Geometric Levelling Data and Some Systematic Faults in Their Treatment

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Abstract

The aim of this article is to illuminate some latent systematic faults in the mathematical treatment of precise levelling data. The first one is associated with the use of the average of both measurements of the height differences between the terminal benchmarks in levelling lines. Another weak point in the classical treatment of levelling data is the incomplete minimization of the impact of the spatial network configuration on the produced mean standard errors of the nodal benchmarks from the adjustment. Generating sixty random paired samples of size 1000, derived from three continuous distributions, e.g. Normal (0, 1), Uniform (-1.732, 1.732) and Gamma (1, 1), it was found that the average of two same distributed and ordered observations is very nearby to the theoretical expectation, in comparison to both observations, only in approximately 27-30% of all cases. Contrary, in other 70-74% of cases, either the "first" or the "second" observation is in close proximity to the expectation. The miss of this fact leads to a statistically significant deterioration of the final accuracy of the levelling networks. In the current study, it is also shown that the minimization of the standard errors of the adjusted normal heights of the nodal benchmarks in the Bulgarian Levelling Network 1980 cannot be achieved with the weights w=const.L-1, which are the most popular and used type of weights in the adjustment of geometric levelling networks. Finally, it is illustrated that taking into account the above marks and applying an appropriate adjustment algorithm, the mean of the standard errors of the adjusted heights of the nodal benchmarks in the analysed network is possible to be less than 1mm. The standard error of the adjusted height of the most remoted benchmark "Pushkarov", which is 598 km far away from the datum point located in Varna, is equal to 1.40mm. The obtained from the adjustment mean standard error for the weight unit is estimated to be 0.164 mm/√km. In comparison, the adjustment mean standard error for the weight unit, but yielded by the classical approach of adjustment of the analysed network, is 1.289 mm/√km or almost 9 times higher. Despite being tedious and time-consuming, it is not on point of discarding the precise geometric levelling as a main geodetic method for solving of a couple of scientific and engineering tasks, where differences in heights have to be determined with the highest accuracy.

Keywords: geometric levelling data, systematic faults, bulgarian levelling network, accuracy

Introduction

The highest order geometric levelling has been used as a main method for establishing of height system for state territories [1-4] or even for a whole continent [1, 4-5] since the last decades of 19th century. The method is also widely applied for validation of gravimetric geoid [5] and quasigeoid models [6], verification of chronometric levelling results [7], monitoring of the recent vertical movements of the Earth's crust [3], verification and calibration of GNSS antennas [8], etc. The importance of the precise geometric levelling for civil engineering activities is also well-known [9].

During the years, the development of the method is based on the modernization of the levels and rods, investigating of the impact of systematic errors on the measured heights and adding of appropriate corrections [7, 10 - 11], improvement of the methodology of measurements, etc.

However, there has been no increase of the levelling accuracy for 50-70 years [2, 11 -12]. Obviously, the nostrum regarding development of the highest order geometric levelling is somewhere else. Since the dawn of this method the mathematical treatment of levelling data has not been changed, supposing that the low of error and discrepancies accumulation, applied weights in the adjustment of levelling networks, etc., are absolutely clear. Some recent studies [12, 13] show some cracks into the classical theory.

The main objective of the current article is to bring to the surface some latent systematic faults in the mathematical processing of the precise levelling data. The first one is associated with the use of the average of both measurements of the height differences between the terminal benchmarks in levelling lines. The second is concerned with the incomplete minimization of the impact of the spatial network configuration on the produced mean standard errors of the nodal benchmarks from the adjustment.

Latent systematic faults and their treatment

The current paper is focused on development of mathematical methodology for processing of precise levelling data in the best possible way, which is based on modern probability theory and its methods, the power of modern computers, deep scientific facts and lack of stereotypes. In order to break the plateau in the estimated accuracy of the precise geometric levelling new points of view are necessary.

