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Dynamical Model of Propagation of Pollutants in a River

1. Introduction

The biochemical oxygen demand (BOD) and the dissolved oxygen (DO) are basic oxygen indices which allow to evaluate the river water quality [3]. The time–varying values of BOD and DO are described by the so–called advection–diffusion partial differential equations. In practice, for headwaters and rivers in theirs middle course, it is being regarded that the advection term dominated the diffusion component. Hence in what follows we shall assume that the diffusion process in negligible (absence of the second order partial derivatives with respect to a spatial variable).

In order to improve the water quality it is being proposed to set aerators on a river [10]. Clearly, this only supports the wastes purification and compensates the results of pollution but it does not cancel its sources. The point waste waters are registered and if they exceed some admissible standards then they have to be purified. However, the surface waste waters run down to a river without any limitations and they may represent even 50% of the whole wastes. Therefore it seems to be reasonable that the aeration would be helpful in keeping oxygen indices within the standards.

Suppose that a number of aerators are working on a river and we have a steady-state. Of course, it has to be proved that it is worth to set aerators, i.e., that they significantly improve the water quality. For that the system has to be identified and some numerical experiments are in order. This can be done with the aid of some computer software like, e.g., WODA package [11, 12, 16]. Now suppose that the equilibrium is lost, e.g., by introducing of an additional amount of waste waters. An idea is then to steer the aerators is such a way that the balance will be recovered under some minimal costs.

This problem is of practical importance and it is also interesting from the control theory viewpoint, if it is formulated as a linear–quadratic (LQ) problem with infinite–time horizon on a Hilbert state space. In [9], a problem of the water quality control has been formulated for the

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first time as the LQ problem with distributed control and distributed observation which are mathematically modeled by bounded control and output operators.

In a real world the control action and measurements are rather of point nature and, mathematically, they can be expressed in terms of unbounded operators.

Recently, a large number of, especially theoretical, publications devoted to LQ problem with infinite–time horizon and with unbounded control and observation operators is available, e.g. [13, 22], however there are only a few papers in which some examples are completely treated. To our knowledge, the first paper providing an example with a complete solution of such a LQ problem (for a distortionless electric $\mathfrak{RLCG}$ transmission line) was [4]. This example was also studied in [17] and [18]. An another example can be found in [23]. The LQ problem for the first order hyperbolic equations with unbounded control and observation has been treated too in [20] and [2, 14].

The paper is organized as follows. In Section 2 we recall a dynamical model of the water quality control in a form of partial differential equations of the advective–type, an equilibrium point of which has been established for a given nominal control and waste waters. Next step is to translate this equilibrium to the origin. In Section 3 the system dynamics is written in its abstract factor form on the Hilbert state space $\mathrm{H} = \mathrm{L}^2[0, a] \oplus \mathrm{L}^2[0, a]$. It is also shown that the semigroup generated by the state operator A decays to zero in a finite time. Further, we prove that our observation operator is A–bounded and admissible and that the system transfer functions is in the space $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathbb{C}^M, \mathbb{C}^K))$. In conclusions (Section 4), we present some conclusions and formulate the LQ problem with infinite–time horizon a solution of which is a prospect for further investigations.

2. Dynamical model

Following [15, 19], consider a dynamical model of the river water quality control governed by the equations

$$\left.\begin{cases} \dfrac{\partial L}{\partial t} = -\upsilon\dfrac{\partial L}{\partial \theta} - (K_1 + K_3)L + J \\[2ex] \dfrac{\partial D}{\partial t} = -\upsilon\dfrac{\partial D}{\partial \theta} - K_2 D + K_1 L + D_B + R - P - U \end{cases}\right\}, t \geq 0, \theta \in [0, a] \tag{1}$$

where:

- a – the length of a given river interval [m],
- t – time [s],
- $L(\theta, t)$ – BOD concentration [mg/m^3],
- $C(\theta, t)$ – DO concentration (dissolved oxygen) [mg/m^3],
- $C_S(\theta, t)$ – saturated value of dissolved oxygen [mg/m^3],

$D(\theta, t) = C_S(\theta, t) - C(\theta, t)$,
υ – velocity of the water flow in a river [m/s],
K_1 – coefficient of biochemical degradation of organic matters [1/s],
K_2 – coefficient of reaeration [1/s],
K_3 – coefficient of sedimentation [1/s],
J – waste waters – index of BDO emission [mg/(m^{3}s)],
U – control function – aeration [mg/(m^{3}s)],
D_B – oxygen consumption by sludge[1)] [mg/(m^{3}s)],
R – oxygen consumption by respiration of plants [mg/(m^{3}s)],
P – oxygen concentration increase due to photo-synthesis [mg/(m^{3}s)].

