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GRAPHS WITH ODD AND EVEN DISTANCES BETWEEN NON-CUT VERTICES

Kateryna Antoshyna and Sergiy Kozerenko

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Abstract. We prove that in a connected graph, the distances between non-cut vertices are odd if and only if it is the line graph of a strong unique independence tree. We then show that any such tree can be inductively constructed from stars using a simple operation. Further, we study the connected graphs in which the distances between non-cut vertices are even (shortly, NCE-graphs). Our main results on NCE-graphs are the following: we give a criterion of NCE-graphs, show that any bipartite graph is an induced subgraph of an NCE-graph, characterize NCE-graphs with exactly two leaves, characterize graphs that can be subdivided to NCE-graphs, and provide a characterization for NCE-graphs which are maximal with respect to the edge addition operation.

Keywords: non-cut vertex, graph distance, line graph, block, strong unique independence tree.

Mathematics Subject Classification: 05C12, 05C05, 05C75.

1. INTRODUCTION

Despite the well-known characterization of “shortest paths” metrics on vertex sets of finite connected graphs [8], metric graph theory is a very promising field of research with its challenges and mesmerizing classes of graphs (see the corresponding survey by Bandelt and Chepoi [2]).

To motivate the research conducted in this work, we need to recall several definitions. A set of vertices in a graph is called independent if no two vertices from this set are adjacent. A graph G is called strong unique independence graph provided it has a unique maximum independent set $A \subset V(G)$ such that its complement $V(G)\setminus A$ is also independent in G.

In [7] Hopkins and Staton characterized strong unique independence trees (shortly, SUITs) as trees which have even distances between their leaves. Interestingly, this class of trees arises from another classical graph construction. Recall that a vertex u in a connected graph G is called a cut vertex if $G-u$ is disconnected. A connected graph without cut vertices is called 2-connected. A block in a graph is its maximal 2-connected subgraph. These notions give birth to several classical unary graph operators. Namely, the block graph $B(G)$ of a graph G is the intersection graph on the class of all blocks in G (see [5]). A closely related construction is called the block-cutpoint-tree $\mathrm{bc}(G)$. Namely, let G be a connected graph. The graph $\mathrm{bc}(G)$ has cut vertices and blocks of G as its vertices with edges of the form $\{u, B\}$, where u is a cut vertex, B is a block in G, and $u \in V(B)$. It is fairly easy to prove that for a connected graph G, the graph $\mathrm{bc}(G)$ is indeed a tree. In fact, it turns out that this graph operator produces exactly SUITs (and vice versa, each SUIT is isomorphic to the block-cutpoint-tree of some connected graph). Since in a tree, the leaves are exactly the non-cut vertices, SUITs provide an example of connected graphs with even distances between its non-cut vertices.

In this work, we study graphs with parity conditions on distances between non-cut vertices. Namely, a connected graph is called an NCE-graph (NCO-graph) if it has even (odd) distances between its non-cut vertices. The paper is organized as follows. After giving basic definitions in Section 2.1, we proceed by stating several well-known results on block graphs and line graphs in Section 2.2. Then, in Section 2.3, we prove several auxiliary results which will be used later in the paper.

Our first main result consists of characterizing NCO-graphs which turned out to be exactly the line graphs of SUITs (see Theorem 3.1). We then conclude Section 3.1 with the inductive construction of SUITs from stars using a simple operation (Theorem 3.3).

In Section 3.2 we at first prove the key observation that NCE-graphs are bipartite (Theorem 3.4) and then use this result to present a criterion for NCE-graphs in Corollary 3.5. We also show that any bipartite graph can be embedded as an induced subgraph of an NCE-graph (Corollary 3.6) and characterize induced subgraphs of NCO-graphs (Corollary 3.9). After we noticed that every non-trivial NCE-graph has at least two leaves, we present a complete characterization of NCE-graphs with exactly two leaves (Theorem 3.13). Further, in Proposition 3.14 we characterize graphs that can be subdivided to NCE-graphs. Finally, Theorem 3.16 proposes a criterion for NCE-graphs which are maximal relative to the operation of adding edges.

We note that some of the results from this paper were announced at International Conference of Young Mathematicians [1] in 2023.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. BASIC DEFINITIONS

In this paper, all graphs are assumed to be finite and simple. That is, by a *graph* G we mean an ordered pair (V, E), where $V = V(G)$ is a finite set (elements of which are called *vertices*) and $E = E(G)$ is a set of some 2-element subsets of $V(G)$ (which are called *edges*). Instead of $\{u, v\} \in E(G)$ we will write $uv \in E(G)$. And in this case, we say that the vertices u, v are *adjacent*.

The *neighborhood* of a vertex $u \in V(G)$ is the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$. The *degree* of u is the number $d_G(u) = |N_G(u)|$. A vertex $u \in V(G)$ is called a *leaf* provided $d_G(u) = 1$, while the corresponding unique vertex from $N_G(u)$ is called the *support vertex* of u. A vertex which is not a leaf will be called an *inner vertex*.

We say that a graph H is a *subgraph* of a graph G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A subgraph H of G is called *induced* provided $E(H) = E(G) \cap \binom{V(H)}{2}$. By $G[A]$ we denote the subgraph of G induced by $A \subset V(G)$.

The *join* operation takes two graphs G, H and produces the new graph $G + H$ with

$$V(G + H) = V(G) \sqcup V(H)$$

and

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

Definition 2.1. Having an edge subset $E' \subset E(G)$, consider the new graph $\mathrm{Sub}(G, E')$ which has the vertex set $V(G) \sqcup \{x_e : e \in E'\}$ and the edge set $(E(G) \setminus E') \cup (\bigcup \{ux_e, x_e v : uv \in E'\})$. We say that $\mathrm{Sub}(G, E')$ is obtained from G by the *subdivision of edges from* E'. Further, a graph H is called a *subdivision* of G, if there is $E' \subset E(G)$ with $\mathrm{Sub}(G, E') \simeq H$.

Remark 2.2. We emphasize that by fixing the set E' from the start, we do not allow multiple subdivisions of the same edges in G (although, the corresponding graphs from these multiple subdivisions of edges in G can be obtained after iterations of our graph operator Sub).

A set of vertices $A \subset V(G)$ in a graph G is called *dominating* provided for any $u \in V(G) \setminus A$ there is $a \in A$ with $au \in E(G)$. A set $A \subset V(G)$ is called *independent* if no two vertices from A are adjacent in G. A *maximum independent* set is an independent set having the largest cardinality.

A graph is called *bipartite* if its vertex set can be partitioned into two independent subsets (each called a *part*). Any such a partition is called its *bipartition*. Note that any connected bipartite graph has a unique bipartition (up to permutation of parts). A *complete bipartite graph* is a bipartite graph in which there is an edge between every two vertices from different parts. By $K_{m,n}$ we denote the complete bipartite graph having parts of cardinalities m, n.

A *tree* is a connected acyclic graph. It is clear that every tree is a bipartite graph. A *star* is a complete bipartite graph $K_{1,n}$ for $n \in \mathbb{N}$.

A graph G is called *strong unique independence graph* provided it has a unique maximum independent set $A \subset V(G)$ such that its complement $V(G) \setminus A$ is also independent in G. It is easy to see that strong unique independence graphs are bipartite. Throughout this paper, strong unique independence trees will be simply called *SUITs*. The following result provides a metric characterization of SUITs.

