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Results in Q-measure

AbstractThis paper introduces the notion of a generalized measure for a sequence of functions with oscillation and concentration effects. This measure is constructed by averaging the sequence of Borel measurable functions using singular or regular perturbations. In this way, the generalized limits of such sequences are conceptualized by enlarging the space of functions to measure spaces. It is a modification of the Young measure. This modified measure was termed a Q-measure. It can be difficult to determine the Young measure for a broad function. The Q-measure can be easily calculated for particular functions. This is one of the advantages of this study. As an application of the measure, we can define another weaker type of Monotone convergence theorem, the Lebesgue-dominated convergent theorem. A notion of average for underlying sequences to define the Q-measure is given, as also its application in signal analysis and atmospheric sciences.

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1. Introduction. Sequences of bounded functions that are oscillatory and concentrated in nature often arise in many practical problems. For example, problems in nonconvex optimization which lead to the nonexistence of a classical minimizer (cf.[10]). In this case, the minimizing sequence, which minimizes the integral, oscillates rapidly. Moreover, this sequence is not pointwise convergent but remains uniformly bounded. Moreover, even weak* convergent subsequences of such a finite sequence quickly oscillate around the weak* limit.

To overcome this problem (cf.[45]), the generalized limits of such sequences are conceptualized by enlarging the space of functions to the measure spaces. The idea is to assign the limit, not as a *usual* function but as a probability measure-valued function, referred to as *Young measure* or *parameterized measure* or *generalized curve*[46, 47]. The Young measure is further investigated in [25, 26, 41, 42] and it is successfully used to capture oscillatory behavior of sequences of function; however, it fails to represent some case of concentration property of the sequence (cf.[42]). Some oscillatory sequences with concentration property in a bounded domain are constructed in [44]. Geometrical interpretation of Young measure is introduced in [36] and later, an elaborate study of the Young measure and the Tangent measure has been examined in (cf. [37]). The theoretical reports concerning the computational studies of Young measure can be found in [30, 19, 31, 32, 33].

Tartar [40] introduces a family of measures named as H-measure (see for more details [16, 34]). An important application of the H-measure is associated with systems of partial differential equations which describe the propagation of oscillations and concentration effects in the solution. A measure with strong L^2 convergence obtained by integrating the H-measure [3] with respect to the Fourier space variable is introduced by Patrick Gérard and termed as micro-local defect measure [18]. One of the modified forms of the H-measure is parabolic H-measure with the application includes in the transport equation as well as Schrödinger Equation. Specifically, it is the tool used to compute the weak limits of quadratic products of oscillating fields. The H- measure is also useful for solving the linear wave equation with smooth coefficients and rapidly oscillating initial data. The H-measure and Defect measure are useful in parabolic and elliptic systems (v. [16, 18, 40, 3]). In particular, the incapability of Defect measure to capture the direction of flow in solution leads to an inappropriate for hyperbolic PDE's.

Recently (cf. [14, 15]), the entropy measure-valued solutions of hyperbolic conservation laws are discussed by using Young Measure. The Young measure is successfully used to capture the oscillatory behavior of sequences of function; however, it is not weakly stable and fails to represent some case of the concentration property of the sequence (cf.[42]. Moreover, if we integrate a given function using Monte Carlo simulation, then the relative error[1] turns out to be more in Young measure as compared to Q-measure.

The major motivation of this work is to find a weakly stable family of probability measure that can capture oscillatory and concentrating behavior of sequence of functions. Q- measure first introduced by Jisha in [23]. Q- measure application in PDE is discussed in this article. This innovative concept of a new measure of a function is given by generating a sequence of functions through singular or regular perturbation. This measure is termed as the Q-measure which can capture highly oscillatory and concentrating nature of the function.

In section 2, we discuss the construction of the barycenter of a sequence of function u_n, $n \in \mathbb{N}$ (eq. (1)) and its application. In the section 3, gives the generalized Q-measure (see definition 3.1), weak convergence of Q-measure(see definition 3.2), construction of a sequence from $u(x)$ and some general form of Q-measure for a particular type of functions are given. The oscillation, concentration, and blow-up of function and corresponding measure are discussed in the subsequent section 4. The generalized Q measure is discussed in the section 5. In section 6 we provide the example for generalized Q-measure.

2. Barycenter construction for the sequence of function

[1] V. appendix C

Throughout this work, we consider the sequence of Borel measurable function (cf. [23]). Let A_i be a disjoint partition of $R = \prod_{i=1}^{l}[a_i, b_i]$, $l = 1, 2$ (see [2]), and (v_i) is a given sequence of Borel measurable function in R and $\overline{u}_i \in L^1(A_i)$, for all A_i. Then we can define the Barycenter of the sequence of functions indeed,

$$\overline{u}_n(x) = \lim_{k\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{v_i \chi_{A_j}(x)}{n}. \tag{1}$$

where

$$u_i = \begin{cases} v_i & \text{if } x \in A_j \\ 0 & \text{if } x \notin A_j \end{cases}. \tag{2}$$

Barycenter of sequence of functions is

$$\overline{u}(x) = \lim_{n\to\infty} \overline{u}_n(x), \tag{3a}$$

$$= \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{u_i \chi_{A_j}(x)}{n}, \tag{3b}$$

The nature of $\overline{u}(x)$ is similar to the Cesaro summability of a series given in [13]. In other words, for a fixed $x \in [a, b]$, $\overline{u}_n(x)$ is the convex combination (arithmetic mean) of the first n partial sums of the series. For example, $u_n = (-1)^n \chi_{[n,n+1]}, n \in \mathbb{N}$. i.e., the characteristic functions on an interval of unit length that escapes to infinity. This sequence does not convergence in any $L^p(\Re)$ norm, $p > 1$. This type of sequence of functions cannot be treated with the Young measure and the Lebesgue measure. Also, sequence u_n is the combination of oscillation and concentration in nature.

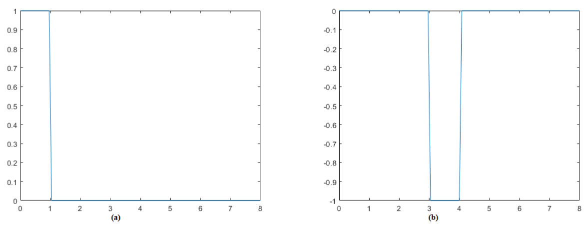

Figure 1: Subfigures (a) and (b) represent $u_1(x)$ and $u_2(x)$, respectively.

Consider the Borel sigma algebra $\mathcal{B}(\Re)$ [2], If x is a random variable and $u(x)$ is a Borel measurable function, then $u(x)$ is also a random variable, but not for a Lebesgue measurable function. In other words, the function $u(x)$ is a Lebesgue measurable function, x is a random variable, but $u(x)$ is not a

[2] which is the smallest sigma-algebra generated by all open sets of the $K \subset \mathbb{R}$.

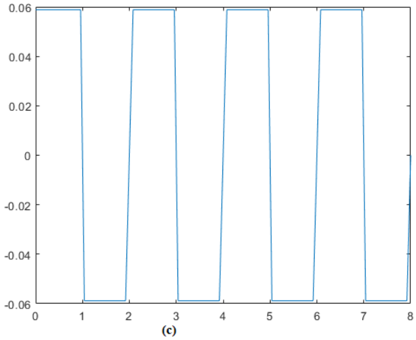

Figure 2: Denoted by $\sum_{i=1}^{17} \frac{u_i(x)}{17}$ of above sequence.

random variable. The sequence of functions is considered as an independent random variable then by the law of large number, $\overline{u}_n(x)$ converges to the expectation of u as n tends to infinity.

In equation (1), u_i is multiplied with characteristic functions and posses a lead role throughout because of its property in reducing the randomness of a variable. The $\overline{u}$ is well-defined for a sequence of function but not for u. The application of $\overline{u}$ is given in the section 6.2. The existence of Q- measure is discussed in [23].

3. General Q-measure.

3.1. Preliminaries.

Definition 3.1 (Q-measure at a point)[28, 11, 17] A family of probability measure $\{\nu_x\}_{x\in K}$ is said to be Q-measure associated with a sequence of function $\overline{u}_j(x)$ at x corresponding to the sequence $u_j(x)$ such that supp(ν_x) $\subset \mathbb{R}^N$, where supp(ν_x) is the support of ν_x.

