

The changing shape of a liquid drop in an electric field

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ABSTRACT

An approximate extension of the slender body theory was used to determine the static shape of a conically ended dielectric fluid drop in an electric field. Using induced surface charge density, hydrostatic pressure and the surface tension of the liquid with interfacial tension stresses and Maxwell electric stresses, a governing equation was obtained for slender geometries for the equilibrium configuration and numerically solved for 3D. For an applied electric field, the electric energy on a spherical drop can be maximized in a weak dielectric by increasing the applied electric field. The minimum dielectric constant ratio needed to produce a conical end is 14.5 corresponding to a cone angle 31.25° . There is a sharp increment of the aspect ratio after reaching the threshold value of the applied field strength and the deformation of the fluid drop increases with the increase in dielectric constant of the fluid drop. For a particular dielectric constant ratio, the threshold electric field producing conical interface increases with the increased surface tension of the liquid. The threshold electric field for a water drop is 1.0854×10^4 units and the corresponding aspect ratio is 15. For the minimum dielectric ratio the cone angle of the drop decreases with applied field making the drop more stable at higher fields.

Key Words

Surface charge density, dielectric constant, cone angle, surface tension, minimum electric field

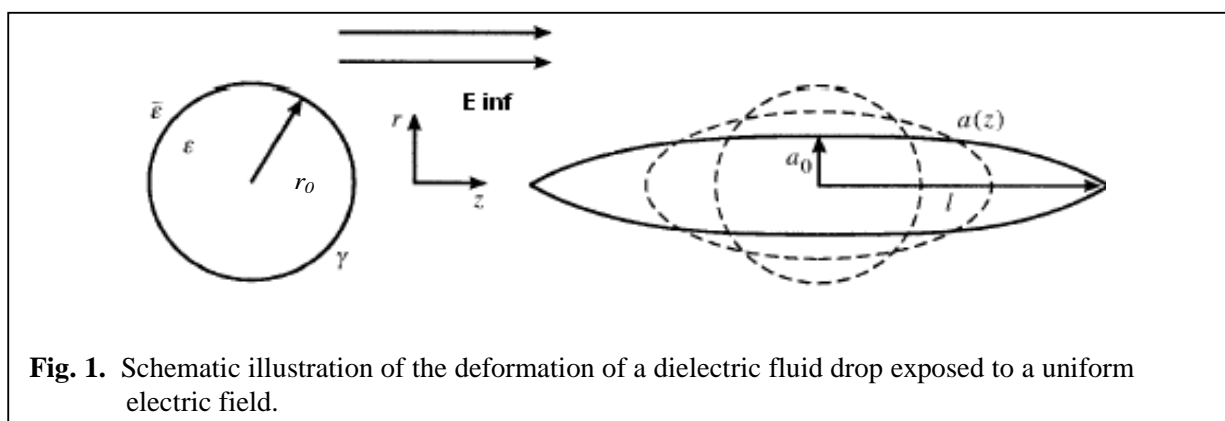
1. INTRODUCTION

The isolated neutral droplet has provided a simple idealized system to investigate fluid motion in an electric field. For drops held at an end of a capillary, the occurrence of conical tips at an interface exposed to an electric field was discovered by Zelney¹. By adapting Rayleigh's stability criterion for a charged sphere and assuming for droplets of equal inside and outside pressure, that the drop lengthened approximately into a form of prolate spheroid, he showed that the disintegration occur due to hydro dynamical instability. Many of early practical approaches were implemented on conical interfaces of soap bubbles held at the end of capillary tubes. Wilson and Taylor examined, the uncharged soap films subjected to a uniform electric field² and nearly conical equilibrium shapes of water drops at an end of a

spherically formed conical electrode³. Similar investigations concerning nearly conical tips and on the sprays of tiny droplets that accompany the conical end have been made due to their use in electro-atomization. A method based on tensor virial providing a basis for a systematic investigation of both equilibrium and stability in the same framework was developed by Chandrasekhar⁴. An appropriate extension of the virial method developed by Chandrasekhar was used by Rosenkilde⁵ to systematically re-examine the equilibrium of incompressible dielectric fluid drops placed in a uniform electric field.

In the absence of fluid motion, the shape of the interface is implemented by the balance between interfacial tension stresses and Maxwell electric stresses in the interior of the droplet. In the present work for studying the equilibrium configurations of a real liquid, in addition to the above stresses, the hydrostatic pressure and the surface tension of the liquid were considered. We have assumed that the drop is static under the electric field and that the gravitational forces are negligible compared to the electric field and small excess pressures. In the absence of an electric field, the dielectric drop was assumed to be a sphere of interfacial tension. The shape of the drop and the electric field were coupled through the normal stress of the interface which balances electric stress, fluid pressure and the interfacial tension of the drop. Simplifying the normal stress balance for slender geometries an integral equation for the electric field was approximated using the slender hypothesis to obtain an ordinary differential equation that couples the electric field to the shape of the drop. Due to the nonlinearity of the equation, the small aspect ratios were neglected and exposing the drop to large electric fields, the electric field inside the dielectric drop was calculated and tested for large aspect ratios. The change in aspect ratios with the applied electric field and the dielectric constant of the drop was tested. The electric field which atomizes the drop was calculated by considering normal electric stress balance and the surface tension of the drop and was examined for different dielectric constants. The variation of the polarized surface charge density and the induced electric energy on the drop with the dielectric constants were computed. The conical ended shapes of drops with higher applied fields were also determined.

2. THE INDUCED CHARGE DENSITY ON THE CONICAL TIP



The shape of the liquid drop can be considered initially spherical. When a dielectric liquid droplet is exposed to an electric field first elongates into prolate shape and then, if the field strength is sufficiently high to a pointed or conical ends (Figure 1).

For the prolate spheroidal configurations, the conditions for the onset of instability can be obtained from elementary considerations. Nevertheless, it is useful to establish these conditions by a systematic approach which can be extended to more complicated situations involving internal currents of the drop. When a spherical dielectric liquid drop is exposed to an electric field, a surface charge due to the polarization of the material is induced and due to this excess electric stress at the interface between the liquid drop and the surrounding medium, the drop starts to elongate (Figure 1). In the absence of fluid motion, the shape of the interface is implemented by the balance between interfacial tension stresses, Maxwell electric stresses in the interior of the droplet as well as hydrostatic pressure and the surface tension of the liquid.

