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Rozważania o metodzie elementów skończonych
Some considerations on the finite element method

Praca dedykowana pamięci Doktora Romana Kamińskiego

Słowa kluczowe: metoda elementów skończonych, płyty, powłoki, historia
Key words: finite element method, plates, shells, history

Wprowadzenie

Metoda elementów skończonych (MES) dominuje niepodzielnie w zastosowaniach inżynierskich od kilkudziesięciu lat. MES jest metodą obliczeniową, której rozwój datuje się na ostatnie 50 lat. Rozwój ten związany jest z niebywałym postępem w technologii komputerowej. Opowiadania sprzed trzydziestu lat o obliczeniach inżynierskich, na które wybierano się jak na biwak, ze śpiworami i prowiantem, brzmią dziś jak bajka, ale bajką nie są. Także twórcy MES są (R.W. Clough, R.L. Taylor) lub do niedawna byli wśród nas (J.H. Argyris, O.C. Zienkiewicz). Obecnie sama metoda jest przytłumiona przez programy obliczeniowe, które na niej bazują. W niniejszej pracy przywołujemy trochę historii i przedstawiamy najważniejsze prace dotyczące zastosowania MES w analizie dźwigarów powierzchniowych.

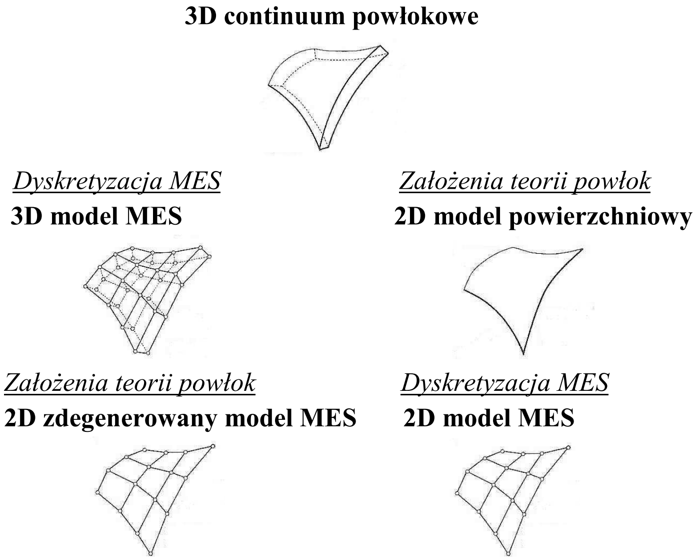
Korzenie

Początki rozwoju sformułowań MES w analizie płyt i powłok należy datować na wczesne lata 60. XX wieku, wskazując na zespół badawczy profesora R.W. Clougha na Uniwersytecie w Berkeley, USA. Pod kierunkiem Clougha jego dyplomant Ari Adini (który pracował w jednym pokoju z późniejszym profesorem Edwardem Wilsonem) wykorzystywał aparat algebry macierzowej do rozwiązywania problemów inżynierskich. Finalizował swoją pracę w zakresie zginania płyt cienkich. Zbudował prostokątny element skończony i wykazał, że tę klasę zagadnień można poprawnie rozwiązywać za pomocą MES. Wyniki pracy opublikowano w raporcie Adini i Clough (1960). Pracę tę należy uważać za pierwszą w zakresie MES do analizy płyt. Co ciekawe praca ta nie została zaakceptowana do wygłoszenia na konferencji American Society of Civil Engineers, gdyż przyjęto już dwie inne prace z Uniwersytetu w Berkeley. Opracowany element był 4-węzłowy o 12 stopniach swobody. Do aproksymacji wykorzystano pełny wielomian trzeciego stopnia, sumując niektóre człony wielomianu. Opracowany element spełniał większość wymagań MES, jednak ciągłość klasy C-1 spełniona była tylko w czterech węzłach. Adini kontynuował pracę, rozwiązując wiele prostych zadań powłokowych, rozszerzając rozważania na stany zgięciowe i membranowe. W 1961 roku obronił pracę doktorską na temat analizy konstrukcji powłokowych za pomocą MES (Adini 1961). Adini, autor pierwszych prac z omawianej dziedziny, nie kontynuował pracy naukowej i jego nazwisko nie jest widoczne w żadnej późniejszej publikacji.

W 1960 roku Jim Tocher, doktorant profesora Clougha, rozpoczął prace w zakresie praktycznych zastosowań MES do zginania płyt. Po dwóch latach przedstawił on dysertację (1961), w której wykazał, że rzeczywiste konstrukcje płytowe mogą być modelowane za pomocą trójkątnych elementów skończonych. Otrzymane wyniki nie były jednak w pełni satysfakcjonujące – element był zbyt „miękki” i nie zawsze uzyskiwano zbieżność wyników do rozwiązania ścisłego. W latach 1962–1963 Tocher uzyskał grant podoktorski w Norwegii i kontynuował prace nad trójkątnym elementem skończonym, będąc w ciągłym kontakcie z profesorem Cloughem. W tym samym czasie w Berkeley student profesora Clougha T.K. Hsieh zaproponował skomplikowany sposób budowy płytowego elementu trójkątnego, który spełnia wszystkie warunki dostosowania. Element ten został włączony do programu obliczeniowego i przetestowany przez Tochera w czasie jego pracy dla koncernu Boeing. Element wykazywał znakomite własności zbieżności i został nazwany HCT od pierwszych liter nazwisk Hsieha, Clougha i Tochera. Wyniki opublikowano w 1965 roku na konferencji Wright-Patterson Conference on Matrix Methods (Clough i Tocher 1965). Element ten był wykorzystywany w przemyśle do obliczeń zginania płyt cienkich przez ponad 20 lat.