The initial condition are zero

$$L(\theta, 0) = 0, \;\; D(\theta, 0) = 0, \;\; \theta \in [0, a],$$

whilst the boundary conditions are of the form

$$L(0, t) = 0, \;\; D(0, t) = 0, \;\; t \geq 0.$$

The waste waters are given by

$$J(\theta) = p + \sum_{j=1}^{N} q_j \delta(\theta - \xi_j), \quad p, q_j \in \mathbb{R}, (j = 1, ..., N),$$

where p stands for the surface wastes (uniform along the whole river length), and the sum represents the point wastes, δ is the Dirac–delta pseudo-function.

Out of technological realization the control is assumed to be exclusively of the point–type

$$U(t, \theta) = \sum_{n=1}^{M} \alpha_n(t) \delta(\theta - \eta_n), \quad 0 < \eta_1 < \eta_2 < ... \eta_M < a.$$

In order to determine the equilibria $L^*(\theta)$ and $D^*(\theta)$ of the system (1) one has to solve the system of equations

$$\begin{cases} L^{*\prime} = -\dfrac{(K_1 + K_3)}{\upsilon} L^* + \dfrac{1}{\upsilon} J(\theta) \\ D^{*\prime} = -\dfrac{K_2}{\upsilon} D^* + \dfrac{K_1}{\upsilon} L^* + \dfrac{1}{\upsilon} W + \dfrac{1}{\upsilon} U_2(\theta) \end{cases} \tag{2}$$

[1)] The parameter D_B is difficult to be determined and in majority of references it is ignored.

where we assumed for simplicity $W = D_B + R - P$ while the control $U_2(\theta)$ is given by

$$U_2(\theta) = \sum_{n=1}^{M} \beta_n \delta(\theta - \eta_n).$$

Solving the first equation with initial condition $L^*(0) = 0$ one obtains

$$L^*(\theta) = \int_0^\theta e^{-\frac{K_1+K_3}{\upsilon}(\theta-\tilde{x})} \frac{1}{\upsilon} J(\tilde{x}) \mathrm{d}\tilde{x}.$$

Inserting the expression describing the waste waters we get

$$L^*(\theta) = \frac{1}{\upsilon}\int_0^\theta e^{-\frac{K_1+K_3}{\upsilon}(\theta-\tilde{x})} \left(p + \sum_{j=1}^{N} q_j \delta(\tilde{x} - \xi_j) \right) \mathrm{d}\tilde{x} =$$

$$= \frac{1}{\upsilon}\int_0^\theta e^{-\frac{K_1+K_3}{\upsilon}(\theta-\tilde{x})} p \mathrm{d}\tilde{x} + \frac{1}{\upsilon}\int_0^\theta e^{-\frac{K_1+K_3}{\upsilon}(\theta-\tilde{x})} \left(\sum_{j=1}^{N} q_j \delta(\tilde{x} - \xi_j) \right) \mathrm{d}\tilde{x},$$

whence

$$L^*(\theta) = \frac{p}{K_1 + K_3}\left(1 - e^{-\frac{K_1+K_3}{\upsilon}\theta} \right) + \frac{1}{\upsilon}\sum_{j=1}^{N} \mathbb{1}(\theta - \xi_j) q_j e^{-\frac{K_1+K_3}{\upsilon}(\theta-\xi_j)},$$

where $\mathbb{1}$ denotes the Heaviside step function. The solution of the second equation of (2) with initial condition $D^*(0) = 0$ is

$$D^*(\theta) = \int_0^\theta e^{-\frac{K_2}{\upsilon}(\theta-\tilde{x})} \left[\frac{K_1}{\upsilon} L^*(\tilde{x}) + \frac{W}{\upsilon} + \frac{U_2(\tilde{x})}{\upsilon} \right] \mathrm{d}\tilde{x},$$

whence

$$D^*(\theta) = \frac{K_1 p}{K_2(K_1 + K_3)}\left(1 - e^{-\frac{K_2}{\upsilon}\theta} \right) -$$