Solving the Laplace's equation for a spherical drop of dielectric constant ε , radius r_0 placed in a medium of dielectric constant $\bar{\varepsilon}$ and a uniform electric field of magnitude E_0 directed along the z -axis (Figure 1) with no free charges inside and outside the sphere, for the boundary conditions at $r = r_0$, we obtain the potential inside ϕ_{in} and outside ϕ_{out} the liquid drop as

$$\phi_{in} = -\left(\frac{3}{\varepsilon/\bar{\varepsilon} + 2}\right)E_0 r \cos \theta \quad (r > r_0) \quad (1)$$

$$\phi_{out} = \left(\frac{\varepsilon/\bar{\varepsilon} - 1}{\varepsilon/\bar{\varepsilon} + 2}\right)\frac{r_0^3}{r^2}E_0 \cos \theta - E_0 r \cos \theta \quad (0 \leq r \leq r_0) \quad (2)$$

Using Maxwell's equations, the electromagnetic force per unit volume acting on the dielectric liquid drop in SI units is⁵

$$\mathbf{f} = \varepsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \left(\frac{1}{\mu_0}\nabla \wedge \mathbf{B} - \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right) \wedge \mathbf{B} \quad (3)$$

ε_0 is the permittivity of free space. The electric stress tensor in a medium with dielectric constant ε is given by

$$T^E = \varepsilon_0 \varepsilon \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) \quad (4)$$

Normal stress of the outer surface T^E_{out} and inner surface T^E_{in} of the drop are given in terms of tangential (E_t) and normal (E_n) components of electric field as⁶,

$$\left(T^E_{out}\right) = \frac{1}{2} \epsilon_0 \bar{\epsilon} \left(\bar{E}_n^2 - \bar{E}_t^2\right), \quad \left(T^E\right)_{in} = \frac{1}{2} \epsilon_0 \epsilon \left(E_n^2 - E_t^2\right). \quad (5)$$

The jump in the normal electric stress across the interface separating two distinct dielectric liquids is

$$\left(\Delta T^E\right) = \frac{1}{2} \epsilon_0 (\epsilon - \bar{\epsilon}) \left[E_t^2 + \left(\frac{\epsilon}{\bar{\epsilon}}\right) E_n^2 \right]. \quad (6)$$

$\epsilon/\bar{\epsilon}$ is assumed to be a fixed parameter. The radius of the drop $a(z)$ in cylindrical coordinates (r, θ, z) aligned with the applied electric field E_0 , is assumed to be uniform at large distances from the interface. At the surface $r = a(z)$, the small radial field E_r is⁶

$$E_r = -\frac{1}{2} a(z) \frac{\partial}{\partial z} (E_z) \quad (7)$$

For slender shapes (i.e., $a_0/l \ll 1$), assuming that $(a_0/l)^2 (\epsilon/\bar{\epsilon}) \ll 1$, the electric stress can be approximated by³

$$\left(\Delta T^E\right) \approx \frac{1}{2} \epsilon_0 (\bar{\epsilon} - \epsilon) E^2. \quad (8)$$

This approximation neglects the effect of the normal components. Assuming that the drop is static, the shape of the drop and the electric field can be coupled through the normal-stress equation at the interface by balancing the electric stress difference and the fluid pressure to interfacial tension. The normal stress balance can be approximated by⁷

$$\frac{1}{2} \epsilon_0 (\bar{\epsilon} - \epsilon) E^2 + \Delta P = \frac{\gamma}{a(z)} \quad (9)$$

Where ΔP is the constant pressure excess inside the drop and γ is the coefficient of the interfacial tension.

In order to find the behavior near a conical surface, the solution of the Laplace's equation can be written in the form of Legendre function $P_\nu(\cos\theta)$. If the limited angular region $0 \leq \theta \leq \beta$, $0 \leq \phi \leq 2\pi$ is bounded by a conical dielectric surface for $\theta = \beta$ as shown in figure 2, the region can be considered as a deep conical hole bored in a dielectric surface. For $\beta > \pi/2$, the region of space is which surrounds a pointed conical

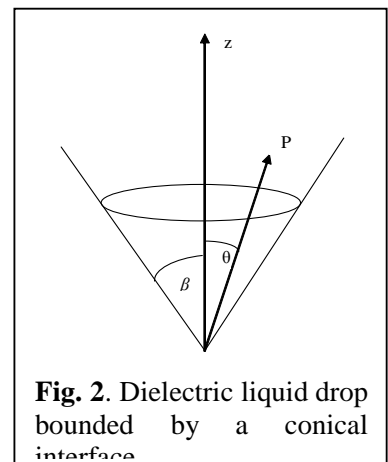


Fig. 2. Dielectric liquid drop bounded by a conical interface

dielectric surface. With the assumption of azimuth symmetry, finite and single-valued solutions in the range of $\cos \beta \leq \cos \theta \leq 1$ are sought. Since the dielectric surface $\theta = \beta$ is at a fixed potential which can be taken as zero, the solution in $\cos \theta$ must vanish at $\theta = \beta$ to satisfy the boundary conditions. For regularity at $x = 1$, it is convenient to make a series expansion around $x = 1$ instead of $x = 0$. Since the potential must vanish at $\theta = \beta$ for all r , the complete solution for the azimuthally symmetric potentials in regions of the conical surface $0 \leq \theta \leq \beta$ and $\beta \leq \theta \leq \pi$ are⁸

$$\Phi(r, \theta) = \sum_{k=0}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos \theta) \quad 0 \leq \theta \leq \beta \quad (10)$$

$$\Phi(r, \theta) = \sum_{k=0}^{\infty} B_k r^{\nu_k} P_{\nu_k}(\cos(\pi - \theta)) \quad \beta \leq \theta \leq \pi \quad (11)$$

Standard boundary conditions for a fluid of dielectric constant ϵ lead to⁷

$$P_{\nu}(\cos \theta_0) P_{\nu}'(-\cos \theta_0) + \frac{\epsilon}{\epsilon'} P_{\nu}'(-\cos \theta_0) P_{\nu}(\cos \theta_0) = 0 \quad (12)$$

For given ϵ and θ , equation (12) has a singular field solution for the range of ν between $0 < \nu < 1$. There is a critical value for dielectric constant, $\epsilon_c = 17.59$ below which $\nu = 1/2$ for all possible angles between $0 \leq \theta \leq \pi/2$, i.e. there is solution with $\nu = 1/2$. For dielectric constants higher than critical value there are two angles corresponding to $\nu = 1/2$. At the critical value, the minimum reaches $1/2$ at $\theta = 30^\circ$. The component of the electric field and the charge density σ on the conical dielectric drop are

$$E_r = -\frac{\partial \Phi}{\partial r} \approx -\nu A r^{\nu-1} P_{\nu}(\cos \theta), \quad E_{\theta} = -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \approx -\nu A r^{\nu-1} \sin \theta P_{\nu}'(\cos \theta)$$

$$\sigma(r) = \frac{1}{4\pi} E_{\theta} |_{\theta=\beta} \approx \frac{A}{4\pi} r^{\nu-1} \sin \beta P_{\nu}'(\cos \beta)$$

