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MIXTURE OF DISTRIBUTIONS AS A LIFETIME DISTRIBUTION OF A BUS ENGINE

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Abstract

We show that a upside-down bathtub failure rate function can be obtained from a mixture of two increasing failure rate function (IFR) models. Specifically, we study the failure rate of the mixture an exponential distribution, and an IFR distribution with strictly increasing failure rate function. Examples of several other upside-down bathtub shaped failure rate functions are also presented. The method are illustrated by a numerical example of the time between the failures for the bus engines.

Keywords: *Bathtub curve, upside-down bathtub curve, failure rate function, mixture of distributions, reliability function*

1. Introduction

The distributions with non-monotonic failure rate functions are considered frequently in the reliability theory and practice. The distributions with a bathtub shape failure functions (BFR) belong to such a class of distributions. The models with BFR are very useful in the reliability theory and practice. We give the definition of a bathtub shape failure rate function below. It is useful, throughout his paper by increasing or decreasing, understood respectively as non – decreasing or non – increasing.

Definition 1: A lifetime T, with failure rate function r(t) is said to have a bathtub shaped failure rate if there exists t^* such that $0 < t^* < \infty$ and r(t) is decreasing for $0 \leq t \leq t^*$ and r(t) is increasing for $t > t^*$.

A brief discussion and summary for such distributions are given in [1] and [12]. However, there are known many examples of applications of distributions with upside-down bathtub shaped (unimodal) failure rate functions (UBFR). In particular cases, the unimodal failure rate function is used in [9] for data of motor bus failures, in [1] and [4] for optimal burn decisions, and in [5] and [7] for ageing property in reliability.

One of the ways of generating distributions with non-monotone failure rate functions is mixing the standard distributions. It is well known result that the mixture of distributions with a decreasing failure rate functions (DFR) has a decreasing failure rate function (see Prochan [11]). Klutke et al. [8] have been studied the mixture of two Weibull distributions and they suggested that this mixture can be the distribution with unimodal failure rate function. However, in [13], it is stated that the considered mixture failure rate function has a decreasing initial period. The mixture of the two Weibull distributions has also been studied in [14]. For the same values of scale parameter all possible types of shape failure rate function are found. However, for the different scale parameter the numerical computing is performed. Block et al. [3] have been studied the mixture of two distributions with increasing linear failure rate functions.

The paper is organized as follows. In Section 2, the model of the mixture of two distributions is introduced and discussed, while, in Section 3, the particular cases are considered. In last section the numerical examples with technical data are presented.

2. The model of mixture distributions

We consider a mixture of two lifetime T_1, T_2 with densities $f_1(t)$, $f_2(t)$, with corresponding reliability functions $R_1(t)$, $R_2(t)$, failure rate function $r_1(t)$, $r_2(t)$ and weights p and $q = 1 - p$, where $0 < p < 1$. The mixed density is then written as

$$f(t) = p\, f_1(t) + (1 - p)\, f_2(t)$$

and mixed reliability functions is

$$R(t) = p\, R_1(t) + (1 - p)\, R_2(t)$$

The failure rate function of the mixture can be written as the mixture [2]

$$r(t) = \omega(t)\, r_1(t) + [1 - \omega(t)]\, r_2(t)$$

where $\omega(t) = pR_1(t) / R(t)$. Moreover, from [2], we have under some mild conditions, that

$$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} \min\{r_1(t), r_2(t)\}$$

In the following propositions, we give some properties for the mixture failure rate function.

Proposition 1: For the first derivative of $\omega(t)$, we have

$$\omega'(t) = \omega(t)\,(1 - \omega(t))\,(r_2(t) - r_1(t))$$

Proposition 2: For the first derivative of $r(t)$, we obtain

$$r'(t) = (1 - \omega(t))\,((-\omega(t)\,(r_2(t) - r_1(t))^2 + r'_2(t)) + \omega(t)\, r'_1(t)$$

Proposition 3: If $R_1(t) = \exp(-\lambda_1 t)$, then

$$r'(t) = (1 - \omega(t))\,((-\omega(t)(r_2(t) - \lambda_1)^2 + r'_2(t))$$

Proposition 4: If $R_1(t) = \exp(-\lambda_1 t)$, then $r'(0) \geq 0$ if and only if

$$r'_2(0) \geq p\,(r_2(0) - \lambda_1)^2$$

We suppose that $r_2(t) = \gamma t + \alpha t^{\alpha-1} / \beta^{\alpha}$, where $\alpha \geq 1$. The reliability function $R_2(t)$ is a particular case of the reliability function given by Gurwich [6] (see also [10]).Without loss generality, we assume that $\beta = 1$. Hence $r_2(0) = 0$ for $\alpha > 1$.Consequently, the reliability function of corresponding toT_2 is

$$R_2(t) = \exp(-½ \gamma t^2 - t^\alpha) \quad \text{for } t \geq 0.$$

Let $h_1(t) = \omega(t)\ (r_2(t) - \lambda_1)^2$, $h_2(t) = r'_2(t)$. Since $\omega(t) \geq 0$ for $t \geq 0$ and $r_2(t)$ is increasing from 0 to ∞, and $\omega(0) = p$, $\omega(\infty) = 1$, we conclude, that the equation $h_1(t) = 0$ has only one solution t_1. We can also examine the ratio of the function $h_1(t)$ and $h_2(t)$, i.e.

