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PROPERTIES FOR SOME SUBCLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY BESSEL FUNCTION

Abstract. In this paper, we introduce and study a new subclass of meromorphic univalent functions defined by Bessel function. We obtain coefficient inequalities, extreme points, radius of starlikeness and convexity. Finally we obtain partial sums and neighborhood properties for the class $\sigma_p^*(\eta, k, \upsilon)$.

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1. Introduction

Let Σ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n\, z^n \tag{1}$$

which are analytic in the punctured unit disc

$$U^* := \{z : z \in \mathbb{C},\ 0 < |z| < 1\} = U \setminus \{0\}. \tag{2}$$

Let $g \in \Sigma$ be given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n\, z^n. \tag{3}$$

Then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n\, b_n\, z^n = (g * f)(z). \tag{4}$$

Let us consider the second order linear homogenous differential equation (see Baricz [3, p. 7]):

$$z^2\, w''(z) + z\, w'(z) + (z^2 - \upsilon^2)\, w(z) = 0, \qquad \upsilon \in \mathbb{C}. \tag{5}$$

The function $w_\upsilon(z)$, which is called the generalized Bessel function of the first kind of order υ, where υ is an unrestricted (real or complex) number, is defined a particular solution of (5). The function $w_\upsilon(z)$, has the representation

$$w_\upsilon(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\,\Gamma(n+\upsilon+1)} \left(\frac{z}{2}\right)^{2n+\upsilon}.$$

Let us define

$$\mathfrak{L}_\upsilon = \frac{2^\upsilon\, \Gamma(\upsilon+1)}{z^{\frac{\upsilon}{2}+1}}\, w_\upsilon(z^{\frac{1}{2}}) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n\, \Gamma(\upsilon+1)}{4^n\, \Gamma(n+1)\, \Gamma(n+\upsilon+1)}\, z^n.$$

The operator $\mathfrak{L}_\upsilon$ is a modification of the of the operator introduced by Deniz [4] for analytic functions.

By using the Hadamard product (or convolution), we define the operator $\mathfrak{L}_\upsilon$ as follows:

$$(\mathfrak{L}_\upsilon f)(z) = \mathfrak{L}_\upsilon(z) * f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \phi_n(\upsilon)\, a_n\, z^n,$$

where $\phi_n(\upsilon) = \frac{(-1)^n\,\Gamma(\upsilon+1)}{4^n\,\Gamma(n+1)\,\Gamma(n+\upsilon+1)}$.

The operator $\mathfrak{L}_\upsilon$ is a modification of the operator introduced by Szasz and Kupan [10] for analytic functions.

It is easy to verify that

$$z\,(\mathfrak{L}_\upsilon f)'(z) = (\upsilon+1)\,(\mathfrak{L}_\upsilon f)(z) - (\upsilon+2)\,(\mathfrak{L}_{\upsilon+1} f)(z). \tag{6}$$

Motivated by Sivaprasad Kumar et al. [9], Atshan et al. [2] and Venkateswarlu et al. [11, 12], now we define a new subclass $\sigma_p^*(\eta, k, \upsilon)$ of $\sum$.

Definition 1. *For $0 \leqslant \eta < 1$, $k \geqslant 0$, we let $\sigma_p^*(\eta, k, \upsilon)$ be the subclass of $\sum$ consisting of functions of the form* (1) *and satisfying the analytic criterion*

$$-\operatorname{Re}\left(\frac{z(\mathfrak{L}_\upsilon f(z))'}{\mathfrak{L}_\upsilon f(z)} + \eta\right) > k\left|\frac{z(\mathfrak{L}_\upsilon f(z))'}{\mathfrak{L}_\upsilon f(z)} + 1\right|. \tag{7}$$

In order to prove our results wee need the following lemmas [1].

Lemma 2. *If η is a real number and ω is a complex number then*

$$\operatorname{Re}(\omega) \geqslant \eta \iff \big|\omega + (1-\eta)\big| - \big|\omega - (1+\eta)\big| \geqslant 0.$$

Lemma 3. *If ω is a complex number and η, k is a real numbers, then*

$$-\operatorname{Re}(\omega) \geqslant k\,|\omega+1| + \eta \iff \forall\ \theta \in [-\pi, \pi] :\ -\operatorname{Re}\big(\omega\,(1 + k\,e^{i\theta}) + k\,e^{i\theta}\big) \geqslant \eta.$$

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, extreme points, radii of meromorphic starlikeness and convexity for the class $\sigma_p^*(\eta, k, \upsilon)$. Further, we obtain partial sums and neighborhood properties for the class also.

2. Coefficient estimates

In this section, we obtain necessary and sufficient condition for a function f to be in the class $\sigma_p^*(\eta, k, \upsilon)$.