$$- \frac{K_1 p}{(K_1 + K_3)(K_2 - (K_1 + K_3))}\left(e^{-\frac{K_1+K_3}{\upsilon}\theta} - e^{-\frac{K_2}{\upsilon}\theta} \right) +$$

$$+ \sum_{j=1}^{N} \mathbb{1}(\theta - \xi_j) q_j \frac{K_1}{\upsilon(K_2 - (K_1 + K_3))}\left(e^{-\frac{K_1+K_3}{\upsilon}(\theta-\xi_j)} - e^{-\frac{K_2}{\upsilon}(\theta-\xi_j)} \right) +$$

$$+ \frac{W}{K_2}\left(1 - e^{-\frac{K_2}{\upsilon}\theta} \right) + \frac{1}{\upsilon}\sum_{n=1}^{M} \mathbb{1}(\theta - \eta_n) \beta_n e^{-\frac{K_2}{\upsilon}(\theta-\eta_n)}.$$

Making translations

$$x_1(t,\theta) := L(t,\theta) - L^*(\theta), \quad x_2(t,\theta) := D(t,\theta) - D^*(\theta),$$

introducing the deviated control

$$V(\theta,t) := U(\theta,t) - U_2(\theta) = \sum_{n=1}^{M} u_n(t)\delta(\theta-\eta_n), \quad u_n(t) := \alpha_n(t) - \beta_n$$

and recalling the definitions of L^* and D^* we get

$$\left\{\begin{array}{l} \dfrac{\partial x_1}{\partial t} = -\upsilon\dfrac{\partial x_1}{\partial\theta} - (K_1+K_3)x_1 \\ \dfrac{\partial x_2}{\partial t} = -\upsilon\dfrac{\partial x_2}{\partial\theta} - K_2x_2 + K_1x_1 + V(t,\theta) \\ x_1(0,t) = 0 \\ x_2(0,t) = 0 \\ x_1(\theta,0) = -L^*(\theta) \\ x_2(\theta,0) = -D^*(\theta), \end{array}\right\}, \quad t \ge 0, \quad \theta \in [0,a] \tag{3}$$

3. Abstract model

Assuming the translated distribution functions (profiles) of BDO and DO in a fixed time $t \ge 0$ as components of the state vector $x(t) = [x_1(\cdot,t), x_2(\cdot,t)]^T$ from the space

$$\mathrm{H} = \mathrm{L}^2(0,a) \oplus \mathrm{L}^2(0,a),$$

equipped in the standard scalar product

$$\langle x, w\rangle_H = \int_0^a \begin{bmatrix} x_1(\theta) \\ x_2(\theta)\end{bmatrix}^T \begin{bmatrix} w_1(\theta) \\ w_2(\theta)\end{bmatrix} \mathrm{d}\theta = \langle x_1, w_1\rangle_{\mathrm{L}^2(0,a)} + \langle x_2, w_2\rangle_{\mathrm{L}^2(0,a)},$$

we can rewrite (3) into its abstract additive form

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + \sum_{n=1}^{M} b_n u_n(t) \\ y(t) = (c^{\#}x)(t) \end{array}\right. \tag{4}$$

with unbounded linear operator $A:(\mathcal{D}(A)\subset H)\to H$,

$$\left\{\begin{array}{l} Ax=-\upsilon x'+Qx, \quad Q=\begin{bmatrix}-(K_1+K_3) & 0\\ K_1 & -K_2\end{bmatrix} \\ \mathcal{D}(A)=\{x\in \mathrm{H}: x'\in \mathrm{H},\, x(0)=0\}=\mathrm{W}_0^{1,2}[0,a]\oplus \mathrm{W}_0^{1,2}[0,a]\end{array}\right\},$$

control vectors

$$b_n=\begin{bmatrix}0\\ \delta(\theta-\eta_n)\end{bmatrix}\notin \mathrm{H}, \quad n=1,\ldots,M$$

and the vector of linear unbounded functionals of observation at the points $\{\gamma_i\}_{i=1}^{K}\subset(0,a)$,

$$\left\{\begin{array}{l} c^{\#}x=\begin{bmatrix}c_1^{\#}x\\ c_2^{\#}x\\ \vdots\\ c_K^{\#}x\end{bmatrix}, \quad c_i^{\#}x=x_2(\gamma_i) \\ \mathcal{D}(c^{\#})=\{x\in \mathrm{H}: x_2 \text{ is continuous at } \gamma_i,\, i=1,\ldots,K\}\end{array}\right\} \tag{5}$$

