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VARIABLE STRUCTURE CONTROL OF SECOND ORDER SYSTEM WITH ACCELERATION CONSTRAINT

SUMMARY

The paper presents a new sliding mode control (SMC) method for the second order system subject to acceleration constraint. The proposed method employs a time-varying switching line which initially passes through the representative point, specified by the initial conditions of the system, in the error state space. Afterwards the line moves smoothly, with a constant velocity and a constant angle of inclination to the origin of the error state space and then, having reached the origin it stops moving and remains fixed. Thus the proposed control algorithm eliminates the reaching phase and forces the representative point of the system always to stay on the switching line. Consequently, insensitivity of the system with respect to external disturbances and model uncertainties is guaranteed from the very beginning of the proposed control action. Parameters of the switching line are selected to minimize two control quality criteria – the integral absolute error (IAE) and the integral of the time multiplied by the absolute error (ITAE) – without violating the system acceleration constraint. Furthermore, the switching line is selected in such a way that the tracking error converges to zero monotonically, without overshoots or oscillations. The proposed method is verified and illustrated by a simulation example.

Keywords: *sliding mode control, switching line design, variable structure systems*

ŚLIZGOWE STEROWANIE OBIEKTEM DRUGIEGO RZĘDU Z OGRANICZENIEM PRZYSPIESZENIA

W artykule zaprezentowano metodę projektowania ślizgowego sterowania obiektem dynamicznym drugiego rzędu przy ograniczeniu jego przyspieszenia. Wykorzystano w tym celu niestacjonarną linię przełączeń, która w chwili początkowej zawiera punkt opisujący dynamikę obiektu w przestrzeni stanu, a następnie – nie obracając się – przesuwa się ruchem jednostajnym do początku tej przestrzeni. Po osiągnięciu początku układu współrzędnych linia zatrzymuje się i w dalszej fazie procesu regulacji pozostaje nieruchoma. W ten sposób wyeliminowana zostaje konieczność osiągania ruchu ślizgowego, a dzięki temu zapewniona jest niewrażliwość obiektu na zakłócenia zewnętrzne i niedokładności określenia modelu już od samego początku trwania procesu regulacji. Linia przełączeń projektowana jest tak, aby jej parametry zapewniały minimalizację dwóch kryteriów jakości regulacji – całki modułu uchybu oraz całki iloczynu czasu i modułu uchybu – przy równoczesnym spełnieniu ograniczenia przyspieszenia obiektu. Ponadto zastosowana w układzie sterowania linia przełączeń zapewnia monotoniczną zbieżność uchybu regulacji do zera bez przeregulowań i oscylacji. Zaproponowana w artykule metoda została zilustrowana i zweryfikowana przykładem symulacyjnym.

Słowa kluczowe: *sterowanie ślizgowe, projektowanie linii przełączeń, sterowanie o zmiennej strukturze*

1. INTRODUCTION

In recent years much of the research in the area of control systems theory focused on the design of a discontinuous feedback which switches the structure of the system according to the evolution of its state vector. This technique, usually called sliding mode control, provides an effective and robust means of controlling nonlinear plants (Utkin 1977; Hung 1993). The main advantage of this technique is that once the system state reaches a sliding surface, the system dynamics remain insensitive to a class of parameter variations and disturbances. However, robust tracking is assured only after the system state hits the sliding surface, i.e. the robustness is not guaranteed during the reaching phase. Since conventional time-invariant sliding plane is considered, the insensitivity of the system is not obtained for some time from the beginning of its motion. Furthermore, usually for the given initial conditions there is an essential trade-off between the short reaching phase and the fast system response in the sliding phase. In order to overcome these problems the idea of the time-varying switching lines applied for the sliding mode control of the second order time-varying, nonlinear systems was introduced in (Choi 1994).

This paper is focused on SMC design and it presents a new method of switching line selection. The switching line introduced in the paper moves with a constant velocity and a constant angle of inclination to the origin of the error state space and it is selected in such a way that at the initial time the system representative point belongs to the line. In this way the reaching phase is eliminated and the system is insensitive with respect to external disturbance and model uncertainty from the very beginning of the control process. Afterwards the line moves to the origin of the error state space and having reached the origin it remains fixed. The main contribution of this work is the procedure for the optimal, in the sense of the integral absolute error (IAE) and the

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integral of the time multiplied by the absolute error (ITAE), selection of the switching line parameters with the system acceleration constraint.