On the Location of the Average of Two Random Observations

The average of two observations is supposed to be $\sqrt{2}$ times more accurate than each of observations under the assumption that both random observations derive from common population with finite variance. In fact, if we generate two random samples of size n from arbitrary chosen distribution with finite variance σ^2 and we create a new sample from the means of the same ordered observations of both original samples, then the variance of the new sample will tend to $\sigma^2/_2$ when n tend to infinity. There is the crucial point, the size of the sample has to tend to infinity, not to 2. The methods of the modern statistics reveal that if the size of a sample is less than n=30 [14, 15] than the average of the sample observation is unstable. Therefore, if we have two observations, their mean is likely to be further away from the expectation of the distribution than either the "first" or the "second" observation. In order to prove this fact, sixty random paired samples of size 1000 derived from three continuous distributions, e.g. Normal (0, 1), Uniform (-1.732, 1.732) and Gamma (1, 1) were generated. These distributions are continuous distributions and are supposed to be the most representative ones regarding measurement errors. Their parameters are intentionally chosen in such manner to ensure that the random samples will have equal variances $\sigma^2 = 1$ and preliminarily known expectations, $\mu=0$ for both the Normal (0, 1) and Uniform (-1.732, 1.732) and Gamma (1, 1) and $\mu=1$ for Gamma (1, 1) distributed samples. The frequencies, which we will roughly call "probabilities" in the text below, of the "first", the "second" observation and their mean, to be the closest one to the theoretical expectation for the analysed distributions are illustrated by Figure 1. According to these charts, in the case of the N(0, 1), the probability of either the "first" or the "second" observation to be near to the expectation $\mu=0$ is approximately 35%. Contrary, the probability the mean of two random observation, derived from N (0, 1) distribution, to be closer to $\mu=0$ is only almost 30%. In the case of strongly skewed distribution like Gamma (1, 1), the probabilities are 36.5%, 36.5% and 27% in favour of the "first" observation, the "second" observation and their mean, respectively. Approximately equal probabilities to be nearby to the theoretical expectation have "first" observation, the "second" observation and their mean in the case of Uniform (-1.732, 1.732), respectively 34%, 33% and 33%. The standard deviations of the above-mentioned frequencies are approximately 1%, which means that if one execute a similar simulation, they will obtain different results, but similar final conclusions. Based on the presented results, it is obvious that the use of the average of both height measurements is not scientifically the best decision. Contrary, in almost 70% of the cases it will lead to worse results than the choice of either the "first" or the "second" measured line elevation, even if the normal distribution of the elevations in the levelling network is supposed.

