SEMI-MARKOV MODEL OF MAINTENANCE
OF URBAN TRANSPORT BUSES

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Abstract

The object of the investigations is maintenance system of urban transport buses. The subject of the investigations is a determined set of the maintenance states of means of transport and their maintenance processes, as well as the relations occurring between the aforementioned elements, and between them and the maintenance process effectiveness. The paper deals with the selected issues related to modelling, forecasting and controlling the maintenance process of a certain class of technical objects being carried out in a complex maintenance system. Supporting a decision maker in the decisions making process concerning the maintenance system under analysis is to forecast the maintenance system behaviour and evaluate the influence of the selected decision making variants on the course of the maintenance process. The purpose of the work is to present a possibility to use a semi-Markov model of maintenance process of technical objects to preliminarily forecast the state of the maintenance system after changing the values of the model input parameters. Changing the value of the input parameters may simulate an impact of various factors on the system behaviour. The paper presents the assumptions to build a model of the process being performed within the investigation object and the method of analysing it. The values of the model parameters were assessed on the basis of the results of the preliminary test carried out in a real means of transport maintenance system. All the considerations have been illustrated by a computational example. Due to the assumed generalization degree of the description, the method to model and forecast the maintenance process presented herein may be used to other maintenance systems than the urban bus transport system.

Keywords: operation and maintenance, maintenance process, modelling, semi-Markov process, urban public transport

1. Introduction

When utilising vehicles, various events occur the results of which affect the courses of their usage and service processes, as well as the economic effect generated by the transport system. The vehicles in the maintenance process may be found in many different maintenance states that create the state space S.

Evaluating, analysing and forecasting the effectiveness of work, as well as reliability and readiness in the transport systems, are connected with the problems of mathematical modelling maintenance processes of technical objects. Therefore, it is necessary to create a formalised description of the processes performed in the means of transport maintenance system. The most important processes are those of changing the technical states of the objects, and those of using and servicing them. Those processes are random ones and depend on one another. The

mathematical models of those processes, by their very nature, shall be simplified. The practical conclusions resulting from studying those models should be formulated carefully.

The selected maintenance states of vehicles, described in the models of their maintenance processes, may change in various sequences, the order of which depends, to the great extent, on the method of carrying out transportation and the vehicle service processes, type and technical condition of the technical objects (including the means of transport), environmental conditions and the decisions made by the operators and the authorities who manage the system.

Due to analysing the results of the investigations performed in a real vehicle maintenance system, it has been found that there are no grounds to accept a hypothesis about the conformity of empiric distributions of times of duration of the analysed maintenance states with the exponential distribution. Therefore, the random process describing the changes of the maintenance states of the vehicles used in the investigation object is not the Markov process used frequently to model the problems of that type. Thus, the development method and possibilities of using the semi-Markov model of maintenance process of technical objects for preliminary forecasting the state of the maintenance system after changing the values of the model input parameters haven been presented herein.

The values of the model parameters have been assessed on the basis of the results of the preliminary tests carried out in a real means of transport maintenance system.

2. Model of maintenance process of a vehicle

The natural model of a bus maintenance process is a random process with a finite space of states S and a set of the parameters R_+. This process is denoted with the symbol $\{X(t) : t \in R_+\}$. It has been assumed that the semi-Markov process is a model which describes the real process of the maintenance state changes relatively well. In the practical applications it is needed to verify, if there are no grounds to reject the assumptions resulting from the mathematical apparatus applied.

The semi-Markov process $\{X(t) : t \geq 0\}$ is defined by a homogeneous, two-dimensional Markov chain:

$$\{\xi_n, \vartheta_n : n \in N_0\} \quad \xi_n \in S_k, \ \vartheta_n \in [0, \infty). \tag{1}$$

such one, that the probabilities of transitions depend on the discrete coordinate:

$$P\{\xi_{n+1} = j, \ \vartheta_{n+1} \leq t \ / \ \xi_n = i, \ \vartheta_n = t_n\} = P\{\xi_{n+1} = j, \ \vartheta_{n+1} \leq t \ / \ \xi_n = i\} \tag{2}$$

and

$$P\{\xi_0 = i, \ \vartheta_0 \leq 0\} = P\{\xi_0 = i\}. \tag{3}$$

The two-dimensional Markov chain defined that way is called the Markov renewal process, and the matrix:

$$Q(t) := [Q_{ij}(t) : i, j \in S], \quad t \geq 0 \tag{4}$$

where:

$$Q_{ij}(t) := P\{\xi_{n+1} = j, \vartheta_{n+1} \leq t / \xi_n = i\} \tag{5}$$

is called the renewal kernel.