Theorem 2.3 ([7, Theorem 3])**.** *A tree T is a SUIT if and only if the distance between any two leaves in T is even.*

2.2. BLOCK GRAPHS AND LINE GRAPHS

A graph is *connected* if it contains a path between every pair of its vertices. A *connected component* in a graph is its maximal connected subgraph. A vertex in a graph G is a *cut vertex* if its deletion from G increases the number of connected components.

We say that a vertex u *separates* two vertices x, y provided x, y lie in the same connected component in G, but in different connected components in $G - u$. Clearly, a vertex is a cut vertex if and only if it separates some pair of vertices. Similarly, an edge is called a *bridge* if its deletion increases the number of connected components.

A graph is called 2-*connected* if it has no cut vertices. A *block* in a graph is its maximal 2-connected subgraph.

Lemma 2.4 ([6, Theorem 3.3]). *In a graph, two vertices lie in the same block with at least three vertices if and only if they lie on a common simple cycle.*

The *block graph* $B(G)$ of a graph G is the intersection graph on the collection of blocks in G. In other words, the vertices of $B(G)$ correspond to the blocks in G with two blocks being adjacent provided they share a common vertex (which must be a cut vertex in G). An abstract graph H is called a *block graph* if it is isomorphic to $B(G)$ for some G. The next characterization of block graphs is often used as their definition.

Theorem 2.5 ([5, Theorems 1 and 2]). *A graph H is a block graph if and only if every block in H is a complete subgraph.*

Note that Theorem 2.5 immediately asserts that any tree is a block graph.

There is another approach which leads to SUITs that is closely related to the construction of block graphs. Namely, for a connected graph G we consider the graph $\mathrm{bc}(G)$ having the cut vertices and blocks in G as its vertices, and two vertices being adjacent provided one of them is a cut vertex u in G and another is a block B in G which contains u. The graph $\mathrm{bc}(G)$ is called the *block-cutpoint-tree* of G. It turns out, that the image of this graph operator consist exactly of SUITs.

Theorem 2.6 ([6, Theorem 4.4]). *For a graph H, there exists a connected graph G with $\mathrm{bc}(G) \simeq H$ if and only if H is a SUIT.*

The *line graph* $L(G)$ of a given graph G is the intersection graph on the collection of edges $E(G)$. In other words, the vertices of $L(G)$ correspond to the edges of G with two edges being adjacent if they share a common vertex. A well-known Whitney isomorphism theorem [10] states that two connected graphs G and H are isomorphic if and only if $L(G)$ and $L(H)$ are isomorphic, unless $\{G, H\} = \{K_3, K_{1,3}\}$. We will use the next characterization of line graphs of trees.

Proposition 2.7 ([6, Theorem 8.5]). *A graph is the line graph of a tree if and only if it is a connected block graph in which each cut vertex lies in exactly two blocks.*

2.3. METRIC GRAPH THEORY

Let G be a connected graph. The *distance* $d_G(u, v)$ between a pair of vertices $u, v \in V(G)$ is the length (i.e., the number of edges) of the shortest path between u and v in G. The *eccentricity* of u is the number

$$\mathrm{ecc}_G(u) = \max\{d_G(u, x) : x \in V(G)\}.$$

The *diameter* and *radius* of G are the numbers

$$\begin{aligned}\operatorname{diam} G &= \max\{\operatorname{ecc}_G(u) : u \in V(G)\} \\ &= \max\{d_G(u,v) : u, v \in V(G)\}\end{aligned}$$

and

$$\operatorname{rad} G = \min\{\operatorname{ecc}_G(u) : u \in V(G)\},$$

respectively.

The *metric interval* between two vertices $u, v \in V(G)$ in a connected graph G is the set

$$[u,v]_G = \{x \in V(G) : d_G(u,x) + d_G(x,v) = d_G(u,v)\}.$$

For a triple of vertices $u, v, w \in V(G)$, we define the corresponding *median set* as

$$M_G(u,v,w) = [u,v]_G \cap [u,w]_G \cap [v,w]_G.$$

Lemma 2.8. *If in a connected graph G, vertices $x, y, z \in V(G)$ have a non-empty median set, then at least one of the distances $d_G(x,y)$, $d_G(x,z)$, $d_G(y,z)$ is even.*

Proof. Assume all the distances $d_G(x,y)$, $d_G(x,z)$, $d_G(y,z)$ are odd. Fix an element $m \in M_G(x,y,z)$. Then

$$\begin{aligned}d_G(x,y) &= d_G(x,m) + d_G(m,y), \\ d_G(x,z) &= d_G(x,m) + d_G(m,z), \\ d_G(y,z) &= d_G(y,m) + d_G(m,z).\end{aligned}$$

Hence,

$$d_G(x,y) + d_G(x,z) + d_G(y,z) = 2d_G(x,m) + 2d_G(y,m) + 2d_G(z,m),$$

an even number. The obtained contradiction proves the lemma. □

A subgraph H of a connected graph G is called *isometric* provided H is connected and $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$. The next folklore result is the cornerstone of many proofs in metric graph theory.

Lemma 2.9. *In a connected graph, any odd cycle of the smallest length is isometric.*

Given a connected graph G, a vertex $u \in V(G)$ and a set of vertices $A \subset V(G)$, the *distance from u to A* is the number

$$d_G(u,A) = \min\{d_G(u,a) : a \in A\}.$$

The projection of u onto A is the set

$$\operatorname{pr}_A(u) = \{a \in A : d_G(u, A) = d_G(u, a)\}.$$

A set $A \subset V(G)$ is called *Chebyshev* provided $|\operatorname{pr}_A(u)| = 1$ for every $u \in V(G)$. Having a Chebyshev set A, we will consider the projection pr_A just as a mapping from $V(G)$ to itself.

Let G be a connected graph, $u \in V(G)$ and $A \subset V(G)$. A vertex $a \in A$ is called a *gate for u in A* provided for all $x \in A$ it holds $a \in [u, x]_G$. For example, any $a \in A$ is a gate in A for itself. It is also can be easily proved that for any $u \in V(G)$ there exists at most one gate in A. This motivates the following definition. A set A is called *gated* if for any $u \in V(G)$ there exists a gate for u in A. For such a set A, by $g_A(u)$ we will denote the unique gate for u in A. One can observe that any gated set is Chebyshev (with $g_A(u)$ being the unique element in $\operatorname{pr}_A(u)$).

It is clear that empty set is not gated, hence the intersection of gated sets is not always gated itself. However, the next result holds.

Lemma 2.10 ([3])**.** *The non-empty intersection of gated sets is a gated set.*

The following lemma specifies an important family of gated sets in connected graphs.