Definition 1

A family $\{T(t)\}_{t\geq 0}\subset L(H)$ *is called the* C_0*–semigroup on the space* H *if the following conditions hold*

$$T(0)=I, \quad T(t+\tau)=T(t)T(\tau) \quad \forall t,\tau\geq 0$$

$$\lim_{t\to 0+} T(t)x=x \quad \forall x\in \mathrm{H}.$$

The linear operator

$$Ax:=\lim_{t\to 0+}\frac{1}{t}[T(t)x-x], \quad \mathcal{D}(A)=\left\{x\in \mathrm{H}:\exists \lim_{t\to 0+}\frac{1}{t}[T(t)x-x]\right\}$$

is said to be the infinitesimal generator of the C_0 *– semigroup* $\{T(t)\}_{t\geq 0}$.

From the semigroup theory (e.g. [6, 21]) we know that the Laplace transform of a semigroup is the resolvent of its infinitesimal generator A, i.e.,

$$\int_0^{\infty} e^{-st}T(t)x\mathrm{d}t=(sI-A)^{-1}x, \quad x\in \mathrm{H}.$$

Theorem 1

The operator A *is the infinitesimal generator of* C_0*–semigroup with* $\{T(t)\}_{t\geq 0}$ *the property:* $T(t)=0$ *for each* $t\geq \dfrac{a}{\upsilon}$.

Proof. Since the matrix Q is diagonally dissipative, i.e., there exists a diagonal matrix $H > 0$ such that $Q^T H + HQ < 0$ (e.g., $H = \mathrm{diag}\left\{\frac{1}{K_2(K_1+K_3)}, \frac{1}{K_1^2}\right\}$), then the fact that A generates an exponentially stable semigroup $\{T(t)\}_{t\geq 0}$ immediately follows from the result of [7].

This result can be sharpened by representing A as the sum of the generator of scaled (t replaced by υt) semigroup of right–shifts $\{S(t)\}_{t\geq 0}$,

$$S(t)x_i = \begin{cases} x_i(\theta - \upsilon t) & \text{dla} \quad a \geq \theta \geq \upsilon t \\ 0 & \text{dla} \quad \theta < \upsilon t \end{cases}, \quad i = 1, 2,$$

and the operator of multiplication by the matrix Q. Due to this the following representation is valid

$$T(t)x = e^{Qt}\begin{bmatrix} S(t)x_1 \\ S(t)x_2 \end{bmatrix}, \quad e^{Qt} = \begin{bmatrix} e^{-(K_1+K_3)t} & 0 \\ \frac{1}{K_2(K_1+K_3)}\left[e^{-(K_1+K_3)t} - e^{-K_2 t}\right] & e^{-K_2 t} \end{bmatrix},$$

and consequently, since $S(t) = 0$ for $t \geq \frac{a}{\upsilon}$, then also $T(t) = 0$ for $t \geq \frac{a}{\upsilon}$. □

For every $\lambda \in \mathbb{C}$ and $y \in \mathrm{H}$ the equation

$$\lambda x(\theta) - Ax(\theta) = y(\theta),$$

taking in H the particular form,

$$\begin{cases} \lambda x_1(\theta) + \upsilon x_1'(\theta) + (K_1 + K_3)x_1(\theta) = y_1(\theta) \\ \lambda x_2(\theta) + \upsilon x_2'(\theta) - K_1 x_1(\theta) + K_2 x_2(\theta) = y_2(\theta) \\ x_1(0) = 0, \quad x_2(0) = 0 \end{cases}$$

has a unique solution in $\mathcal{D}(A)$ with components

$$x_1(\theta) = \int_0^\theta \frac{1}{\upsilon} e^{-\frac{\lambda + K_1 + K_3}{\upsilon}(\theta - \tilde{x})} y_1(\tilde{x})\mathrm{d}\tilde{x} \tag{6}$$

$$x_2(\theta) = \int_0^\theta e^{-\frac{\lambda + K_2}{\upsilon}(\theta - w)} \left[\frac{K_1}{\upsilon^2}\int_0^w e^{-\frac{\lambda + K_1 + K_3}{\upsilon}(w - \tilde{x})} y_1(\tilde{x})\mathrm{d}\tilde{x} + \frac{1}{\upsilon} y_2(w)\right]\mathrm{d}w \tag{7}$$

Formula (6) and (7) define for any $\lambda \in \mathbb{C}$ the resolvent of A and, consequently, the operator A has empty spectrum.