Theorem 4. *Let $f \in \sum$ be given by (1). Then $f \in \sigma_p^*(\eta, k, \upsilon)$ if and only if*

$$\sum_{n=1}^{\infty} \big[n\,(k+1) + (k+\eta)\big]\, \phi_n(\upsilon)\, a_n \leqslant (1-\eta). \tag{8}$$

Proof. Let $f \in \sigma_p^*(\eta, k,\ \upsilon)$. Then by Definition 1 and using Lemma 3, it is enough to show that

$$-\operatorname{Re}\left\{ \left(\frac{z(\mathfrak{L}_\upsilon f(z))'}{\mathfrak{L}_\upsilon f(z)} \right) (1 + k\, e^{i\theta}) + k\, e^{i\theta} \right\} \geqslant \eta, \qquad -\pi \leqslant \theta \leqslant \pi. \tag{9}$$

For convenience

$$\begin{aligned} C(z) &= -\big[z(\mathfrak{L}_\upsilon f(z))'\big]\,(1 + k\, e^{i\theta}) - k\, e^{i\theta}\, \mathfrak{L}_\upsilon f(z), \\ D(z) &= \mathfrak{L}_\upsilon f(z). \end{aligned}$$

That is, the equation (9) is equivalent to

$$-\operatorname{Re}\left(\frac{C(z)}{D(z)} \right) \geqslant \eta.$$

In view of Lemma 2, we only need to prove that

$$\big|C(z) + (1-\eta)\, D(z)\big| - \big|C(z) - (1+\eta)\, D(z)\big| \geqslant 0.$$

Therefore

$$\big|C(z) + (1-\eta)\, D(z)\big| \geqslant (2-\eta)\, \frac{1}{|z|} - \sum_{n=1}^{\infty} \big[n\,(k+1) + (k+\eta-1)\big]\, \phi_n(\upsilon)\, a_n\, |z|^n$$

and

$$\big|C(z) - (1+\eta)\, D(z)\big| \leqslant \eta\, \frac{1}{|z|} + \sum_{n=1}^{\infty} \big[n\,(k+1) + (k+\eta+1)\big]\, \phi_n(\upsilon)\, a_n\, |z|^n.$$

It is to show that

$$\begin{aligned}&\left|C(z)+(1-\eta)\,D(z)\right|-\left|C(z)-(1+\eta)\,D(z)\right|\geqslant\\&\qquad\geqslant 2\,(1-\eta)\,\frac{1}{|z|}-2\sum_{n=1}^{\infty}\big[n\,(k+1)+(k+\eta)\big]\,\phi_n(\upsilon)\,a_n\,|z|^n\geqslant 0,\end{aligned}$$

by the given condition (8). Conversely suppose $f\in\sigma_p^*(\eta,k,\upsilon)$. Then by Lemma 2, we have (9).

Choosing the values of z on the positive real axis the inequality (9) reduces to

$$\operatorname{Re}\left\{\frac{(1-\eta)\,\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}\big[n\,(1+k\,e^{i\theta})+(\eta+k\,e^{i\theta})\big]\,\phi_n(\upsilon)\,z^{n-1}}{\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}\phi_n(\upsilon)\,a_n\,z^{n-1}}\right\}\geqslant 0.$$

Since $\operatorname{Re}(-e^{i\theta})\geqslant -|e^{i\theta}|=-1$, the above inequality reduces to

$$\operatorname{Re}\left\{\frac{[1-\eta]\,\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}\big[n\,(1+k)+(\eta+k\big]\,\phi_n(\upsilon)\,a_n\,r^{n-1}}{\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}\phi_n(\upsilon\,r^{n-1}}\right\}\geqslant 0.$$

Letting $r\to 1^-$ and by the mean value theorem, we have obtained the inequality (8). □

Corollary 5. *If $f\in\sigma_p^*(\eta,k,\upsilon)$ then*

$$a_n\leqslant\frac{(1-\eta)}{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}. \tag{10}$$

Theorem 6. *If $f\in\sigma_p^*(\eta,k,\upsilon)$ then for $0<|z|=r<1$:*

$$\frac{1}{r}-\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\,r\leqslant|f(z)|\leqslant\frac{1}{r}+\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\,r. \tag{11}$$

This result is sharp for the function

$$f(z)=\frac{1}{z}+\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\,z. \tag{12}$$

Proof. Since $f(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}a_n\,z^n$, we have

$$|f(z)|=\frac{1}{r}+\sum_{n=1}^{\infty}a_n\,r^n\leqslant\frac{1}{r}+r\sum_{n=1}^{\infty}a_n. \tag{13}$$