Lemma 2.11. *The vertex set of any connected union of blocks in a connected graph is a gated set.*

Proof. Let G be a connected graph and $B_1, \dots, B_m$ be its blocks such that $H = \bigcup_{i=1}^m B_i$ is a connected subgraph of G. Put $A = V(H)$. To the contrary, assume that A is not gated. Then there is a vertex $x \in V(G)$ such that for all $a \in A$ there exists $b \in A$ with $a \notin [x, b]_G$. Clearly, $x \notin A$. Let x be such a vertex with the smallest distance $d_G(x, A)$. Consider any vertex $a \in \operatorname{pr}_A(x)$ and the corresponding vertex $b \in A$. Fix three shortest paths: P_1 – from x to a in G, P_2 – from x to b in G, and P_3 – from a to b in H (here we use the connectedness of H). It is easy to see that $V(P_1) \cap V(P_2) = \{x\}$ (otherwise, there would be a contradiction with the minimality of $d_G(x, A)$) and $V(P_1) \cap V(P_3) = \{a\}$ (otherwise, we obtain a contradiction with the condition $a \in \operatorname{pr}_A(x)$). Now fix a vertex $b' \in V(P_2) \cap V(P_3)$ with the smallest distance $d_{P_2}(x, b')$ (in other words, b' is the first vertex on a path P_2 that lies on P_3). Concatenate the following paths: P_1 (from x to a) with the part of P_3 from a to b' and the part of P_2 from b' to x. The resulted closed walk would be a simple cycle (see Figure 1). Hence, by Lemma 2.4, we obtain that x and a lie in the same block B_i in G. Thus, $x \in A$. The obtained contradiction proves the lemma. □

Lemma 2.12. *Let G be a graph and $v \in V(G)$ be a cut vertex lying in blocks $B_1, \dots, B_m$. Then for each $i \in \{1, \dots, m\}$ there exists a non-cut vertex $u_i \in V(G)$ such that $v = g_{A_i}(u_i)$ for the gated set $A_i = \bigcup_{j=1, j\neq i}^{m} V(B_j)$.*

Proof. Clearly, A_i is the vertex set of a connected union of blocks in G. Hence, by Lemma 2.11, A_i is a gated set. Further, for any $i \in \{1, \dots, m\}$ consider the connected component G_i in $G-v$ which contains the set $V(B_i) \setminus \{v\}$ (clearly, this is a connected set in G). As G_i is a connected graph, it has a non-cut vertex $u_i \in V(G_i)$. It is easy to see that u_i is also a non-cut vertex in G. Moreover, for any $x \in A_i$ every shortest $u_i - x$ path in G contains v which yields the equality $v = g_{A_i}(u_i)$. □

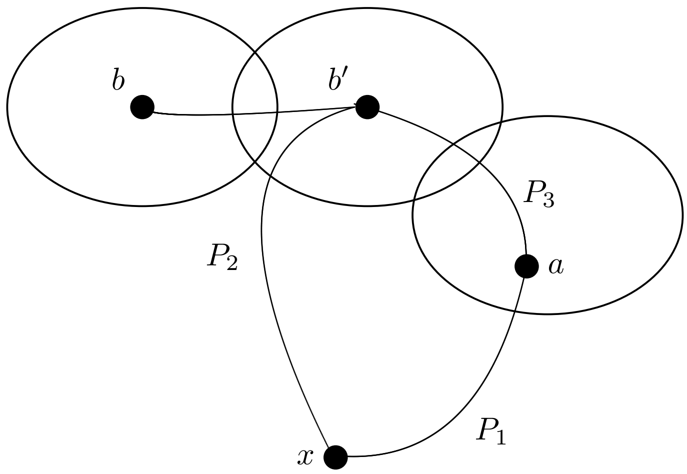

Fig. 1. An illustration for the proof of Lemma 2.11

3. MAIN RESULTS

3.1. NCO-GRAPHS AND SUITS

At first, we present the criterion for a connected graph to have odd distances between any two of its different non-cut vertices. Subsequently, we will refer to such graphs as *NCO-graphs.*

Theorem 3.1. *A connected graph is an NCO-graph if and only if it is a line graph of a SUIT.*

Proof. Sufficiency. Immediately follows from the fact that for each pair of edges $xy, uv \in E(T)$ in any tree T it holds

$$d_{L(T)}(xy, uv) = \max\{d_T(x, u), d_T(x, v), d_T(y, u), d_T(y, v)\} - 1.$$

Now assuming T is a SUIT and taking xy, uv to be the non-cut vertices in $L(T)$, without loss of generality, we can conclude that $x, u \in \text{Leaf}(T)$. Hence, the maximum above equals $d_T(x, u)$, which is an even number. Therefore, $d_{L(T)}(xy, uv) = d_T(x, u) - 1$ is odd.

Necessity. Let G be an NCO-graph. We break down the proof in this direction into three separate claims.

Claim 1. G is a block graph.

By contradiction, we assume G is not a block graph. Then it contains a non-complete block B, implying there exist two vertices $u, v \in V(B)$ with $d_B(u, v) = d_G(u, v) = 2$. Fix a vertex $w \in V(B)$ with $w \in [u, v]_G$. It is clear that u and v cannot be non-cut vertices simultaneously. Further we consider two cases.

Case 1. Both u, v are cut vertices.

By Lemma 2.12, there exist two different non-cut vertices $x, y \in V(G)$ with $\mathrm{pr}_B(x) = u$ and $v = \mathrm{pr}_B(y)$. If w is also a non-cut vertex, then $d_G(x, w) + d_G(w, y) = d_G(x, y)$ implying these three distances cannot be odd simultaneously. If w is a cut vertex, then again, invoking Lemma 2.12, we can ensure the existence of a non-cut vertex $z \in V(G)\backslash\{x, y\}$ having $\mathrm{pr}_B(z) = w$. Hence, $w \in M_G(x, y, z)$ implying that, by Lemma 2.8, the distances $d_G(x, y)$, $d_G(x, z)$, $d_G(y, z)$ cannot be odd simultaneously.

Case 2. Exactly one of the vertices u, v is a cut vertex.

Without loss of generality, suppose u is a cut vertex and v is not. As before, there is a non-cut vertex $x \in V(G)$ with $\mathrm{pr}_B(x) = u$. If w is non-cut, then $d_G(x, w)$ and $d_G(x, v)$ are cannot be odd simultaneously. If w is a cut vertex, then we fix $z \in V(G)\backslash\{x\}$ with $\mathrm{pr}_B(z) = w$. We have $w \in M_G(x, v, z)$. Thus, in this case, the distances $d_G(x, z)$, $d_G(x, v)$ and $d_G(z, v)$ cannot be odd simultaneously as well. Therefore, G is a block graph.

Claim 2. G is a line graph of a tree.

According to Proposition 2.7, we only need to show that each cut vertex of G lies exactly in two blocks. To the contrary, assume there exists a cut vertex $v \in V(G)$ which lies in blocks $B_1, \ldots, B_m$, where $m \geq 3$. By Lemma 2.12, there exist non-cut vertices $u_1, u_2, u_3 \in V(G)$ such that $v = g_{A_1}(u_1) = g_{A_2}(u_2) = g_{A_3}(u_3)$, where $A_i = \bigcup\limits_{j=1, j\neq i}^{m} V(B_j)$ for $i \in \{1, \ldots, m\}$. In particular, $v \in M_G(u_1, u_2, u_3)$. Also, we claim that u_1, u_2, u_3 are pairwise different. Indeed, if, for example, $u_1 = u_2$, then fix the vertex $w \in N_G(v)$ on some shortest $u_1 - v$ path. By construction, $w \in B_1 \cap B_2$ which is a contradiction. Finally, by Lemma 2.8, one of the distances $d_G(u_1, u_2)$, $d_G(u_2, u_3)$, $d_G(u_1, u_3)$ is even and non-zero. The obtained contradiction proves that G is a line graph of a tree.

Claim 3. G is a line graph of a SUIT.