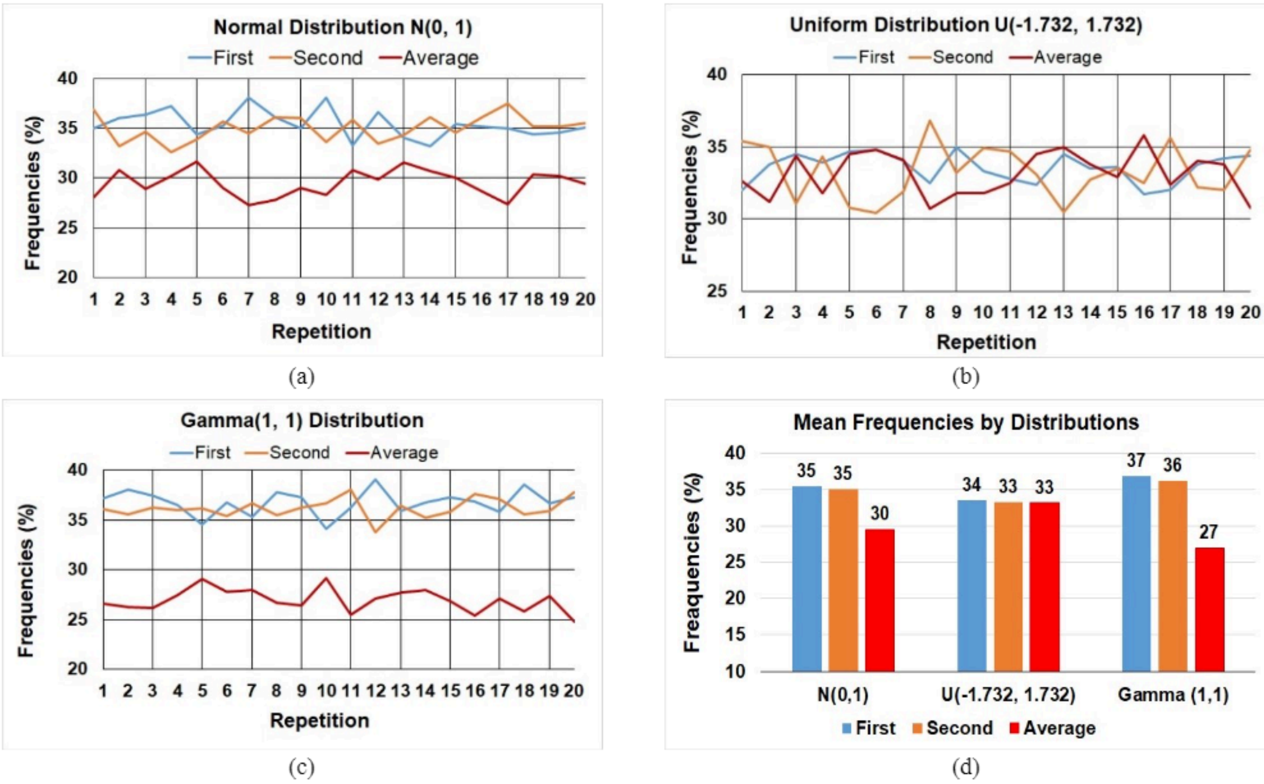

Fig. 1. Frequencies of the nearest location to the theoretical expectation of the "first", the "second" or their mean regarding different continuous distributions, based on independent paired random samples of size 1000: a) Normal distribution - N(0, 1), b) Uniform distribution - U(-1.732, 1.732), c) Gamma distribution – Gamma (1, 1), d) Mean frequencies by the analyzed distributions.

On the Impact of the Network Configuration on the Final Accuracy

Final accuracy of the adjusted height Hi of an arbitrary chosen nodal benchmark with index i, presented by its standard error σ_{H_i}, yielded by the parametric adjustment of a levelling network can be given by equation (1).

$$\sigma_{H_i} = \mu.\sqrt{Q_{i,i}} \quad (1)$$

In equation (1) μ is the standard error per unit weight and $Q_{i,i}$ is the i^{th} member of the main diagonal of the matrix Q. The matrix Q is function of the matrix of the weights of the measured line elevation W and the information matrix A. The matrix W is a symmetric matrix with members of the main diagonal equal to the weight of concrete line elevation in the levelling network. All other its members are equal to 0. The information matrix A contains information about the levelling lines in the leveling network. Its value are -1 for the initial benchmark of a line, 1 for the terminal benchmark of a line and zeros for all other adjusted benchmarks. If we present the matrix Q by equation (2), we can rewrite equation (1) as an equation (3).

$$Q = (A^TWA)^{-1} \quad (2)$$

$$\sigma_{H_i} = \mu . \sqrt{((A^T W A)^{-1})_{i,i}} \quad (3)$$

Equation (3) clearly shows that the standard errors of the adjusted heights in a geometric levelling network depend on not only from the accuracy of measurements, but also from the applied weights and the configuration of the network. If we suppose that all line elevations have equal accuracy, we can write equation (4).

$$\sigma_{H_i} = \mu . \sqrt{((A^T A)^{-1})_{i,i}} \quad (4)$$

Therefore, in the case of equal weighting the standard errors of the adjusted heights in our geometric levelling network depend on from the accuracy of measurements and from the configuration of the network.

It is a common practice the weights w_n to be given as a function of the length of a levelling line L_n [1, 9, 11] similar to equation (5) despite other weights are more statistically relevant [12, 13].

$$w_n = \frac{const.}{L_n} \quad (5)$$

Equation (5) is a particular case of the inverse distance weighting equation (6).

$$w_n = \left(\frac{const.}{L_n}\right)^p \quad (6)$$

Equations (3) and (6) give the relation between the power parameter p and standard errors of the adjusted heights of the nodal benchmarks in a levelling network. Moreover, the standard error per unit weight μ is also affected by the Inverse Distance Weighting power parameter p. This relation is given by equation (7).

$$\mu^2 = \frac{\sum_{i=1}^{n} w_n . V_n^2}{n-k} \quad (7)$$

In equation (7) n denotes the number of the levelling lines in the adjusted network, k is the number of the benchmarks, which heights are being adjusted, and v_n are the corrections of the measured line elevations.