Q-measure is the limit of a probability distribution of the estimations of barycenter(i.e. $\overline{u}_j(x)$) of the $\{u_j(x)\}_{j=1}^{n}$, provided each neighboring points x are taken randomly in K. Let $B_r(x)$ denotes the ball of radius $r > 0$, centered at x and $E \in \Re^N$ be any measurable set, then

$$\nu_x(E) = \lim_{r\downarrow 0} \lim_{j\to\infty} \frac{\mu\left(\{y \in B_r(x) : \overline{u}_j(y) \in E\}\right)}{\mu(B_r(x))}.$$

♣

The η' represent $\overline{u}(x)^{-1}$ in figure 3.

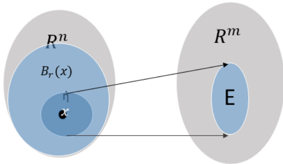

Figure 3: The graphical representation of Q- measure,

Definition 3.2 [Q- measure through projection map] A positive measure ν on $K \times \Re$ is called a Q- measure if for every Borel subset, B of K satisfies

$$\nu(B \times \Re) = \mu(B), \tag{4}$$

i.e. Q- measure is a measure such that the measure of every box $B \times \Re$ is determined by the projection of the box on to the set B in K. ♣

Definition 3.3 (Q- measure associated to u) Let $u : K \to \Re$ be a Borel measurable function in $L^2(K)$ and $\overline{u}$ is given by equation (1). The ν is said to be Q- measure associated to u, if for every continuous and bounded function $\beta : K \times \Re \to \Re$ the associated Q measure ν^u satisfies the following condition,

$$\int_{K \times \Re} \beta(x, y) d\nu^u(x, y) = \int_K \beta(x, \overline{u}(x)) d\mu. \tag{5}$$

♣

Definition 3.4 (Slicing a measure) Let ν be a positive measure on $K \times \Re$ and σ be its projection onto K (i.e., $\sigma(B) = \nu(B \times \Re)$). The ν is sliced into measures $(\sigma_x)_x \in K$ if it satisfies the following conditions

1. The ν_x is a probability measure.
2. The mapping $x \to \int_\Re \beta(x, y) d\nu_x(x, y)$ is measurable for every continuous function β and satisfies

$$\int_{K \times \Re} \beta(x, y) d\nu(x, y) = \int_K \int_\Re \beta(x, y) d\nu_x(y) d\sigma(x).$$

Q- measure is also a slicing measure related to Borel measurable function $u(x)$. For the Q- measure ν^u associated to u, the measure σ in definition 3.3 is μ and hence

$$\int_{K \times \Re} \beta(x, y) d\nu^u(x, y) = \int_K \int_\Re \beta(x, y) d\nu^u_x(y) d\mu(x).$$

On the other hand,

$$\int_{K\times\Re}\beta(x,y)d\nu^u(x,y) = \int_K \beta(x,\overline{u}(x))d\mu(x,y), \quad (6)$$
$$= \int_K\int_{\Re}\beta(x,y)d\nu_x^u(y)d\mu(x),$$

where μ^u sliced to $\mu_x^u = \delta_{\overline{u}(x)}$.

♣

3.2. Weak Convergence of Q- measure.

DEFINITION 3.5 (WEAK CONVERGENCE OF Q-MEASURE) Let ν_n, ν be Q-measures on $(K, \mathcal{B}(\Re))$ where $K \subseteq \Re$, and $\mathcal{B}(\Re)$ is a Borel sigma algebra. The ν_n weakly converges to ν as $n \to \infty$ if for any bounded function $u : K \to \Re$,

$$\int_K u(x)\nu_n(dx) \to \int_K u(x)\nu(dx), \text{as } n \to \infty. \quad (7)$$

♣

REMARK 3.6 Let ν_n and ν are Q- measure, u is continuous on a compact set K and satisfy equation (7) then ν_n converges to ν weakly.

3.3. Construction of the sequence $u_n(x)$ from $u(x)$ [23] To construct Q- measure we need the sequence $u_n(x)$, that we constructed from $u(x)$. For this, we need a sequence of function corresponds to $u(x), x \in K \subseteq \Re^N$. In this work, to construct a sequence of the function $u_n(x)$, we chose two types of perturbation, namely singular and regular perturbations, where the sequence $\epsilon_n > 0$ for all $n \in \mathbb{N}$ as follows:

DEFINITION 3.7 Let $u_n \in L^2(K), K \subset \mathbb{R}$. The function $u(x)$ is said to be regularly perturbed with respect to $L^2(K)$ norm if it satisfied for all positive $\epsilon_n, n = 1, 2....$

$$||u(x - \epsilon_n) - u(x)|| \to 0 \text{ as } \epsilon_n \to 0 \ .$$

Otherwise it is said to be singularly perturbed.

♣

3.1.1. Construction of sequence $u_n(x)$ from singularly perturbed function $u(x)$

To construct a sequence of the function $u_n(x)$, we choose a sequence

$\epsilon_n > 0,\ n = 1, 2, ...$ with $\sum_{i=1}^{\infty} \epsilon_i < \infty$ such that for $x \in K, c > 0$, $x \pm \epsilon_n + c \in K \subset \mathbb{R}$ and $u(x \pm \epsilon_n + c) \nrightarrow u(x)$ as ϵ_n tends to 0.

$$u_{2m+1}(x) = \begin{cases} u(x - \epsilon_n + c) & \text{if } x - \epsilon_n + c \in K \\ u(x) & \text{if } x - \epsilon_n + c \notin K, \end{cases} \tag{8}$$

$$u_{2m}(x) = \begin{cases} u(x + \epsilon_n + c) & \text{if } x + \epsilon_n + c \in K \\ u(x) & \text{if } x + \epsilon_n + c \notin K, \end{cases} \tag{9}$$

where $m = 1, 2, 3, ..., n$.

3.1.2. Construction of the sequence $u_n(x)$ from regularly perturbed function $u(x)$

In this case, we choose a sequence $\epsilon_n > 0$ with $\sum_{i=1}^{\infty} \epsilon_i < \infty$ such that for $x \in K$, $x \pm \epsilon_n \in K$, define the sequences $u_n(x)$.

$$u_{2m+1}(x) = \begin{cases} u(x - \epsilon_n) & \text{if } x - \epsilon_n \in K \\ u(x) & \text{if } x - \epsilon_n \notin K, \end{cases} \tag{10}$$

$$u_{2m}(x) = \begin{cases} u(x + \epsilon_n) & \text{if } x + \epsilon_n \in K \\ u(x) & \text{if } x + \epsilon_n \notin K, \end{cases} \tag{11}$$

where $m = 1, 2, 3...n$. For example, $u(x) = \sin(x)$.

$$u_{2m+1}(x) = \begin{cases} \sin(x - \frac{1}{2^n}) & \text{if } x - \frac{1}{2^n} \in K \\ \sin(x) & \text{if } x - \frac{1}{2^n} \notin K, \end{cases} \tag{12}$$

$$u_{2m}(x) = \begin{cases} \sin(x + \frac{1}{2^n}) & \text{if } x + \frac{1}{2^n} \in K \\ \sin(x) & \text{if } x + \frac{1}{2^n} \notin K. \end{cases} \tag{13}$$

We applied average concept $\overline{u}_n(x)$ of $u_n(x)$ in the Young measure concept. We remark that if $\overline{u}(x) = u(x)$ in a bounded domain (space), Q-measure is equal to Young measure almost everywhere. The Q-measure is possible to find in the unbounded domain provided Borel measurable function lies in $L^2(K), K \subseteq \mathbb{R}$.

To construct the sequences of function in equation (10) and (11) is useful for various fields especially calculus of variation[24], signals analysis problems and atmospheric sciences.

3.4. Q-measure associated to a bounded piecewise differentiable sequence of function. Throughout this work, we consider a Borel measurable function $u \in L^2(K)$, u is a piecewise differentiable and measurable function $u : \Omega \to K$, where $\Omega \subseteq \Re^d$ and $K \subseteq \Re^l$ be a smallest connected

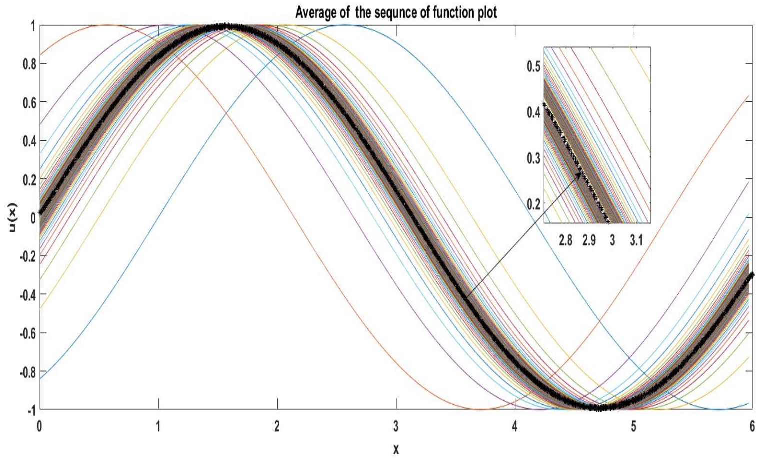

Figure 4: Mid black color graph denote the average $\overline{u}$ of the sequence of function corresponding to $u(x) = \sin(x), x \in [0, 6]$ and $\epsilon_n = \frac{1}{2^n}, n = 1, 2, \cdots, 1000$.

compact set.
Let

$$M = \mu\left(\{y | y \in K, u : \Omega \to K\}\right), M > 0 \tag{14}$$

and $d\mu(x) = \frac{1}{M}dx$ (given in [30]), where dx is the normalized Lebesgue measure.