A drop with conical ends can be constructed by matching a spheroid with two cones angle θ . For a drop with a conical tip, the electric field must diverge as $1/\sqrt{r}$. Therefore the electric field near the conical tips can be found when $\nu = 1/2$.

$$E_r \approx -\frac{1}{2} A r^{\frac{-1}{2}} P_{\frac{1}{2}}(\cos \theta), \quad E_{\theta} \approx -\frac{1}{2} A r^{\frac{-1}{2}} \sin \theta P_{\frac{1}{2}}'(\cos \theta) \quad (13)$$

$$\sigma(r) \approx \frac{A}{4\pi} r^{-\frac{1}{2}} \sin \beta P_{\frac{1}{2}}'(\cos \beta) \quad (14)$$

The induced charge density of the dielectric liquid drop can be obtained from potential given in equation 1 and 2 by using the boundary conditions,

$$E_{in} = e_r \left(\frac{3}{\epsilon/\bar{\epsilon} + 2} \right) E_0 \cos \theta \quad \text{and} \quad E_{out} = e_r \left(2 \left(\frac{\epsilon/\bar{\epsilon} - 1}{\epsilon/\bar{\epsilon} + 2} \right) + 1 \right) E_0 \cos \theta$$

where e_r is unit vector towards the direction of r . The polarization surface charge density σ_{pol}

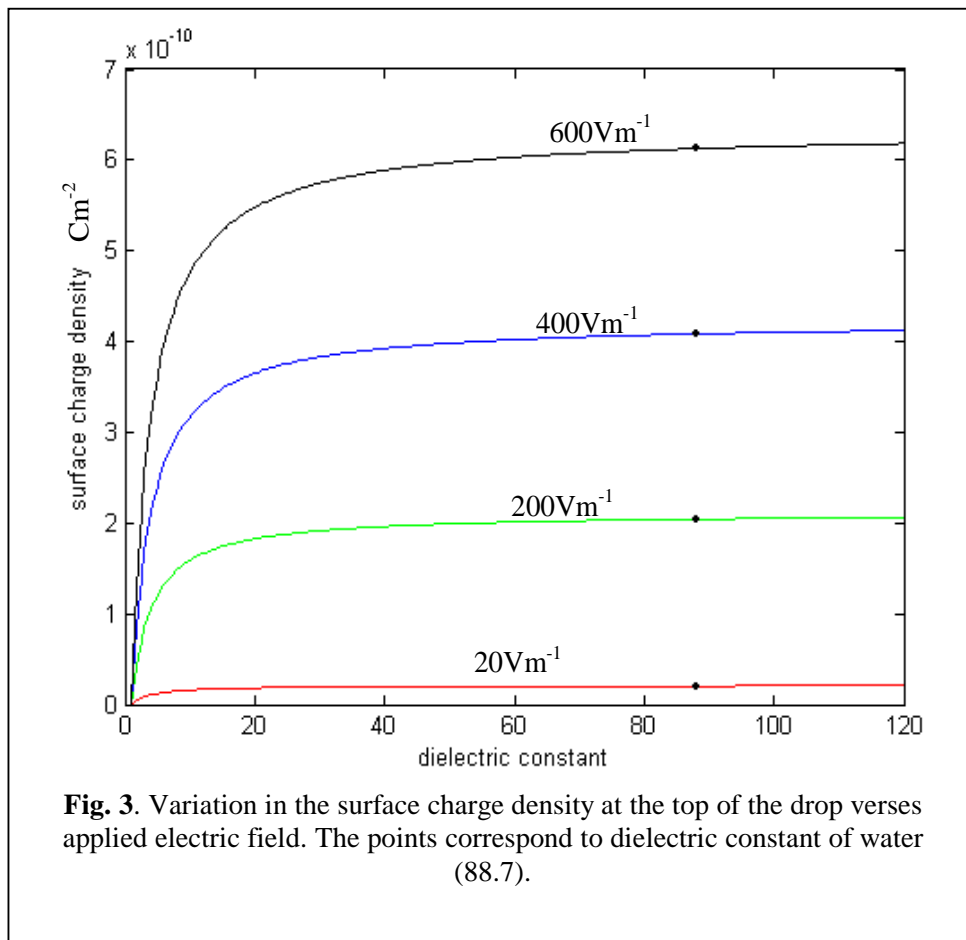


Fig. 3. Variation in the surface charge density at the top of the drop verses applied electric field. The points correspond to dielectric constant of water (88.7).

$$E_{out} - E_{in} = \frac{4\pi}{\epsilon_0} \sigma_{pol}$$

$$\sigma_{pol} = \frac{3\epsilon_0}{4\pi} \left(\frac{\epsilon/\bar{\epsilon} - 1}{\epsilon/\bar{\epsilon} + 2} \right) E_0 \cos \theta \quad (15)$$

The surface charge densities calculated for a dielectric fluid drop for different dielectric constant and applied electric fields of 20, 200, 400, and 600Vm⁻¹ are shown in figure 3. There are two main regions in the graph, a linear region and a saturate region. For an applied electric field, there is a limiting charge density for every dielectric

constant corresponding to the finite maximum of the charge separation of the liquid drop.

3. ENERGY AND ASPECT RATIO

In the absence of an electric field, the shape of the liquid drop is spherical due to minimization of surface energy caused by imbalance of adhesive forces and cohesive forces. In the presence of a uniform electric field due to induced electric charges inside the liquid drop, an excess electric energy is accumulated perturbing the shape of the drop. The electric energy stored in a spheroidal shape liquid drop can be obtained by evaluating the integral

$$E = \frac{1}{2} \int \sigma_{pol} \phi_{in} da = \frac{3\epsilon_0}{2} E_0^2 \left(\frac{\epsilon/\bar{\epsilon} - 1}{(\epsilon/\bar{\epsilon} + 2)^2} \right) r_0^2 = \frac{3\epsilon_0}{2} E_0^2 \left(\frac{\epsilon/\bar{\epsilon} - 1}{(\epsilon/\bar{\epsilon} + 2)^2} \right) \left(\frac{3v}{4\pi} \right)^{\frac{2}{3}} \quad (16)$$