$$\lim_{t\to\infty} \frac{h_1(t)}{h_2(t)} = \infty$$

Since there are t' such that, for all $t > t'$, we have $h_1(t) > h_2(t)$, where as, we have $h_1(t_1) = 0 < h_2(t_1)$. Hence the equation $h(t) = h_1(t) - h_2(t) = 0$ has at least one solution.

3. The particular case of mixture

We shall give the conditions under which the failure rate of the mixture of an exponential distribution and the distribution with failure rate $r_2(t)$ has an UBFR. In this section, we consider three particular cases of a failure rate $r_2(t)$.

Proposition 5: If $2 \leq \alpha \leq 6$ and $p\,\lambda_1^2 \leq \gamma$ then $r(t) \in$ UBFR.

Proof: It is know that the equation $h(t) = 0$ has at least one a solution for $t > t_1$. We consider the ratio

$$u(t) = \frac{(r_2(t) - \lambda_1)^2}{r'_2(t)}$$

It is easy that $u(t_1) = 0$ and $\lim_{t\to\infty} u(t) = \infty$. For the first derivative, we have

$$u'(t) = \frac{r_2(t) - \lambda_1}{[r'_2(t)]^2}\{2[r'_2(t)]^2 - r''_2(t)(r_2(t) - \lambda_1)\}$$

Let $u_1(t) = 2\,[r'_2(t)]^2 - r''_2(t)\,(r_2(t) - \lambda_1)$ and

$$u_1(t) = 2\gamma^2 + \gamma\alpha(\alpha - 1)t^{\alpha-2}(6 - \alpha) + \alpha^3(\alpha - 1)t^{2\alpha-4} + \lambda_1\alpha(\alpha - 1)\,(\alpha - 2)t^{\alpha-3}$$

If $2 \leq \alpha \leq 6$ then $u_1(t) > 0$ and $u'(t) > 0$ for $t \geq t_1$.

By Proposition 1 $\omega(t)$ is increasing for $t \geq t_1$ and $\omega(t)\,u(t)$ is increasing for $t \geq t_1$. Hence the equation $h(t) = 0$ has only one solution and $r(t) \in$ UBFR.

4. The numerical examples

In this section, we consider four examples to illustrate the theoretical research given in the previous sections.

Example 1: We consider the exponential distribution with failure rate function $r_1(t) = \lambda_1 = 1$ and Gurvich distribution given in the section II with parameters $\beta = 1$, $\gamma = 1$, $\alpha \in \{ 2.5, 3, 4, 5, 6\}$and mixing proportion $p = 0.8$. Thus $r(t)$ have an upside – down bathtub shaped. Figure 1 shows the five plots of $r(t)$.