Actually, making use of some results presented in [1], it can be shown even more – the resolvent $(\lambda I - A)^{-1}$ is a compact Volterra operator.

Substituting $\lambda = 0$ in (6) and (7) we obtain the inverse of A:

$$A^{-1}\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = -\frac{1}{\upsilon}\begin{bmatrix} \int_0^\theta e^{-\frac{K_1+K_3}{\upsilon}(\theta-\tilde{x})} Y_1(\tilde{x})\mathrm{d}\tilde{x} \\ \int_0^\theta e^{-\frac{K_2}{\upsilon}(\theta-w)}\left[K_1\int_0^w e^{-\frac{K_1+K_3}{\upsilon}(w-\tilde{x})} Y_1(\tilde{x})\mathrm{d}\tilde{x} + Y_2(w)\right]\mathrm{d}w \end{bmatrix}.$$

The abstract model (4) can be analysed in the frames of the so–called well–posed, regular linear systems developed by Salamon and Weiss (Weiss *et al.* 1997). In this theory, the state equation is being interpreted as an equation on a larger space than the state space H (usually denotes as H_{-1} or $[\mathcal{D}(A^*)]'$), to which all control vectors b_n belong. An important ingredient of the theory is a proof that the state operator A naturally extends, onto this larger space, to a linear unbounded operator A_{ext} with the domain $\mathcal{D}(A_{ext}) = \mathrm{H}$.

An alternative approach is the theory of the so–called abstract factor models developed by Grabowski and Callier [5, 8]. In this theory all objects of the abstract model are defined within the state space H, but a price paid for this simplification is that now the control action does not enter the model in an additive form "$+\sum_{n=1}^{M} b_n u_n(t)$" but in the factor form and the resulting state equation is

$$\dot{x} = A\left[x + \sum_{n=1}^{M} d_n u_n(t)\right].$$

In order to ensure a consistency of both two models one has to assume $d_n := A_{ext}^{-1} b_n$, and $d_n \in H$ are so–called factor control vectors which generally do not satisfy $d_n \notin \mathcal{D}(A)$.

Since A^{-1} is an integral operator then it has a natural extension to the Dirac–delta distribution in the form commonly known as the Dirac–delta "sifting property"

$$A_{ext}^{-1} b_n = -\frac{1}{\upsilon}\begin{bmatrix} 0 \\ \int_0^\theta e^{-\frac{K_2}{\upsilon}(\theta-w)} \delta(w-\eta_n)\mathrm{d}w \end{bmatrix} =$$

$$= -\frac{1}{\upsilon}\begin{bmatrix} 0 \\ \mathbb{1}(\theta-\eta_n) e^{-\frac{K_2}{\upsilon}(\theta-\eta_n)} \end{bmatrix} := d_n(\theta)$$

and all elements of the factor model with observation

$$\left\{\begin{array}{l}\dot{x}=A\left[x+\sum_{n=1}^{M} d_n u_n(t)\right] \\ y(t)=(c^{\#} x)(t)\end{array}\right\} \tag{8}$$

are now determined.

Definition 2

An observation operator $c^{\#} \in \mathbf{L}(\mathcal{D}(A), \mathbb{R}^K)$ *is called admissible if there exists* $\varepsilon > 0$ *such that*

$$\int_0^{\infty}\left|c^{\#} T(t) x_0\right|^2 \mathrm{d}t \leq \varepsilon\left\|x_0\right\|_{\mathrm{H}}^2 \quad \forall x_0 \in \mathcal{D}(A). \tag{9}$$

