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Computers Methods for Chaos Diagnostic

1. Introduction

In this article some computer methods for chaos diagnostic were presented. These methods are based on showing some chaotic properties of systems. To understand these properties some basic definitions shoud be known. They were presented in the first section, then in next sections chaotic properties, like existence of strange attractor, transitivity and sensitive dependence on initial conditions were described. After short explanation of the main goal of each feature, the numerical examples and tests were presented to ilustrate it. In the end of this article, the short conclusions were presented.

First, the definitions of continuous and discrete dynamic systems were introduced (Def. 2).

Definition 1. *The set X is called metric space, if for each two points x,y,z from the set X, there exists a real value $d(x,y) \geqslant 0$ (called measure), such that* [4]:

1) $d(x,y) = 0$ only if $x = y$,
2) $d(x,y) = d(y,x)$,
3) $d(x,y) + d(y,z) \geqslant d(x,z)$. ■

Definition 2. *Let X be a metric space and $\{S^t\}_{t\geqslant 0}$ be a dynamical (or semidynamical) system on X, that is* [9]:

1) $S^t : X \to X$, for $t \geqslant 0$,
2) $S^0 = \mathrm{Id}$, $\quad S^{t+s} = S^t \circ S^s$, for $t,s \geqslant 0$ (where $\circ$ means function composition),
3) $S^t : [0,\infty) \times X \to X$ is a continuous function of (t,x).

If $S : X \to X$ is the map, then the sequence of its iterates $\{S^n\}_{n=0}^{\infty}$ is a discrete dynamical system [9]. ■

For topological transitivity, some definitions from topologic theory are necessary, like open and close set, closure and neigbourhood.

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Definition 3. *$\overline{A}$ is called a closure of set A if it fulfills the below conditions (called closure axioms)* [4]:

1) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
2) $A \subset \overline{A}$,
3) $\overline{\varnothing} = \varnothing$,
4) $\overline{\overline{A}} = \overline{A}$. ■

Definition 4. *The set A is called closed set if $\overline{A} = A$* [4]. ■

Definition 5. *The set A is called open set if its complement is the closed set, that is $\overline{X \backslash A} = X \backslash A$* [4]. ■

Definition 6. *The set A is called dense set if $\overline{A} = X$* [4]. ■

Definition 7. *The open ball $B(p, \varepsilon)$ centered at a point p is the set of points x, such that the measure between p and x is smaller than ε* [4]:

$$B(p, \varepsilon) = \{x : d(x, p) < \varepsilon\}.$$ ■

2. Strange attractor

In 1973 D. Ruelle and F. Takens suggested that chaos is connected with existence of strange attractor [1]. This criterium is very popular, because its simple to see strange attractor, but in mathematical way is very hard to proof that it exists. For example the Lorenz attractor is one of the most popular attractor, but their mathematical proof has only a couple of year [1]. The Lorenz attractor is presented in Figure 1 and the Lorenz system is given by (1) [6]:

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 - x_2x_1x_3 \\ \dot{x}_3 &= -cx_3 + x_1x_3 \end{aligned} \tag{1}$$

where: $a = 10$, $b = 28$ and $c = 8/3$.

Formal definitions of attractor and strange attractor are given by definitions 8 and 9.

Definition 8. *The compact subset $A \subset X$ is called the attractor of system $\{S^t\}_{t \geqslant 0}$, if there exists an open set U, such that A is a subset of U, $S^t(\overline{U}) \subset U$ for $t > 0$, and $A = \bigcap_{t>0} S^t(U)$* [3]. ■

Definition 9. *The attractor is called strange if it is a fractal set, i.e. if it has a different topological and Hausdorff dimensions* [9]. ■