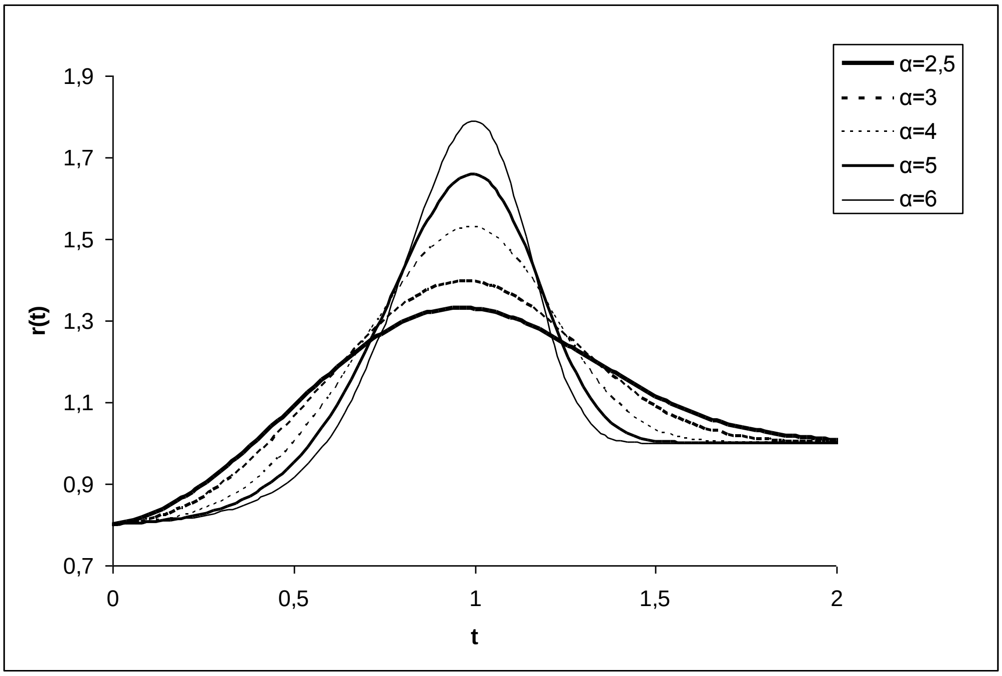

Fig. 1. Mixture failure rate of exponential and Gurwich distribution for $\alpha \in \{2.5, 3, 4, 5, 6\}$ with UBFR shape

Example 2: We consider two failure rate functions namely $r_1(t) = \lambda = 1$ and a failure rate of Gurwich distribution with $\alpha = 2$, $\beta = 1$, $\gamma \in \{ 2, 3, 4, 5, 6 \}$. The mixing proportion p = 0.8. Figure 2 shows the plots of r(t) for different values of parameter γ.

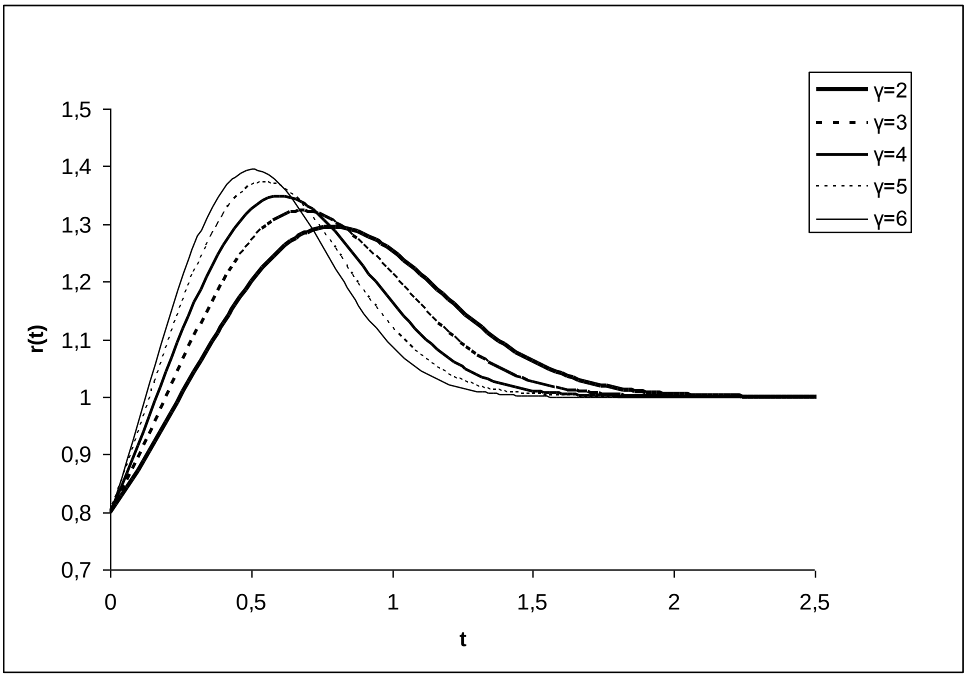

Fig. 2. Mixture failure rate of exponential and IFR distribution for $\gamma \in \{2, 3, 4, 5, 6\}$, with UBFR shape

Example 3: In this example, we consider the mixture of exponential distribution with $\lambda = 2$ and Gurwich distribution with $\alpha = 3$, $\beta = 1$ and $p \in \{0.4, 0.3, 0.2, 0.15, 0.1\}$. Figure 3 contains five plots of failure rate functions for different values of p.