Again, we prove it by contradiction. Let T be a tree with $G \simeq L(T)$ and $u, v \in \mathrm{Leaf}(T)$ such that $d_T(u, v)$ is odd. Let $x, y \in V(T)$ be support vertices for u and v, respectively. Then ux and yv are non-cut vertices in $L(T)$, and hence in G. However, $d_{L(T)}(ux, yv) = d_T(u, v) - 1$ is even. The obtained contradiction proves the theorem. □

Since for any tree T and a pair of its leaves $u, v \in \mathrm{Leaf}(T)$ with the respective support vertices u', v' it holds $d_{T\backslash \mathrm{Leaf}(T)}(u', v') = d_T(u, v) - 2$, the next proposition is clear.

Proposition 3.2. *If T is a SUIT, then $T\backslash \mathrm{Leaf}(T)$ is also a SUIT.*

Now we show that any SUIT can be obtained from a star by repeatedly applying one simple operation. To make this precise, we need one more definition. By an *inner even vertex* we mean an inner vertex that has even distances to all the leaves. By parity, the distances between inner even vertices are always even. And clearly, an inner even vertex cannot be a support vertex.

Theorem 3.3. *Let T be a tree with* $\operatorname{diam} T \geq 4$*. Then T is a SUIT if and only if T is inductively obtained from $K_{1,m}, m > 1$ by the following operation: each leaf must become a support vertex for new leaves and each inner even vertex may become a support vertex for new leaves.*

Proof. Sufficiency. We use induction on the number of such operations.

For the base case, since $K_{1,m}$ does not have inner even vertices, the only operation that can be applied here is adding leaves on each leaf of $K_{1,m}$, i.e., we obtain T such that $T \setminus \operatorname{Leaf}(T) = K_{1,m}$. Clearly, for all $u, v \in \operatorname{Leaf}(T)$ it holds $d_T(u, v) = 4$. Hence, T is a SUIT.

Now let T be a tree obtained from $K_{1,m}$ by $q \geq 2$ operations. Denote by T' the tree obtained by the respective $q - 1$ operations. Let $x_1, \ldots, x_k$ be those inner even vertices of T' that would become support vertices in q-th operation. Fix two leaves $u, v \in \operatorname{Leaf}(T)$. We have the following cases:

1. u, v are new leaves attached to the leaves u', v' of T', respectively. Then $d_T(u, v) = d_{T'}(u', v') + 2$ is an even number.
2. u is a new leaf attached to the leaf u' of T' and v is a leaf-neighbor of x_i for some $i \in \{1, \ldots, k\}$. Then $d_T(u, v) = d_{T'}(u', x_i) + 2$ is an even number.
3. u is a leaf-neighbor of x_i, v is a leaf-neighbor of x_j for some $i, j \in \{1, \ldots, k\}$. Then $d_T(u, v) = d_{T'}(x_i, x_j) + 2$ is an even number.

Necessity. Again, we use induction on $\operatorname{rad} T$ to show that T is obtained from $K_{1,m}$ by applying exactly $\operatorname{rad} T - 1$ operations.

For the base case, assume $\operatorname{diam} T = 4$. Then the tree $T \setminus \operatorname{Leaf}(T)$ has diameter two. Thus, $T \setminus \operatorname{Leaf}(T) \simeq K_{1,m}$ for $m > 1$. Clearly, T is obtained from $T \setminus \operatorname{Leaf}(T)$ by adding new leaves to each leaf in $T \setminus \operatorname{Leaf}(T)$.

Now let $\operatorname{diam} T \geq 5$. Similarly, we consider the tree $T' = T \setminus \operatorname{Leaf}(T)$, which is also a SUIT (see Proposition 3.2). It is clear that $\operatorname{rad} T' = \operatorname{rad} T - 1$. Thus, by induction assumption, T' is obtained from $K_{1,m}$ by applying $\operatorname{rad} T' - 1$ operations.

Let $S \subset V(T')$ be the set of all support vertices for leaves in T. Clearly, $\operatorname{Leaf}(T') \subset S$. We claim that any vertex $v \in S \setminus \operatorname{Leaf}(T')$ is an inner even vertex in T'. Indeed, let v be the support vertex for the leaf $u \in \operatorname{Leaf}(T)$. If v is not an inner even vertex in T', then there is a leaf $x \in \operatorname{Leaf}(T')$ with $d_{T'}(v, x)$ being odd. However, since $x \in S$, there is a leaf $y \in \operatorname{Leaf}(T)$ with x being its support vertex. This implies

$$d_T(u, x) = 1 + d_T(v, x) + 1 = d_T(v, x) + 2 = d_{T'}(v, x) + 2$$

is odd as well. A contradiction, which proves that T is indeed obtained from $K_{1,m}$ by applying exactly $\operatorname{rad} T' - 1 + 1 = \operatorname{rad} T' = \operatorname{rad} T - 1$ operations. □

3.2. NCE-GRAPHS

In this section, we study connected graphs in which the distance between any two non-cut vertices is even. For convenience, we will refer to them as *NCE-graphs*. Thus, SUITs are exactly the NCE-trees. The crucial step in obtaining the criterion for NCE-graphs is the following observation.

Theorem 3.4. *Every NCE-graph is bipartite.*

Proof. To the contrary, assume that G is an NCE-graph which is not bipartite. Hence, G has an odd cycle. Let C be an odd cycle of the smallest length in G. Then C is an isometric subgraph in G (see Lemma 2.9). Consider the following vertex 2-coloring of C: $f\colon V(C) \to \{0,1\}$, $f(u) = 1$ for all cut vertices u in G and $f(u) = 0$, otherwise. Since C has an odd length, there is an edge $uv \in E(C)$ with $f(u) = f(v)$. If $f(u) = f(v) = 0$, then both u and v are non-cut vertices in G at the distance $d_G(u,v) = 1$, which is a contradiction. Hence, we have $f(u) = f(v) = 1$ implying that u and v are both cut vertices in G.

Further, fix the vertex $x \in V(C)$ with $d_C(x,u) = d_C(x,v)$ (in other words, the vertex x is the "opposite" to the edge uv on C). By Lemma 2.12, there is a non-cut vertex $a \in V(G)$ with u being the gate for a in $V(C)$. A similar argument ensures the existence of a non-cut vertex $b \in V(G)$ with v being the gate for b in $V(C)$ (see Figure 2).

Since G is an NCE-graph, $d_G(a,b)$ is an even number. Also,

$$d_G(a,b) = d_G(a,u) + d_G(u,v) + d_G(v,b) = d_G(a,u) + 1 + d_G(v,b)$$

implying that $d_G(a,u)$ or $d_G(v,b)$ is an odd number. Without loss of generality, assume that $d_G(a,u)$ is odd. In this case, $d_G(v,b)$ is even.

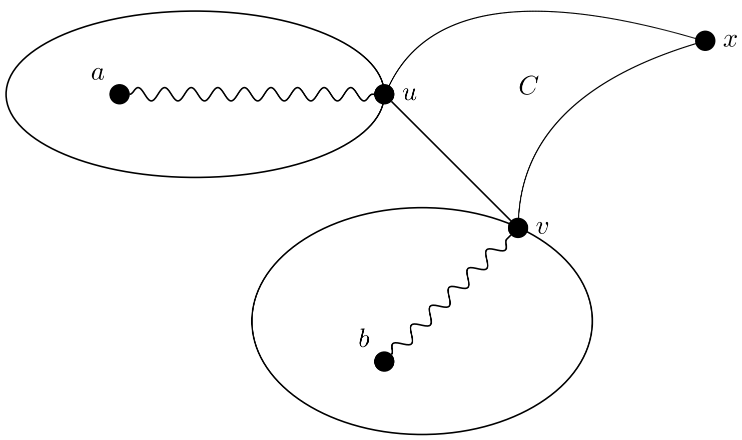

Fig. 2. An illustration for the proof of Theorem 3.4

If x is a non-cut vertex in G, then both distances $d_G(a,x)$ and $d_G(b,x)$ are even. On the other hand,

$$\begin{aligned} d_G(a,x) &= d_G(a,u) + d_G(u,x) = d_G(a,u) + d_G(v,x) \\ &= d_G(a,u) + d_G(b,x) - d_G(v,b) \end{aligned}$$

is an odd number (as $d_G(a,u)$ is odd and $d_G(b,x), d_G(b,v)$ are even numbers), which is a contradiction.