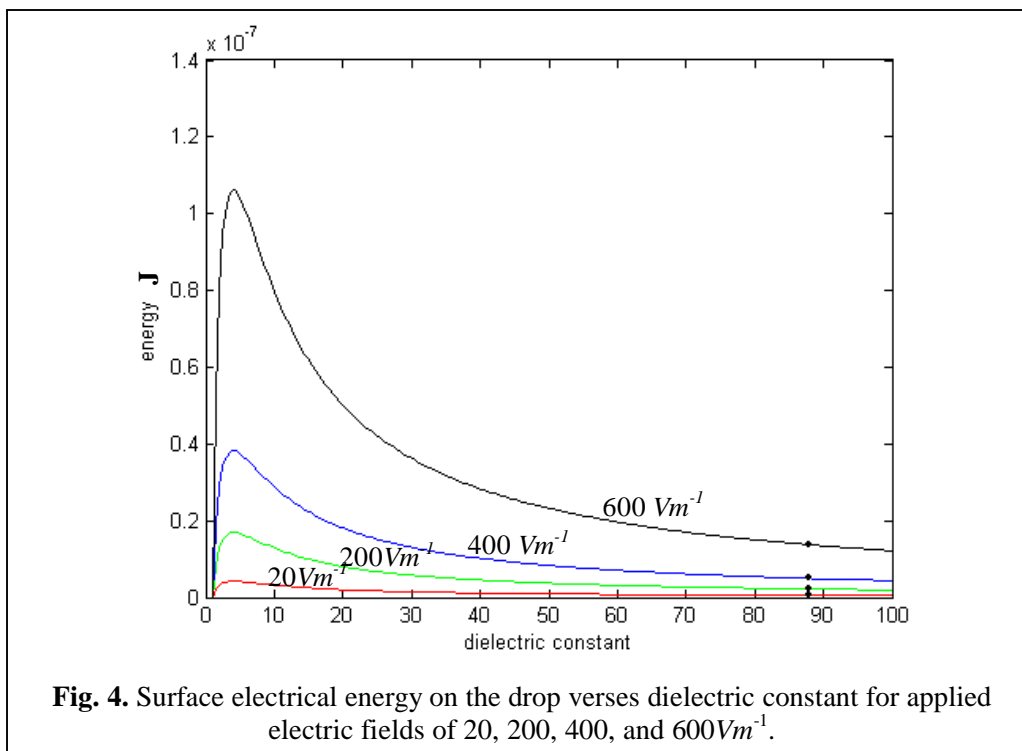


Fig. 4. Surface electrical energy on the drop versus dielectric constant for applied electric fields of 20, 200, 400, and 600Vm⁻¹.

where $v = 4\pi a_0^2 l / 3$ is the volume of the drop. a_0 and l are constant for spheroids. Figure 4 is a plot of the energy stored in a spheroidal dielectric liquid drop of unit volume as a function of dielectric constant for applied electric fields of 20, 200, 400, and $600Vm^{-1}$. The figure indicates that the electric energy on a spherical drop can be maximized in a weak dielectric by increasing the applied electric field.

A slender dielectric drop when exposed to an electric field appears from the outside as a line distribution of charges. Using Gauss's law, total induced charge Q and thereby the potential generated at a point (r,z) external to the drop due to line charge can be obtained. At the surface of the spheroid, $z = l$ and $r = a_0$. The potential for $a_0 \lll l$

$$\phi(r, z) = \frac{\epsilon}{4\bar{\epsilon}} \frac{d}{dz} (a^2 E) \ln \left| \frac{4l}{a_0} \right|. \quad (17)$$

At this point a new parameter termed, aspect ratio $R = l / a_0$ is defined. The induced axial electric field is

$$E_{ic} = \frac{\epsilon}{2\bar{\epsilon}} \ln \left| \frac{4l}{a_0} \right| \frac{d^2}{dz^2} (a(z)^2 E) \quad (18)$$

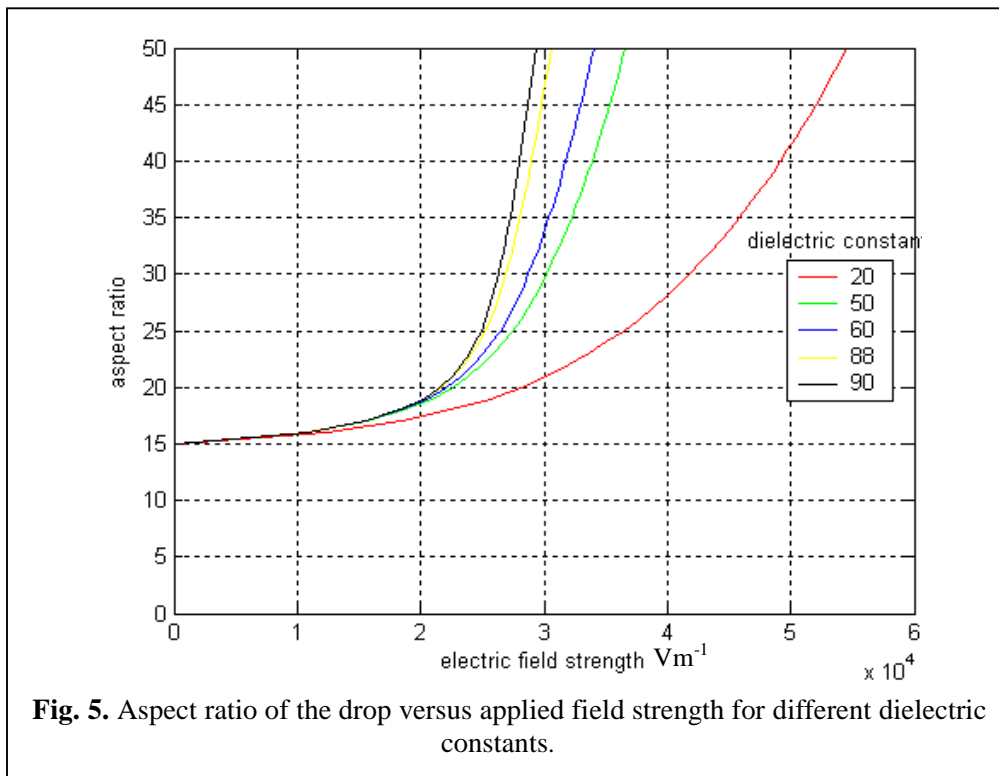
The resultant field inside the drop

$$E(z) = E_0 - E_{ic} = E_0 - \frac{\epsilon}{2\bar{\epsilon}} \ln \left| \frac{4l}{a_0} \right| \frac{d^2}{dz^2} (a(z)^2 E) \quad (19)$$

With the assumption that electric field inside the drop is still a uniform electric field is

$$E(z) = \frac{E_0}{\left(1 + \frac{\epsilon}{2\bar{\epsilon}} \ln \left| \frac{4l}{a_0} \right| \frac{d^2}{dz^2} (a(z)^2) \right)}. \quad (20)$$