The mapping $\beta \to \beta'$ defined by β'(x)= $\beta(x, .)$ is an isometrically isomorphism between $Car(\Omega, K; \mathbb{R})$ and $L^1(\Omega, K)$ (cf.[37, 21]), where β is the Caratheodory function[3]. Let $i : U \to L^1(\Omega, C(K)^*)$, where U be a collection of sequence of Borel measurable function and $\overline{u}(x)$ from equation (1) (average of the function $u(x)$) then,

$$\langle i(\overline{u}), \beta \rangle = \int \beta(x, \overline{u}(x)) d\mu(x) = \frac{1}{M} \int \beta(x, \overline{u}(x)) dx.$$

The weak* closure of $i(\overline{u})$ in $L^1(\Omega, C(\Omega)^*)$ is denoted by $\mathcal{Y}(\Omega, K)$
i.e., $\mathcal{Y}(\Omega, K) = \{\eta \in L^1(\Omega, C(K)^*); \exists (\overline{u}_k) \subset U$ for u_k such that $i(u_k) \xrightarrow{weak*} \eta\}$

$$|\langle i(u), \beta \rangle| = \left| \int_K \beta(x, \overline{u}(x)) dx \right| \leq \int_K \sup_{s \in K} |\beta(x, s) ds| = \|\beta\|_{L^1(K, C(K))}.$$

The general definition of Q- measure is given in section 5. For practical convenience, we are taking the following definition.

[3]if $f \in Car(K, S)$ and $K \in \Re^n$ be a compact set, then $f(., k)$ is measurable function for all $k \in S$ and $f(x, .)$ is continuous function for almost all x.

Definition 3.8 Let $u : \Omega \to K$ be a bounded piecewise differentiable function, where $\Omega \subseteq \Re^d$ and $K \subseteq \Re^l$ are connected compact sets. A family of probability measures $\{\nu_x\}_{x \in K}$ is said to be the Q- measure corresponding to u, if for every continuous function $\beta : \Omega \to \Re$,

$$\int_K \beta(\lambda) d\nu_x(\lambda) = \int_\Omega \beta(\overline{u}(x)) d\mu(x), \tag{15}$$

provided $\overline{u}$ is exist and $\beta(x_1, \ldots, i, \ldots, x_n)$ is integrable.

♣

Q- measure is a weak* limit of the average of sequence of Borel measurable functions. The limit is the weak* measurable map $\nu : K \to \mathbf{P}(\Re^N)$, where $K \subseteq \mathcal{B}(\Re^N)$, $\mathcal{B}(\Re^N)$ denotes the Borel sigma algebra in $\Re^N$ and $\mathbf{P}$ stands for the probability measure.

Note 3.9 [37] For almost all $x \in \Omega$, ν_x is absolutely continuous with respect to the Lebesgue measure on K. According to Radon–Nikodym's Theorem, there is a density $d_{\nu_x} \in L^1(K)$ such that $\nu_x(ds) = d_{\nu_x}(s)ds$.

Definition 3.10 The atomic measure is defined by $\nu_x = \nu$ for any $x \in \Omega$ i.e, the family $(\nu_x)_{x \in \Omega}$ is a linear combination of Dirac delta $\delta_{u(x)}$, where $u(x)$ is a measurable function in $(K, \mathcal{B}(\Re^N), \nu)$.

♣

The atomic measure is an important concept which is useful in barycenter concept and applied to various partial differential equations.

Next, we discuss the similar results as proved for Young measure in [30].

Proposition 3.11 *Let $u(x) : [a, b] \to [c, d]$ be a strictly monotonic and differentiable function where $a, b, c, d \in \mathbb{R}$. Q- measure is absolutely continuous with respect to the Lebesgue measure on K. Then density of Q- measure associated with the function $u(x)$ is $|\overline{u}'(x)^{-1}|$.*

For details, see appendix B.

Proposition 3.12 *Let $u(x) : [a, b] \to [c, d]$, $u(x) = \sum_{j=1}^{n} u_i(x)\chi_{A_j}$ be a continuous function and corresponding average function $\overline{u}(x)$ be a strictly monotonic differentiable function on K ,where $K = [c, d] \subseteq \mathbb{R}$ and A_j be a disjoint partition of $[a, b]$. Then Q-measure associated to the function $u(x)$, $\overline{u}(x)$ is absolutely continuous corresponding to the Lebesgue measure on K. The density $g(x)$ is equal to $\sum_{i=1}^{n} |\overline{u}_i'(x)^{-1}|$.*

The proof is given in appendix B.

Now, we consider the sequence of function u_i as a piecewise constant function.

THEOREM 3.13 *(v. [20])*

i. *Let $(K, \mathcal{F}, \mu)$ be a measurable space and let $A_1, A_2, \cdots, A_n$ be disjoint elements of $\mathcal{F}$, $\cup_{i=1}^{n} A_i = A$ and let $a_1, \cdots a_n$ are real numbers. Let $u(x)$ be a simple measurable function i.e.,*

$$u(x) = \sum_{i=1}^{n} a_i \chi_{A_i},$$

then the Q- measure, $\nu_x = \frac{1}{M} \sum_{i=1}^{n} m_i \delta_{a_i}$, and $\sum_{i=1}^{n} m_i = 1$ where M, m_i is the Lebesgue measure of the interval I and I_i respectively.

ii. *Let $(K, \mathcal{F}, \mu)$ be a measurable space and $A_1, A_2, \cdots, A_n$ be disjoint elements of $\mathcal{F}$ and let $a_1, \cdots a_n$. are real numbers. Let $u(x)$ be a function of the form*

$$u(x) = \sum_{i=1}^{n} (a_i x + b_i) \chi_{A_i},$$

then, the Q- measure $\nu = \dfrac{1}{M} \displaystyle\sum_{i=1}^{n} \left| \frac{1}{a_i} \right| \chi_{w_i(I_i)}$, where $w_i(x) = (a_i x + b_i)$ is the density function.

iii. *Let $(K, \mathcal{F}, \mu)$ be a measurable space with $A_1, A_2, \cdots, A_n$ be disjoint intervals in $\mathcal{F}$. Moreover $u_1, \cdots, u_n$ be a functions in A_i. If u be the function of form $u = \displaystyle\sum_{i=1}^{n} u_i \chi_{A_i}$, then the Q measure*

$$\nu_x = \sum_{i=1}^{n} \frac{1}{m_i} |J_{(\overline{u}_i)^{-1}}|, \text{ where } \frac{1}{m_i} = 1,$$

and u is the Jacobin matrix.