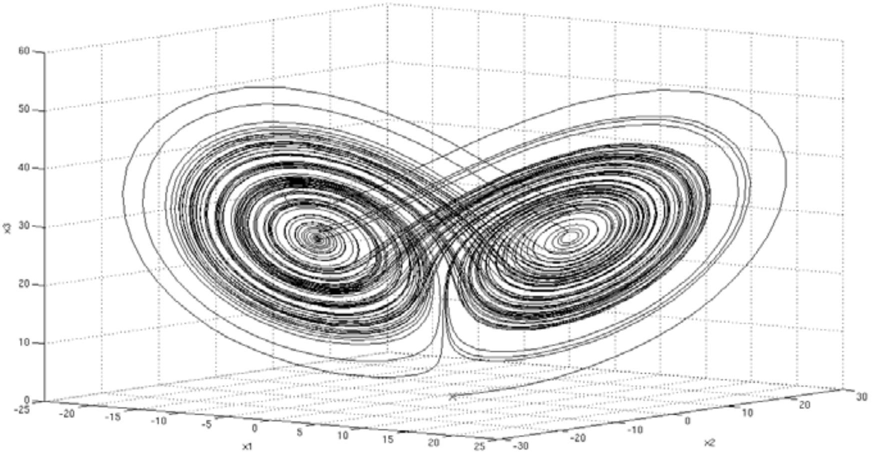

Fig. 1. The Lorenz attractor

Drawing an attractor is simple method to see if it exists, but it can be misleading. Good example of this misleading is the LC ladder system presented and detailed described in [7]. It is given by equation (2):

$$\ddot{x}(t) + Ax(t) = 0, \qquad x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T, \tag{2}$$

where:

$$A = \omega^2 \begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & & & & & \\ 0 & \ldots & 0 & 0 & -1 & 2 \end{pmatrix}_{n \times n}, \qquad \omega^2 = \frac{1}{LC} \cdot \frac{1}{(n+1)^2}.$$

For numerical simulations presented below, $LC = 1$ and $n = 10$ were assumed. Projection of phase trajectory on $(x_1, \dot{x}_1)$ surface is shown in Figure 2. It can be interpreted as chaotic behaviour, but as it can be read in [7], the frequency analysis can expose the countable set of frequencies.

This is the reason why some additionaly tests should be done. Above-mentioned test is frequency analyze. Because of high dimension of system space, the norm of signal, given by formula (3) will be analyzed.

$$norm = \sqrt{\sum_{i=1}^{n} (x_i^2 + \dot{x}_i^2)} \tag{3}$$

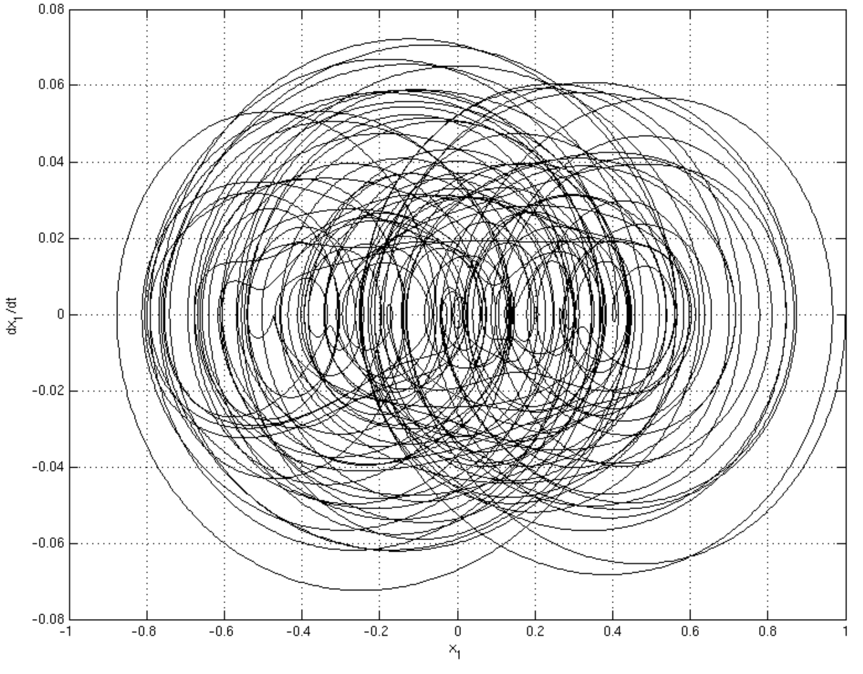

Fig. 2. Projection of phase trajectory on $(x_1, \dot{x_1})$ surface

The norm for LC system was shown in Figure 3.