Obviously, the minimization of the sum of the standard errors of the adjusted benchmark heights goes through finding of an appropriate power parameter p, which is specific for each network and depends on its configuration. An illustration of the above logic is given by Figure 4, where the network pictured in Figure 3 is adjusted.

Adjustment of the Bulgarian Levelling Network 1980 by Taking Into Account the Above Notes

In order to test the above-commented mathematical facts with real data, we will use the data of the Bulgarian Levelling Network 1980, which was part of the United Precise Levelling Network of socialistic republics in Eastern Europe – the Second Realization [4]. The configuration of the network is shown by Figure 3.

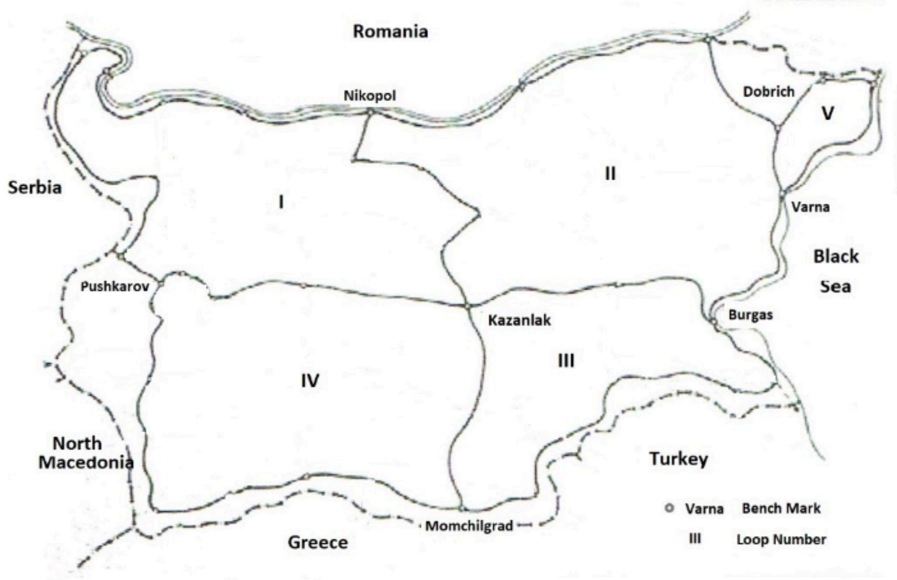

Fig. 2. Configuration of the Bulgarian Levelling Network - 1980.

General levelling data are given in table 1. As can be seen, the Bulgarian part of UPLN – Second Edition consists of 11 lines grouped in 5 loops. The total length of the network is 3438.13 km. All measurement were performed during 1975 to 1980 year.

Tab.1. Results of the Bulgarian Levelling Network - 1980.

From	To	Upward Elevation (m)	Downward Elevation (m)	Average (m)	Distance (km)
Varna	Dobrich	161.17537	161.13759	161.15648	198.68
Varna	Dobrich	161.18833	161.17403	161.18118	51.73
Varna	Burgas	17.06735	17.07590	17.07162	151.93
Burgas	Kazanlak	294.02308	293.94574	293.98441	210.91
Burgas	Momchilgrad	182.62402	182.59601	182.61002	566.12
Momchilgrad	Kazanlak	111.30933	111.29822	111.30378	187.27
Nikopol	Kazanlak	191.12593	191.06141	191.09367	273.83
Nikopol	Dobrich	41.24309	41.22124	41.23217	383.88
Kazanlak	Pushkarov	176.86510	176.85955	176.86233	235.07
Momchilgrad	Pushkarov	288.23428	288.16329	288.19879	539.39
Nikopol	Pushkarov	368.01370	367.94470	367.97920	639.31

The analyzed levelling network was adjusted by three different approaches, namely:

- Variant 1 – Parametric adjustment by classical approach, which uses the averages of both measurements of elevations in each lines. The weights applied in this adjustment variant are calculated by equation (5), which is the same as equation (6) with power parameter *p*=1.
- Variant 2 - Based on the fact shown above that the average is closer to the theoretical expectation only in almost 30% than either the "first" or the "second" measurement result, we performed 3^{11} or 177 147 independent parametric adjustments, which is a full combination of usage of the data in table 1, columns 3-5. The weights used in this variant of adjustment were calculated by equation (6) with power parameter *p*=1. That is to say, the classical weights used in the adjustment of geometric levelling networks. As a result, it was found, that the values of measured line elevations colored in red, see table 1, fit in the best manner the analyzed levelling network.
- Variant 3 – This variant extends Variant 2 by applying of Inverse Distance Weighting procedures with different values of the power parameter *p*. In other words, we use only the red colored values in table 1, columns 3-5, and perform IDW with them. Results, obtained by this variant are illustrated by Figure 2.

The nodal benchmark, located in Varna, was chosen as a datum point in all variants of adjustment.

Results

Comparison among the results produced by Variants 1-3 are presented by Figure 3

Looking at Figure 3, one can see that:

- Variant 1 produced the worst results. The standard errors of the adjusted heights of the nodal benchmarks in the Bulgarian Part of UPLN – Second Edition vary from 8.1 mm to 22.9 mm. The mean error per unit weight μ, yielded by the adjustment, is equal to 1.289 mm/$\sqrt{km}$.
- Variant 2 produced statistically significant better results than Variant 1. A paired two-sample for means t-Test based on the samples of the standard errors of the adjusted heights of the nodal benchmarks, obtained by Variant 1 and Variant 2, rejects the null hypothesis $H_0 : \mu_1 = \mu_2$ in favour of the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ at level higher than 99.9 %. The mean error per unit weight μ, yielded by Variant 2, is equal to 0.244 mm/$\sqrt{km}$.
- Variant 3 gave the best results. The process of minimizing of the sum of the standard errors of the nodal benchmarks by performing IDW is illustrated by Figure 4. As can be seen, the minimization was achieved when the power parameter *p* is equal to 6.6. A paired two-sample for means t-Test based on the samples of the standard errors of the adjusted heights of the nodal benchmarks, obtained by Variant 2 and Variant 3, rejects the null hypothesis $H_0 : \mu_2 = \mu_3$ in favour of the alternative hypothesis $H_1 : \mu_2 \neq \mu_3$ at level higher than 99.9 %. The actual p-value of the two-tail variant of the test is 0.00024. The mean error per unit weight μ, yielded by Variant 3, is equal to 0.164 mm/$\sqrt{km}$. As a result, the standard error of Pushkarov benchmark, which is almost 600 km remoted from the datum point Varna, is only 1.4 mm. The worst determined height of the nodal benchmark is that of Nikopol benchmark, which has standard error below 2.0 mm.

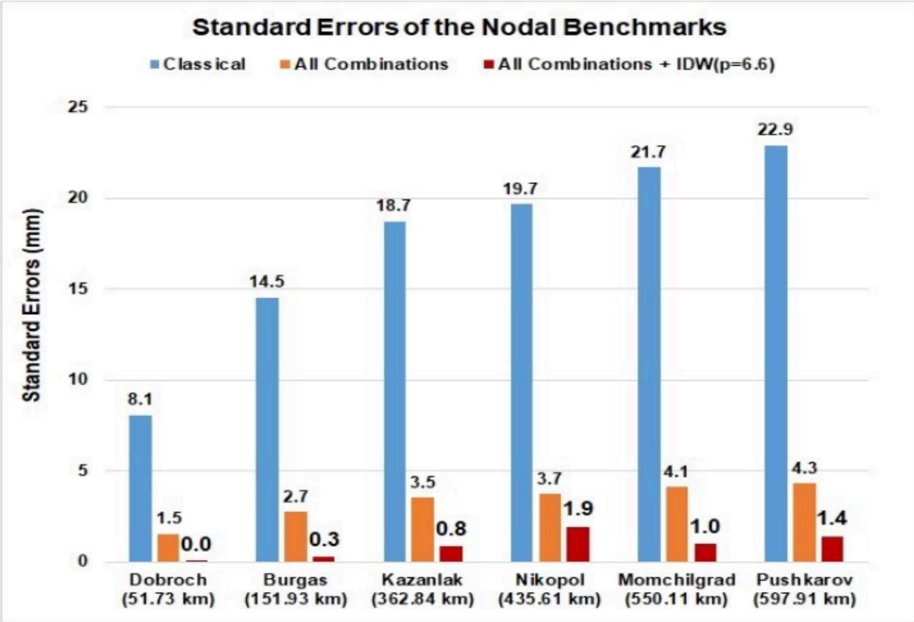

Fig. 3. Standard Errors of the Nodal Benchmarks in the Bulgarian Levelling Network – 1980, yielded by different approaches.