Koncepcje

Patrząc historycznie na rozwój płytowych i powłokowych elementów skończonych, można wyróżnić dwa schematy postępowania, przedstawione na rysunku 1. W obu schematach punktem wyjścia jest trójwymiarowe ciało o cechach powłoki lub płyty. Koncepcja klasyczna polega na zbudowaniu w pierwszej kolejności dwuwymiarowej teorii powłok (płyt), a następnie zastosowaniu formalizmu MES. Koncepcja tzw. zdegenerowanych elementów powłokowych polega na sformułowaniu MES w odniesieniu do ciała trójwymiarowego i zmodyfikowanie sposobu aproksymacji MES tak, aby nadać modelowi cechy powłokowe przez wprowadzenie rotacyjnych stopni swobody jako parametrów węzłowych. Obie koncepcje mogą prowadzić do różnych elementów skończonych. Szczegółową analizę różnic i podobieństw obu



RYSUNEK 1. Dwie koncepcje budowy elementów skończonych płytowych/powłokowych
FIGURE 1. Two ways of development of plate/shell finite elements

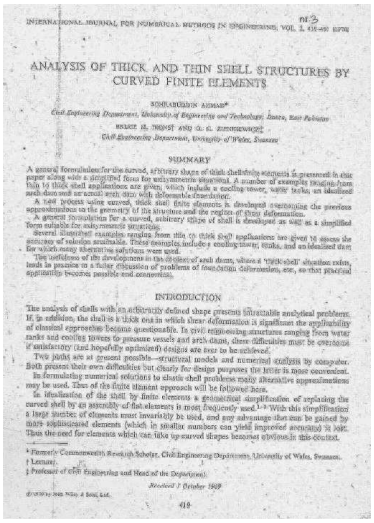
sformułowań przedstawili ostatnio Bishoff i inni (2004). Koncepcja zdegenerowanych izoparametrycznych elementów powłokowych była przez wiele lat dominująca w literaturze przedmiotu.

Kroki milowe

Obserwując na przestrzeni 50 lat, rozwój MES w zakresie budowy elementów skończonych płyt i powłok, można zdefiniować najważniejsze prace, które stanowią swego rodzaju kroki milowe tego rozwoju. Poza pracami przedstawionymi wcześniej, warto zauważyć kilka innych.

Jako pierwszą należy wymienić fundamentalną pracę Ahmada i innych (1970) – rysunek 2, w której opracowano element powłokowy jako odpowiednią modyfikację izoparametrycznego elementu trójwymiarowego. Modyfikacja ta polegała na wprowadzeniu do opisu MES hipotezy odpowiadającej założeniom kinematycznym Mindlina––Reissnera. Wkrótce przyjęła się nazwa sformułowania „zdegenerowany izoparametryczny element powłokowy” lub „element Ahmada”. W niedługim czasie po opublikowaniu artykułu stwierdzono bardzo wolną zbieżność sformułowania dla płyt i powłok cienkich oraz zjawisko zbyt dużej sztywności elementu. W mechanice komputerowej zjawisko to znane jest pod nazwą blokada przemieszczeń (ang. *locking*). W literaturze definiuje się kilka typów blokad, związanych z różnymi parametrami geometrycznymi i fizycznymi

Pawsey i Clough (1971) oraz równocześnie Zienkiewicz i inni (1971) – rysunek 3, zauważyli pozorny paradoks, że własności zbieżności zdegenerowanych elementów powłokowych można znacząco polepszyć przez obniżenie rzędu



INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 1, 419–451 (1970)

nr 3

ANALYSIS OF THICK AND THIN SHELL STRUCTURES BY CURVED FINITE ELEMENTS

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SUMMARY

A general formulation for the curved, arbitrary shape of thick shell finite elements is presented in this paper along with a simplified form for axisymmetric situations. A number of examples ranging from thin to thick shell applications are given, which include a cooling tower, water tanks, an idealized arch dam and an actual arch dam with deformable foundation.

A new process using curved, thick shell finite elements is developed overcoming the previous approximations to the geometry of the structure and the neglect of shear deformation.

A general formulation for a curved, arbitrary shape of shell is developed as well as a simplified form suitable for axisymmetric situations.

Several illustrated examples ranging from thin to thick shell applications are given to assess the accuracy of solution attainable. These examples include a cooling tower, tanks, and an idealized dam for which many alternative solutions were used.

The usefulness of the development in the context of arch dams, where a 'thick shell' situation exists, leads in practice to a fuller discussion of problems of foundation deformation, etc., so that practical application becomes possible and economical.

INTRODUCTION

The analysis of shells with an arbitrarily defined shape presents intractable analytical problems. If, in addition, the shell is a thick one in which shear deformation is significant the applicability of classical approaches become questionable. In civil engineering structures ranging from water tanks and cooling towers to pressure vessels and arch dams, these difficulties must be overcome if satisfactory (and hopefully optimized) designs are ever to be achieved.

Two paths are at present possible—structural models and numerical analysis by computer. Both present their own difficulties but clearly for design purposes the latter is more convenient.

In formulating numerical solutions to elastic shell problems many alternative approximations may be used. That of the finite element approach will be followed here.

In idealization of the shell by finite elements a geometrical simplification of replacing the curved shell by an assembly of flat elements is most frequently used.[1–3] With this simplification a large number of elements must invariably be used, and any advantage that can be gained by more sophisticated elements (which in smaller numbers can yield improved accuracy) is lost. Thus the need for elements which can take up curved shapes becomes obvious in this context.