PROOF We construct a sequence of function $\overline{u}(x)$ from $u(x)$ by equation (10) and (11)

i: By construction

$$\overline{u}(x) = \lim_{m \to \infty} \lim_{k \to \infty} \sum_{l=1}^{2m} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{a_i}{2m} \chi_{k_j \cap A_l} = \lim_{k \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{k} a_i \chi_{k_j \cap A_i},$$

where k_j be a disjoint interval,

$$\begin{aligned}\int_K \beta(\lambda)d\nu_x(\lambda) &= \int_{[b,c]} \beta(\overline{u}(x))d\mu(x), \quad (16)\\ &= \lim_{k\to\infty}\frac{1}{M}\sum_{i=1}^{n}\sum_{j=1}^{k}\int \beta(a_i)\chi_{k_j\cap A_i}dx,\\ &= \lim_{k\to\infty}\frac{1}{M}\sum_{j=1}^{k}\frac{1}{m_i}\int_K \beta(y)d\delta_{a_i}.\\ \text{While comparing both side, } \nu_x &= \frac{1}{M}\sum_{i=1}^{n} m_i\delta_{a_i} \text{ and } \sum_{i=1}^{n} m_i = 1.\end{aligned}$$

Where M, m_i be the Lebesgue measure on the interval I and I_i, respectively.

ii: As constructed earlier,

$$\overline{u}(x) = \lim_{m\to\infty}\lim_{k\to\infty}\sum_{l=1}^{2m}\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{a_l(x-\epsilon_l)+b-(a_l(x+\epsilon_l)+b)}{2m}\chi_{k_{jl}\cap A_l},$$

$k_i, i = 1, 2, ..., k$ be a disjoint infinitesimal partition of A_l

$$\overline{u}(x) = \lim_{m\to\infty}\lim_{k\to\infty}\sum_{l=1}^{2m}\sum_{i=1}^{n}\sum_{j=1}^{k}(a_l x + b)\chi_{k_{jl}\cap A_l}.$$

$$\begin{aligned}\int_K \beta(\lambda)d\nu_x(\lambda) &= \int_{[b,c]} \beta(\overline{u}(x))d\mu(x),\\ &= \frac{1}{M}\sum_{l=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{k}\int \beta(a_i x + b)\chi_{k_{jl}\cap A_l}dx,\end{aligned}$$

$\nu = \dfrac{1}{M}\displaystyle\sum_{i=1}^{n}\left|\frac{1}{a_i}\right|\chi_{w_i(I_i)}$, where $w_i(x) = (a_i x + b_i)$ is the density function.

iii. From induction principle, from previous result we consider ν_x hold for i=1. For $i = (n+1)$.

$$\nu_x = \sum_{i=1}^{n+1}\frac{1}{m_i}|J_{(\overline{u}_i)^{-1}}|,$$

$$\begin{aligned} \int_K \beta(\lambda) d\nu_x(\lambda) &= \int_K \beta(\overline{u}(x) d\mu(x) = \int_K \frac{1}{M} \sum_{i=1}^{n+1} \beta(y) |J_{(\overline{u}_i)^{-1}}|, \\ d\nu_x(\lambda) &= \frac{1}{M} |\sum_{i=1}^{n} J_{(\overline{u}_i)^{-1}}| dy, \\ \nu_x &= \frac{1}{M} \sum_{i=1}^{n} \frac{1}{m_i} |J_{(\overline{u}_i)^{-1}}|, \nu_x = \sum_{i=1}^{n} \frac{1}{m_i} |J_{(\overline{u}_i)^{-1}}|. \end{aligned}$$

Theorem 3.14 *Let u be a continuously differentiable function such that $\overline{u}(x)^{-1}$ exist. Then, Q- measure associated with $u(x)$ is a measure that is absolutely continuous with respect to the Lebesgue measure on K. It's density equals to $\frac{1}{M}|J_{(\overline{u})^{-1}}|$.*

Proof We construct a sequence of function $\overline{u}(x)$ from $u(x)$ by equation (10) and (11), and get

$$\int_K \beta(\lambda) d\nu_x(\lambda) = \int_K \beta(\overline{u}(x) d\mu(x) = \int_K \frac{1}{M} \beta(y) |J_{(\overline{u})^{-1}}|,$$

By comparison, we get

$$\begin{aligned} d\nu_x(K) &= \frac{1}{M} |J_{(\overline{u})^{-1}}| dy, \\ \nu_x &= \frac{1}{M} |J_{(\overline{u})^{-1}}|. \end{aligned}$$

Remark 3.15 If the domain is compact and the function $u(x)$ is continuous and measurable with respect to Borel sigma algebra, then Q- measure is equal to Young measure for the regularly perturb function $u(x)$.

The proof can be followed using the similar arguments as given in [30, 31].

4. Concentration and Oscillation concept in Q- measure. The concentration and rapid oscillation property of function influenced the convergence of a function. So it play an important role in variational problem.

4.1. Concentration. Theorem 1 and theorem 2 in [9] tell us that any conceivable oscillation or concentration behaviour of function can be realized by a sequence of weak solutions of partial differential equation. We have demonstrated via an example that if $\overline{u}_n$ in (1) exists for large $n \in \mathbb{N}$ then, one can define Q- measure for the concentration of sequence in the unbounded domain. In our construction, we consider $\overline{u}$ corresponding to given Borel measurable function u. It is the average of functions in a different sense as used in equation (1). Such understood average is useful in a signal processing for reducing low frequency noise. According to this average sense, the intensity of concentration nature of function is reduced whenever the sequence has less oscillation, concentration, and no more singularities. In this case $\beta(u_n)$ does not converge strongly to $\beta(u)$ but we obtained better convergence compare to the weak convergences if the sequence of function $\{u_n\}$ weakly converges to u.

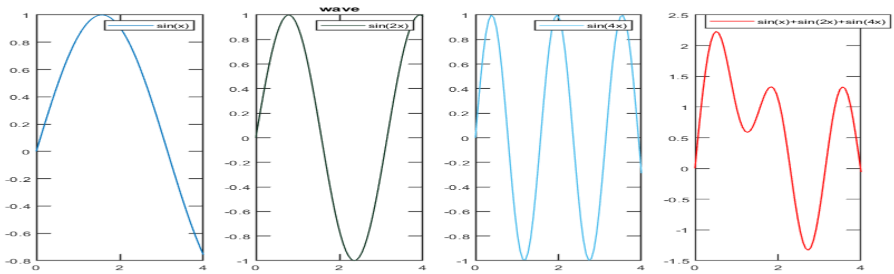

Figure 5: Wave formation via combination of functions

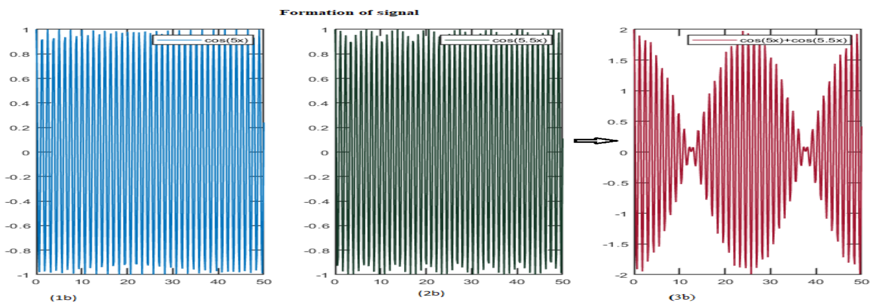

Figure 6: Signal formation via combination of functions

Corollary 4.1 *Let $u(x) = a, \quad \forall x \in [b, c]$, where a, b and c are constants then corresponding Q-measure is δ_a.*

Proof Let $k_n = [b_n, c_n]$, where $b_1 = b, c_1 = c, n = 1, 2, ..., k$ be a disjoint infinitesimal partition of $[b,c]$, then

$$\overline{u}(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{k}\frac{a}{n}\chi_{k_j} = a\sum_{j=1}^{k}\chi_{k_j},$$

$$\int_K \beta(\lambda)d\nu_x(\lambda) = \int_{[b,c]}\beta(\overline{u}(x))d\mu(x) = \sum_{j=1}^{k}\int \beta(a)\chi_{k_j}d\mu(x),$$
$$= \int_K \beta(y)d\delta_a,$$

Whereas, $d\nu = \delta_a dy \Rightarrow \nu = \delta_a$. ∎

4.2. Oscillation Let u_j be a real-valued function of a real variable. The oscillations of u_j on an interval I in its domain is the contrast between the supremum and infimum of u_j. The sequence of rapidly oscillating functions provide an example of weakly but not strongly converging sequences.

In the case of variational problem, we are unable to find a solution because of it's rapidly oscillatory property. However, Young measure and Q-measure can capture such oscillations. It is well known that, every periodic signal a sum of a sinusoids of various frequencies. From the Figure 6, in the accompanying hypothesis we can find corresponding Q-measure. Herein, first three signal in Figure 5 are the decomposition of red shading signals. This decomposition is useful in signal analysis problem.

We can utilize following hypothesis in AM signal decoding.