Finally, assume that x is a cut vertex. Using Lemma 2.12, we can conclude that there is a non-cut vertex $c \in V(G)$ with x being the gate for c in $V(C)$. We have that $d_G(a,c)$ and $d_G(b,c)$ are even numbers. However, if $d_G(c,x)$ is odd, then $d_G(u,x) = d_G(v,x)$ is even implying that $d_G(b,c) = d_G(b,v) + d_G(v,x) + d_G(x,c)$ is odd. Similarly, if $d_G(c,x)$ is even, then $d_G(a,c)$ is odd. The obtained contradictions show that G cannot have odd cycles, hence, G is bipartite. □

Having proved Theorem 3.4, the criterion for NCE-graphs follows easily.

Corollary 3.5. *A connected graph G is an NCE-graph if and only if G is bipartite having all its non-cut vertices in a common part of the corresponding bipartition.*

Proof. Immediately follows from Theorem 3.4 and the fact that in a connected bipartite graph the distance between two vertices is even if and only if they lie in the same part of the bipartition. □

Corollary 3.6. *Any bipartite graph is an induced subgraph of an NCE-graph.*

Proof. Let H be a bipartite graph and $V(H) = A \sqcup B$ be a bipartition of H. Put $A' = \{u \in A \colon u \text{ is a non-cut vertex in } H\}$. In order to construct the desired NCE-graph G, take H and for any vertex u in A' add a new leaf u' adjacent to u:

$$V(G) = V(H) \sqcup \{u' \colon u \in A'\} \quad \text{and} \quad E(G) = E(H) \cup \{uu' \colon u \in A'\}.$$

It is clear that now all the non-cut vertices are in part B and leaves $\{u' \colon u \in A'\}$ also join the part B in a bipartition of G. Also, all connected components of G are NCE-graphs. To make G a connected graph, add a vertex x and connect it to one representative of each "non-cut"-part of each connected component. Again, all the non-cut vertices in the resulting graph lie in the same part, hence it is an NCE-graph by Corollary 3.5. □

We illustrate the construction in the proof of Corollary 3.6 with the following example.

Example 3.7. Consider a graph H with

$$V(H) = \{1,2,3,4,5,a,b,c,d\} \quad \text{and} \quad E(H) = \{13,14,23,24,25,ac,bc,bd\}.$$

Clearly, H is bipartite having two connected components. Put $A = \{1,2,a,b\}$ and $B = \{3,4,5,c,d\}$ (see Figure 3). Then $A' = \{1,a\}$. The corresponding connected NCE-graph G, which contains H as its induced subgraph, is depicted in Figure 4.

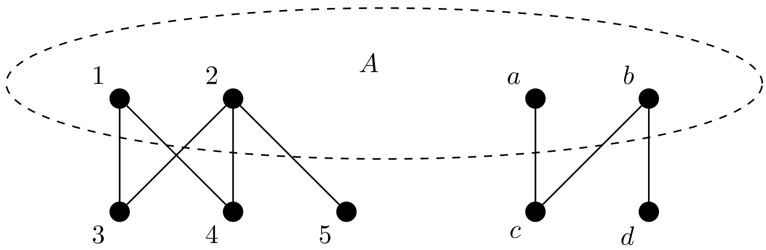

Fig. 3. The bipartite graph H from Example 3.7

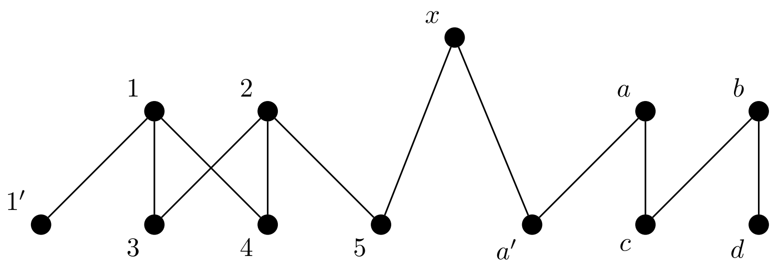

Fig. 4. An NCE-graph G having H as an induced subgraph

Corollary 3.8. *Every tree is a subtree of a SUIT.*

Proof. Directly follows from the proof of Corollary 3.6 as the addition of new leaves cannot produce a cycle or disconnect a tree. □

Combining Theorem 3.1 and Corollary 3.8, we can easily characterize induced subgraphs of NCO-graphs.

Corollary 3.9. *A graph is an induced subgraph of an NCO-graph if and only if it is a line graph of a tree.*

Proof. Necessity. Let H be an induced subgraph of an NCO-graph G. By Theorem 3.1, G is a line graph of a SUIT. Proposition 2.7 asserts that G is a block graph in which every cut vertex lies in two blocks. It is clear that H is also a block graph with the same property. Hence, by Proposition 2.7, H is a line graph of some tree (not necessarily a SUIT).

Sufficiency. Assume $H \simeq L(T_0)$ for some tree T_0. By Corollary 3.8, T_0 is a subtree of some SUIT T. Thus, Theorem 3.1 implies that $H \simeq L(T_0)$ is an induced subgraph of an NCO-graph $L(T)$. □

It is not a coincidence that the NCE-graph in Figure 4 has two leaves. We call a block B in a graph G a *leaf block* if it contains a unique cut vertex from G (equivalently, if B is a non-cut vertex in the block graph $B(G)$).

Proposition 3.10. *Every leaf block in an NCE-graph is a leaf edge.*

Proof. Assume that B is a leaf block in an NCE-graph G and $u \in V(B)$ is its unique cut vertex. If $|V(B-u)| = 1$, then B is a leaf edge. Otherwise, $B-u$ would have an edge $xy \in E(B-u)$. However, x and y are non-cut vertices in G. A contradiction. □

Remark 3.11. Finite multigraphs (with multiple edges allowed, but no loops) that satisfy the property described in Proposition 3.10 find applications in the study of smooth functions, particularly Morse functions, on closed manifolds. Specifically, a digraph D is said to have a *good orientation* if it is acyclic, and all its sources and sinks are leaves. Sharko proved in [9] that a multigraph G serves as a Reeb graph of some smooth function with isolated critical points on a closed smooth manifold M^n if and only if G admits a good orientation.

Later, in [4], Gelbukh further clarified this result by showing that such multigraphs G are precisely those in which every leaf block is an edge. This finding was extended in [4] to multigraphs G that allow the so-called S-good orientations (characterized by the structure of blocks in G). Here, given $S \subset \mathbb{Z}_+$, a digraph D is said to have an *S-good orientation* if it is acyclic, and the degrees of its sources and sinks belong to S.