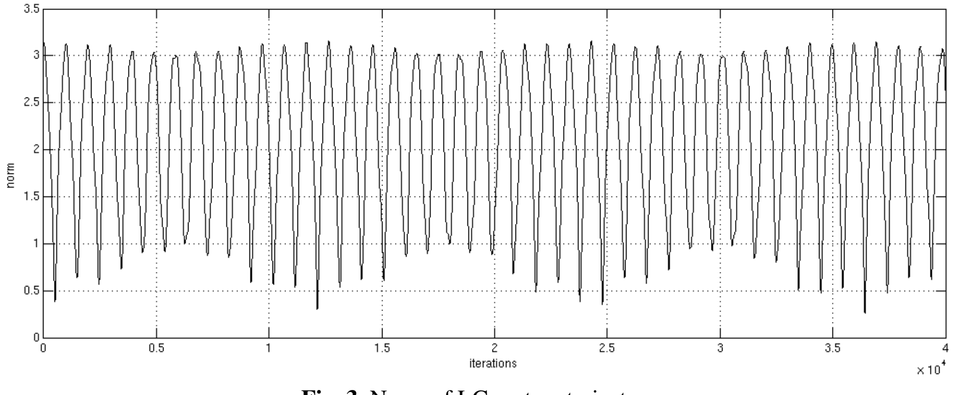

Fig. 3. Norm of LC system trajectory

The frequency analysis is in fact the discrete Fourier transformation, and it is calculated using the Matlab function *fft*. Of course their precision depends on the number of samples. Obtained frequency analysis is shown in Figure 4. It was calculated for 5000 samples with step equals 0.1.

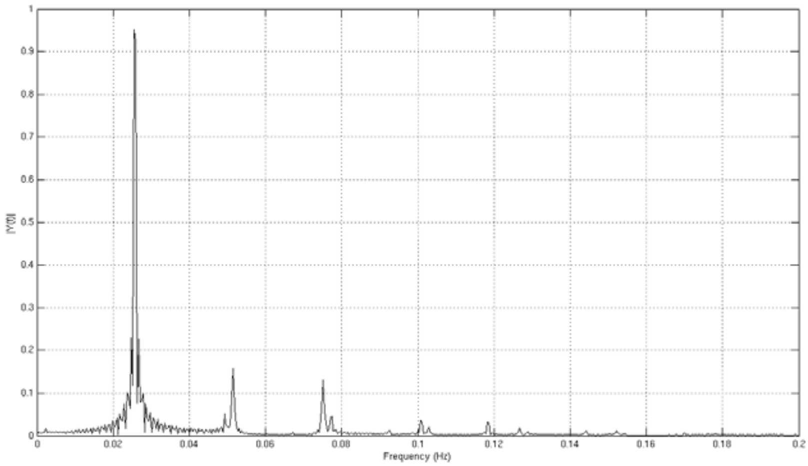

Fig. 4. Discrete Fourier transformation for norm of LC system

It is simple to see that this analysis exposes the couple dominating frequencies. The exactness of this calculation can be checked. According to [7], the main frequencies can be obtained with the formulas (4) and (5):

$$\omega_i = \frac{1}{n+1}\frac{2}{\sqrt{LC}}\sin\frac{\varphi_i}{2}, \tag{4}$$

where:

$$\varphi_i = \frac{i\pi}{n+1}, \qquad i = 1, 2, \ldots, n. \tag{5}$$

For $n = 10$ the calculated ω_i are given in Table 1 in first row. In second row are the successive maximum values from Figure 4.

Table 1
Calculated frequencies and obtained maximum values

i	1	2	3	4	5	6	7	8	9	10
ω_i	0.0259	0.0512	0.0755	0.0983	0.1191	0.1374	0.1530	0.1654	0.1745	0.1800
max_i	0.0259	0.0515	0.0755	0.1007	0.1186	0.1442	0.1526	0.1701	0.1781	0.1934

The first maximum values are the same or close to frequencies, then they are more unconvergence, because of simulation precision. To comparision, in Figures 5 and 6 was presented the norm and their frequency analyze for Lorenz system, which is well-known as chaotic system.