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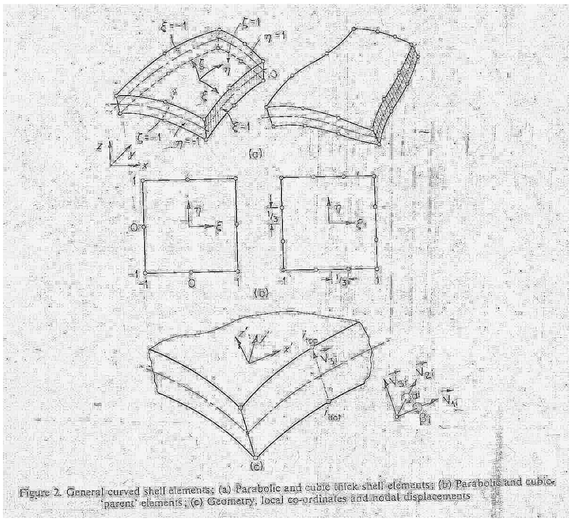
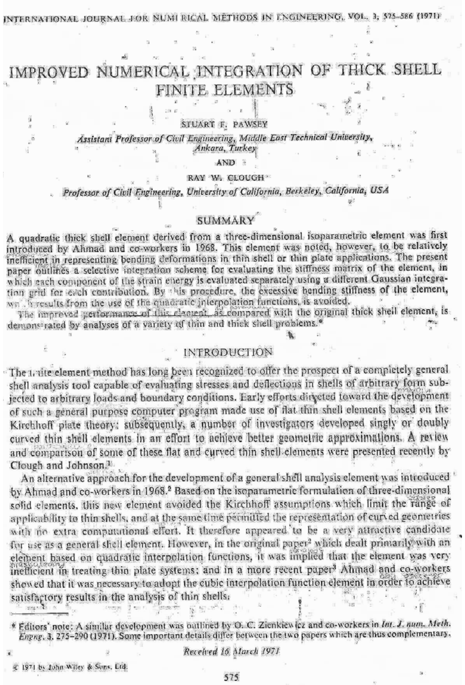


Figure 2. General curved shell elements; (a) Parabolic and cubic thick shell elements; (b) Parabolic and cubic 'parent' elements; (c) Geometry, local co-ordinates and nodal displacements

RYSUNEK 2. Zdegenerowane elementy powłokowe Ahmada
FIGURE 2. Degenerated shell concept – Ahmad finite elements



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IMPROVED NUMERICAL INTEGRATION OF THICK SHELL FINITE ELEMENTS

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SUMMARY

A quadratic thick shell element derived from a three-dimensional isoparametric element was first introduced by Ahmad and co-workers in 1968. This element was noted, however, to be relatively inefficient in representing bending deformations in thin shell or thin plate applications. The present paper outlines a selective integration scheme for evaluating the stiffness matrix of the element, in which each component of the strain energy is evaluated separately using a different Gaussian integration grid for each contribution. By this procedure, the excessive bending stiffness of the element, which results from the use of the quadratic interpolation functions, is avoided.

The improved performance of this element, as compared with the original thick shell element, is demonstrated by analyses of a variety of thin and thick shell problems.*

INTRODUCTION

The finite element method has long been recognized to offer the prospect of a completely general shell analysis tool capable of evaluating stresses and deflections in shells of arbitrary form subjected to arbitrary loads and boundary conditions. Early efforts directed toward the development of such a general purpose computer program made use of flat thin shell elements based on the Kirchhoff plate theory; subsequently, a number of investigators developed singly or doubly curved thin shell elements in an effort to achieve better geometric approximations. A review and comparison of some of these flat and curved thin shell elements were presented recently by Clough and Johnson.[1]

An alternative approach for the development of a general shell analysis element was introduced by Ahmad and co-workers in 1968.[2] Based on the isoparametric formulation of three-dimensional solid elements, this new element avoided the Kirchhoff assumptions which limit the range of applicability to thin shells, and at the same time permitted the representation of curved geometries with no extra computational effort. It therefore appeared to be a very attractive candidate for use as a general shell element. However, in the original paper[2] which dealt primarily with an element based on quadratic interpolation functions, it was implied that the element was very inefficient in treating thin plate systems; and in a more recent paper[3] Ahmad and co-workers showed that it was necessary to adopt the cubic interpolation function element in order to achieve satisfactory results in the analysis of thin shells.

* Editors' note: A similar development was outlined by O. C. Zienkiewicz and co-workers in *Int. J. num. Meth. Engng*, 3, 275–290 (1971). Some important details differ between the two papers which are thus complementary.

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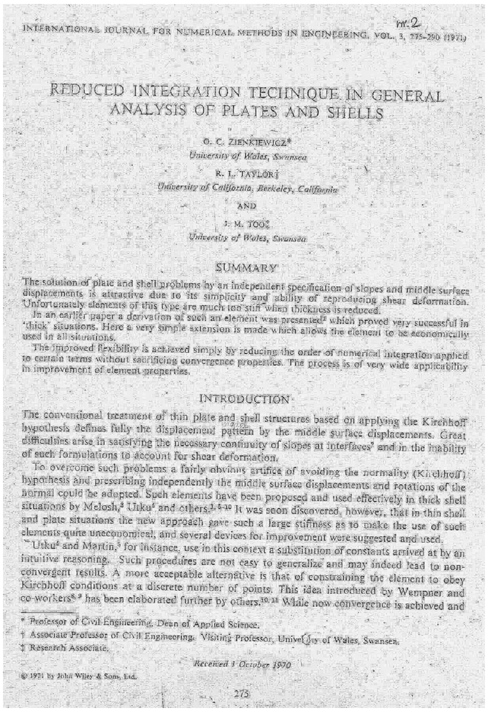

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 3, 275–290 (1971)

nr 2

REDUCED INTEGRATION TECHNIQUE IN GENERAL ANALYSIS OF PLATES AND SHELLS

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SUMMARY

The solution of plate and shell problems by an independent specification of slopes and middle surface displacements is attractive due to its simplicity and ability of reproducing shear deformation. Unfortunately elements of this type are much too stiff when thickness is reduced.

In an earlier paper a derivation of such an element was presented[1] which proved very successful in 'thick' situations. Here a very simple extension is made which allows the element to be economically used in all situations.

The improved flexibility is achieved simply by reducing the order of numerical integration applied to certain terms without sacrificing convergence properties. The process is of very wide applicability in improvement of element properties.

INTRODUCTION

The conventional treatment of thin plate and shell structures based on applying the Kirchhoff hypothesis defines fully the displacement pattern by the middle surface displacements. Great difficulties arise in satisfying the necessary continuity of slopes at interfaces[2] and in the inability of such formulations to account for shear deformation.