2. PROBLEM FORMULATION

We consider the time-varying and nonlinear, second order system described by the following equations:

$$\dot{x}_1 = x_2,\ \ \dot{x}_2 = f(\boldsymbol{x},t) + \Delta f(\boldsymbol{x},t) + b(\boldsymbol{x},t)u + d(t) \tag{1}$$

where x_1, x_2 are the state variables and $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)]^T$ is the state vector, t denotes time, u is the input signal, b, f – are a priori known, bounded functions of time and the system state, Δf and d are functions representing the system uncertainty and external disturbance, respectively. It is assumed that there exists a strictly positive constant δ which is the lower bound of the absolute value of $b(\boldsymbol{x}, t)$, i.e. $0 < \delta = \inf\{|b(\boldsymbol{x}, t)|\}$. Furthermore, functions Δf and d are unknown and bounded. Therefore, there exists a constant μ which for every pair $(\boldsymbol{x}, t)$ satisfies the following inequality $|\Delta f(\boldsymbol{x}, t) + d(t)| \le \mu$. The initial conditions of the system are denoted as x_{10}, x_{20} where $x_{10} = x_1(t_0)$, $x_{20} = x_2(t_0)$. System (1) is supposed to track the desired trajectory given as a function of time $\boldsymbol{x}_d(t) = [x_{1d}(t)\ \ x_{2d}(t)]^T$ where $x_{2d}(t) = \dot{x}_{1d}(t)$ and $x_{2d}(t)$ is a differentiable function of time. The trajectory tracking error is defined by the following vector $\boldsymbol{e}(t) = [e_1(t)\ e_2(t)]^T = \boldsymbol{x}(t) - \boldsymbol{x}_d(t)$. Hence, we have $e_1(t) = x_1(t) - x_{1d}(t)$ and $e_2(t) = x_2(t) - x_{2d}(t)$. It is assumed that at the initial time $t = t_0$, the tracking error and the error derivative can be expressed as $e_1(t_0) \neq 0$ and $e_2(t_0) = 0$. The switching line is shifted in the state space with a constant angle of inclination. At the beginning the line moves uniformly (i.e. with a constant velocity) in the state space and then it stops at a time instant $t_f > t_0$. Consequently, for any $t \in \langle t_0, t_f\rangle$ the switching line is described by the following equation:

$$s(\boldsymbol{e}, t) = e_2(t) + ce_1(t) + A + Bt = 0 \tag{2}$$

where c, A and B are constants. The selection of c, A and B will be presented further in the paper. Since the considered line becomes motionless at the time t_f, for any $t \ge t_f$ it is fixed and can be described as follows:

$$s(\boldsymbol{e}, t) = e_2(t) + ce_1(t) = 0 \tag{3}$$

In order to ensure the stability of the sliding motion on the line described by equations (2) and (3), parameter c must be strictly positive. Furthermore, in order to actually eliminate the reaching phase, and consequently, to ensure insensitivity of the considered system from the very beginning of its motion, constants A, B and c should be chosen in such a way that the representative point of the system at the initial time $t = t_0$ belongs to the switching line. For that purpose, the following condition must be satisfied:

$$s\left[\boldsymbol{e}(t_0), t_0\right] = e_2(t_0) + ce_1(t_0) + A + Bt_0 = 0 \tag{4}$$

The following control signal ensures the stability of the sliding motion on line (2):

$$u = \frac{-f(\boldsymbol{x},t) - ce_2(t) + \dot{x}_{2d}(t) - B - \gamma \operatorname{sgn}\left[s(\boldsymbol{e},t)\right]}{b(\boldsymbol{x},t)} \tag{5}$$

where $\gamma = \eta + \mu$ and η is a strictly positive constant. Substituting relations (1) and (5) into the product $s(\boldsymbol{e},t)\dot{s}(\boldsymbol{e},t)$, we get the following inequality $s(\boldsymbol{e},t)\dot{s}(\boldsymbol{e},t) \le -\eta|s(\boldsymbol{e},t)|$ which proves the existence and stability of the sliding motion on the plane described by equations (2) and (3). Taking into account condition (4) and the assumption that $t_0 = 0$ we obtain:

$$A = -\,ce_1(0) \tag{6}$$

Solving equation (2) with the introduced initial condition, assuming that $t_0 = 0$ and considering relation (6) we get the tracking error and its derivative for the time $t \in \langle 0, t_f\rangle$:

$$e_1(t) = -Bc^{-2}e^{-ct} + e_1(0) + Bc^{-2} - Bc^{-1}t \tag{7}$$

$$e_2(t) = Bc^{-1}\left(e^{-ct} - 1\right) \tag{8}$$

Notice that for the time $t \ge t_f$ the switching line is fixed and passes through the origin of the error state space. This leads to the condition $A + Bt_f = 0$. From this equation and relation (6) we obtain

$$t_f = \frac{ce_1(0)}{B} \tag{9}$$

The fact that c and t_f are greater than zero implies that constants $e_1(0)$ and B always have the same signs. Next, we will analyze the behavior of the system in the second phase of its motion, that is when the switching line does not move. The time invariant switching line is described by relation (3). The initial conditions which are necessary to solve equation (3) can be determined from (7) and (8) whose values are evaluated at the time instant t_f. Solving equation (3) with those initial conditions, we obtain:

$$e_1(t) = Bc^{-2}\left(-1 + e^m\right)e^{-ct} \tag{10}$$

$$e_2(t) = Bc^{-1}\left(1 - e^m\right)e^{-ct} \tag{11}$$

where

$$m = \frac{c^2 e_1(0)}{B} \tag{12}$$

is a strictly positive constant. Notice that the tracking error described by (7) and (10) does not exhibit any overshoots. Next in the paper a new method of the switching line selection will be proposed.

In order to select the optimal switching line parameters c, A and B we minimize two criteria:

1) the integral absolute error (IAE)

$$J_{\text{IAE}} = \int_{t_0}^{\infty} |e_1(t)|\, dt \tag{13}$$

2) the integral of the time multiplied by the absolute error (ITAE)

$$J_{\text{ITAE}} = \int_{t_0}^{\infty} t\,|e_1(t)|\, dt \tag{14}$$

Since the tracking error converges monotonically, substituting equations (7) and (10) into criteria (13) and (14), we obtain that IAE is given by:

$$J_{\text{IAE}} = \frac{c\left[e_1(0)\right]^2}{2|B|} + \frac{|e_1(0)|}{c} \tag{15}$$

and ITAE can be presented as follows:

$$J_{\text{ITAE}} = \frac{|e_1(0)|}{6}\left\{\frac{6}{c^2} + \frac{3e_1(0)}{B} + \frac{c^2\left[e_1(0)\right]^2}{B^2}\right\} \tag{16}$$

On the other hand, calculating c from (12) we obtain:

$$c = \sqrt{\frac{mB}{e_1(0)}} \tag{17}$$

In order to facilitate the minimization process, further in the paper we will consider the quality criteria as functions of variables m and B, instead of c and B. Substituting (17) into (15) and (16), we obtain the IAE criterion expressed as:

$$J_{\text{IAE}}(m, |B|) = \frac{|e_1(0)|^{3/2}}{\sqrt{|B|}}\left(\frac{\sqrt{m}}{2} + \frac{1}{\sqrt{m}}\right) \tag{18}$$

and the ITAE criterion equivalent to:

$$J_{\text{ITAE}}(m, |B|) = \frac{\left[e_1(0)\right]^2}{6|B|}\left(\frac{6}{m} + 3 + m\right) \tag{19}$$

Next we will present how to find the optimal (in the sense of the IAE and ITAE criteria) switching line parameters, subject to acceleration constraint.