From Proposition 3.10 we immediately obtain the next corollary.

Corollary 3.12. *Any NCE-graph with at least two vertices has at least two leaves.*

Moreover, we can characterize NCE-graphs having exactly two leaves. To do this, by $B_2(G)$ we denote the subgraph of the block graph $B(G)$ which is induced by all bridges in G.

Theorem 3.13. *A graph G is an NCE-graph with $|\operatorname{Leaf}(G)| = 2$ if and only if G is a path with an odd number of vertices, or G is not a path and satisfies the following conditions:*

1. *the block graph $B(G)$ is a path with at least two vertices,*
2. *each block in G is isomorphic to K_2 or $K_{2,m}$ for some $m \geq 2$,*
3. *each block $B \simeq K_{2,m}$, $m \geq 2$ in G has exactly two cut vertices from G which form a part of the bipartition of B,*
4. *each connected component in $B_2(G)$ which does not contain leaves from $B(G)$ has an even number of vertices,*
5. *each connected component in $B_2(G)$ containing leaves from $B(G)$ has an odd number of vertices.*

Proof. Necessity. If G is a path, it is clear that in order to be an NCE-graph, G must have an odd number of vertices. Hence, in what follows we assume that G is not a path. We prove each condition separately.

1. From Proposition 3.10 and the condition $|\operatorname{Leaf}(G)| = 2$ it follows that $B(G)$ has exactly two non-cut vertices. Therefore, the block graph $B(G)$ must be a path with at least two vertices.

2. Let B be a block of G. If B is a leaf block, then $B \simeq K_2$. Otherwise, B contains exactly two cut vertices from G (as $B(G)$ is a path), say $u, v \in V(B)$. As B is

2-connected, u and v lie on a cycle C in B. Since $V(C)\backslash\{u,v\}$ contains only non-cut vertices in G, C must have length at most four (otherwise, C would contain two adjacent non-cut vertices). If C is a triangle, then G is not bipartite implying it is not an NCE-graph (see Theorem 3.4). This implies that C is of length four, say $C = \{u-x-v-y-u\}$. Moreover, it is clear that C is an induced subgraph in G. Finally, for any $w \in V(B)\backslash\{u,v\}$ we have $N_B(w) \subset \{u,v\}$. However, as B is 2-connected, we have $N_B(w) = \{u,v\}$. This proves that $B \simeq K_{2,m}$.

3. If a block $B \simeq K_{2,m}$ in G contained at most one cut vertex from G, then B would have two adjacent non-cut vertices from G. A contradiction.

4. Assume there exists a connected component H in $B_2(G)$ with an odd number of vertices and which does not contain leaves in $B(G)$. It is clear that H is a path. Denote by H_1 and H_2 its two leaves. Also, there are two blocks $B_1, B_2 \in V(B(G))\backslash V(B_2(G))$ which are adjacent in $B(G)$ with H_1 and H_2, respectively. This means that there are two cut vertices from G, say $x, y \in V(G)$ with $x \in V(H_1) \cap V(B_1)$ and $y \in V(H_2) \cap V(B_2)$. Since B_1 and B_2 are not bridges in G, they contain two non-cut vertices from G, say $u \in V(B_1)$ and $v \in V(B_2)$ (see Figure 5). From condition 2 it follows that $ux, vy \in E(G)$. And since each vertex from H is a bridge in G, we obtain $d_G(u,v) = 1 + |V(H)| + 1 = 2 + |V(H)|$ is an odd number. This is a contradiction.

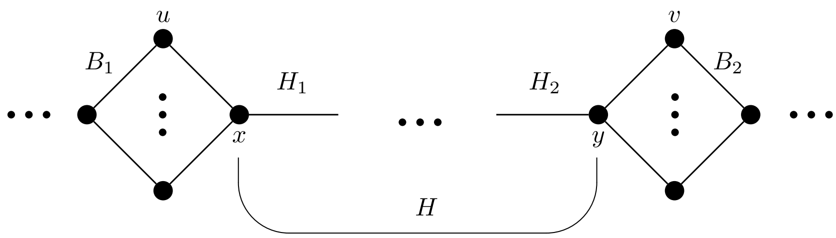

Fig. 5. The distance between two non-cut vertices u and v is odd

5. Fix one of the two connected components H in $B_2(G)$ which contains a leaf B_0 in $B(G)$. Since G is not a path, it contains a block which is not a bridge. By the condition 2, all such blocks are isomorphic to $K_{2,m}$, $m \geq 2$. Hence, let $B \in V(B(G))\backslash V(B_2(G))$ be such a block which is adjacent to a leaf B' from H. This implies the existence of a cut vertex $x \in V(G)$ with $x \in V(B') \cap V(B)$ (see Figure 6). Fix a non-cut vertex $u \in V(B)$ in G and a leaf $v \in V(B_0)$. Then $d_G(u,v) = d_G(u,x) + |V(H)| = 1 + |V(H)|$ implying that $|V(H)|$ is an odd number.

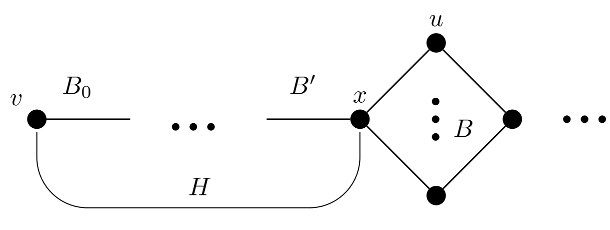

Fig. 6. The distance between a non-cut vertex u and a leaf v is even

Sufficiency. If G is a path with an odd number of vertices, then clearly G is an NCE-graph. Otherwise, suppose that G is not a path satisfying conditions 1–5. Condition 1 immediately ensures that G is connected.

Aiming towards contradiction, we assume that G is not an NCE-graph. Then there is a pair of non-cut vertices in G at the odd distance. Let $u, v \in V(G)$ be such a pair with the smallest $d_G(u, v)$. Also, let B_1 and B_2 be the (unique) blocks containing u and v, respectively. If $B_1 = B_2 = B$, then condition 2 implies that $B \simeq K_{2,m}$ for some $m \geq 2$. Thus, condition 3 asserts that u and v lie in the same part of the bipartition of B. But this means that $d_G(u, v) = d_B(u, v) = 2$.

Hence, suppose $B_1 \neq B_2$. Fix the shortest path $P : u = x_0 - \ldots - x_d = v$ between u and v (here $d = d_G(u, v) \geq 3$). From the minimality of $d_G(u, v)$ it follows that any vertex x_i, $1 \leq i \leq d-1$ is a cut vertex in G. From conditions 2–3 we can conclude that any edge $x_i x_{i+1}$, $1 \leq i \leq d-2$ is a bridge in G. Clearly, these bridges form a connected subgraph H in $B_2(G)$. We proceed considering two cases.

Case 1. Neither B_1 nor B_2 is a leaf in $B(G)$.

Here H is a connected component in $B_2(G)$. Indeed, if this is not the case, there is another bridge $B \in V(B_2(G)) \backslash V(H)$ adjacent to some bridge in H. Since $B \notin \{B_1, B_2\}$, there would be some cut vertex x_i, $1 \leq i \leq d-1$, lying in at least three different blocks in G (the blocks being $x_{i-1}x_i$, $x_i x_{i+1}$ and B provided $1 < i < d-1$; B_1, $x_1 x_2$ and B provided $i = 1$; B_2, $x_{d-1}x_{d-2}$ and B provided $i = d-1$). Hence, $B(G)$ would contain a triangle, which contradicts condition 1.