To overcome such problems a fairly obvious artifice of avoiding the normality (Kirchhoff) hypothesis and prescribing independently the middle surface displacements and rotations of the normal could be adopted. Such elements have been proposed and used effectively in thick shell situations by Melosh,[3] Utku[4] and others.[1,5–10] It was soon discovered, however, that in thin shell and plate situations the new approach gave such a large stiffness as to make the use of such elements quite uneconomical, and several devices for improvement were suggested and used.

Utku[4] and Martin,[5] for instance, use in this context a substitution of constants arrived at by an intuitive reasoning. Such procedures are not easy to generalize and may indeed lead to non-convergent results. A more acceptable alternative is that of constraining the element to obey Kirchhoff conditions at a discrete number of points. This idea introduced by Wempner and co-workers[8,9] has been elaborated further by others.[10,11] While now convergence is achieved and

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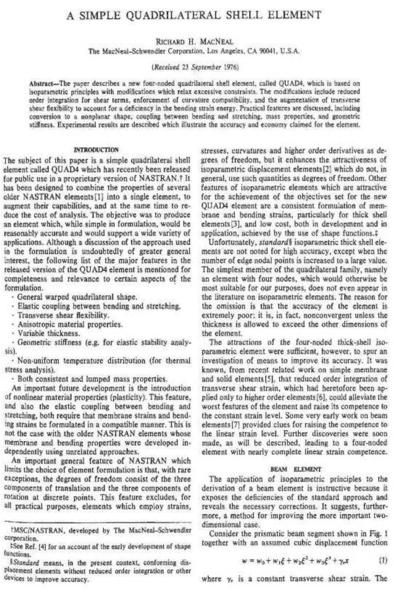
RYSUNEK 3. Całkowanie zredukowane i selektywne
FIGURE 3. Reduced and selective integration

kwadratury Gaussa–Legandre'a podczas całkowania macierzy sztywności. Technika całkowania zredukowanego stała się bardzo popularnym „lekarstwem" na blokady elementów płytowych i powłokowych. Wkrótce stwierdzono jednak, że całkowanie zredukowane (i jego modyfikacja – całkowanie selektywne) prowadzi do nadosobliwości macierzy sztywności elementów, która, w zależności od warunków brzegowych zadania i zastosowanej siatki podziału na elementy, może prowadzić do osobliwości lub quasi-osobliwości zadania.

Znaczny postęp w rozwoju efektywnych elementów skończonych płyt/ /powłok związany jest z zastosowaniem specjalnych, ulepszonych schematów interpolacji wybranych składowych stanu odkształcenia. Pierwszym pełnym sformułowaniem tego zagadnienia był opublikowany przez MacNeala (1978) element QUAD4 (rys. 4), gdzie w sposób specjalny potraktowano poprzeczne odkształcenia postaciowe.

Bathe i Dvorkin (1986) opracowali 4-węzłowy element (MITC4) i jego 8- -węzłowy odpowiednik (MITC8), które bazują na koncepcji mieszanej interpolacji składowych tensorowych (rys. 5). W elemencie zakrzywionym (MITC8) konieczne było wprowadzenie specjalnego opisu odkształceń błonowych w celu uniknięcia zjawiska blokady membranowej. Rodzina płytowych i powłokowych elementów skończonych (MITCn, gdzie n oznacza liczbę węzłów) zapoczątkowała koncepcję budowy elementów skończonych o tzw. założonych odkształceniach, na bazie mieszanego sformułowania wariacyjnego. Park i Stanley (1986) opublikowali, pod nazwą założonych naturalnych odkształceń (ang. *assumed natural strains* – ANS),



A SIMPLE QUADRILATERAL SHELL ELEMENT

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(*Received* 23 *September* 1976)

Abstract—The paper describes a new four-noded quadrilateral shell element, called QUAD4, which is based on isoparametric principles with modifications which relax excessive constraints. The modifications include reduced order integration for shear terms, enforcement of curvature compatibility, and the augmentation of transverse shear flexibility to account for a deficiency in the bending strain energy. Practical features are discussed, including conversion to a nonplanar shape, coupling between bending and stretching, mass properties, and geometric stiffness. Experimental results are described which illustrate the accuracy and economy claimed for the element.

INTRODUCTION

The subject of this paper is a simple quadrilateral shell element called QUAD4 which has recently been released for public use in a proprietary version of NASTRAN.† It has been designed to combine the properties of several older NASTRAN elements[1] into a single element, to augment their capabilities, and at the same time to reduce the cost of analysis. The objective was to produce an element which, while simple in formulation, would be reasonably accurate and would support a wide variety of applications. Although a discussion of the approach used in the formulation is undoubtedly of greater general interest, the following list of the major features in the released version of the QUAD4 element is mentioned for completeness and relevance to certain aspects of the formulation.

· General warped quadrilateral shape.
· Elastic coupling between bending and stretching.
· Transverse shear flexibility.
· Anisotropic material properties.
· Variable thickness.
· Geometric stiffness (e.g. for elastic stability analysis).
· Non-uniform temperature distribution (for thermal stress analysis).
· Both consistent and lumped mass properties.

An important future development is the introduction of nonlinear material properties (plasticity). This feature, and also the elastic coupling between bending and stretching, both require that membrane strains and bending strains be formulated in a compatible manner. This is not the case with the older NASTRAN elements whose membrane and bending properties were developed independently using unrelated approaches.

An important general feature of NASTRAN which limits the choice of element formulation is that, with rare exceptions, the degrees of freedom consist of the three components of translation and the three components of rotation at discrete points. This feature excludes, for all practical purposes, elements which employ strains, stresses, curvatures and higher order derivatives as degrees of freedom, but it enhances the attractiveness of isoparametric displacement elements[2] which do not, in general, use such quantities as degrees of freedom. Other features of isoparametric elements which are attractive for the achievement of the objectives set for the new QUAD4 element are a consistent formulation of membrane and bending strains, particularly for thick shell elements[3], and low cost, both in development and in application, achieved by the use of shape functions.‡

Unfortunately, *standard*§ isoparametric thick shell elements are not noted for high accuracy, except when the number of edge nodal points is increased to a large value. The simplest member of the quadrilateral family, namely an element with four nodes, which would otherwise be most suitable for our purposes, does not even appear in the literature on isoparametric elements. The reason for the omission is that the accuracy of the element is extremely poor; it is, in fact, nonconvergent unless the thickness is allowed to exceed the other dimensions of the element.