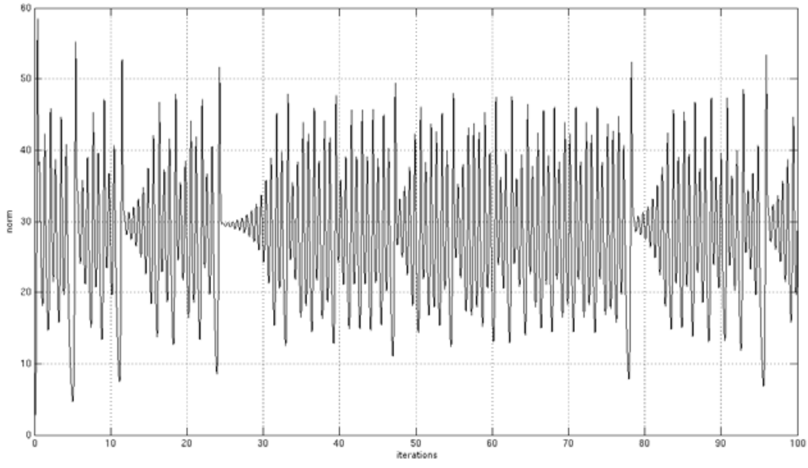

Fig. 5. Norm of Lorenz system trajectory

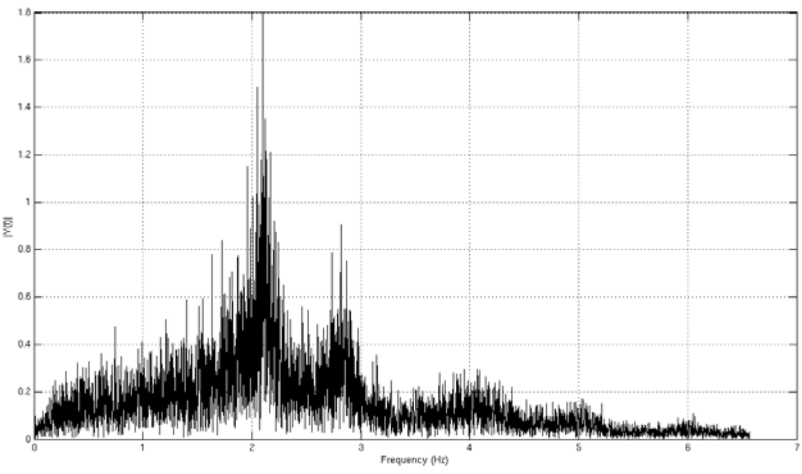

Fig. 6. Discrete Fourier transformation for norm of Lorenz system

The second test is autocorrelation of the norm. Autocorrelation of signal g is nothing more than cross-correlation of this signal with itself. It was calculated according to formula (6):

$$r_k = \frac{\sum_i [(g_{k+i} - \bar{g}) \cdot (g_i - \bar{g})]}{\sum_i (g_i - \bar{g})^2} \tag{6}$$

Interpretation of autocorrelation plot is simple: if the oscilations are small (no more than $0.2 - 0.3$) the signal is non periodic, if the oscilations are bigger, especially close to 1 and -1, then the signal has some periodicity, so its no chaotic. In Figure 7 the autocorrelation plot was presented. According to earlier explanation, because of the oscilations are big, there is a periodicity in norm signal.

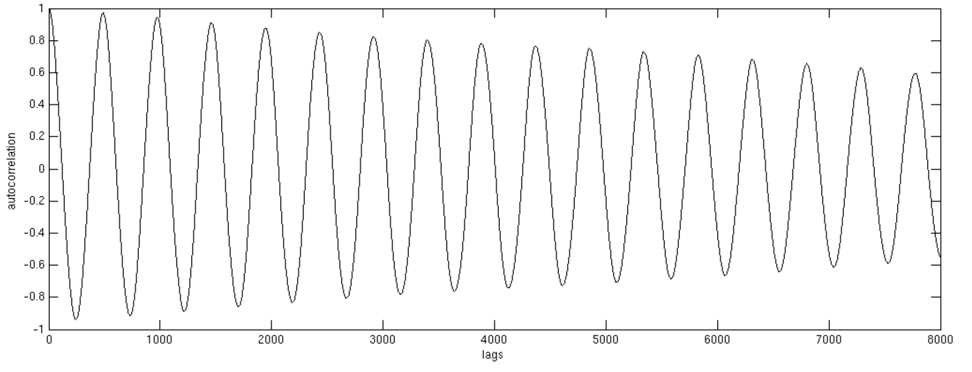

Fig. 7. Autocorrelation plot for norm of LC system

And again to comparision in Figure 8 the autocorrelation for norm of Lorenz system was presented.