Therefore, H is a connected component in $B_2(G)$ not containing the leaves from $B(G)$. Condition 4 then implies that $|V(H)| = d-2$ is even. Consequently, d is also even, which contradicts our assumption.

Case 2. One of B_1, B_2 is a leaf in $B(G)$.

Without loss of generality, suppose B_1 is a leaf in $B(G)$. Then B_2 is not a leaf in $B(G)$. Indeed, assuming they both are leaves in $B(G)$ and invoking conditions 2–3, we obtain that $B_1 \simeq B_2 \simeq K_2$. Therefore, each block in G is a bridge. Combining this with condition 1, we obtain that G is a path, which is a contradiction.

Thus, B_2 is not a leaf in $B(G)$. We claim that in this case $V(H) \cup \{ux_1\}$ induces a connected component H' in $B_2(G)$. Indeed, as in the previous case, it is clear that H' is a connected subgraph. Again, assuming H' is not a connected component in $B_2(G)$ leads to the existence of a bridge $B \in V(B_2(G)) \backslash V(H')$ which is adjacent to some bridge in H'. But this would imply the existence of a cut vertex x_i, $1 \leq i \leq d-1$ lying in at least three different blocks in G. Again, a contradiction with condition 1.

Therefore, H' is a connected component in $B_2(G)$ containing a leaf B_1 from $B(G)$. Condition 5 now asserts that $|V(H')| = d-1$ is odd. Therefore, d is even again. The obtained contradiction finishes the proof. □

It is easy to see that every tree can be subdivided to an NCE-graph. Indeed, if we subdivide all the edges in a given tree, then the resulting tree clearly will be a SUIT. We now generalize this result by providing the full characterization of graphs which can be subdivided to NCE-graphs.

Proposition 3.14. *A connected graph G with $|V(G)| \geq 3$ can be subdivided to an NCE-graph if and only if the set of its non-cut vertices is independent.*

Proof. Necessity. To the contrary, suppose G contains a pair of adjacent non-cut vertices u and v. In this case, $d_G(u, v) = 1$ and the edge uv must be subdivided by a vertex x in order to make this distance even. However, in any such a subdivision H of G the vertices u and x (since $|V(G)| \geq 3$) are adjacent non-cut vertices. Thus, H is not an NCE-graph.

Sufficiency. Subdivide each edge $e \in E(G)$ between every pair of cut vertices in G by a new vertex x_e. Denote the obtained graph as H. It is clear that non-cut vertices in H are precisely the non-cut vertices of G with those new vertices x_e for which the respective edge e is not a bridge. Since the set of non-cut vertices in G is independent, we obtain that H is a bipartite graph with one part containing all cut vertices from G and second part containing all non-cut vertices from G and all x_e. Since all non-cut vertices of H lie in a common part of the bipartition, by Corollary 3.5, H is an NCE-graph. □

Corollary 3.15. *If a graph can be subdivided to an NCE-graph, then it has at least two leaves.*

Proof. Follows from Corollary 3.12 and the fact that subdivision of edges cannot produce new leaves. □

We say that an NCE-graph is *minimal* if it does not contain a strictly smaller spanning NCE-graph. Similarly, an NCE-graph is called *maximal* if it is not a spanning subgraph of a strictly bigger NCE-graph. It is easy to observe that minimal NCE-graphs are exactly SUITs. Indeed, on the one hand, it is clear that each SUIT is a minimal NCE-graph. On the other hand, from Corollary 3.5 it follows that the deletion of a non-bridge edge from any NCE-graph results in an NCE-graph as well. Hence, each edge in a minimal NCE-graph G is a bridge. This means that G is a tree, implying that G is a SUIT.

In the next result, we characterize maximal NCE-graphs. To do this, we introduce several auxiliary definitions. Let G be a connected graph. A vertex $u \in V(G)$ is called *nc-even* (*nc-odd*) if it has even (odd) distances to all non-cut vertices in G. For instance, inner even vertices in trees provide an example of nc-even vertices. The *block degree* $\mathrm{bd}_G(u)$ of a vertex u in a connected graph G is the number of connected components in $G - u$. Thus, u is a cut vertex if and only if $\mathrm{bd}_G(u) \geq 2$.

Theorem 3.16. *An NCE-graph G is maximal if and only if $G \simeq K_{1,m}$ or the next conditions hold:*

1. *each block in G is a complete bipartite graph;*
2. *for every nc-odd vertex $u \in V(G)$ it holds:*
   (a) $\mathrm{bd}_G(u) = 2$,
   (b) *any cut vertex from $N_G(u)$ is separated by u from other vertices in $N_G(u)$.*

Proof. Necessity. Let G be a maximal NCE-graph and B be its block. By Corollary 3.5, B is bipartite with all the non-cut vertices in G from $V(B)$ located in a common part of the corresponding bipartition $V(B) = X \sqcup Y$. If there were two non-adjacent vertices $x \in X$, $y \in Y$, then adding the edge xy to G would not affect the block structure of G and, consequently, result in a strictly larger NCE-graph. Thus, B must be complete bipartite.

Now suppose that G is not a star and let $u \in V(G)$ be an nc-odd vertex. Since G is an NCE-graph, we can conclude that u is a cut vertex. Further, G is not a star, which implies that there exists a vertex $v \in V(G)$ with $d_G(u, v) = 2$. Fix a vertex $x \in N_G(u) \cap N_G(v)$. Note that v is also a cut vertex in G (and x can be a cut as well as a non-cut vertex).

If $\mathrm{bd}_G(u) \geq 3$, then there exist two different blocks B_1, B_2 of G which contain u, but do not contain the edge ux. Let us fix a vertex $y \in V(B_1) \cap N_G(u)$. It is clear that $yv \notin E(G)$. Add the edge yv to G in order to obtain the graph G' (see Figure 7).

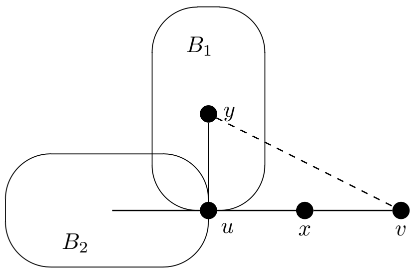

Fig. 7. By adding the missing edge yv to G we can make a bigger NCE-graph G'

We claim that G' is an NCE-graph as well. At first, one can observe that G' is a bipartite graph (with the same bipartition as G). Further, it is clear that the addition of a new edge does not turn non-cut vertices into cut vertices. And since $y - u - x - v$ is the only simple path between y and v in G, the addition of an edge yv to G could turn, possibly, a cut vertex x into a non-cut vertex. If x was a non-cut vertex in G, then G' has the same non-cut vertices and hence, G' is a NCE-graph by Corollary 3.5. Hence, suppose that this is not the case. Let x be a cut vertex in G and a non-cut vertex in G'. We have that x lies in the opposite part to u in G', which is the same part where all the non-cut vertices in G are. This means that G' is also a connected bipartite graph having all of its non-cut vertices in the same part. Corollary 3.5 again implies that G is an NCE-graph. The obtained contradiction shows that $\mathrm{bd}_G(u) = 2$ and hence, the condition 2(a) holds.