Let $\quad \tau_0 = 0, \tau_n := \vartheta_1 + \ldots + \vartheta_{n1}, k \in N$ oraz $\quad v(t) := \max\{n : \tau_n \leq t\}$. The random process $\{X_k(t) : t \geq 0\}$ defined by the formula:

$$X(t) = \xi_{v(t)} \tag{6}$$

is called the semi-Markov process generated by the renewal kernel Q(t).

As it results from this definition, the semi-Markov process takes constant values in the random ranges:

$$X(t) = \xi_n \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}), \quad n \in N_0 , \tag{7}$$

Moreover, the sequence of random variables $\{X_k(\tau_n) = \xi_n : n \in N_0\}$ is a homogeneous chain with the transition probability matrix P:

$$P=[p_{ij}] = Q_{ij}(\infty) : i, j \in S, \tag{8}$$

which is called the Markov chain embedded in the semi-Markov process.

The symbol T_{ij} is used to denote the time which is elapsing from the moment of the process entering $\{X(t) : t \in R_+\}$ the state $i \in S$ until the first change of that state into the state $j \in S$. For the distribution F_{ij} of the random variable T_{ij} there are the following relations:

$$F_{ij}(t) = P\{T_{ij} < t\} = P\{\tau_{n+1} - \tau_n \le t | X(\tau_{n+1}) = j, X(\tau_n) = i\} = \frac{Q_{ij}(t)}{p_{ij}} \quad \text{when } p_{ij} > 0. \tag{9}$$

If $p_{ij} = 0$ it is assumed that:

$$F_{ij}(t) = \begin{cases} 0 & for \ t < 1 \\ 1 & for \ t \ge 1 \end{cases}. \tag{10}$$

The symbol T_i is used to denote the time which is elapsing from the moment of entering the state $i \in S$ until the moment of the first change of that state:

$$G_i(t) = P\{T_i < t\} = P\{\tau_{n+1} - \tau_n \le t \ | X(\tau_n) = i\} = \sum_{j \in S} Q_{ij}(t). \tag{11}$$

Let $P_j(t) = P\{X(t) = j\}; j \in S$ stands for a one-dimensional distribution of the semi-Markov process. If the analysed process $\{X(t), i \ge 0\}$ is an ergodic one, then [4]:

$$P_{ij} = \lim_{t \to \infty} P_{ij}(t) = P_j = \lim_{t \to \infty} P_j(t) = \frac{g_j m_j}{\sum_{i \in S} g_i m_i}, \quad i, j \in S, \tag{12}$$

where:

g_i - ergodic probabilities of the Markov chain,

and:

$$m_i = \int_0^\infty [1 - G_i(t)]dt \tag{13}$$

is a stationary distribution of the embedded Markov chain $\{X(\tau_n) : n \in N_0\}$. The stationary probabilities are the only solutions of the system of equations:

$$\sum_{i \in S} g_i p_{ij} = g_j, \quad j \in S, \quad \sum_{j \in S} g_j = 1. \tag{14}$$

3. Essential assumptions for the model of maintenance process of a bus

It has been assumed that the general mathematical model of the bus maintenance process (maintenance state change process) is a stochastic process $\{X(t), t \ge 0\}$. The analysed stochastic process $\{X(t), t \ge 0\}$ has a finite phase space S, $S=\{S_1, S_2, ..., S_n\}$.

As a result of identification of the bus maintenance system in an urban transport system and the maintenance process being performed therein, the following bus maintenance states significant for the analysis of the investigated system operation have been selected:

- the state in which a technical object is used (realization of a transport task),
- the state in which a technical object waits for a corrective service to be performed in the maintenance system environment (waiting for so called technical emergency service units),
- the state in which a service is performed in the maintenance system environment – the state in which a bus is a subject to corrective service processes performed by so called technical emergency service units due to occurrence of a vehicle damage when realizing the transport tasks,
- the state in which a technical object waits for a corrective service to be performed in the maintenance system,
- the state in which a corrective service is performed on a technical object in the maintenance system,