Stationary liquid drop acting under the influence of capillary forces assumes an exact spherical shape and can be expected to be nearly spheroidal in a small electric field. The stable configurations can be assumed to have an elliptical boundary of the form



$$\frac{a(z)^2}{a_0^2} + \frac{z^2}{l^2} = 1 \quad (21)$$

For convenience, initially these boundaries have been chosen to be two dimensional coordinate spaces. This assumption has the advantage of greatly simplifying the boundary value problem associated with electric field. Therefore equation (20) simplifies to

$$E(z) = \frac{E_0}{\left(1 + \frac{\epsilon}{\bar{\epsilon}} \left(\frac{1}{R}\right)^2 \ln|4R|\right)} \quad (22)$$

The normal stress balance equation (9) can be coupled with equation (22) for a stable static drop volume $V = 4\pi a_0^2 l / 3$ of a spheroid. With the symmetry requirements for $E(z)$ and $a(z)$ coupled equation read

$$\left(\gamma \left(\frac{4\pi R}{3} \right)^{\frac{1}{3}} - \Delta P \right) \left(1 + \frac{\epsilon}{\bar{\epsilon}} \left(\frac{1}{R} \right)^2 \ln |4R| \right)^2 = \epsilon_0 (\epsilon - \bar{\epsilon}) E_0^2 \quad (23)$$

The above equation was solved numerically.

Regardless of the orientation of the spheroid, aspect ratio R can be changed by changing the applied electric field. Figure 5 shows aspect ratio for a unit volume of a liquid drop as a function of applied electric field strength for various dielectric constants. Aspect ratio is significantly affected by interfacial tension and the excess pressure of the drop. Therefore the analysis was done for selected

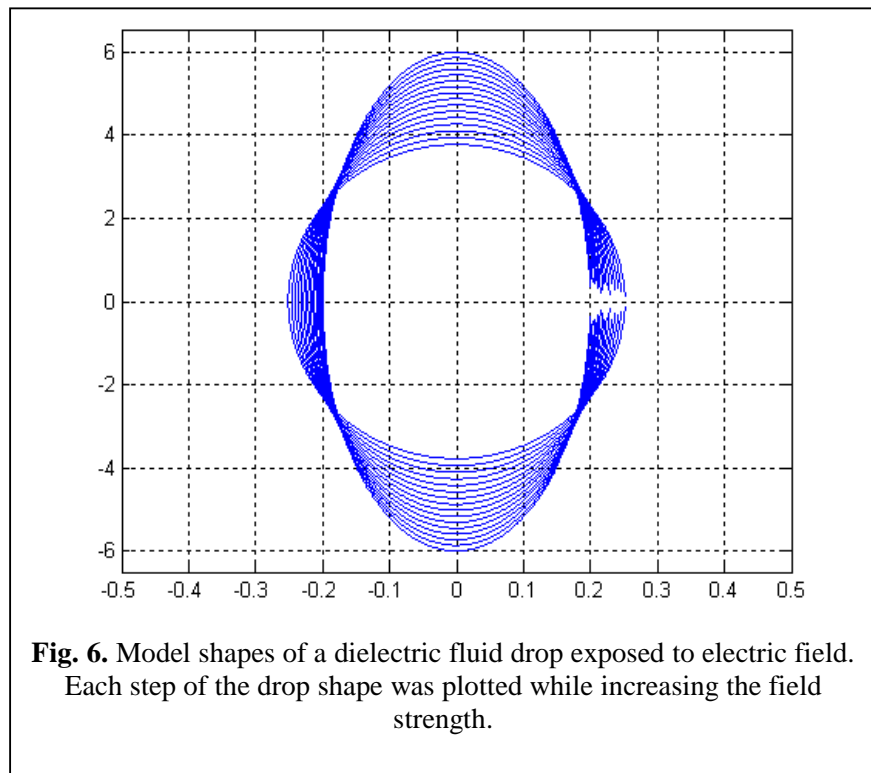


Fig. 6. Model shapes of a dielectric fluid drop exposed to electric field. Each step of the drop shape was plotted while increasing the field strength.

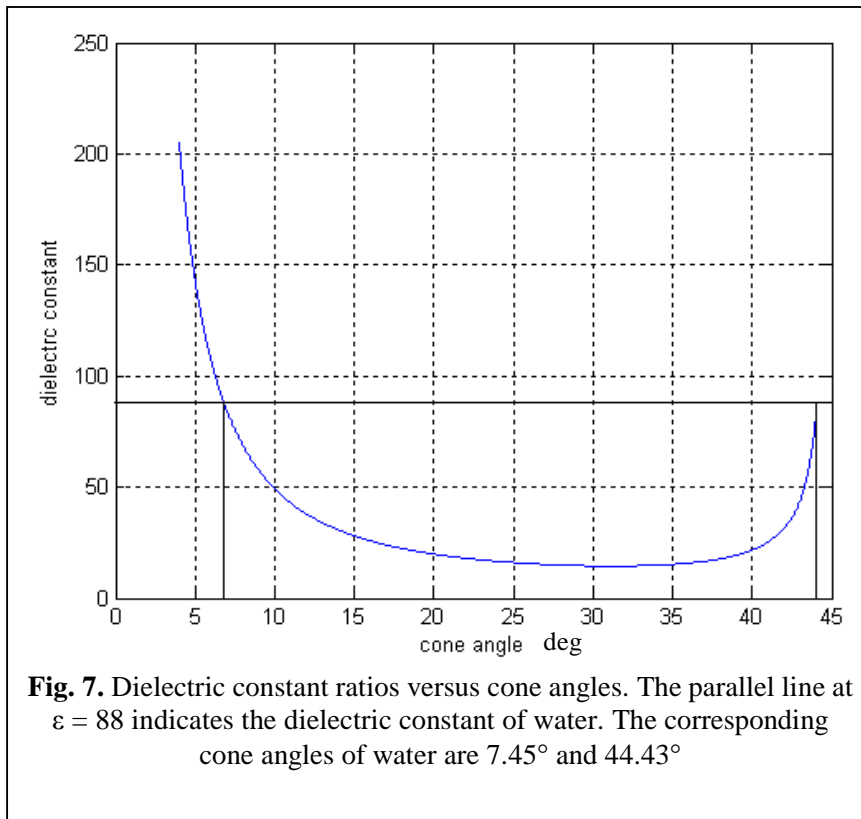
values of interfacial tension, namely interfacial coefficient between water and air and excess pressure of one unit. Small aspect ratios of the drop was neglected in the calculation by the assumption $a_0 \lll l$. Figure 5 shows that the shape of the drop gets further elongated with increased electric field strengths. When the dielectric constant of the drop is decreased, the field strength which is necessary to obtain the same aspect ratio has to be increased. There is a sharp increment of the aspect ratio after the threshold value of the applied field strength. The deformation of the fluid drop increases with the increase in dielectric constant of the fluid drop. For high applied electric fields, the drop still endures its configuration properties, interfacial tension and electric bond number.