The attractions of the four-noded thick-shell isoparametric element were sufficient, however, to spur an investigation of means to improve its accuracy. It was known, from recent related work on simple membrane and solid elements[5], that reduced order integration of transverse shear strain, which had heretofore been applied only to higher order elements[6], could alleviate the worst features of the element and raise its competence to the constant strain level. Some very early work on beam elements[7] provided clues for raising the competence to the linear strain level. Further discoveries were soon made, as will be described, leading to a four-noded element with nearly complete linear strain competence.

BEAM ELEMENT

The application of isoparametric principles to the derivation of a beam element is instructive because it exposes the deficiencies of the standard approach and reveals the necessary corrections. It suggests, furthermore, a method for improving the more important two-dimensional case.

Consider the prismatic beam segment shown in Fig. 1 together with an assumed cubic displacement function

$$w = w_0 + w_1\xi + w_2\xi^2 + w_3\xi^3 + \gamma_e x \quad (1)$$

where γ_e is a constant transverse shear strain. The

†MSC/NASTRAN, developed by The MacNeal–Schwendler corporation.

‡See Ref. [4] for an account of the early development of shape functions.

§*Standard* means, in the present context, conforming displacement elements without reduced order integration or other devices to improve accuracy.

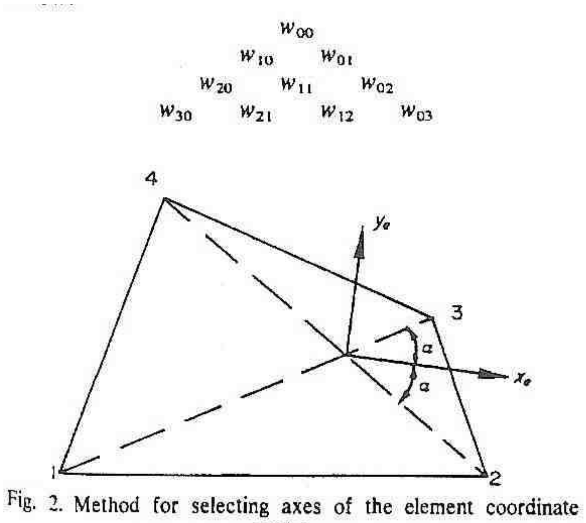
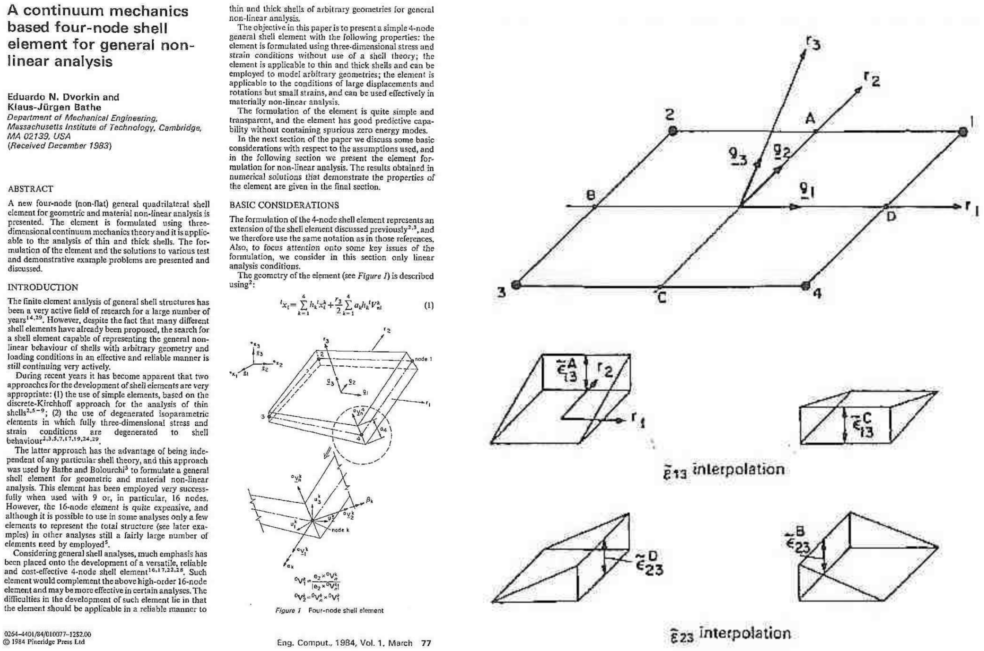


Fig. 2. Method for selecting axes of the element coordinate system.

RYSUNEK 4. Element typu QUAD4
FIGURE 4. QUAD4 finite element



A continuum mechanics based four-node shell element for general non-linear analysis

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ABSTRACT

A new four-node (non-flat) general quadrilateral shell element for geometric and material non-linear analysis is presented. The element is formulated using three-dimensional continuum mechanics theory and it is applicable to the analysis of thin and thick shells. The formulation of the element and the solutions to various test and demonstrative example problems are presented and discussed.

INTRODUCTION

The finite element analysis of general shell structures has been a very active field of research for a large number of years[14,29]. However, despite the fact that many different shell elements have already been proposed, the search for a shell element capable of representing the general non-linear behaviour of shells with arbitrary geometry and loading conditions in an effective and reliable manner is still continuing very actively.

During recent years it has become apparent that two approaches for the development of shell elements are very appropriate: (1) the use of simple elements, based on the discrete-Kirchhoff approach for the analysis of thin shells[2,5–9]; (2) the use of degenerated isoparametric elements in which fully three-dimensional stress and strain conditions are degenerated to shell behaviour[2,3,5,7,17,19,24,29].