Theorem 2

The observation operator $c^{\#}$ *is admissible.*

Proof. For a proof that $c^{\#} \in \mathbf{L}(\mathcal{D}(A), \mathbb{R}^K)$ it suffices to demonstrate that there exist $h_i \in \mathrm{W}^{1,2}(0, a) \oplus \mathrm{L}^2(0, a), i=1, 2, \ldots, K,$

$$h_i=\begin{bmatrix}h_{i1} \\ h_{i2}\end{bmatrix}, \quad h_{i2}(\theta)=\left\{\begin{array}{ll}h_{i2}^{-} \in W^{1,2}[0, \gamma_i], & 0 \leq \theta \leq \gamma_i \\ h_{i2}^{+} \in W^{1,2}[\gamma_i, a], & \gamma_i \leq \theta \leq a\end{array}\right\}$$

such that

$$c_i^{\#} x=x_2(\gamma_i)=\langle A x, h_i\rangle_{\mathrm{H}} \quad \forall x \in \mathcal{D}(A), \tag{10}$$

i.e., $\left.c_i^{\#}\right|_{\mathcal{D}(A)}=h_i^{*} A.$ If then integrating–by–parts we obtain

$$\begin{aligned}
&\langle A x, h_i\rangle_{\mathrm{H}}=\int_0^a\left[-\upsilon x_1'(\theta)-(K_1+K_3) x_1(\theta)\right] h_{i1}(\theta) \mathrm{d}\theta- \\
&-\int_0^{\gamma_i} \upsilon x_2'(\theta) h_{i2}^{-}(\theta) \mathrm{d}\theta-\int_{\gamma_i}^a \upsilon x_2'(\theta) h_{i2}^{+}(\theta) \mathrm{d}\theta- \\
&-\int_0^a K_2 x_2(\theta) h_{i2}(\theta) \mathrm{d}\theta+\int_0^a K_1 x_1(\theta) h_{i2}(\theta) \mathrm{d}\theta= \\
&=-\upsilon h_{i1}(a) x_1(a)+\upsilon h_{i1}(0) \underbrace{x_1(0)}_{=0}+\int_0^a x_1(\theta)\left[\upsilon h_{i1}'(\theta)-(K_1+K_3) h_{i1}(\theta)\right] \mathrm{d}\theta- \\
&-\upsilon h_{i2}^{-}(\gamma_i) x_2(\gamma_i)+\upsilon h_{i2}^{-}(0) \underbrace{x_2(0)}_{=0}+\int_0^{\gamma_i} x_2(\theta) \upsilon \frac{\mathrm{d}}{\mathrm{d}\theta}\left[h_{i2}^{-}(\theta)\right] \mathrm{d}\theta- \\
&-\upsilon h_{i2}^{+}(a) x_2(a)+\upsilon h_{i2}^{+}(\gamma_i) x_2(\gamma_i)+\int_{\gamma_i}^a x_2(\theta) \upsilon \frac{\mathrm{d}}{\mathrm{d}\theta}\left[h_{i2}^{+}(\theta)\right] \mathrm{d}\theta+ \\
&+\int_0^a K_1 x_1(\theta) h_{i2}(\theta) \mathrm{d}\theta-\int_0^a K_2 x_2(\theta) h_{i2}(\theta) \mathrm{d}\theta.
\end{aligned}$$

Hence the condition (10) would hold if

$$\upsilon h_{i1}'(\theta)-(K_1+K_3)h_{i1}(\theta)+K_1h_{i2}(\theta)=0,\quad 0\le\theta\le a \tag{11}$$

$$\upsilon h_{i2}'^{-}(\theta)-K_2h_{i2}^{-}(\theta)=0,\quad 0\le\theta\le\gamma_i \tag{12}$$

$$\upsilon h_{i2}'^{+}(\theta)-K_2h_{i2}^{+}(\theta)=0,\quad \gamma_i\le\theta\le a \tag{13}$$

$$-\upsilon h_{i2}^{-}(\gamma_i)+\upsilon h_{i2}^{+}(\gamma_i)=1 \tag{14}$$

$$h_{i1}(a)=0 \tag{15}$$

$$h_{i2}^{+}(a)=0 \tag{16}$$

From (13) and (16) we get h_{i2} on the interval $\gamma_i\le\theta\le a$ while from (12) and (14) we determine h_{i2} on $0\le\theta\le\gamma_i$. Hence

$$h_{i2}(\theta)=\begin{cases}-\dfrac{1}{\upsilon}e^{\frac{K_2}{\upsilon}(\theta-\gamma_i)}, & 0\le\theta\le\gamma_i\\ 0, & \gamma_i\le\theta\le a\end{cases}.$$