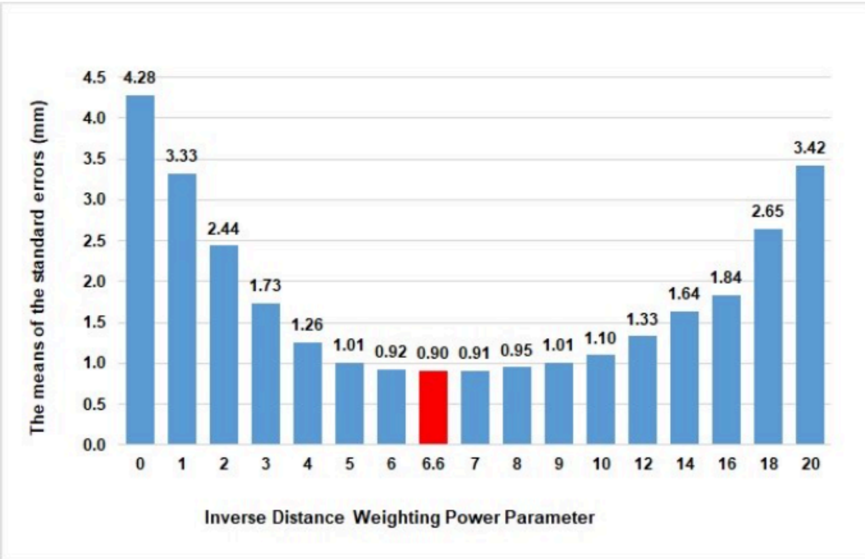

Fig. 4. Adjustment results of the Bulgarian Levelling Network – 1980 by applying IDW in Variant 3.

Discussion

Results presented in this text show that the classic variant of adjustment of geometric levelling networks is not the best one. This approach don't take into account the fact that the average of two observations is only up to 30% nearby the theoretical expectation than either of the "first" or the "second" observation. Owing to this fact, accepting automatically the mean as the approximation of the expectation, leads to deterioration of the initial adjustment data. A good example are values of measured elevations in both lines Varna – Dobrich, which form loop V in the analyzed network. One can see that we have two measurements, which are close to each other, these of the upward measurement along the route with length 198.7 km and the downward of the route with length 51.7 km. If we use these measurements as initial data in adjustment, the closing error of loop V will be 1.33 mm. Appling the classical approach we have two means 161.15648 and 161.18118, respectively for the longer and for shorter route, which produce closing error of 24.7 mm. As a result, the impact on the standard errors of the adjusted heights of the nodal points in the network will be significant. One can calculate that this will increase the values of the standard errors, which are colored in orange in Figure 3, more than twice. The mean error for unit weight will increase from 0.244 mm/$\sqrt{km}$ to 0.547 mm/$\sqrt{km}$. Assuming that the measurements associated with the shorter route are probably more accurate, one can see the effect of the outlier value 161.13759. But it is another question. The above illustrate mechanism of deterioration of the initial adjustment data is more obscure in loops formed from more lines, but exists.

Appling 3^n independent adjustments, in order to find these observations which fit the network in the best way, is computational expensive approach, but support the findings about the accuracy and the location of the means of two measurements. These finding were fully confirmed by the real data of the Bulgarian part of UPLN – Second Edition.

Concerning the use of Inverse Distance Weighting in order to minimize additionally the standard errors of the nodal benchmarks in the analyzed network, there were not expected any surprises. Analogical results were obtained with the data of the Second Levelling of Finland and the Third Levelling of Bulgaria [13].

The efficiency of the above explained algorithm for treatment of the highest order geometric levelling data will be demonstrate with the data of another European country in the nearest future.