The latter approach has the advantage of being independent of any particular shell theory, and this approach was used by Bathe and Bolourchi[3] to formulate a general shell element for geometric and material non-linear analysis. This element has been employed very successfully when used with 9 or, in particular, 16 nodes. However, the 16-node element is quite expensive, and although it is possible to use in some analyses only a few elements to represent the total structure (see later examples) in other analyses still a fairly large number of elements need by employed[5].

Considering general shell analyses, much emphasis has been placed onto the development of a versatile, reliable and cost-effective 4-node shell element[16,17,22,28]. Such element would complement the above high-order 16-node element and may be more effective in certain analyses. The difficulties in the development of such element lie in that the element should be applicable in a reliable manner to thin and thick shells of arbitrary geometries for general non-linear analysis.

The objective in this paper is to present a simple 4-node general shell element with the following properties: the element is formulated using three-dimensional stress and strain conditions without use of a shell theory; the element is applicable to thin and thick shells and can be employed to model arbitrary geometries; the element is applicable to the conditions of large displacements and rotations but small strains, and can be used effectively in materially non-linear analysis.

The formulation of the element is quite simple and transparent, and the element has good predictive capability without containing spurious zero energy modes.

In the next section of the paper we discuss some basic considerations with respect to the assumptions used, and in the following section we present the element formulation for non-linear analysis. The results obtained in numerical solutions that demonstrate the properties of the element are given in the final section.

BASIC CONSIDERATIONS

The formulation of the 4-node shell element represents an extension of the shell element discussed previously[2,3], and we therefore use the same notation as in those references. Also, to focus attention onto some key issues of the formulation, we consider in this section only linear analysis conditions.

The geometry of the element (see *Figure 1*) is described using[2]:

$$^t x_i = \sum_{k=1}^{4} h_k \, ^t x_i^k + \frac{r_3}{2} \sum_{k=1}^{4} a_k h_k \, ^t V_{ni}^k \qquad (1)$$

Figure 1 Four-node shell element

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RYSUNEK 5. Rodzina elementów typu MITn
FIGURE 5. MITn family of finite elements

sformułowanie elementu powłokowego 9-węzłowego, wolnego od zjawisk blokady poprzecznego ścinania i blokady membranowej (rys. 6).

Inną owocną techniką uniknięcia zjawiska blokady przemieszczeń w MES jest koncepcja zastępczych pól odkształceń (ang. *enhanced assumed strain* – EAS) (Simo i Rifai 1990 – rys. 7). Podczas gdy technika ANS obniża stopień wielomianu interpolacyjnego dla wybranych składowych stanu odkształcenia, to w EAS standardowe pole odkształceń, wynikające z różniczkowania przemieszczeń jest „zszywane" z niezależnie budowanym wzbogaconym polem odkształceń. Warunek ortogonalności zastępczego pola odkształceń i pola naprężeń pozwala uniknąć dodatkowego członu w wyrażeniu na energię sprężystą. Simo i Rifai (1990) oparli budowę modelu EAS na trójpolowej zasadzie wariacyjnej Hu–Washizu. Jednak w późniejszych pracach stwierdzono, że EAS nie jest techniką mieszaną MES, lecz raczej zmodyfikowanym sformułowaniem przemieszczeniowym.

Kolejną techniką budowy elementów skończonych, którą należy uznać za kamień milowy w opisie płyt i powłok, jest cała gama sformułowań mieszanych oraz hybrydowych. W sformułowaniach mieszanych niezależnej aproksymacji podlegają pola przemieszczeń i odkształceń lub naprężeń. W modelach hybrydowych dodatkowej, niezależnej interpolacji podlegają wielkości na brzegu element skończonego. Prace w tym

A Curved C^0 Shell Element Based on Assumed Natural-Coordinate Strains

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A curved C^0 shell element is presented, which corrects several deficiencies in existing quadratic shell elements. The improvements realized in the present element include rank sufficiency without transverse shear locking, consistent membrane strain interpolation that admits inextensional bending without reduced integration, and adequate representation of curvature effects to capture the important membrane-bending coupling. The element can be constructed either by a nine-point integration rule or by a four-point integration rule with the proper rank compensating terms. Numerical experiments with the present element on several benchmark problems indicate that the element yields accurate and reliable solutions without any ostensible deficiency. The element is recommended for production analysis of shell structures.

1 Introduction

Structural engineers often prefer simple elements in performing plate and shell analyses, which leads to the adoption of triangular or four-node elements for most circumstances. This popularity is reflected in recent developments of four-node C^0 elements by Hughes et al. (1977, 1981), MacNeal (1978, 1982), Wempner et al. (1982), Crisfield (1983) and Dvorkin and Bathe (1984).

In order to make four-node elements more efficient while retaining all of their desirable properties intact, an element synthesis technique was developed that led to the formulation of a rank-sufficient one-point integrated quadrilateral plate element (Park and Flaggs, 1985) and a shell element (Park et al., 1985) which, unlike uniformly reduced-integrated elements, maintains the accuracy of fully integrated (four-point rule) elements. The four-node shell element is a strong candidate for production applications in which the strains vary *smoothly*.

There are circumstances, however, where curved elements are considered to be advantageous. These include shell structures in which the stresses vary rapidly or in which inextensional bending deformations dominate the solution, and for modeling curved boundaries and free edges by distorted element shapes as in curved panels with holes and stiffeners. For these problems, the performance of existing 8 and 9-node quadratic shell elements has been disappointing (Bathe et al., 1983), and as a result analysts have been forced to utilize the substantially more expensive 16-node elements (Ramm and Sattele, 1981). This has motivated the present authors to reexamine existing quadratic plate and shell elements, correct the deficiencies and provide a quadratic shell element as an advantageous alternative to 4 and 16-node elements. Although the improvements presented in the paper are applicable to other types of shell elements, we focus on the 9-node Lagrange shell element for reasons discussed below.