- the state in which a technical object waits for pre-repair diagnosis,
- the state in which pre-repair diagnoses are performed on a technical object,
- the state in which a technical object waits for post-repair diagnosis,
- the state in which post-repair diagnoses are performed on a technical object,
- the state in which a service is performed on the day on which a technical object is used (so called daily service),
- the state in which a technical object waits to undertake realization of a transport task (after bringing back the task serviceability state a technical object does not perform the scheduled transport task due to the method of organizing the transport tasks, e.g. in an urban bus transport maintenance system determined by the existing transport task schedule, so called "timetable"),
- the state in which a technical object waits due to unserviceability of the environment,
- the state of a scheduled standstill of a technical object (no transport tasks e.g. in an urban bus transport maintenance system resulting from the schedule of the transport tasks, including a night transport break).

In order to illustrate the considerations, the following bus maintenance states, out of the aforementioned ones, have been analysed:

S_1 - the state of a scheduled standstill of a bus,
S_2 - the state in which a vehicle is used (realization of a transport task),
S_3 - the state in which a bus is serviced in the system environment,
S_4 - the state in which a corrective service is performed on a vehicle in the maintenance system,
S_5 - the state of a corrective service performed during a scheduled standstill of a bus (at so called bus depot)

It has been assumed that the set T is a sum of intervals of the bus maintenance time. It has been assumed that the codes of the selected vehicle maintenance states correspond to the codes of the states of the analysed stochastic process $\{X_t, t \in T\}$. The possible transitions between the selected bus maintenance states have been determined. A matrix of the process state change intensity has been assessed for the selected maintenance states.

The random variable expressing the duration of the state $i \in S$ of the process, when the next state is $j \in S$ with the distribution determined by the distribution function $F_{ij}(t)$, is denoted with the symbol T_{ij}.

Due to the structure of the data achieved on the basis of the results of investigations of a real system, being indispensable to determine the semi-Markov process $\{X_t, t \in T\}$, it is convenient to define such a process by determining the following three $(p, P, F(t))$ elements:

- $p = [p_i : i \in S]$ - stochastic vector of the initial distribution process $\{X_t, t \in T\}$,

- $P = [p_{ij} : i, j \in S]$ - matrix of transition probabilities of the Markov chain embedded in the process $\{X_t, t \in T\}$,

- $F(t) = [F_{ij}(t) : i, j \in S]$ - matrix of distribution functions of the conditional distributions of the random variables T_{ij}, i,j $\in$ S of the times remaining in the states.

To simplify further investigations it has been assumed that $F_{ij}(t) = G_i(t)$, i, j $\in$ S. The function $G_i(t)$ is a distribution function of distribution of the duration time of the state $i \in S$. This assumption means that the duration of the current state does not depend on the next process state.

The state space S consists of five states S = {S_1, S_2, S_3, S_4, S_5} in the considered example. The possible transitions between the selected bus maintenance states, presented by the relation (15), were determined on the basis of identification of a real investigation object (a maintenance system of an urban bus transport system) and the maintenance processes performed therein.

$$P = \begin{bmatrix} 0 & p_{12} & 0 & p_{14} & p_{15} \\ p_{21} & 0 & p_{23} & p_{24} & 0 \\ p_{31} & p_{32} & 0 & p_{34} & 0 \\ p_{41} & 0 & 0 & 0 & 0 \\ p_{51} & 0 & 0 & p_{54} & 0 \end{bmatrix} \tag{15}$$

The transition matrix of the embedded Markov chain estimated on the basis of the results from the performed preliminary maintenance investigations is formulated as follows:

$$P = \begin{bmatrix} 0 & 0,9998 & 0 & 0,0001 & 0,0001 \\ 0,986 & 0 & 0,013 & 0,001 & 0 \\ 0,11 & 0,84 & 0 & 0,05 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0,9999 & 0 & 0 & 0,0001 & 0 \end{bmatrix} \tag{16}$$

The values of the estimators of the maximum credibility, formulated as stated below, have been adopted as statistical estimations of the transition probabilities of the embedded Markov chain:

$$\widehat{p}_{ij} = \frac{n_{ij}}{\sum\limits_{j \in S} n_{ij}} \tag{17}$$

where:

n_{ij} - the number of the direct chain state changes, from the state i into the state j in its finite process realization.

The determined limiting distribution of the Markov chain embedded in the considered semi-Markov process is presented in the Table 1.