THEOREM 4.2 *Let $u_n \in K$ be a strictly bounded and differentiable function given by $u_n(x) = du(c_n x) + b$, for all $x \in A = [a,b]$ where d, b are constants greater than zero and c_n is an increasing sequence in $\Re$. Let $u_n(x) = u(c_n x)$ and $u(x) = u(c_n x)$, for all n. Then, the density of u_n is $\frac{1}{dM}(u^{-1}(\frac{y-b}{d}))'$.*

PROOF Let u_n be a given function. Choose $[a_i', b_i'] = A_i, i = 1, 2, ..., n$ be a disjoint partition of $[a,b]$,

$$\begin{aligned}
\lim_{n\to\infty}\overline{u}_n(x) = \overline{u}(x) &= \lim_{n\to\infty}\lim_{k\to\infty}\sum_{i=1}^{n}\sum_{j=i}^{k}\frac{u_i\chi_{A_j}(x)}{n},\\
&= \lim_{n\to\infty}\lim_{k\to\infty}\sum_{i=1}^{n}\sum_{j=i}^{k}\frac{du(c_n x)+b}{n}\chi_{A_j}(x),\\
&= \lim_{n\to\infty}\lim_{k\to\infty}\sum_{i=1}^{n}\sum_{j=i}^{k}\frac{du(x)+b}{n}\chi_{A_j}(x),\\
&= \lim_{k\to\infty}\sum_{j=i}^{k}(du(x)+b)\chi_{A_j}(x).
\end{aligned}$$

Then, from equation (1) we have

$$\begin{aligned}\int_K \beta(\lambda) d\nu_x(\lambda) &= \int_a^b \beta(\overline{u}_n(x)) d\mu(x),\\ &= \lim_{k\to\infty} \sum_{j=i}^{k} \int_{k_j} \beta(du(x)+b) d\mu(x),\\ &= \lim_{k\to\infty} \frac{1}{Md} \int_K \beta(y)(u^{-1}(\frac{y-b}{d}))' dx,\end{aligned}$$

ν_x is absolutely continuous with respect to the Lebesgue measure on $[a,b]$, by

Radon-Nikodym theorem there is a density $f(x) = \frac{1}{Md}(u^{-1}(\frac{y-b}{d}))'$. ■

Corollary 4.3 *Let $u_n(x)$ be a sequence of bounded and differentiable functions in the form of $u_n(x) = du(c_n x) + b$ on $(A, \mathcal{B}(A), \mu)$ where $A = [a,b]$ and satisfy above mentioned theorem. Then the Q- measure ν_n associated to u_n converges to ν and ν is a constant for any Borel function u on A.*

Corollary 4.4 *Let $u : [0,1] \to [0,1]$ be a periodic function with period T and the sequence*

$$u_n(x) = u(nx), n \in \mathbb{N},$$

clearly $u_n(x)$ will be a sequence of periodic function with period $\frac{T}{n}$. Then corresponding Q- measure is $\frac{1}{M}(u^{-1})'$.

Proof Let us choose $d = 1, b = 0$ in theorem B.1(see appendix). The proof can be followed in a similar way as done in [39]. ■

Let us recall some remarks from the theory of Young measure.

Remark 4.5 Let $(u_j) \in L^1(K; \Re^N)$ with ν is Q measure generated by corresponding (u_j);

- Given (u_j) is equintegrable, then ν is not homogeneous atomic measure or it has no concentration part (ν is not a combination of δ_x).
- Given that (u_j) converges locally in measure to $u \in L^1(K; \Re^N)$ if and only if $\nu = \delta_{u(x)}$.
- Let (u_j) with ν is a Q- measure generated by corresponding (u_j) then the (u_j)converges globally in measure to $u \in L^1(K; \Re^N)$ iff $\nu = \sum_i a_i \delta_{b_i}$, where $\sum_i a_i = 1$ and b_i is the value of u.

- Strong convergence of sequence is equivalent to the absence of oscillation, concentration and blow up.
- By Vittali convergence theorem in[36], limit of nonlinear expression of non-oscillating (point wise or convergence in measure), non-concentrating (equi-integrable) and non-blowup (converging in norm) sequence converge to corresponding nonlinear expression of limit of sequence.

5. Generalized Q-measure The general limiting behavior of a weakly convergent sequences with oscillation and concentration effects has been denoted by the notion of a generalized Young measure ([1, 8, 9, 14]). Similarly, we need a notation of general limiting behavior of a weakly convergent sequence having oscillation and concentration effects that can be denoted by the notion of a generalized Q- measure in the space of $\mathcal{M}(\Re^d)$

Let us consider $u(x) \in BV(K, R^m)$, then corresponding average $\overline{u}(x) \in BV(K; R^m)$. We write the K as a disjoint union;

$$K = D_u \cup \mathcal{J}_u \cup C_u \cup N_u,$$

where D_u denotes the set of points at which u is approximately differentiable, J_u denotes the set of jump points of u, C_u denotes the set of points where u is approximately continuous but not approximately differentiable, and N_u is the collection of normal of u.

By Radon Nikodym Decomposition theorem, finite Q- measure can be decomposed to

$$D\nu = D^a\nu + D^s\nu,$$

$D^a\nu$ is a absolute continuous part and $D^s u$ is a singular part of derivative

$$D^a u(x) = \int_{-\infty}^{x} p(y)dy,$$

$p(y)$ is density function.

$$D^s\nu = D^j\nu + D^c\nu,$$

$D^j\nu$ is the jump part(blow up) and $D^c\nu$ is the cantor part or pure part.

$$D^j\nu = \sum_{x \in J_u} (u^+(x) - u^-(x))\delta_x,$$

where j_u be the jump set[4] , it is an $N-1$ rectifiable hyper surface(H). Also, u^+ and u^- are positive and negative part of the function u.

$$Du(x) = D^a u(x) + \sum_{x \in J_u} (u^+(x) - u^-(x))\delta_x + C(u),$$

[4]jump part also denotes pure part of measure

$C(u)$ is the cantor part or discrete part such that

$$C(u(x)) = \sum_{x_i<x} p(x_i),\ p(x_i) \geq 0,\ \sum_i p(x_i) \leq 1,$$

$\tilde{\mathscr{U}}$ is the extension of the integral of the function $u(x)$.

We extends the classical theory to the framework of functions of bounded variation. A generalized gradient Q- measure ν is defined as a the triplet of measures $\nu = (\nu^{ab}, \nu^b, \nu^c)$ where ν^b is a positive bounded Radon measure on K. Also, ν^{ab} is a absolute continuous part, this measure part corresponds to oscillation part of the function and ν^c cantor part of ν
Let $\nu \in \mathbb{R}^N$ a family of probability measures on $\Re^N$ and ν^∞ is a family of probability measures on S^{n-1}, the unit sphere of $\mathbb{R}^N$. The integral notation representation of generalized Q- measure is given as:

$$\tilde{\mathscr{U}}[u] = \int_K \int_{\Re^n} u(x,y)d\nu(y)dx + \int_{\overline{K}} \int_{\mathbb{S}^{n-1}} u^\infty(x,y)d\nu^\infty(y)dx,$$

where u^∞ is the recession function according to the decomposition flip Rindler(for details, see [35]).

6. Important examples of generalized Q- measure. Given a sequence of functions u_n which converges pointwise to some limit function u, it is not always true that

$$\lim_{n\to\infty} \int u_n dx = \int \lim_{n\to\infty} u_n dx.$$

The Monotone Convergence Theorem (MCT), Dominated Convergence Theorem (DCT), and Fatou's Lemma are the three noteworthy results were coined in the theory of Lebesgue integration tells about the interchangeability of integral and limit. The DCT and MCT tell us that with the certain restriction on u_n and u, integral can be interchanged. As an application of the Q-measure, we can define an another weaker type of Monotone convergence theorem and Lebesgue dominated convergent theorem.

$$\lim_{n\to\infty} \int u_n d\nu_n = \int \lim_{n\to\infty} u_n d\nu.$$

Consider the following examples which are the explanation behind the origin of Monotone convergent theorem and Dominated convergent theorem with respect to Lebesgue measure. In this way, we consider the following examples in the sense of Q- measure associated to a bounded function.