The shapes of the drop modeled for a unit volume by taking the interfacial tension between the water and air for small excess pressures for several aspect ratios by using numerical values of electric field is shown in figure 6. This model was applied for large aspect ratios of the drop to avoid the domination of $(1/R)^2$ term in equation (23) leading to nonlinear equations.

The range of aspect ratio of the drop was taken between $15 \leq R \leq 30$. We have shown that a drop with conical tips must have an electric field that diverges as $r^{-1/2}$ with amplitude determined by the force balance equation (9). This amplitude depends on the applied field and the shape of the drop. In order to find external fields that give rise to conical tips, a liquid drop with cone angle $2\alpha_0$ was considered. As $z \rightarrow l$, $a_0 \rightarrow 0$, static equilibrium require that the local electric field diverges as $a(z)^{-1/2}$. As $z \rightarrow l$, $a(z) = (l - z) \tan \alpha_0$. With $R = a_0/l = 1/\tan \alpha_0$ equation (19) read

$$\frac{E_0}{E(z)} = \left(1 - \frac{3}{8} \frac{\epsilon}{\bar{\epsilon}} \tan^2 \alpha_0 \ln \left(\frac{1}{\tan \alpha_0} \right) \right) \quad (22)$$

When the electric field ratio between the outer surface of the drop and the inner surface is equal to 2, equation (22) read



$$\frac{\varepsilon}{\bar{\varepsilon}} = -\frac{8}{3} \left[\tan^2 \alpha_0 \ln \left(\frac{1}{\tan \alpha_0} \right) \right]^{-1} \quad (23)$$

This ratio gives a good approximation of the minimum dielectric constant needed to produce conical tips at the end of the drop. Figure 7 shows the dielectric ratio with respect to cone angles. The minimum dielectric constant ratio is 14.4975 and the cone angle is 31.25° . Equation (23) also gives reasonable results for two larger cone angles. The two singular points in the graph in the limit $\varepsilon/\bar{\varepsilon} \rightarrow \infty$ correspond to $\alpha_0 \rightarrow 45^\circ$ and $\alpha_0 \rightarrow 0$. Water has a dielectric constant (88) greater than the critical value for producing conical tips. The conical interface for water is at two particular half angles (7.450 and 44.430). One angle is stable while the other is unstable.

4. MINIMUM ELECTRIC FIELD NECESSARY TO PRODUCE CONICAL ENDS

The minimum electric field required to sustain a drop with conical tips is a function of the dielectric constant ε . The units of quantities were removed by introducing dimensionless variables, electric capillary l_{cap} and electric Bond number B ,

$$l_{cap} = \frac{2\gamma}{\varepsilon\bar{\varepsilon}E_0^2} \quad \text{and} \quad B = \frac{r}{l_{cap}} = \frac{\varepsilon_0\bar{\varepsilon}r}{2\gamma} E_0^2 \quad (24)$$

The volume of the liquid drop may be reduced to a convenient form

$$V = \int_{-l}^l a(z)^2 dz = \frac{4}{3} B^3 \left[\frac{2}{\ln R} \right]^{\frac{1}{2}} \left(\frac{\varepsilon}{\bar{\varepsilon}} \right)^{\frac{5}{2}}$$

The electric bond number can be eliminated using equation (24)

$$\frac{4}{3} \left(\frac{\varepsilon\bar{\varepsilon}r}{2\gamma} E_0^2 \right)^3 \left[\frac{2}{\ln R} \right]^{\frac{1}{2}} \left(\frac{\varepsilon}{\bar{\varepsilon}} \right)^{\frac{5}{2}} = V$$

$$E_0^6 (e)^{\frac{5}{2}} = \frac{3}{4} V \left[\frac{2}{\ln R} \right]^{-\frac{1}{2}} \left(\frac{\varepsilon_0\bar{\varepsilon}r}{2\gamma} \right)^{-3} = \beta^6 \quad (25)$$

where $V = 1$, $\frac{\varepsilon}{\bar{\varepsilon}} = e$, $\beta = \left(\frac{3}{4} V \right)^{\frac{1}{6}} \left[\frac{2}{\ln R} \right]^{-1/12} \left(\frac{\varepsilon_0\bar{\varepsilon}r}{2\gamma} \right)^{-1/2}$.

In order to find the minimum electric field E_{\min} which produces conical tips, the finite minimum volume of the drop must be found. This can be obtained by minimizing β . In terms of new parameters, the electric field is $\tilde{E} = E(z)/E_0$, radial distance $\tilde{a}(\tilde{z}) = (\epsilon/\bar{\epsilon} - 1)a(z)/l_{cap}$, axial distance $\tilde{z} = [2(\epsilon/\bar{\epsilon} - 1)/\ln R]^{1/2} z/l_{cap}$ and the internal excess pressure is $\tilde{p} = (l_{cap} / \gamma(\epsilon/\bar{\epsilon} - 1))\Delta P$. Then the governing equations reduces to the simpler form

$$\tilde{E} - \frac{d^2}{d\tilde{z}^2}(\tilde{a}(\tilde{z})\tilde{E}) = 1 \quad (a), \quad \tilde{E}^2 + \tilde{p} = \tilde{a}^{-1} \quad (b), \quad \int_{\tilde{z}}^{\tilde{l}} \tilde{a}^2 d\tilde{z} = \frac{4}{3} B^3 \left[\frac{2}{\ln R} \right]^{1/2} \left(\frac{\epsilon}{\bar{\epsilon}} \right)^{5/2} \quad (c). \quad (26)$$

For given value of internal excess pressure \tilde{p} , equation (26 (a) and(b)) determine the scaled shape of the drop. The corresponding electric field or equivalently the electric bond number is necessary to find the shape of the drop. In order to determine the drop shape numerically, a new parameter v is defined as $v = \tilde{a}^2 \tilde{E}$,

$$v dv = \frac{(\tilde{E} - 1)(\tilde{p} - 3\tilde{E}^2)}{(\tilde{E}^2 + \tilde{p})^3} d\tilde{E} \Rightarrow v^2 = \frac{3\tilde{E}^2 - 2\tilde{E} + \tilde{p}}{(\tilde{E}^2 + \tilde{p})^2} + c \quad (27)$$

At the conical end $v \rightarrow 0$ and the constant of integration $c = 0$. The electric field \tilde{E} , which is symmetric about $\tilde{z} = 0$, may be determined by integrating the first order differential equation⁹

$$\frac{d\tilde{E}}{d\tilde{z}} = \frac{(3\tilde{E}^2 - 2\tilde{E} + \tilde{p})^{1/2} (\tilde{E}^2 + \tilde{p})}{3\tilde{E}^2 - \tilde{p}} \quad (28)$$

If $\tilde{E} = 0$ is attained, then the shape of the drop follows from equation (26 a).