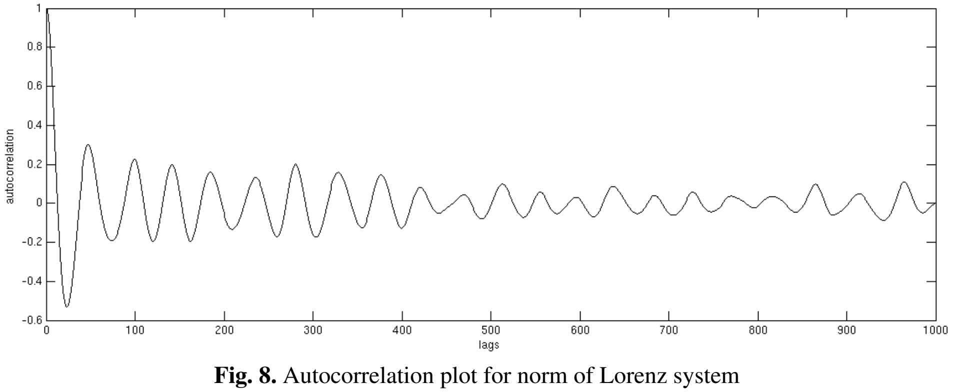

Fig. 8. Autocorrelation plot for norm of Lorenz system

3. Transitivity

In this and the next section the discrete dynamical system will be considered. The most popular discrete system, which can generate chaos is the logistic map. It is given by formula:

$$x_{n+1} = \mu x_n(1 - x_n) \tag{7}$$

This system project section $[0,1]$ on itself for proportional coefficint $\mu \in [0,4]$. In Figure 9 the biffurcation diagram was presented. It shows doubling periodicity in function of parameter μ and in consequence chaos for $\mu = 4$.

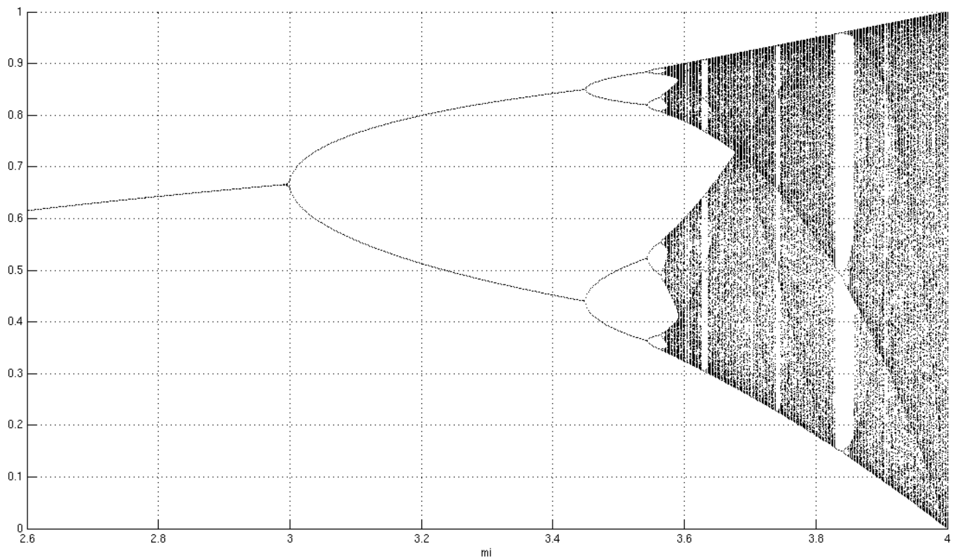

Fig. 9. The biffurcation diagram for logsitic map

In some chaos definitions the transitivity as a necessary condition was given (for example Devaney's and Auslanedr-Yorke definitions).

Definition 10. *Let consider discrete system* $\{S^n\}_{n=0}^{\infty}$. *It is transitive if for any nonempty, open sets* $U_1, U_2 \subset X$, *there exists* $n > 0$ *such that* $S^n(U_1) \cap U_2 \neq \varnothing$ [8]. ■

The transitivity condition can be replaced by existence of dense trajectory (its equivalent) [9]. Intuitively, dense trajectory means that this trajectory visits each part of space. Good measure of this property is invariant measure.