Employing the solution h_{i2} jointly with (11) and (15) we find

$$h_{i1}(\theta)=\begin{cases}\dfrac{K_1}{\upsilon(K_1+K_3-K_2)}\left(e^{\frac{K_1+K_3}{\upsilon}(\theta-\gamma_i)}-e^{\frac{K_2}{\upsilon}(\theta-\gamma_i)}\right), & 0\le\theta\le\gamma_i\\ 0, & \gamma_i\le\theta\le a.\end{cases}$$

Hence there exists a uniquely determined vector $h_i\in \mathrm{H}$ satisfying (10). By (10),

$$\|c^{\#}x\|_{\mathbb{R}^K}^2=\sum_{i=1}^{K}\left|c^{\#}x\right|^2=\sum_{i=1}^{K}\left|\langle Ax,h_i\rangle_{\mathrm{H}}\right|^2\le\|Ax\|_{\mathrm{H}}^2\sum_{i=1}^{K}\|h_i\|_{\mathrm{H}}^2,\quad x\in\mathcal{D}(A)$$

which means that $c^{\#}\in\mathbf{L}(\mathcal{D}(A),\mathbb{R}^K)$.

Now we show the admissibility of $c^{\#}$. Let $x_0\in\mathcal{D}(A)$. Making use of the explicit expression for the semigroup and employing evident inequalities

$$\left|e^{-(K_1+K_3)t}-e^{-K_2t}\right|\le 1,\quad \left|e^{-K_2t}\right|\le 1,\quad \forall t\ge 0,$$

we have

$$\int_0^\infty \left|c_i^{\#}T(t)x_0\right|^2 \mathrm{d}t =$$

$$= \int_0^{\gamma_i/\upsilon} \left| \underbrace{\frac{K_1(e^{-(K_1+K_3)t} - e^{-K_2 t}}{K_2-(K_1+K_3)} x_0^1(\gamma_i - \upsilon t)}_{\Xi} + \underbrace{e^{-K_2 t} x_0^2(\gamma_i-\upsilon t)}_{\Upsilon} \right|^2 \mathrm{d}t \le$$

$$\le 2\int_0^{\gamma_i/\upsilon} \left(|\Xi|^2 + |\Upsilon|^2\right)\mathrm{d}t \le 2\tilde{k}\left[\int_0^{\gamma_i/\upsilon} \left|x_0^1(\gamma_i-\upsilon t)\right|^2 \mathrm{d}t + \int_0^{\gamma_i/\upsilon}\left|x_0^2(\gamma_i-\upsilon t)\right|^2 \mathrm{d}t\right] \le$$

$$\le \frac{2\tilde{k}}{\upsilon}\int_0^{\gamma_i}\left[\left|x_0^1(\theta)\right|^2 + \left|x_0^2(\theta)\right|^2\right]\mathrm{d}\theta \le \frac{2\tilde{k}}{\upsilon}\|x_0\|_{\mathrm{H}}^2,$$

where

$$\tilde{k} := \max\left\{\left(\frac{K_1}{K_2-(K_1+K_3)}\right)^2, 1\right\}$$

Hence

$$\int_0^\infty \left\|c^{\#}T(t)x_0\right\|_{\mathbb{R}^K}^2 \mathrm{d}t = \sum_{i=1}^{K}\int_0^\infty \left|c_i^{\#}T(t)x_0\right|^2 \mathrm{d}t \le \frac{2K\tilde{k}}{\upsilon}\|x_0\|_{\mathrm{H}}^2$$

and $c^{\#}$ is admissible with the admissibility constant ε of Definition 2 equal to $\frac{2K\tilde{k}}{\upsilon}$. □

Since in the examined system the spaces of controls and outputs are, respectively, $\mathbb{R}^M$ and $\mathbb{R}^K$, i.e., they are finite–dimensional, then to describe influence of the n-th component u_n of the control vector u onto i-th component y_i of the output (observation) vector y one can apply the theory of SISO abstract factor control systems presented in [8] with some complements in [5]. In accordance with this theory, for every $x_0 \in \mathrm{H}$ and $u_n \in \mathrm{W}^{1,2}(0,\infty)$ the following version of the variation–of–constants formula

$$x(t) = T(t)x_0 + \sum_{n=1}^{M} A\int_0^t T(t-\tau)d_n u_n(\tau)\mathrm{d}\tau.$$

expresses a weak solution to the initial–value problem associated with (8)[2)]

$$\left\{\begin{matrix} \dot{x} = A\left[x + \sum_{n=1}^{M} d_n u_n\right] \\ x(0) = x_0 \end{matrix}\right\},$$