Conclusion

In this paper was demonstrated that the precise geometric levelling can reach accuracy from 0.164 mm/$\sqrt{km}$. This means that the standard errors of the heights of the benchmarks, which are located more than 4000 km from a datum point, can be less than 10 mm. According to the results given in [8] and [7], regarding the precision of the GNSS vertical accuracy and the last achievements of chronometric levelling, respectively, this study shows that the precise geometric levelling is still the most accurate method for determining of elevation differences. Despite being tedious and time-consuming this method can continue to be used as a main method for establishing of height systems for state territories and even continents, validation of gravimetric geoid and quasigeoid models, monitoring of the recent vertical movements of the Earth's crust, etc.

References

1. M. Sacher, J. Ihde, G. Liebsch and J. Mäkinen, "EVRF2007 as Realization of the European Vertical Reference System", Presented at the Symposium of the IAG Sub-commission for Europe (EUREF) in Brussels, June 18–21 2008.
2. Ł. Borowski, B. Kubicki and J. Gołąb, "Implementation of the EVRF2007 height reference frame in Poland" Journal of Applied Geodesy, 2023, https://doi.org/10.1515/jag-2023-0020 .
3. S. Gospodinov, E. Peneva and P. Penev, "A specific approach to least squares adjustment of the state levelling network", 22nd International Multidisciplinary Scientific GeoConference SGEM, 2022, https://doi.org/10.5593/sgem2022/2.1/s09.20 .
4. National Institute of Geodesy and Photogrammetry, "United precise levelling network of socialistic republics in Eastern Europe, cooperation with Poland, Czechoslovakia, East Germany, Hungary, USSR and Romania", 1980. (in Bulgarian).
5. X. Li, "Modeling the North American vertical datum of 1988 errors in the conterminous United States" Journal of Geodetic Science, vol. 8, no. 1, 2018, pp. 1-13. https://doi.org/10.1515/jogs-2018-0001 .
6. L. E. Sjöberg, "Geoid model validation and topographic bias", Journal of Geodetic Science, vol. 12, no. 1, 2022, pp. 38-41. https://doi.org/10.1515/jogs-2022-0133 .
7. Y. Tanaka and Y. Aoki, "A Geodetic Determination of the Gravitational Potential Difference Toward a 100-km-Scale Clock Frequency Comparison in a Plate Subduction Zone". In: International Association of Geodesy Symposia. Springer, Berlin, Heidelberg, 2022, https://doi.org/10.1007/1345_2022_147 .
8. L. Borowski, J. Kudrys, B. Kubicki, M. Slámová and K. Maciuk, "Phase Centre Corrections of GNSS Antennas and Their Consistency with ATX Catalogues." Remote Sens. 2022, 14, 3226. https://doi.org/10.3390/rs14133226.
9. A. Angelov, "Geodetic methods in the study for deformation process of high buildings and engineering facilities", Monographic, Second Edition, 2022, https://uacg.bg/filebank/att_22864.pdf , (in Bulgarian).
10. S. Enman and V. Enman, "Systematic errors in leveling of mountainous areas", Bull. Geod., 58, 1984, pp. 475-493.
11. V. Saaranen, P. Lehmuskoski, M. Takalo and P. Rouhiainen, "The Third Precise Levelling of Finland", FGI Publications No. 161, Kirkkonummi, 2021, The Third Precise Levelling of Finland (helsinki.fi).
12. V. Cvetkov, "Two adjustments of the second levelling of Finland by using nonconventional weights" Journal of Geodetic Science, vol. 13, no. 1, 2023, https://doi.org/10.1515/jogs-2022-0148.
13. V. Cvetkov and S. Gospodinov, "Inverse Absolute Height Weighting in the Highest Order Levelling Networks", EGU General Assembly 2023, Vienna, Austria, 24–28 Apr 2023, EGU23-4219, https://doi.org/10.5194/egusphere-egu23-4219 , 2023.
14. F. M. Dekking, C. Kraaikamp, H. P. Lopuhaä and L. E. Meester, "A Modern Introduction to Probability and Statistics", London, Springer – Verlag, 2005.
15. V. Cvetkov, „Alternative measures of central tendency", Deutsche Internationale Zeitschrift Für Zeitgenössische Wissenschaft, 38, pp. 4–8, 2022, https://doi.org/10.5281/zenodo.7002877