3. ALGORITHM FOR FINDING SWITCHING LINE PARAMETERS

In this section, in order to find the optimal switching line parameters, we minimize IAE and ITAE with the following constraint:

$$|\dot{e}_2(t)| \le a_{\max} \tag{20}$$

where $a_{\max}$ is the maximum admissible value of the system acceleration. Calculating the maximum value of $|\dot{e}_2(t)|$ (Bartoszewicz 2009), we obtain that relation (20) holds if the following inequality is satisfied:

$$|B| \le a_{\max} \tag{21}$$

Notice that for any given value of m, the minimum of criteria (18) and (19), is obtained for the greatest value of $|B|$ satisfying constraint (21). Consequently, minimization of the IAE and ITAE criteria, as two variable functions $J_{\text{IAE}}(m, |B|)$ and $J_{\text{ITAE}}(m, |B|)$ with constraint (21) may be replaced by the minimization of the following (single variable) functions without any constraint:

$$J_{\text{IAE}}(m) = \frac{|e_1(0)|^{3/2}}{\sqrt{a_{\max}}}\left(\frac{\sqrt{m}}{2} + \frac{1}{\sqrt{m}}\right) \tag{22}$$

and

$$J_{\text{ITAE}}(m) = \frac{\left[e_1(0)\right]^2}{6a_{\max}}\left(\frac{6}{m} + 3 + m\right) \tag{23}$$

Minimizing criterion (22), we obtain that it reaches its minimum value for $m = 2$. Consequently, the optimal value of parameter B is calculated from the following relation:

$$B_{opt} = a_{\max}\,\text{sgn}(e_0) \tag{24}$$

Then, the other switching line parameters, calculated from relations (17), (6) and (9), are presented below:

$$c_{opt} = \sqrt{\frac{2a_{\max}}{|e_1(0)|}} \tag{25}$$

$$A_{opt} = -\text{sgn}\left[e_1(0)\right]\sqrt{2a_{\max}|e_1(0)|} \tag{26}$$

$$t_{fopt} = \sqrt{\frac{2|e_1(0)|}{a_{\max}}} \tag{27}$$

In this way we select the switching line parameters when IAE is minimized.

On the other hand, from the minimization of criterion (23) as a single variable function, we get that the minimum value of this criterion is obtained for $m = \sqrt{6}$. Then, parameter B_{opt} is given by equation (24). In this case, i.e. when ITAE is minimized, the other switching line parameters have the following forms:

$$c_{opt} = \sqrt{\frac{\sqrt{6}a_{\max}}{|e_1(0)|}} \tag{28}$$

$$A_{opt} = -\text{sgn}\left[e_1(0)\right]\sqrt{\sqrt{6}a_{\max}|e_1(0)|} \tag{29}$$

$$t_{fopt} = \sqrt{\frac{\sqrt{6}|e_1(0)|}{a_{\max}}} \tag{30}$$

This set of the optimal (in the sense of ITAE) parameters ends the proposed switching line design procedure.

4. SIMULATION EXAMPLE

In order to verify the proposed method of the switching line design we consider the following second order system $\dot{x}_1 = x_2$, $\dot{x}_2 = (1+\varepsilon)\sin x_1 + 0.5 - \exp\left(-x_2^2\right) + u + d(t)$ with model uncertainty $\varepsilon = 0.39$ and external disturbance $d(t) = 0.6\sin(10t)$. As a consequence, we choose $\gamma = 1$. The initial conditions are specified as $x_{10} = 1$, $x_{20} = 0$. Furthermore, the considered system is supposed to track the following desired trajectory $x_{1d}(t) = -\cos t$. The maximum admissible value of system acceleration is equal to $a_{\max} = 0.5$. Calculating the parameters of the switching line according to the proposed method we obtain the following switching line parameters when IAE is minimized: $B_{opt} = 0.5$, $c_{opt} \approx 0.71$, $A_{opt} \approx -1.41$, $t_{f\,opt} \approx 2.83$ and from minimization of ITAE we get: $B_{opt} = 0.5$, $c_{opt} \approx 0.78$, $A_{opt} \approx -1.56$, $t_{f\,opt} \approx 3.13$.

Figures 1 and 3 show evolution of the tracking error and its derivative in both cases. It can be seen from these figures that, no matter which criterion is minimized, the tracking error converges to zero monotonically and furthermore, the system is insensitive with respect to the external disturbance and the model uncertainty from the very beginning of the control action. Figures 2 and 4 present that the acceleration constraint is always satisfied and finally, Figure 5 demonstrates phase trajectories of the considered system.