[2)] Conditions under which x is the classical solution are given in [5].

i.e., x is a continuous function of t taking values in H and such that for every $w \in \mathcal{D}(A^*)$, the domain of the adjoint operator A^*, the scalar function $t \mapsto \langle x(t), w \rangle_{\mathrm{H}}$ is absolutely continuous and for almost all t there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} = \left\langle x(t) + \sum_{n=1}^{M} d_n u_n(t), A^* w \right\rangle_{\mathrm{H}}.$$

Thanks to this the Laplace transform of the state vector reads as

$$\begin{aligned}\hat{x}(s) &= (sI - A)^{-1}x_0 + \sum_{n=1}^{M} A(sI-A)^{-1} d_n \hat{u}_n(s) = \\ &= (sI - A)^{-1}x_0 + \sum_{n=1}^{M}\left[s(sI-A)^{-1}d_n - d_n\right]\hat{u}_n(s).\end{aligned}$$

Continuing the construction presented in [8], we have to verify whether the so–called compatibility conditions holds: $d_n \in \mathcal{D}(c_i^{\#})$ for $i = 1, ..., K$, $n = 1, ..., M$. Directly by definitions of $c_i^{\#}$ and d_n,

$$d_n \in \mathcal{D}(c_i^{\#}) \Leftrightarrow \gamma_i \neq \eta_n, \quad i = 1, ..., K, \quad n = 1, ..., M,$$

i.e., the observations are not driven in the same points as controls are applied. Then

$$c_i^{\#} d_n = \begin{cases} -\frac{1}{\upsilon} e^{-\frac{K_2}{\upsilon}(\gamma_i - \eta_n)} & \gamma_i > \eta_n \\ 0 & \gamma_i < \eta_n. \end{cases}$$

The compatibility conditions enables us to determine the Laplace transform of the output y,

$$\hat{y}(s) = c^{\#}\hat{x}(s) = c^{\#}(sI-A)^{-1}x_0 + \sum_{n=1}^{M}\left[sc^{\#}(sI-A)^{-1}d_n - c^{\#}d_n\right]\hat{u}(s), \quad s \in \mathbb{C}.$$

With $x_0 = 0$ we can define the matrix–valued transfer function of (8):

$$\hat{G}(s) = \left[\hat{G}_{in}(s)\right]_{i=1,...,K,\, n=1,...,M},$$

where

$$\hat{G}_{in}(s) = sc_i^{\#}(sI-A)^{-1}d_n - c_i^{\#}d_n.$$

Applying (6) and (7) we find

$$\widehat{G}_{in}(s) = \begin{cases} \frac{1}{\upsilon} e^{-\frac{s+K_2}{\upsilon}(\gamma_i - \eta_n)}, & \gamma_i > \eta_n \\ 0, & \gamma_i < \eta_n \end{cases}.$$

$\widehat{G}_{in} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathbb{C}^M, \mathbb{C}^K))$, where denotes the Hardy space of functions analytic and bounded on $\mathbb{C}^+$. This is the case, as $\widehat{G}_{in}$ are entire functions of $s \in \mathbb{C}$ and for $s \in \mathbb{C}^+ \cup j\mathbb{R}$ we clearly have

$$\left|\widehat{G}_{in}(s)\right| < \frac{1}{\upsilon} e^{-\frac{K_2}{\upsilon}(\gamma_i - \eta_n)}.$$

4. Conclusions

We have shown that the dynamical model of propagation of pollutants in a river with M point controls realized by aerators and with K measurement points can be transformed into its abstract factor control form on a suitable Hilbert state space. The semigroup generated by the state operator A decays in a finite–time, the observation operator is admissible, the system transfer functions belongs to the space $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathbb{C}^M, \mathbb{C}^K))$.

The results we have obtained are, in the prospect of further studies, a starting point to solve the standard LQ problem of minimization the quadratic performance index

$$\|y\|^2_{\mathrm{L}^2(0,\infty,\mathbb{R}^K)} + \|u\|^2_{\mathrm{L}^2(0,\infty,\mathbb{R}^M)}$$

over trajectories of (8), jointly with a construction of an optimal linear feedback controller. Such a performance index has a reasonable interpretation: its first component represent a penalty that the values of D deviate, in some selected measurement points, from its nominal state D^*, while the second component represents control costs (costs of aeration).

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