Drop volume was numerically obtained for different values of \tilde{p} from equation 26(c) (Figure.8). According to the figure, the dimensionless drop volume tends to a finite minimum as $\tilde{p} \rightarrow 1/3$ and to infinity as $\tilde{p} \rightarrow -1$. By symmetry of \tilde{E} about the centre of the drop and from equation (28), $\tilde{E}(0)$ can be shown as the larger root of $3\tilde{E}^2 - 2\tilde{E} + \tilde{p} = 0$,

$$\tilde{E}(0) = \frac{1}{3} \left(1 + (1 - 3\tilde{p})^{1/2} \right) \quad \text{and} \quad \tilde{a}(0) = \frac{9/2}{1 + 3\tilde{p} + (1 - 3\tilde{p})^{1/2}} \quad (29)$$

The length and volume are given by the integrals

$$\tilde{l} = \int_{\tilde{E}(0)}^{\infty} \frac{3\tilde{E}^2 - \tilde{p}}{(3\tilde{E}^2 - 2\tilde{E} + \tilde{p})^{1/2} (\tilde{E}^2 + \tilde{p})^2} d\tilde{E}, \quad V = \int_{\tilde{E}(0)}^{\infty} \frac{3\tilde{E}^2 - \tilde{p}}{(3\tilde{E}^2 - 2\tilde{E} + \tilde{p})^{1/2} (\tilde{E}^2 + \tilde{p})^4} d\tilde{E} \quad (30)$$

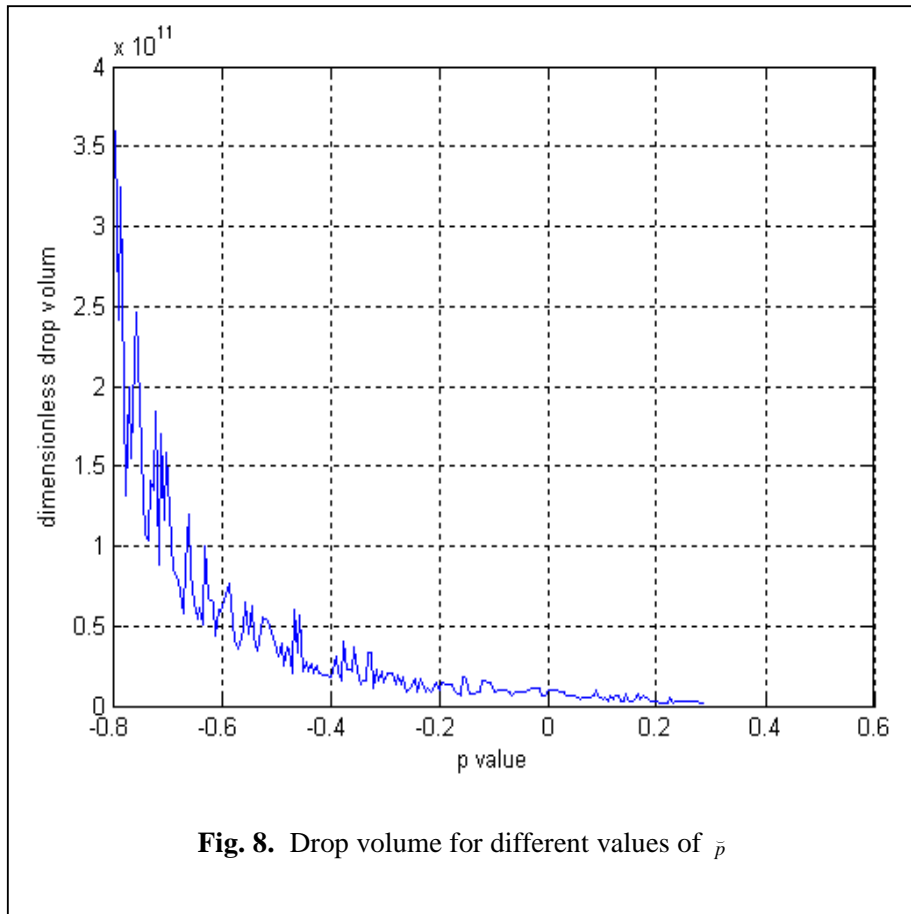
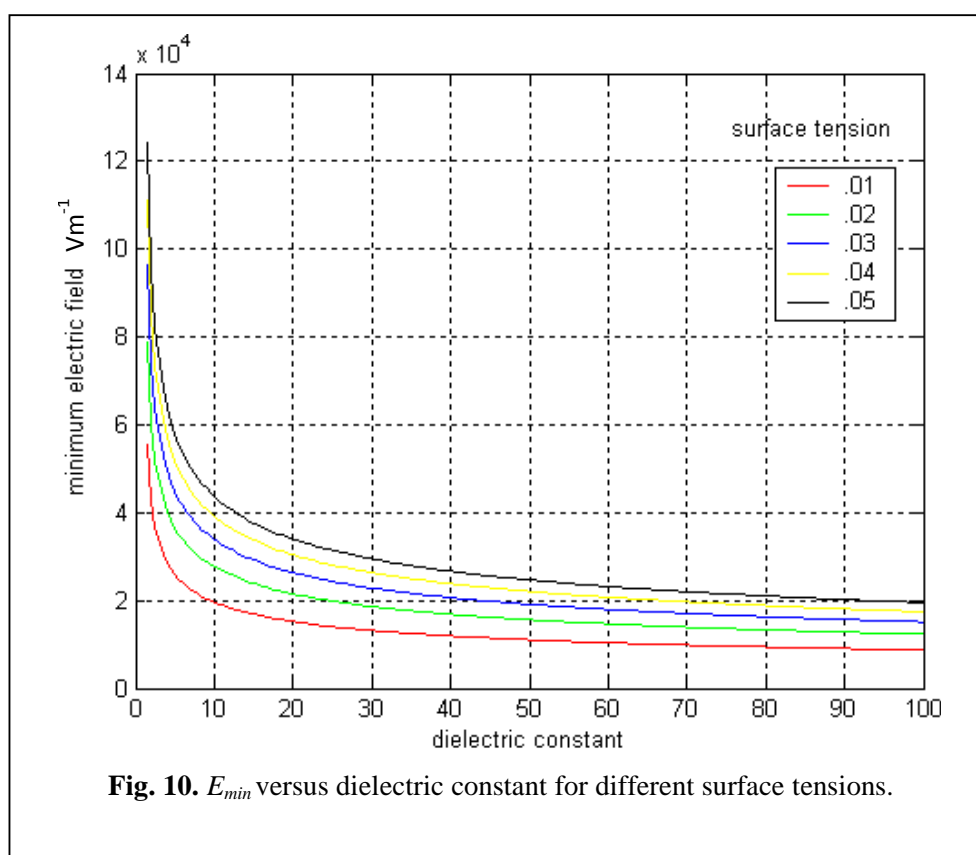