EXAMPLE 6.1 Let $u_n = (-1)^n \chi_{[n,n+1]}, n \in \mathbb{N}$ i.e. the characteristic functions on an interval of unit length which escapes to infinity. This sequence does not convergent with any $L^p(\Re)$ norm, $p > 1$. This type of convergence can not be treated with Young measure and Lebesgue measure. Also, u_n is the combination of oscillation and concentration. Refer figure 2. From equation (1), using the regular perturbation theory,

$$\overline{u}_n(x) = \lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=i}^{k} \frac{u_i(x)\chi_{A_j\cap B_i}(x)}{n}.$$

Using the Q- measure concept in a finite domain, we get

$$\int_K \beta(\lambda)d\nu_x(\lambda) = \int_K \beta(\overline{u}_n(x))\ dx.$$

$$\int_K \beta(\lambda)d\nu_x(\lambda) = \lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=i}^{k} \int_{A_j\cap B_i} \beta(\frac{-1^n}{n})\ dx,$$

from above equation

$$\int_K \beta(\lambda)d\nu_x(\lambda) = \frac{n}{2}\int_0^n \beta(y)d\delta_{\frac{1}{n}} + \frac{n}{2}\int_0^n \beta(y)d\delta_{\frac{-1}{n}},$$

$$\nu_n = \frac{n}{2}\delta_{\frac{1}{n}} + \frac{n}{2}\delta_{\frac{-1}{n}},$$

$$\overline{u}(x) = \lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=i}^{k} \frac{u_i\chi_{A_j}}{n},$$

$$\overline{u}(x) = \lim_{n\to\infty}\lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=i}^{k} \frac{(-1)^i\chi_{A_j\cap B_i}}{n},\ \text{where } B_i = [i, i+1].$$

Here we can not use above mentioned results because of the unbounded nature of interval limit $n \to \infty$. Here we use direct definition of weak convergence and get δ_0(refer about the convergence in [43]) which is the Q- measure corresponding to u. It is the generalized Q- measure which has concentration part δ_0 and oscillation part $\delta_{\frac{-1}{n}}$ and $\delta_{\frac{1}{n}}$. Now, we consider the following integral, $\lim_{n\to\infty}\int u_n d\nu_n$ and $\int \lim_{n\to\infty} u_n d\nu$

$$\begin{aligned}\lim_{n\to\infty}\int u_n d\nu_n &= \lim_{n\to\infty}\int u_n(x)d(\frac{n}{2}\delta_{\frac{1}{n}} + \frac{n}{2}\delta_{\frac{-1}{n}}(x)) \\ &= \lim_{n\to\infty}\frac{n}{2}u_n(\frac{1}{n}) + \frac{n}{2}u_n(\frac{-1}{n}) = 0.\end{aligned} \tag{17}$$

$$\int \lim_{n\to\infty} u_n d\nu = \int \lim_{n\to\infty} u_n d\delta_0(x) = 0.$$

Example 6.2 Let $u_n = \frac{1}{n}\chi_{[0,n]}$ this sequence converge strongly in $L^p, p > 1$ this is a sequence escape width to infinity.

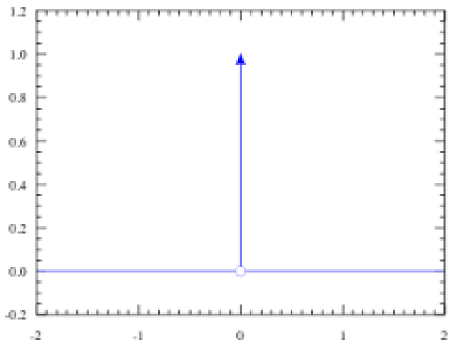

Figure 7: δ_0

Let us choose $A_j = [\frac{j}{2}, \frac{j+1}{2}), j = 0, 1..., ..2n$ be a partition of $[0, n]$. From equation (1)

$$\overline{u}_n(x) = \sum_{i=1}^{n}\sum_{j=1}^{2n} \frac{u_i(x)}{n}\chi_{A_j}(x) \approx \sum_{i=1}^{n}\sum_{j=1}^{2n} \frac{1}{n}\left(ln(n) + \frac{1}{2} + \frac{1}{2n}\right)\chi_{A_j}(x),$$

$$\text{since} \sum_{i=1}^{n} \frac{1}{n} \approx ln(n) + \frac{1}{2} + \frac{1}{2n}.$$

corresponding Q- measure of $u_n(x)$ is $n\delta_{\frac{ln(n)}{n}+\frac{1}{2n}+\frac{1}{2n^2}}$.

$$\overline{u}(x) = \lim_{n\to\infty} \overline{u}_n(x) = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{2n} \frac{u_i(x)}{n}\chi_{A_j}(x),$$

$$\overline{u}(x) = \begin{cases} \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} & if x \in [0,1]\cap A_1 \\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{i} & if x \in [0,1]\cap A_2 \\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=2}^{n}\frac{1}{i} & if x \in [0,2]\cap A_3 \\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=2}^{n}\frac{1}{i} & if x \in [0,2]\cap_{A_4} \\ \vdots & \\ \lim_{n\to\infty} \frac{1}{n^2} & if x \in [0,n]\cap A_{2n-1} \\ \lim_{n\to\infty} \frac{1}{n^2} & if x \in [0,n]\cap A_{2n}. \end{cases}$$

$\overline{u}(x) = 0,\ \forall x \in [0, n]$.
From [27], we observe that harmonic partial sum can not be an integer, and

$$\int_K \beta(\lambda)d\nu_x(\lambda) = \int_K \beta(0)dy = \int_K \beta(y)d(\delta_0),$$

then

$$d\nu = d\delta_0, \text{ or } \nu = \delta_0.$$

Example 6.3 Let $u_n = \frac{1}{\sqrt{n}}\chi_{[\frac{-1}{n},\frac{1}{n}]}$. Choose $A_n = [\frac{-1}{n}, \frac{-1}{n+1}) \cup (\frac{1}{n+1}, \frac{1}{n}]$. From equation (1) we have

$$\overline{u}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} u_i(x)$$

$$= \begin{cases} \lim_{n\to\infty} \frac{1}{n}(1) & \text{if } x \in [-1, \frac{-1}{2}] \cup [\frac{1}{2}, 1] \\ \lim_{n\to\infty} \frac{1}{n}(1 + \frac{1}{\sqrt{2}}) & \text{if } x \in [\frac{-1}{2}, \frac{-1}{3}] \cup [\frac{1}{3}, \frac{1}{2}] \\ \lim_{n\to\infty} \frac{1}{n}(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}) & \text{if } x \in [\frac{-1}{3}, \frac{-1}{4}] \cup [\frac{1}{4}, \frac{1}{3}] \\ \lim_{n\to\infty} \frac{1}{n}(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}) & \text{if } x \in [\frac{-1}{4}, \frac{-1}{5}] \cup [\frac{1}{5}, \frac{-1}{4}] \\ \vdots & \\ \lim_{n\to\infty} \frac{1}{n}(\sum_{i=1}^{n} \frac{1}{\sqrt{i}}) = 0 & \text{if } x \in [\frac{-1}{n}, \frac{-1}{n+1}] \cup [\frac{1}{n+1}, \frac{1}{n}] \\ \vdots & \end{cases}$$

Q- measure corresponding to u_n is $\displaystyle\sum_{i=1}^{n} \frac{2}{n}\delta_{\frac{a_i}{n}}$ and corresponding to u is δ_0. We evaluate the following integral and get,

$$\int \lim_{n\to\infty} u_n d\nu = \int \lim_{n\to\infty} u_n d\delta_0(x) = 0$$

$$\lim_{n\to\infty} \int u_n d\nu_n = \int \lim_{n\to\infty} u_n d\nu \int u d\nu = 0.$$

Example 6.4 (Typewriter sequence), Let $u_n = \chi_{[\frac{n-2^k}{2^k}, \frac{n-2^k-1}{2^k}]}$, $k \geq 0$ and $2^k \leq n < 2^{k+1}$. This is a sequence of indicator functions of intervals of decreasing length, walking over the unit interval $[0, 1]$ again and again. The typewriter sequence is an suitable example of a sequence which converges to zero in measure but does not converge to zero a.e. on comparing, we get $\nu = \delta_0$.

Example 6.5 Let $u_0(x)$, $u_1(x)$, $u_2(x)$ be given as on Figure 8.

$$u_n(x) = \begin{cases} a & x \in [\ \frac{2k}{n}, \frac{2k+1}{n}],\ k \in \mathbb{Z} \\ b & \text{else where} \end{cases}$$

The straight forward calculation leads to, $\nu = \delta_0$ which is the corresponding Q- measure (v. Figure 8).

Weak stability of Q- measure is discussed in [23].