The first candidate quadratic shell element considered was the 8-node Serendipity element (Ahmad et al., 1970). Although the locking-free transverse shear strain interpolation procedure presented in Park (1985) is applicable to this element, it was concluded that the resulting element would be too stiff without introducing a bubble function as shown by Crisfield (1984). In addition, the corresponding membrane strain would have to be augmented by a similar bubble function. This not only complicates the element formulation, but yields a rank-deficient membrane stiffness matrix when the 4-point Gaussian integration rule is used. There is another unsettling aspect of the 8-node element: in shell dynamics, where the membrane response becomes important, it is better to have membrane degrees of freedom that lie on a uniform lattice. It is believed that the absence of the translational degrees of freedom at the center of the Serendipity element will force the energy associated with the membrane strains to be numerically dispersed to the side nodes. This numerical dispersion within the element is not desirable, especially in wave propagation problems. It is noted that the heterosis element (Hughes, 1978) adopts the same shape functions for the translational degrees of freedom as those of the Serendipity element.

Precursors of the present element may be traced to the following works: Hrennikoff (1941) for his adoption of a framework analogy to solve elasticity problems; Turner, Clough, Martin, and Topp (1956) for their assumed strain approach; Taig (1961) for his isoparametric (iso-P) representation of quadrilateral shapes; Wilson (1963) and Argyris (1964) for their boundary stress-strain modeling along flat triangular element sides; Fraeijs de Veubeke for his limitation principle

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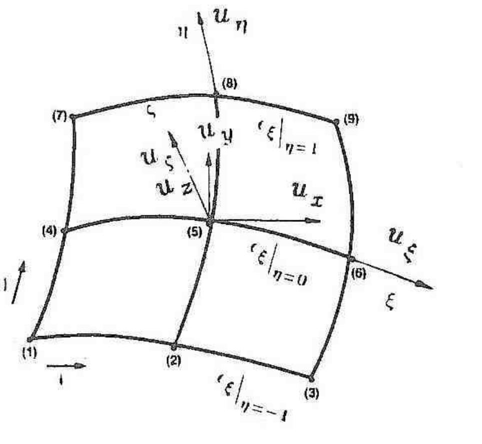

$$\epsilon_{\xi\xi} = N_1(\eta)\cdot\epsilon_\xi\big|_{\eta=-1} + N_2(\eta)\cdot\epsilon_\xi\big|_{\eta=0} + N_3(\eta)\cdot\epsilon_\xi\big|_{\eta=1}$$

Fig. 1 Six reference lines in nine-node element and assumed natural-coordinate strain, $\epsilon_{\xi\xi}$

RYSUNEK 6. Koncepcja założonych naturalnych odkształceń
FIGURE 6. Assumed Natural Strain concept

INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 29, 1595–1638 (1990)

A CLASS OF MIXED ASSUMED STRAIN METHODS AND THE METHOD OF INCOMPATIBLE MODES*

'*. . . two wrongs do make a right in California*' G. STRANG (1973)
'*. . . two rights make a right even in California*' R. L. TAYLOR (1989)

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SUMMARY

A three-field mixed formulation in terms of displacements, stresses and an *enhanced strain* field is presented which encompasses, as a particular case, the classical method of incompatible modes. Within this framework, incompatible elements arise as particular 'compatible' mixed approximations of the enhanced strain field. The conditions that the stress interpolation contain piece-wise constant functions and be L_2-orthogonal to the enhanced strain interpolation, ensure satisfaction of the patch test and allow the elimination of the stress field from the formulation. The preceding conditions are formulated in a form particularly convenient for element design. As an illustration of the methodology three new elements are developed and shown to exhibit good performance: a plane 3D elastic/plastic QUAD, an axisymmetric element and a thick plate bending QUAD. The formulation described herein is suitable for non-linear analysis.

1. INTRODUCTION AND MOTIVATION

In recent years, considerable attention has been devoted to the development of low order quadrilateral elements (typically bilinear) which exhibit *high accuracy in coarse meshes*; particularly in *bending dominated situations*. Furthermore, many of these low order elements do not exhibit the well-known 'spurious locking' in the nearly incompressible limit and are, therefore, particularly attractive in general purpose finite element analysis programs. It should be pointed out that, despite the enhanced accuracy of these quadrilaterals, from a numerical analysis standpoint no improvement on the standard (*asymptotic*) interpolation error estimates is typically obtained (see e.g. the analysis in Strang and Fix (Reference 20, p. 174)).

Broadly speaking, recent design and development of low order quadrilaterals with enhanced coarse mesh accuracy falls within the scope of two alternative approaches, referred to as *assumed strain* and *assumed stress* methods in what follows. Representative of the first approach is the pioneering work of Nagtegaal *et al.*[10] and Willan,[28] the related B-bar methods of Hughes[6] and Simo *et al.*,[19] and the mode-decomposition and Hu–Washizu methods of Belytschko and co-workers; see e.g. Belytschko and Bachrach,[4] among others. As noted in Simo and Hughes,[13] all of these assumed strain methods can be cast into a three-field variational framework, and