Lets define as $N(\xi, \varepsilon, M)$ the number of elements of finite sequence $\{x_1, x_2, \ldots, x_M\}$, which have the value in $[\xi, \xi + \varepsilon)$. The limes [3]:

$$P(\xi, \varepsilon) = \lim_{M \to \infty} \frac{N(\xi, \varepsilon, M)}{M} \tag{8}$$

means probability of the elements of sequence (7) are in $[\xi, \xi + \varepsilon)$. Now, the density of probability distribution $p(\xi)$ can be considered [3]:

$$p(\xi) = \lim_{\varepsilon \to 0} \frac{P(\xi, \varepsilon)}{\varepsilon}. \tag{9}$$

According to [3] for given μ and $x \in [0,1]$, the density $p(x)$ is proportional to blackening level of section $[0,1]$ in figure 9. Now, if we consider a random point x_0 and describe as $p_0(\xi)$ the density of probabilities, that the value of x_0 is equal ξ, then the density of probabilities $p_1(\xi)$ that the value of x_1 is equal ξ can be obtained from Perron-Frobenius formula [3]:

$$p_1(\xi) = \sum_i \frac{1}{|f'(z_i)|} p_0(z_i), \tag{10}$$

where sum is for all points z_i, such that $f(z_i) = \xi$. If the sequence $p_0(\xi), p_1(\xi), p_2(\xi), \ldots$ is convergence to limes p_∞, independently on p_0, then the function $p_\infty(\xi)$ is identicaly with (9), its called the density of invariant measure and its very important feature of systems with chaotic dynamic [3].

The numerical obtained density of invariant measure for the system (7) is presented in Figure 10 for $\mu = 4$ (chaos) and in Figure 11 for $\mu = 3.6$ (no chaos). Its simple to see, that for $\mu = 3.6$ a lot of values are never obtained, so trajectory cannot be dense.

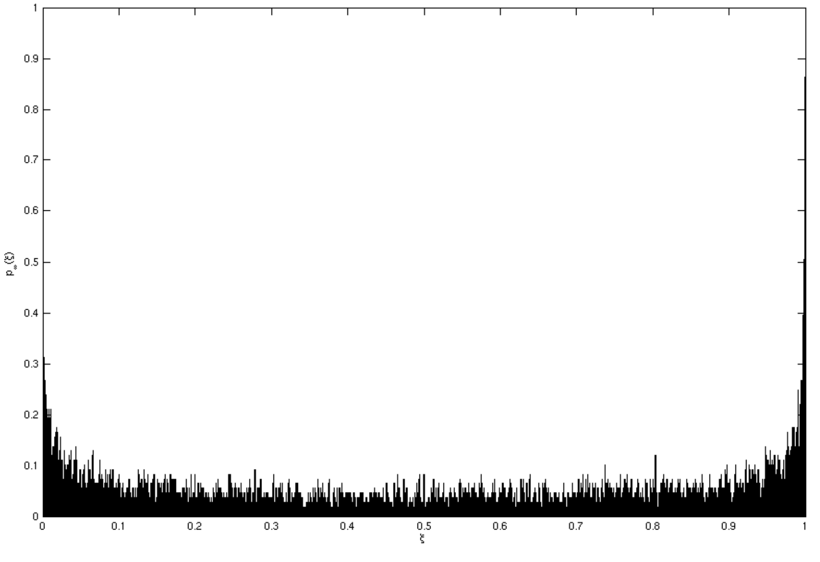

Fig. 10. Density of invariant measure for logistic map ($\mu = 4$)

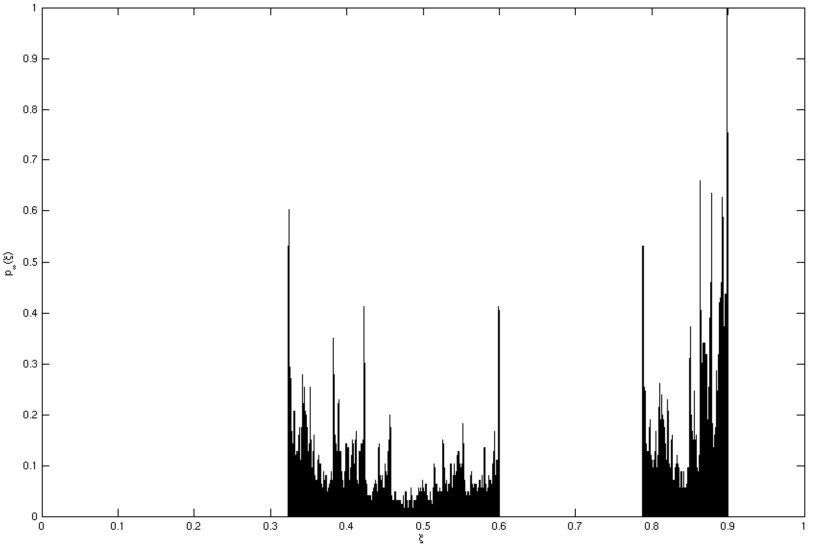

Fig. 11. Density of invariant measure for logistic map ($\mu = 3.6$)