Table 1. Limiting distribution of the embedded Markov chain

g_1	g_2	g_3	g_4	g_5
0,49362066	0,49897069	0,00648662	0,00087267	0,00004936

The expected values $E(T_i)$, $i \in S$ of the distributions determined by the empiric distribution function are the estimations Π_i of the expected value of the time T_i of duration of the state $i \in S$:

$$\Pi_i = \frac{1}{n_i} \sum_{m=1}^{n_i} (t_i^{(m)}) \tag{18}$$

where:

$n_i = \sum\limits_{j \in S} n_{ij}$,

$t_i^{(1)}, ..., t_i^{(n_i)}$ - independent realizations of the random variable T_i which stands for the duration of the state i.

The expected times of duration Π_i of the considered maintenance states (expressed in hours) estimated on the basis of the preliminary data are presented in the Table 2.

Table 2. Expected values of times of duration of the analysed maintenance states

Π_1	Π_2	Π_3	Π_4	Π_5
6	15,7608	0,0456	2,16	0,0336

The determined limiting distribution of the semi-Markov process constituting the model of the maintenance process of a single bus is presented in the Table 3.

Table 3. Limiting distribution of the analysed process

P_1	P_2	P_3	P_4	P_5
0,27352245	0,72627600	0,00002732	0,00017408	0,00000015

4. Model of maintenance process of a set of vehicles

Assuming that n of homogenous vehicles with the codes $k \in N$, forming the set Z of vehicles, are used in the maintenance system, a simplified model of the state change process of the elements in this set has been developed.

It has been assumed that the independent semi-Markov processes $\{X_k(t) : t \geq 0\}$, $k \in N = \{1, ..., n\}$ determined by identical kernels Q(t) described with the relation (4) are the model of the bus maintenance process. The assumption that the processes are independent is just an approximation of the real maintenance processes. In fact, some bus states may have influence over the maintenance states of other buses.

The random vector:
$$\mathbf{X}(t) = [X_1(t), X_2(t), ..., X_n(t)] \tag{19}$$
describes the process of changes of maintenance states of the set of buses of the considered vehicle maintenance system. With such an assumption it is possible to determine the total distribution of this random vector, that is:
$$P\{X_1(t) = i_1, X_2(t) = i_2, ..., X_n(t) = i_n\} = P_{i_1}(t) \, P_{i_2}(t) \, ... \, P_{i_n}(t), \tag{20}$$
where: $i_1, i_2 ..., i_n \in S$.

The random variable:
$$N_i(t) = \#\{k \in N : X_k(t) = i\}, \ t \geq 0, \ i \in S \tag{21}$$
stands for the number of buses in the state i at the moment t.

The random vector $\{(N_1(t), N_2(t), ... , N_8(t)) : t \geq 0\}$ has multinomial distribution, which means:
$$P\{N_1(t) = i_1, N_1(t) = i_2, ..., N_5(t) = i_5\} = \frac{n!}{i_1!i_2!...i_5!}[P_1(t)]^{i_1}[P_2(t)]^{i_2},...,[P_5(t)]^{i_5} \tag{22}$$

where: $i_1 + i_2 + ... + i_5 = n$.

The equation [7] results from the limit theorem for the semi-Markov processes:

$$\lim_{t \to \infty} P\{N_1(t) = i_1, N_1(t) = i_2, ..., N_5(t) = i_5\} =$$

$$= \frac{n!}{i_1!i_2!...i_5!}\lim_{t \to \infty}[P_1(t)]^{i_1}[P_2(t)]^{i_2},...,[P_5(t)]^{i_5} = \frac{n!}{i_1!i_2!...i_5!}P_1^{i_1}P_2^{i_2},...,P_5^{i_5} =$$

$$= \frac{n!\pi_1^{i_1}m_1^{i_1}\pi_2^{i_2}m_2^{i_2}...\pi_5^{i_5}m_5^{i_5}}{i_1!i_2!...i_5!\left(\sum_{i \in S}\pi_i m_i\right)} \tag{23}$$

The expected value of the considered random vector is formulated as:
$$E[(N_1(t), N_2(t),..., N_5(t))] = (nP_1(t), nP_2(t),..., nP_5(t)). \tag{24}$$
The coordinates of this vector stand for the expected number of buses in the analysed maintenance states i, i = 1, 2, ..., 5, at the moment t. Therefore, the value: $E[(N_i(t)] = nP_i(t)$, $i \in S$, $t \geq 0$ stands for the expected number of buses in the state i at the moment t.