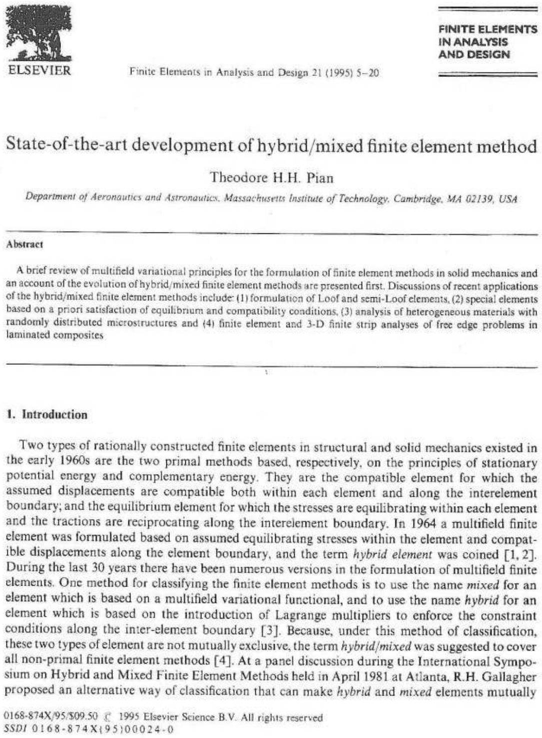
2.1.1. *'Enhanced' strain field, and modified variational formulation.* Next, we introduce a reparametrization of the strain fields in the form given in (1). Consequently, any 'admissible strain variation' is also written as

$$\boldsymbol{\gamma} = \underbrace{\nabla^s\boldsymbol{\eta}}_{\text{compatible}} + \underbrace{\tilde{\boldsymbol{\gamma}}}_{\text{enhanced}}; \quad \text{with} \quad \boldsymbol{\eta}\in\mathcal{V} \qquad (7)$$

The last variational equation $(5)_3$ then takes the form

$$\int_{\mathscr{B}} \tilde{\boldsymbol{\gamma}}\cdot[-\boldsymbol{\sigma} + \partial_\varepsilon W(\mathbf{x}, \nabla^s\mathbf{u} + \tilde{\boldsymbol{\varepsilon}})]\,\mathrm{d}V + \int_{\mathscr{B}} \nabla^s\boldsymbol{\eta}\cdot[-\boldsymbol{\sigma} + \partial_\varepsilon W(\mathbf{x}, \nabla^s\mathbf{u} + \tilde{\boldsymbol{\varepsilon}})]\,\mathrm{d}V = 0 \qquad (8)$$

RYSUNEK 7. Wzbogacone pola odkształceń
FIGURE 7. Enhanced Assumed Strains



ELSEVIER Finite Elements in Analysis and Design 21 (1995) 5–20

FINITE ELEMENTS IN ANALYSIS AND DESIGN

State-of-the-art development of hybrid/mixed finite element method

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Abstract

A brief review of multifield variational principles for the formulation of finite element methods in solid mechanics and an account of the evolution of hybrid/mixed finite element methods are presented first. Discussions of recent applications of the hybrid/mixed finite element methods include: (1) formulation of Loof and semi-Loof elements, (2) special elements based on a priori satisfaction of equilibrium and compatibility conditions, (3) analysis of heterogeneous materials with randomly distributed microstructures and (4) finite element and 3-D finite strip analyses of free edge problems in laminated composites

1. Introduction

Two types of rationally constructed finite elements in structural and solid mechanics existed in the early 1960s are the two primal methods based, respectively, on the principles of stationary potential energy and complementary energy. They are the compatible element for which the assumed displacements are compatible both within each element and along the interelement boundary; and the equilibrium element for which the stresses are equilibrating within each element and the tractions are reciprocating along the interelement boundary. In 1964 a multifield finite element was formulated based on assumed equilibrating stresses within the element and compatible displacements along the element boundary, and the term *hybrid element* was coined [1, 2]. During the last 30 years there have been numerous versions in the formulation of multifield finite elements. One method for classifying the finite element methods is to use the name *mixed* for an element which is based on a multifield variational functional, and to use the name *hybrid* for an element which is based on the introduction of Lagrange multipliers to enforce the constraint conditions along the inter-element boundary [3]. Because, under this method of classification, these two types of element are not mutually exclusive, the term *hybrid/mixed* was suggested to cover all non-primal finite element methods [4]. At a panel discussion during the International Symposium on Hybrid and Mixed Finite Element Methods held in April 1981 at Atlanta, R.H. Gallagher proposed an alternative way of classification that can make *hybrid* and *mixed* elements mutually

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RYSUNEK 8. Sformułowania mieszane i hybrydowe
FIGURE 8. Mixed and Hybrid Formulations

zakresie prowadzone są od lat 70. XX wieku przez różnych autorów. Syntetyczny opis różnych sformułowań, także swojego autorstwa, podał Pian (1995) – rysunek 8. Można wykazać, że niektóre sformułowania mieszane są równoważne modelom przemieszczeniowym z zastosowaniem „trików numerycznych", takich jak np. całkowanie zredukowane.

Podsumowanie

Pomimo 50-letnich wysiłków badaczy z całego świata „najlepszy" powłokowy element skończony nie został dotąd opracowany.

Tysiące publikacji w obszarze sformułowań MES w teoriach płyt i powłok można klasyfikować i analizować w różny sposób. Wymieńmy na koniec kilka ciekawych i trudnych zagadnień:

– modele MES dla powłok cienkich *versus* sformułowania w teoriach powłok o średniej grubości (klasy ciągłości C-0 i C-1),
– problemy obliczeniowe w analizie nieliniowej geometrycznie i fizycznie,
– powłoki kompozytowe, warstwowe i sandwiczowe,
– problem 6. (kątowego) stopnia swobody w elementach powłokowych,
– koncepcja fizycznych funkcji kształtu w analizie MES płyt i powłok.

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Streszczenie

Rozważania o metodzie elementów skończonych. Budowa elementów skończonych cienkich, średniej grubości i grubych płyt i powłok stanowi obszar aktywnych badań naukowców od ponad 50 lat. Zaowocowało to setkami prac i publikacji. W systemach MES istnieje bardzo wiele elementów płytowych lub powłokowych, lecz „najlepszy" element powłokowy nadal czeka na swojego odkrywcę. Praca stanowi spojrzenie historyczne na rozważany problem od wczesnych lat 60. minionego wieku do dnia dzisiejszego.

Summary

Some considerations on the finite element method. Development of finite elements for thin, moderately thick as well as thick plates and shells is one of the active areas of the finite element technology for 50 years, followed by hundreds of publications. A variety of plate and shell elements exist in FE programmes, but "the best" finite element is still to be discovered. The work deals with a historical view into the subject from the first papers in early sixties of the last century until now.