4. Sensitive dependence on initial conditions

Sensitive dependence on initial conditions is one of the most important feature characteristics for chaotic systems [5]. Intuitively it means, that even small difference in initial conditions can be a purpose to the large difference in trajectory behavior. This phenomenon was also called „the butterfly effect" and it became famous thanks to Lorenz and his works, in which he describes a special system of differential equations, relevant with the earth atmosphere model [5].

In mathematical language this property can be formulated as in Definition 11.

Definition 11. *System* $\{S^t\}_{t\geqslant 0}$ *is sensitive dependent on initial conditions, if there exists a constant* $\delta > 0$ *such that for each point* $x \in X$ *and for each* $\varepsilon > 0$ *there exist a point* $y \in B(x, \varepsilon)$ *and* $t > 0$ *such that* $d(S^t(x), S^t(y)) \geqslant \delta$ [9]. ■

As it was said previously this property is very important feature of chaotic systems. It often can be found in chaos definitions (like in definitions given by Auslander-Yorke) and it can be treat like a necessary condition, but it is not enough to diagnose choas.

Let's consider how this feature can be checked through the numerical methods. Of course the simplest way is plot two trajectories with close initial conditions, like in Figure 12 or plot the distance between them, like in Figure 13 and look on it.

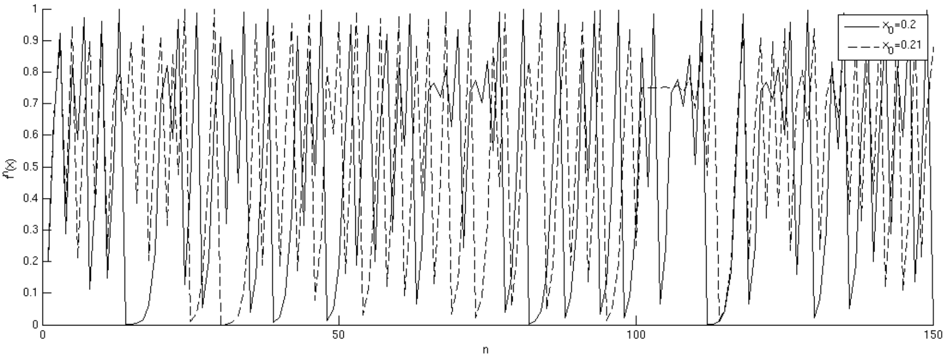

Fig. 12. Two trajectories for logistic map with close initial conditions: $x_0 = 0.2$ and $x_0 = 0.21$

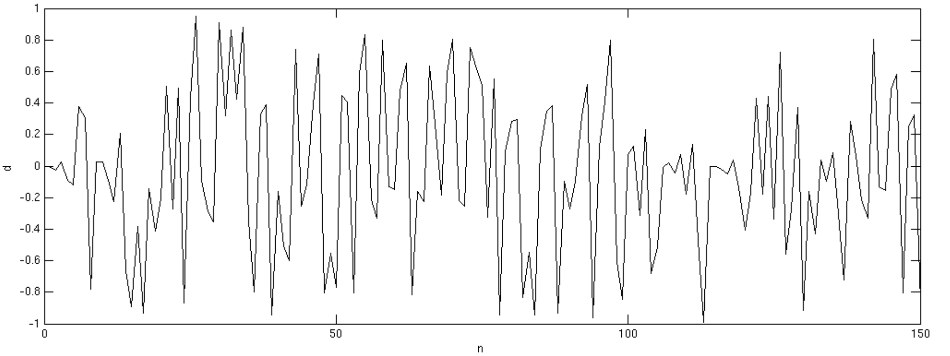

Fig. 13. The distance between two trajectories for logistic map with close initial conditions: $x_0 = 0.2$ and $x_0 = 0.21$

To obtain some more precision result the cross-correlation between this two trajectories can be calculated.