Now let $w \in N_G(u)$ be a cut vertex and $v \in N_G(u)$ such that w and v are not separated by u in G. This means that w, v, u lie in a common block in G. Denote this block as B. Since w is a cut vertex, we can fix a vertex $t \in N_G(w) \backslash V(B)$. Let us add the edge tv to G in order to obtain a graph G'' (see Figure 8). Since $d_G(t, v) = 3$ is odd, G'' is a bipartite graph as well. Moreover, as u do not separate w and v in G, the vertex u remains a cut vertex in G''. Since u is an nc-odd vertex in G and w is adjacent to u in G, the vertex w lies in the same part where all the non-cut vertices in G are. Hence, even if w becomes a non-cut vertex in G'', Corollary 3.5 ensures that G'' is an NCE-graph as well. The obtained contradiction justifies the condition 2(b).

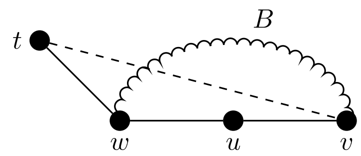

Fig. 8. By adding the missing edge tv to G we can make a bigger NCE-graph G''

Sufficiency. To the contrary, assume that G is not a maximal NCE-graph which satisfies both conditions. Let H be some maximal NCE-graph containing G as its spanning subgraph. Fix an edge $uv \in E(H)\backslash E(G)$.

Again, since G is connected, u, v lie on a cycle in H. Since H is an NCE-graph, by Theorem 3.4, any such a cycle is of even length. This implies that the distance $d_G(u,v)$ is odd. Hence, $d_G(u,v) \geq 3$. Fix the shortest path between u and v in G: $u - u_1 - \ldots - u_m = v$, where $m = d_G(u,v)$. Since u, v lie on a cycle in H, they lie in a common block in H. However, each block in G (and hence, in H) is complete bipartite implying that $uu_3 \in E(H)$. As was mentioned before, the deletion of a non-bridge edge from an NCE-graph results in an NCE-graph. Hence, deleting corresponding edges from H, we can conclude that adding an edge uu_3 to G also results in an NCE-graph. Denote this graph by G'. Further we consider two cases.

Case 1. either u_1 or u_2 is a non-cut vertex in G.

Without loss of generality, suppose u_1 is a non-cut vertex in G. Then u_2 is an nc-odd vertex in G. By condition 2(a), we have $\mathrm{bd}_G(u_2) = 2$. If u_2 does not separate u_1 and u_3 in G, then u_1, u_2 and u_3 lie in a common block B_1 in G. Similarly, as u_1 is a non-cut vertex in G, it cannot separate vertices u and u_2 in G. Thus, u, u_1 and u_2 also lie in a common block B_2 in G. This means that u and u_3 lie in a common block $B_1 = B_2$ in G (as both B_1 and B_2 contain an edge u_1u_2). However, $d_G(u,u_3) = 3$ contradicts the condition 1. This means that u_2 separates u_1 and u_3 in G (see Figure 9). Hence, u_2 does not separate these two vertices in G' (as G' is obtained from G by adding exactly the edge uu_3). Combining this fact with the equality $\mathrm{bd}_G(u_2) = 2$, we can conclude that u_2 is a non-cut vertex in G'. But this is a contradiction, since we have two adjacent non-cut vertices u_1 and u_2 in an NCE-graph G'.

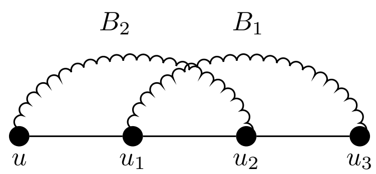

Fig. 9. The vertex u_2 must separate u_1 and u_3 in G

Case 2. both u_1 and u_2 are cut vertices in G.

Since G is an NCE-graph, Corollary 3.5 guarantees that either u_1 or u_2 is an nc-odd vertex in G. Without loss of generality, let us assume that u_2 is an nc-odd vertex in G. From condition 2(a), it follows that $\mathrm{bd}_G(u_2) = 2$. Moreover, based on condition 2(b), we observe that u_2 separates the cut vertex u_1 from u_3 in G. Consequently, when forming G', u_2 becomes a non-cut vertex. However, as u_2 was an nc-odd vertex in G, it lies in the opposite part compared to where all the non-cut vertices in G (and consequently in G') are located. Using Corollary 3.5 and the uniqueness of the bipartition for connected bipartite graphs, we conclude that G' is not an NCE-graph. This contradiction establishes that G is indeed a maximal NCE-graph. □

Remark 3.17. Note that the condition 2 in Theorem 3.16 implies that an nc-odd vertex in a maximal NCE-graph can be adjacent to at most two cut vertices.

Corollary 3.18. *In a maximal NCE-graph, any vertex with block degree at least three is an nc-even vertex.*

Proof. Immediately follows from the condition 2(a) of Theorem 3.16. □

Corollary 3.19. *A graph is a minimal and maximal NCE-graph simultaneously if and only if it is a SUIT in which every vertex of degree at least three is an nc-even vertex.*

Proof. Follows from Corollary 3.18 and the fact that in a tree T we have $d_T(u) = \mathrm{bd}_T(u)$ for all $u \in V(T)$. □

Corollary 3.20. *Any NCE-graph with exactly two leaves is a maximal NCE-graph.*

Proof. Let G be an NCE-graph with $|\,\mathrm{Leaf}(G)| = 2$. Theorem 3.13 implies that each block in G is complete bipartite. Hence, G satisfies the condition 1 from Theorem 3.16. Furthermore, the equality $|\,\mathrm{Leaf}(G)| = 2$ clearly implies that $\mathrm{bd}_G(u) = 2$ for all cut vertices $u \in V(G)$. In particular, this holds for all nc-odd vertices, implying condition 2(a) from Theorem 3.16. Finally, if an nc-odd vertex u is adjacent to a cut vertex w, then the conditions 2 and 3 from Theorem 3.13 imply that the edge uw is a bridge in G. Thus, condition 2(b) from Theorem 3.16 also holds. Hence, by Theorem 3.16, G is a maximal NCE-graph. □

Example 3.21. Consider the graph G depicted in Figure 10. It can be easily verified that G is an NCE-graph; however, it is not maximal. Indeed, G satisfies conditions 1 and 2(a) of Theorem 3.16, but the condition 2(b) is violated at the vertex u. Namely, u is an nc-odd vertex in G that does not separate the cut vertex w from v. If we add the edge tv to G, then the resulting graph would be a maximal NCE-graph.

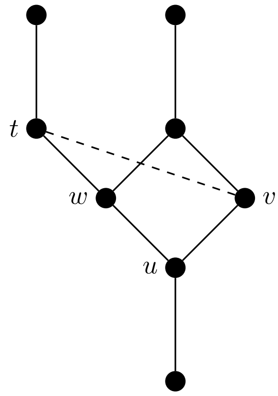

Fig. 10. A non-maximal NCE-graph G:
an nc-odd vertex u does not separate the cut vertex w from v

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Kateryna Antoshyna
https://orcid.org/0009-0005-9221-1351

Institute of Mathematics of the National Academy of Sciences of Ukraine
01024 Kyiv, Ukraine

Sergiy Kozerenko (corresponding author)
kozerenkosergiy@ukr.net
https://orcid.org/0000-0002-5716-3084

Department of Mathematics
National University of Kyiv-Mohyla Academy
04070 Kyiv, Ukraine

Kyiv School of Economics
03113 Kyiv, Ukraine

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