6.1. Application of $\overline{u}_n$ in atmospheric science Monthly temperature: The atmospheric boundary layer (ABL)[cf. [48]] is the lowest portion

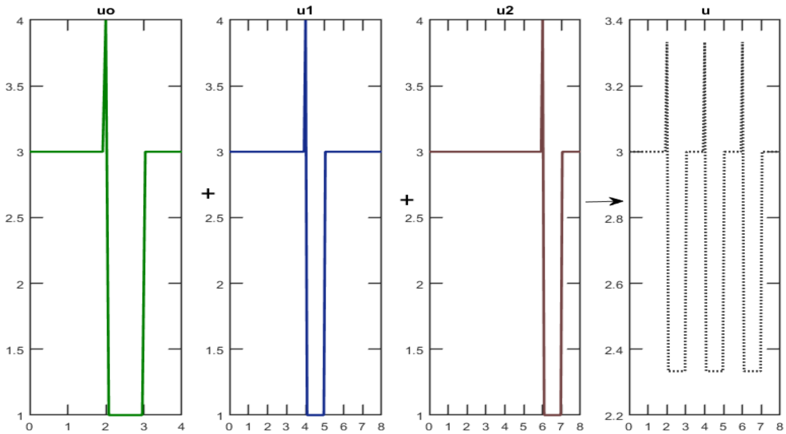

Figure 8: The pictures present functions $u_0(x)$, $u_1(x)$, $u_2(x)$ and $\overline{u}(x)$, where $a = 1, b = 3, k = 4, n = 3$

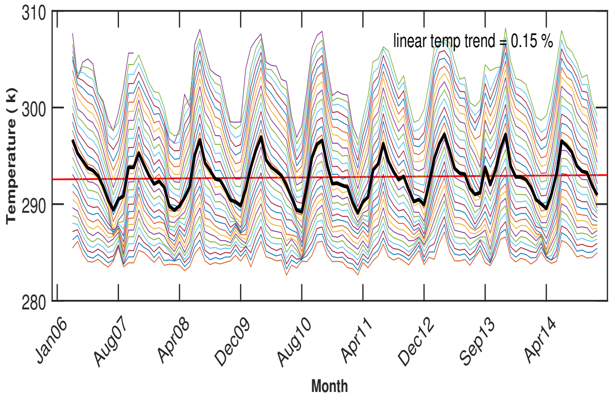

Figure 9: The time series of monthly mean temperature from an attitude $100m\breve{}3000m$ during Apr 2006 – Dec 2014 and linear temperature trend for average temperature (solid black line) over Chennai is observed from radiosonde.

of the earth's atmosphere and it plays an important role in the exchange of the pollutants, heat flux, moisture from surface to free atmosphere through the turbulence. Recent studies shows that the warming trends observed in the lower portion of the atmosphere are due to surface pollutants and greenhouse gasses. For typical example, the linear temperature trend of ABL is calculated over Chennai from 2006 to 2014 is shown in Figure 9. The ABL considered as from the surface up to 3 km and the monthly mean temperature from an attitude of 100 m up to 3000 m is averaged using $\overline{u}_n$. The linear trend has been depicted in Figure 9. It has been observed that over a period of nineyear, 0.15% of increasing trend (cf.[29]) in temperature is obtained.

6.2. Application of $\overline{u}$ based on u.

6.2.1. Denoising of signal: The construction of $\overline{u}$ is useful in Empirical Mode Decomposition (EMD). The following pictures 10-12 are an example of the wave, in which noises can be removed using $\overline{u}_n$. The noisy residue signal is composed of components with several frequencies. From this composed of components, we consider a sequence of functions u_n and define $\overline{u}_n$. Then, we use the average function of $\overline{u}_n$ to reduce high and low level frequency wises. More detailed study can be found in [22].

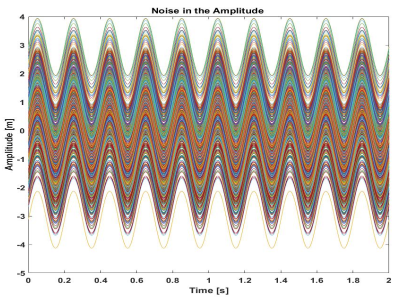

Figure 10: Decomposed noisy signal $y = sin(10\pi x)$, number of points $N = 1000$

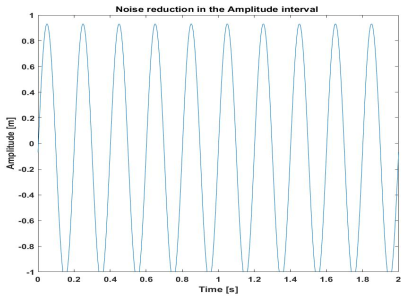

Figure 11: Noise reduction using $\overline{u}$

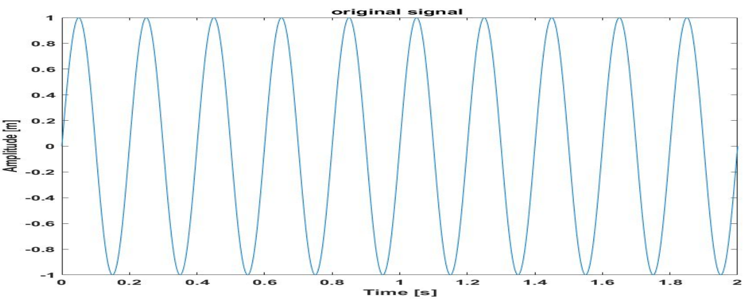

Figure 12: Original signal

7. Conclusion. A notion of average for underlying sequences to define the Q-measure is given and also applied in and atmospheric sciences also discussed. Above discussed results related to the Q- measure is helpful to computing the Q- measure easily for particular types of functions.

Supplementary Materials: All data used in our experiments have been produced with MATLAB random number generators and no external datasets have been used. The datasets generated and analysed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest: The authors declare that they have no confict of interest.

Appendices

A. Basic concepts

Let μ be a Borel measure [6] on the $\mathcal{R}^N$ be the sigma field generated by rectangles then

$$\mu[x : a_i \leq x_i \leq b_i, i - 1, 2, \cdots n] = \prod_{i=1}^{n}(b_i - a_i).$$

This measure denotes ordinary volume eg. it is a length $(k = 1)$, area $(k = 2)$, volume $(k = 3)$, or hyper volume $(k \geq 4)$.

We recall the following definitions as discussed in [4, 43, 14].

Definition A.1 A probability measure on the sample space K is a function, denoted $\mathcal{P}$, from subsets of K to the real numbers $\mathbb{R}$, such that the following hold

i. $\mathcal{P}(A)$ lies between 0 and 1 for every $A \in \mathcal{F}$.
ii. $\mathcal{P}(\phi) = 0$ & $\mathcal{P}(K) = 1$.
iii. $\mathcal{P}$ Satisfies countable additivity.

♣

Definition A.2 (**Dirac measure** δ_x) Let $(K, \mathcal{F}, \mu)$ be a space

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in K \backslash A \end{cases}$$

In particular, $\delta_x(g) = g(x)$ for $g \in C_0(K)$. We recall the following definitions as discussed in [4, 43, 14].

♣

Definition A.3 Let $u_n(x)$ be a sequence of measurable functions from X to $\Re$, on a measure space $(X, \mathcal{F}, \nu)$. The sequence $u_n(x)$is said to converge in measure to $u(x)$ locally if for every $\eta > 0$, for every $E \in \mathcal{F}$ with $\nu(\mathcal{F}) < \infty$

$$\lim_{n\to\infty} \nu(\{x \in E : |u_n(x) - u(x)| > \eta\}) = 0. \tag{18}$$

♣

DEFINITION A.4 Let $u_n(x)$ be a sequence of measurable functions from X to $\Re$, on a measure space $(X, \mathcal{F}, \nu)$. The sequence $u_n(x)$ is said to converge in measure to $u(x)$ globally if for every $\eta > 0$,

$$\lim_{n\to\infty} \nu(\{x \in X : |u_n(x) - u(x)| > \eta\}) = 0. \tag{19}$$

♣

REMARK A.5 (OSCILLATION) Sequence $\{u_j\}_{j\in\mathcal{N}} \in L^p(K, \Re^N)$ converge in measure to u written $u_j \underrightarrow{m} u$, it fail, then the sequence oscillates.

DEFINITION A.6 (UNIFORMLY INTEGRABLE) If sequence $\{u_j\}_{j\in\mathcal{N}}$ is uniformly integrable then

$$\lim_{M\to\infty} \sup_j \int_{\{|u_j|\geq M\}} |u_j(x)| dx = 0.$$

♣

REMARK A.7 (CONCENTRATES) If sequence $\{u_j\}_{j\in\mathcal{N}} \in L^p(K, \Re^N)$ is not uniformly integrable, then sequence is said to be **concentrates**.

DEFINITION A.8 (REGULARITY) A measure μ is outer regular on a Borel-measurable set E if

$$\mu(E) = \inf\{\mu(U) : U \supset E, \text{ for every open set } U\},$$

and μ is called inner regular on Borel-measurable set E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, \text{ where K is a compact set }\}.$$

The measure μ is called regular if it is both inner and outer regular on all Borel sets.