Cross-correlation is a measure of a correspondence between two signals. Value close to 1 or -1 means that signals are similarity, and value close to 0 indicate not related signals. For discrete signals g and h cross-correlation can be calculated according to formula (11):

$$r_k = \frac{\sum_i [(g_{k+i} - \bar{g}) \cdot (h_i - \bar{h})]}{\sum_i [\sqrt{(g_i - \bar{g})^2} \cdot \sqrt{(h_i - \bar{h})^2}]} \tag{11}$$

Calcutated cross-correlation for previous presented trajectories is shown on Figure 14.

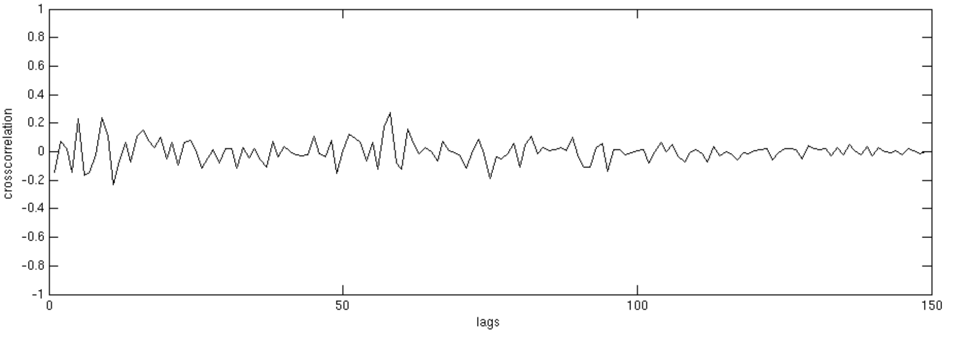

Fig. 14. Cross-correlation for two trajectories with close initial conditions: $x_0 = 0.2$ and $x_0 = 0.21$

The other test to check dependencies on initial conditions is the value of Lyapunov exponent λ. It characterizes sensitivity of trajectory on the initial perturbation and, for system described by function f, it is given by formula (12) [3]:

$$\lambda(x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log_2 |f'(x_k))|, \tag{12}$$

where: x_0 is the initial condition, and $f'(x)$ is diverative of f. If the value of $\lambda > 0$, then the trajectory is sensitive and chaotic [3]. Figure 15 shows the values of Lyapunov exponent in function of $\mu \in [3.5;4]$. From this figure the values of μ can be read, for which the system is chaotic ($\lambda > 0$).

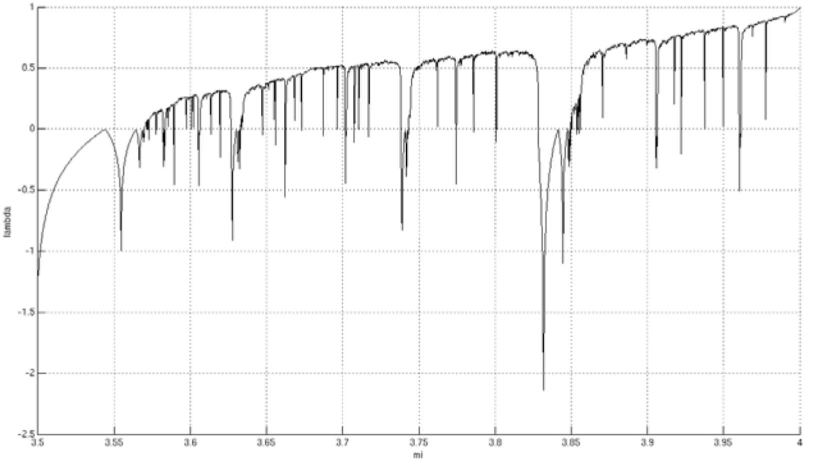

Fig. 15. Lyapunov exponent values for $\mu \in [3.5;4]$

5. Conclusions

In this paper the computers methods for chaos diagnostic were presented. The features which appear in chaos definitions are analyzed and some computer tests were calculated to ilustrated it.

Every computers simulation can be treat only as some approximation, because of numerical errors. If system is stable then small difference in initial condition have no important influence for trajectory behaviour, but in chaotic systems, which are very sensitivity on initial conditions, this small difference can have significant influence.

It is important to remember, that all computer simulations of this kind of systems, should be considered only in short time period, because after this time the numerical errors are too big to expect correct results (more in [2]).

All presented tests are only numarical simulations, so their results can't be treat as proof of existence or non existence of chaos. This tests should be some help tool to quick check, what can be expected from particular system.

References

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