Fig. 8. Drop volume for different values of \bar{p}

Equations (26)(a) and (b) were solved numerically for dimensionless shapes of the drop for various \bar{p} values. Deformation of the drop can be increased by increasing \bar{p} .

Assuming there is a minimum drop volume for which a solution is possible, the minimum electric field E_{min}

$$E_{min}^6 \left(\frac{\epsilon}{\bar{\epsilon}} \right)^{5/2} = \text{constant} \quad (31)$$



The proportionality constant of the equation (31) depends with the interfacial tension between the drop and the surrounding medium. E_{min} as a function of dielectric constant for different aspect ratios is shown in figure 9. All the curves have overlapped. There is also logarithm term of the aspect ratio which has no significant effect on the minimum electric field. We can predict that the minimum electric field which should be applied to produce conical interfaces will not be affected by the aspect ratio of the drop.

The minimum electric field necessary to produce conical interface varies proportionally with the surface tension γ in the form $\gamma^{-1/2} \epsilon_c > 17.59$. Figure 10 shows the variation of the minimum electric field necessary to produce a conical drop with the surface tension.

Surface tension of the water at the 0 °C was assumed to be 0.03 and a unit volume of the drop was considered. Figures 11 and 12 represent the changing configuration of the drop when the applied electric field strength is increased to the minimum value which produces conical interface. The figure 12 shows the modeled shapes for various applied electric fields. The conical tips arise when the aspect ratio of the drop is equal to 15. If the field strength is increased further it will caused more deformation. The cone angles of the

drop are between is 7.45° and 44.48° . The minimum electric field required to produce conical ends for a water drop was 1.0854×10^4 units. The figure 12 shows that the cone angle of the drop is decreasing with increasing applied field strengths making the drop more stable at higher fields.

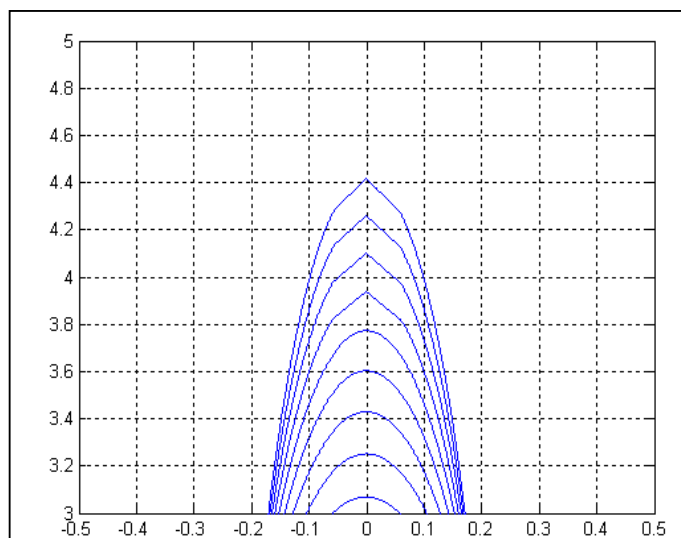


Fig. 11. Changing configuration of a water drop to conical tips with applied electric field strength

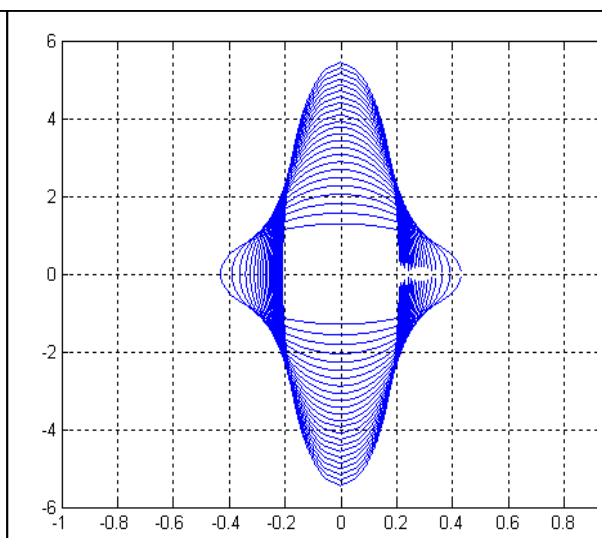


Fig. 12. For dielectric constant ratio $\epsilon_c = 14.4975$, highly deformed shapes form conical ends.

5. CONCLUSIONS

The governing equations using the assumption of slenderness gave analytic characteristics of the drop shape and the minimum electric field necessary to obtain conical drops. The 3D calculations showed that there is a limiting charge density for every dielectric constant corresponding to the finite maximum of the charge separation of the liquid drop for and applied electric field and the electric energy on a spherical drop can be maximized in a weak dielectric by increasing the applied field. The minimum dielectric constant ratio need to produce a conical tip at the end of the drop is 14.5 corresponding to a cone angle 31.25° . The liquid drop volume changes with internal excess pressure tending to a finite minimum. Deformation of the drop can be increased by increasing internal excess pressure. Aspect ratio which is significantly affected by interfacial tension and the excess pressure of the drop has a sharp increment after the threshold value of the applied field is reached. With the decrease in dielectric constant of the drop the field strength which is necessary to obtain the same aspect ratio has to be increased. For a particular dielectric constant ratio, the threshold electric field producing conical interfaces will not be affected by the aspect ratio of the liquid drop but increases with the increased surface tension of the liquid. The minimum electric field varies proportionally with the surface tension in the form $\gamma^{-1/2} \epsilon_c > 17.59$. The conical interface of water has two half angles 7.45° and 44.43° . For the minimum dielectric ratio the cone angle of the drop decreases with applied field making the drop more stable at higher fields. The minimum electric field producing a conical ends of a water drop was 1.0854×10^4 units.

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