Austrian physicist with Irish citizenship who developed a number of fundamental results in quantum theory. The Schrödinger equation provides a way to calculate the wave function of a system and how it changes dynamically in time. He won a Nobel Prize in physics in 1933 (shared with Paul Dirac).

where $\hat{H}$ is hamiltonian energy operator (60) and E is energy eigenvalue.

Dirac equation

Dirac[23] *equation* derived in 1928 is an attempt to unify relativity with quantum mechanics [12], i.e.,

$$(i\hbar\gamma^{\mu}\partial_{\mu} - m\,c)\psi = 0 \quad (66)$$

where i is the imaginary unit, ψ is the wave function for a particle of rest mass m (fermion mass), γ^{μ} is gamma 4×4 matrices with index μ = 0,1,2,3 (Pauli[24] matrices), ∂_{μ} is derivative in 4 dimensions and $\hbar$ is the h-bar constant as given by eqn (57).

Dirac equation describes the phenomenon of quantum connection, which alleges that if two separate systems interact with each other over a certain period of time and then separate, they can be described as two different systems, but they will already exist as one unique system. What happens to one will continue to affect the other, regardless of the distance between them. It's called *quantum intertwining* or *quantum connection*. Two particles that were at some point connected remain connected forever, even if they are light-years apart. For this reason, sometimes Dirac equation is called *love equation*.

Eqn (66) also implies the existence of *antimatter*. For every particle there exists a corresponding *anti-particle*, exactly matching the particle but with opposite charge. Paul Dirac predicted the possibility of *antielectrons* in 1928 with this equation, but it wasn't until 1932 that Carl D. Anderson[25] discovered the first anti-particle: *positron*. An anti-particle has the same mass but opposite charge as compared to ordinary matter (antimatter).

Precursor to the Dirac equation was the Klein[26]-Gordon[27] equation:

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0 \quad (67)$$

The first two terms on the left hand side is the wave equation (14).

There are two problems in this equation. First is that if the solutions of this equation are considered, we get particles that have negative energy (negative energy solutions). The second is the negative probability density.

Dirac worked on solving these two problems and combining special relativity and quantum mechanics. He derived equation (66) that did solve the problem of the negative probability density but still had negative energy solutions in it. Dirac said that all the negative energy states are already occupied. This description of the vacuum as a "sea" of electrons is called the *Dirac sea*. Dirac further reasoned that if the negative-energy states are incompletely filled, each unoccupied state – called a hole – would behave like a positively charged particle.

Carl Anderson discovery of antimatter (1932), first predicted theoretically, actually corresponds to the negative energy solutions.

Conclusions

Hamilton's principle (principle of least action) and Euler-Lagrange equation links mechanical and electrical engineering. Hamilton's principle relates to mechanical systems, while Larmor's principle is extension of Hamilton's principle on electromechanical systems [1]. Larmor's principle uses an analogous variational energy principle of least action for electromagnetic fields. Euler-Lagrange equations describes both linear and rotary motions in mechanical drives systems and conservation of charge and energy in electric circuits. In electrical and electromechanical engineering, Euler-Lagrange equations are applied to electric circuits with concentrated parameters, while Maxwell's equations to electric circuits with distributed parameters. Ohm's laws and Kirchhoff's laws strictly related to electric circuits with concentrated parameters.

It is a pity that most Polish Universities have removed Euler-Lagrange equations and Maxwell's equations from the curricula of undergraduate studies.

In general, all electrical engineering is contained in and can be derived from Euler-Lagrange equation and Maxwell's equations.

The basis of mechanical engineering are Newton's laws of motion, Hamilton's principle, Euler-Lagrange equation, wave equation, laws of thermodynamics, Bernoulli's equation and Einstein's energy-mass relation.

The knowledge of quantum mechanics, particle physics and universe requires understanding of the Einstein field equations , Planck's equation, Heisenberg's uncertainty principle, Schrödinger wave equation, and Dirac equation.

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[12]Thaller B., The Dirac Equation, *Physical Review.*, 28 (1926), No. 6, 1049-1070

[23] Paul Adrien Maurice Dirac (1902 – 1984) was an English theoretical physicist who is regarded as one of the most significant physicists of the 20th century. He completely reshaped *quantum mechanics* with the astounding Dirac equation. Dirac and Schrödinger shared the 1933 Nobel Prize in Physics.

[24] Wolfgang Ernst Pauli (1900 – 1958) was an Austrian theoretical physicist and one of the pioneers of quantum physics. He received Nobel Prize in Physics in 1945.

[25] Carl David Anderson (1905 – 1991) was an American physicist who discovered the *positron* in 1932 and the *muon* in 1936. He received Nobel Prize in Physics the 1936.

[26] Oskar Benjamin Klein (1894 – 1977) was a Swedish theoretical physicist.

[27] Walter Gordon (1893 – 1939) was a German theoretical physicist.