The variance of the random variables N_i takes the form:

$$V[(N_i(t)] = nP_i(t)(1 - P_i(t)), \qquad i \in S, t \geq 0, \tag{25}$$

while the covariance is expressed with the relation:

$$\text{Cov}[(N_i(t), (N_j(t)] = -nP_i(t)(1 - P_j(t)), \qquad i,j \in S, i \neq j, t \geq 0. \tag{26}$$

The expected value of the random variable N_i which stands for the number of buses in the state i in a system working for a long time may be determined using the relation:

$$E[N_i] = \lim_{t \to \infty} E[(N_i(t))] = nP_i, \quad i \in S. \tag{27}$$

The expected value of the random vector $[N_1, N_2, ..., N_5]$ in the considered model is:

$$E[(N_1, N_2,..., N_5)] = (nP_1, nP_2,..., nP_5). \tag{28}$$

The coordinates of this vector stand for the expected number of buses in the determined states in a system working for a long time. The statistical estimation of the elements of this vector is:

$$E(N_i) = n P_i, \quad i \in S. \tag{29}$$

The limiting variance for the state $i \in S$ is:

$$V[N_i] = n P_i (1 - P_i), \quad i \in S, \tag{30}$$

while the limiting covariance of the state pairs is expressed with the relation:

$$\text{Cov}[(N_i(t), (N_j(t)] = -nP_i(1 - P_j), i,j \in S, i \neq j. \tag{31}$$

The determined expected numbers of buses in the specified states in a system working for a long time for the investigated bus maintenance system and the adopted data and $n = 203$ are presented in the Table 4.

Table 4. Statistical estimation of the expected number of buses in the specified states in a system working for a long time (n = 203)

$E(N_1)$	$E(N_2)$	$E(N_3)$	$E(N_4)$	$E(N_5)$
55,53	147,43	0,01	0,04	0,00

It is possible to simulate such an event that the scope of the transport service realization is reduced. Such a situation may take place when it is not profitable to realize transportation by means of specific bus lines or in case of a failure to win another tender for the transportation tasks performed before. If it is decided not to handle one of the bus lines, the number of the necessary vehicles would be reduced by 17 vehicles. For such a variant, the determined expected values of the number of buses in the specified states in a system working for a long time are presented in the Table 5. The calculation results may serve to preliminarily forecast the demand for the corrective service realization both in the system (service station) and in its environment (technical emergency service units), as well as the system operation costs and the number of vehicles necessary to provide continuous transportation.

Tab. 5. Statistical estimation of the number of buses in the specified states in a system working for a long time (n = 186)

$E(N_1)$	$E(N_2)$	$E(N_3)$	$E(N_4)$	$E(N_5)$
50,88	135,09	0,01	0,03	0,00

5. Summary

The purpose of the investigations presented herein was, among others, to show a possibility to use selected stochastic processes for mathematical modelling the system and the process of vehicle maintenance.

The considered example of the model of maintenance process of urban transport buses is characterised by a considerable simplification (due to the nature of the work). However, the presented way to create models of that type and analyse them indicates that they may be used to preliminarily forecast the system state, and to simulate changes of the internal and external interactions affecting the system and to support the decision makers in the decision making process concerning control of the maintenance process and the system in which it is performed.

The analysis of results of the investigations performed shows that the developed model is susceptible to a change of the input parameter values.

The mathematical models of the maintenance processes, performed in complex systems, are by their very nature a considerable simplification of the real processes. The consequence of this is a necessity to carefully formulate practical conclusions resulting from investigating those models. However, in the opinion of the author, it does not limit the purposefulness of developing and analysing models of that type. The analysis of the results of investigating those models, for the model parameter values, determined on the basis of the results of maintenance tests performed in a real transport system, makes it possible to formulate conclusions and opinions both of a qualitative and (to the limited extent) quantitative nature.

Appropriate formulation of the investigation goals, expected results and variants of simulation of behaviour of the analysed system makes it possible, due to analysing the models of that type, to perform, among other things, preliminary evaluation of influence of the considered decision variants (concerning such things as the number of vehicles required to assure effective realization of the transport tasks, frequency and scope of the technical services performed, etc.) on the course and effectiveness of the investigated maintenance process.

The presented method to build models of changes of vehicle maintenance states and the semi-Markov process used to model the maintenance process of urban transport buses may be, in a relatively simple way, applied for other maintenance systems than the considered one.

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