♣

DEFINITION A.9 (RADON MEASURE) A Radon measure is a finite Borel measure that is outer regular on all Borel sets and inner regular on open sets.

♣

THEOREM A.10 *Let $(X, \mathcal{B}(X), \mu)$ be a measure space where $X \in \Re^N$ and μ is a Lebesgue measure. Also, $\beta(x)$ is a continuous function in $\Re^N$, we define*

$$\nu(A) = \int_A \beta(x) d\mu, \; A \in \mathcal{B}(X),$$

then ν is a measure.

PROOF We prove only additivity all other conditions are trivial. Let $A_n \in \mathcal{B}(X)$ be disjoint collection of sets, then

$$\begin{aligned}\nu(\cup_n A_n) &= \int_{\cup_n A_n} \beta(x) d\mu = \int \chi_{\cup_n A_n} \beta(x) d\mu = \sum_{i=1}^{n} \int \chi_{A_n} \beta(x) d\mu, \\ &= \sum_{i=1}^{n} \int_{A_n} \beta(x) d\mu = \sum_{i=1}^{n} \nu(A_n).\end{aligned}$$

REMARK A.11 Let $u_k(x), x \in E \subseteq \mathbb{R}$ be a sequence of continuous function, $\mu(E) < \infty$ and each continuous function $u_k(x)$ is bounded by M then both average concept $u_k^*(x)$ and $\overline{u_k}(x)$ have a bounded difference, with $u_k^*(x) = \frac{1}{\mu(E)} \int_E u_k(x) d\mu$, $\mu(E)$be a lebesgue measure of E and

$$\overline{u}(x) = \lim_{n \to \infty} \overline{u_n}(x) = \lim_{n \to \infty} \lim_{k \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{k} \frac{u_i \chi_{A_j}(x)}{n}$$

as previously defined.

It is worth mentioning that $\overline{u}(x)$ is advantageous over $u_k^*(x)$; as it can be defined for both the continuous and discrete function. Also, $\mu(E)$ can take the values as zero. On the other hand, in $u_k^*(x)$, we can only use u_k as a sequence of continuous functions and $\mu(E) \neq 0$

DEFINITION A.12 A Young measure from $D \in \Re^k$ to $\Re^N$ is a function which maps $z \in D$ to a probability measure on $\Re^N$. Young measure is a weak*-measurable map $\nu' : D \to \mathcal{B}(\Re^N)$.

Therefore, the mapping $z \to \langle \nu'_z, g \rangle$ is Borel measurable for every $g \in C_0(\Re^N)$, where $C_0(\Re^N)$ is the set of all Young measures. It is denoted by $\mathcal{Y}(K, \Re^N)$.

B. Proves.

B.1. Selected proves of the propositions from Sec. 3.4. Due to the importance of the claims cited, in Section 3.4 we also include proofs of the most important ones.

PROOF (PROPOSITION 3.11) Construct a sequence of function $\overline{u}(x)$ from $u(x)$ by equation (10) and (11)

$$\overline{u}(x) = \lim_{n\to\infty} \lim_{k\to\infty} \sum_{i=1}^{2n} \sum_{j=1}^{k} \frac{u_i(x)}{2n} \chi_{k_j},$$

where k_j be a disjoint interval. Substitute $\overline{u}$ in equation (15).

$$\begin{aligned}\int_K \beta(\xi) d\nu_x(\xi) = \int_{[b,c]} \beta(\overline{u}(x)) d\mu(x) &= \frac{1}{M} \int_{[b,c]} \beta(\overline{u}(x)) dx, \\ &= \frac{1}{M} \int_K \beta(y) |\overline{u}'(x)^{-1}| dy,\end{aligned}$$

On comparing, we get

$$d\nu = \frac{1}{M} |\overline{u}'(x)^{-1}| dy,$$

where M takes from the equation as in (14). ■

PROOF (PROPOSITION 3.12) Construct a sequence of function $\overline{u}(x)$ from $u(x)$ by equation (10) and (11)

$$\overline{u}(x) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=1}^{2n} \sum_{j=1}^{m} \sum_{k=1}^{n1} \frac{u_i(x)}{n} \chi_{k_j \cap A_k}.$$

where k_j be a disjoint interval. We prove it by induction on k.
For $i = 1$ it is true by proposition 3.11. Suppose it is true for $k = l$ we get $g_l(x) = \sum_{i=1}^{l} |\overline{u}_i'(x)^{-1}|$. For $k = l + 1$,

We consider $g_{l+1}(x) = \sum_{i\,=\,1}^{l+1} |\overline{u}_i'(x)^{-1}|$

$$\begin{aligned}\int_K \beta(\xi) d\nu_x(\xi) &= \sum_{i=1}^{l+1} \int_{A_k} \beta(\overline{u}_i(x)) d\mu(x), \\ &= \sum_{i=1}^{l} \int_{A_k} \beta(\overline{u}_i(x)) d\mu(x) + \int_{A_{l+1}} \beta(\overline{u}_i(x)) d\mu(x),\end{aligned}$$

Using induction principle,

$$\begin{aligned}\int_K \beta(\xi) d\nu_x(\xi) &= \frac{1}{M} \sum_{i=1}^{l} \int_{A_k} \beta(y) |\overline{u}_i'(x)^{-1}| \chi_{k_i} dx + \frac{1}{M} \int \beta(\overline{u}_{l+1}) \chi_{k_{l+1}} dx \\ &= \frac{1}{M} \sum_{i=1}^{l+1} \int_{A_i} \beta(y) |\overline{u}_i'(x)^{-1}| dy = \frac{1}{M} \sum_{i=1}^{l+1} \int_k \beta(y) |\overline{u}_i'(x)^{-1}| dy,\end{aligned}$$

On comparing, we get

$$d\nu = \frac{1}{M}\sum_{i=1}^{l+1} |\overline{u}'(x)^{-1}| dy,$$

which shows that it is true for $k = 1, 2, ..., n$. ■

Theorem B.1 Evans (2010) Let $1 \leq p \leq \infty$ and $u \in L^p(U)$ be a T periodic function $u_n(x) = u(nx)$ $1 \leq p < \infty$ as $n \to \infty$ then,

$$u_n \rightharpoonup \frac{1}{|T|}\int_0^T u(y)dy, n = 1, 2, \cdots$$

and if $p = \infty$ as $n \to \infty$

$$u_n \rightharpoonup^* \frac{1}{|T|}\int_0^T u(y)dy, n = 1, 2, \cdots$$

C. Convergence of Q- measure in Weak* topology.

Definition C.1 (Measures convergence) A sequence of Q measure $\nu_n \in (K, \mathcal{F}, \mathcal{B}(M))$ of $\mathbb{R}^N$-valued measures converges weakly* to a measure $\nu \in (K, \mathcal{F}, \mathcal{B}(M))$ if and only if

$$\int_K g(x)d\nu_n(x) = \int_K g(x)d\nu(x), \ \forall g \in C_0(K, \Re^N).$$

If the convergence of measure is strong then corresponding sequence of function $u_n(x)$ strongly converges to $u(x)$, where K is a compact subset of $\Re^N$.

♣

We recall convergence of probability measure as defined in Billingsley (1999), Sagitov (2020).

Definition C.2 (Probability distribution) Let X, X_n be a random element defined on the probability space $(K, \mathcal{B}(\Re), P)$. Then a probability measure P on $K \subset \Re$ is the probability distribution of X, if $P(A) = P(X \in A)$ for all $A \subset K$, where $K \subset \Re$

♣

DEFINITION C.3 Let X_n, X are random elements defined on the probability spaces $(K_n, \mathcal{F}_n, P_n)$, $(K, \mathcal{F}, P)$. We say X_n converge in distribution to X as $n \to \infty$ if for any bounded continuous function $u : K \to R$,

$$E_n(u(X_n)) \to E(u(X)), \text{as } n \to \infty.$$

This is equivalent to the weak convergence $P_n \Rightarrow P$ of the respective probability distributions, where $' \Rightarrow '$ denotes the convergence in distribution.

♣

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Results in Q-measure.

CR Jisha

Streszczenie W tym artykule autor wprowadza nową miarę, którą nazywaną miarą Q, reprezentującą słabą* granicę barycentrum ciągu funkcji borelowskich. Omawia niektóre wyniki związane z tą miarą, co jest pomocne przy wyznaczaniu miary Q dla poszczególnych typów funkcji. Ponadto omówiono zastosowanie koncepcji średniej w analizie sygnałów i naukach o atmosferze.

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