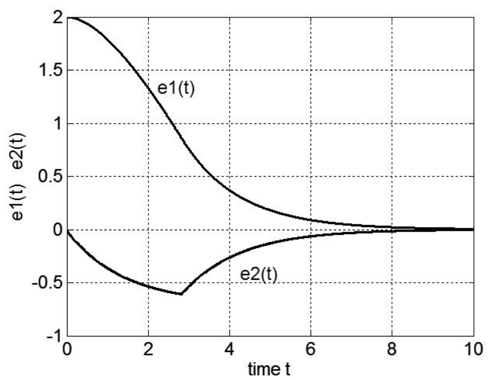

Fig. 1. Tracking error and its derivative (IAE minimization)

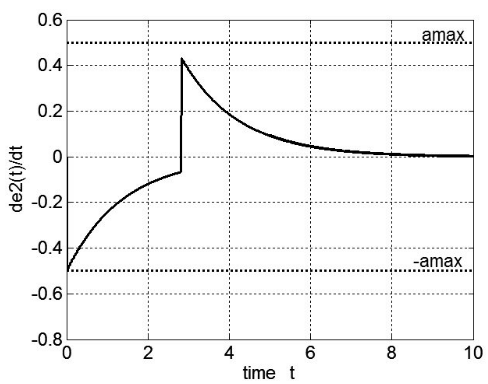

Fig. 2. System acceleration (IAE minimization)

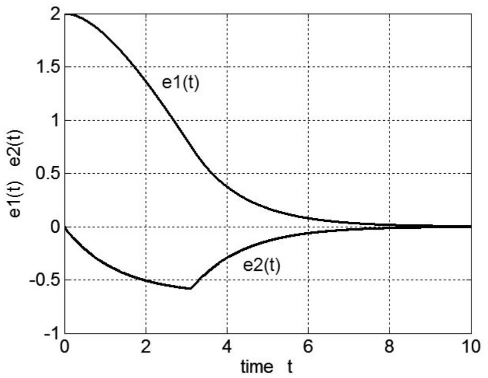

Fig. 3. Tracking error and its derivative (ITAE minimization)

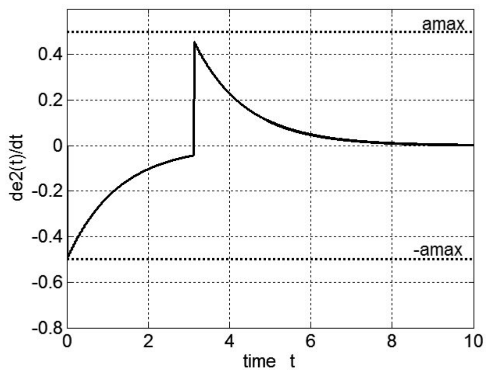

Fig. 4. Tracking error and its derivative (ITAE minimization)

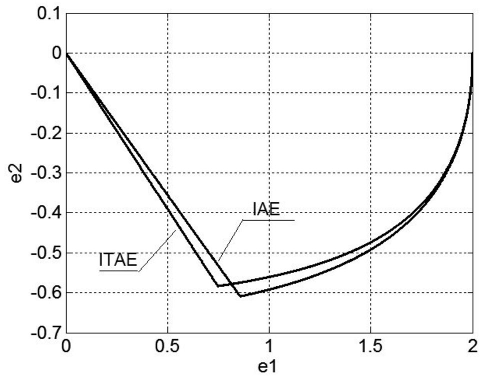

Fig. 5. Phase trajectory

5. CONCLUSIONS

In this paper a new SMC algorithm has been proposed. This algorithm employs a time-varying switching line which moves with a constant velocity and a constant angle of inclination to the origin of the error state space. Parameters of the line are chosen in two ways, i.e. when two criteria are minimized subject to the system acceleration constraint. Firstly, the minimization of the integral absolute error is taken into account and next, the integral of the time multiplied by the absolute error is considered. The switching line designed in both cases guarantees that the reaching phase is eliminated and in this way the system insensitivity with respect to external disturbances and model uncertainties from the very beginning of the control process is ensured. Furthermore, in both cases the tracking error converges to zero monotonically.

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