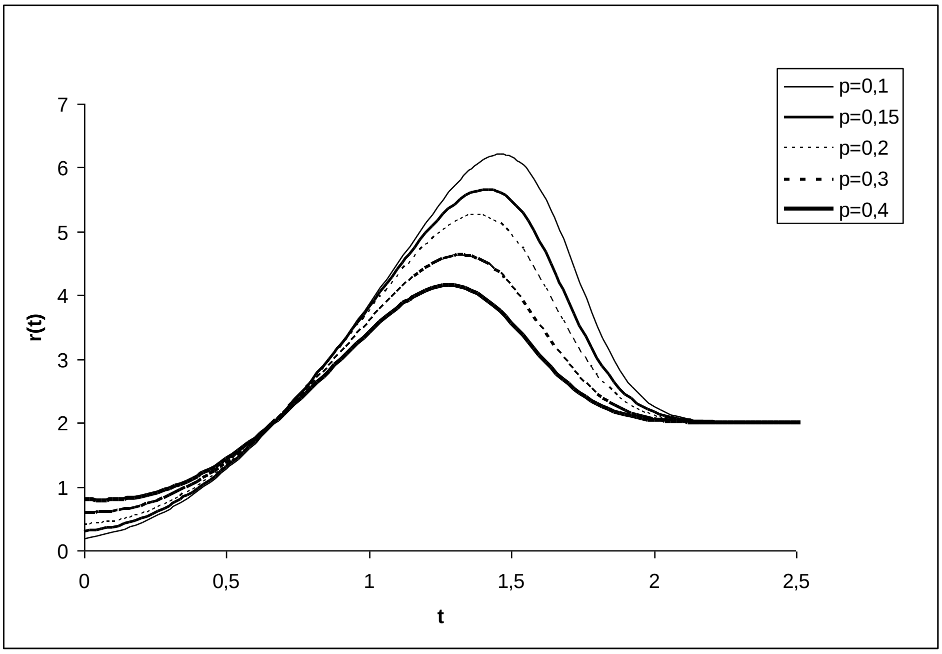

Fig. 3. Mixture of exponential distribution and Gurwich distribution for $p \in \{0.4, 0.3, 0.2, 0.15, 0.1\}$

Example 4: In this example, we consider a real lifetime data. The object of the investigation is a real municipal bus transport system within a large agglomeration. The analyzed system operates and maintains 210 municipal buses of various marks and types. For the investigation purpose, 35 buses of the same make were selected. The data set contains n = 1081 times between successive failures of the engine of the bus. We estimate the parameters p, α, γ, β, λ of the model with the reliability function

$$R(t) = p \exp(-\lambda t) + (1 - p) \exp(-0.5\,\gamma t^2 - (t/\beta)^{\alpha})$$

By maximizing the logarithm of likelihood function for grouped data, we calculate p =0.76, α=2.71, γ =15.56, β= 99.03, λ=0.082. For these values of parameters, we prove Pearson's test of fit and compute associated p–value is 0.46. The reliability function R(t) sufficiently precisely describes the empirical reliability function. By Proposition 5, we conclude that the failure rate is UBFR.

5. Conclusions

Sometimes, we have upside-down bathtub estimated failure rates from model which do not have theoretical UBFR. In this paper we ha ve presented flexible and practical model for UBFR. The purpose of this paper is to present a new UBFR as a mixture of two distributions for the first time. The model of UBFR presented in this paper is fully adaptive to the available failure data and this distributions gives reliability engineers and biostatisticians another option for modeling the lifetime. The numerical examples for life time of an engine system of a bus shows that the mixture can be useful to practical applications.

References

[1] Block HW, Savits TH. Burn–in. Statistical Science 1997; 12:1–13.

[2] Block HW, Joe H. Tail behavior of the failure rate function of mixtures. Lifetime Data Analysis 1997; 3:269–288.
[3] Block HW, Savits T H, Wondmagegnehu ET. Mixtures of distributions with increasing linear failure rates. Journal Application Probability 2003; 40:85–504.
[4] Chang DS. Optimal burn – in decision for products with an unimodal failure rate function. European Journal Operations Research 2000; 126:584–640.
[5] Gupta GL, Gupta RC. Aging characteristics of the Weibull mixtures. Probability in the Engineering and Information Science 1996; 10:591–600.
[6] Gurvich MP, Dibenedetto AT, Range SV. A new statistical distribution for characterizing the random strength of brittle materials. Journal of Material Science 1997;32:2559– 2564.
[7] Jiang R, Ji P, Xiao X. Aging property of unimodal failure rate models. Reliability Engineering System Safety 2003; 79:113–116.
[8] Klutke GA, Kiessler PC, Wortman MA. A critical look at the bathtub curve. IEEE Transactions on Reliability 2003; 52:125–129.
[9] Mudholkar GS, Srivastava DK, Feimer M. The exponentiated Weibull family: a reanalysis of the bus – motor failure data. Technometrics 1995:37; 436–445.
[10] Nadarajah S, Kotz S. On Some Recent Modifications of Weibull Distribution. IEEE Transactions on Reliability 2005; 54:560–561.
[11] Proschan F. Theoretical explanation of observed decreasing failure rate. Technometrics 1963; 5:375–383.
[12] Rajarshi S, Rajarshi MB. Bathtub distributions: A review. Communications Statistics, Theory and Methods1988; 17:2597 –2621.
[13] Wondmagegnehu ET, Navarro J, Hernandez PJ. Bathtub shaped failure rates from mixtures: A practical point of view. IEEE Transactions on Reliability 2005; 54:270–275.
[14] Wondmagegnehu ET, On the behavior and shape of mixture failure rates a family of IFR Weibull distributions, Naval Research Logistics 2004;51:491-500.