Since $n\geqslant 1$, $(2\,k+\eta+1)\,\phi_1(\upsilon)\leqslant\big[n\,(k+1)+(k+\eta)\big]\,\phi_n(\upsilon)$, using Theorem 4, we have

$$\big(2\,k+\eta+1\big)\,\phi_1(\upsilon)\sum_{n=1}^{\infty}a_n\leqslant\sum_{n=1}^{\infty}\big[n\,(k+1)+(k+\eta)\big]\,\phi_n(\upsilon)\leqslant(1-\eta)$$
$$\implies\quad\sum_{n=1}^{\infty}a_n\leqslant\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}.$$

Using the above inequality in (13), we have

$$|f(z)|\leqslant\frac{1}{r}+\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\,r\quad\text{and}\quad|f(z)|\geqslant\frac{1}{r}-\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\,r.$$

The result is sharp for the function $f(z)=\frac{1}{z}+\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\,z$. □

Corollary 7. *If $f\in\sigma_p^*(\eta,k,\upsilon)$ then*

$$\frac{1}{r^2}-\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}\leqslant|f'(z)|\leqslant\frac{1}{r^2}+\frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}.$$

The result is sharp for the function given by (12)

3. Extreme points

Theorem 8. *Let $f_0(z)=\frac{1}{z}$ and*

$$f_n(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{(1-\eta)}{[n\,(1+k)+(\eta+k)\,]\,\phi_n(\upsilon)}\,z^n,\qquad n\geqslant 1. \tag{14}$$

Then $f\in\sigma_p^*(\eta,k,\upsilon)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}u_n\,f_n(z),\qquad u_n\geqslant 0\quad\textit{and}\quad\sum_{n=1}^{\infty}u_n=1. \tag{15}$$

Proof. Suppose $f(z)$ can be expressed as in (15). Then

$$f(z)=\sum_{n=0}^{\infty}u_n\,f_n(z)=u_0\,f_0(z)+\sum_{n=1}^{\infty}u_n\,f_n(z)=$$
$$=\frac{1}{z}+\sum_{n=1}^{\infty}u_n\,\frac{(1-\eta)}{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}\,z^n.$$

Therefore

$$\sum_{n=1}^{\infty}u_n\,\frac{(1-\eta)}{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}\,\frac{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}{(1-\eta)}\,z^n=$$
$$=\sum_{n=1}^{\infty}u_n=1-u_0\leqslant 1.$$

So by Theorem 4, $f\in\sigma_p^*(\eta,k,\upsilon)$.

Conversely suppose that $f\in\sigma_p^*(\eta,k,\upsilon)$. Since

$$a_n\leqslant\frac{(1-\eta)}{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)},\qquad n\geqslant 1.$$

We set $u_n=\frac{[n\,(1+k)+(\eta+k)]\,\phi_n(\upsilon)}{(1-\eta)}\,a_n$, $n\geqslant 1$ and $u_0=1-\sum\limits_{n=1}^{\infty}u_n$. Then we have $f(z)=\sum\limits_{n=0}^{\infty}u_n\,f_n(z)=u_0\,f_0(z)+\sum\limits_{n=1}^{\infty}u_n\,f_n(z)$. Hence the results follows. □

4. Radii of meromorphically starlike and convexity

Theorem 9. *Let $f\in\sigma_p^*(\eta,k,\upsilon)$. Then f is meromorphically starlike of order δ $(0\leqslant\delta\leqslant 1)$ in the unit disc $|z|<r_1$, where*

$$r_1=\inf_n\left[\frac{(1-\delta)}{(n+2-\delta)}\,\frac{[n\,(1+k)+(\eta+k)]\,\phi_n(\upsilon)}{(1-\eta)}\right]^{\frac{1}{n+1}},\qquad n\geqslant 1.$$

The result is sharp for the extremal function $f(z)$ given by (14).

Proof. The function $f \in \sigma_p^*(\eta,k,\upsilon)$ of the form (1) is meromorphically starlike of order δ is the disc $|z|<r_1$ if and only if it satisfies the condition

$$\left|\frac{z\,f'(z)}{f(z)}+1\right| < (1-\delta). \tag{16}$$

Since

$$\left|\frac{z\,f'(z)}{f(z)}+1\right| \leqslant \left|\frac{\sum\limits_{n=1}^{\infty}(n+1)a_n\,z^{n+1}}{1+\sum\limits_{n=1}^{\infty}a_n\,z^{n+1}}\right| \leqslant \frac{\sum\limits_{n=1}^{\infty}(n+1)|a_n|\,|z|^{n+1}}{1-\sum\limits_{n=1}^{\infty}|a_n|\,|z|^{n+1}}.$$

The above expression is less than $(1-\delta)$ if $\sum\limits_{n=1}^{\infty}\frac{(n+2-\delta)}{(1-\delta)}\,a_n\,|z|^{n+1}<1$. Using the fact that $f(z)\in\sigma_p^*(\eta,k,\upsilon)$ if and only if

$$\sum_{n=1}^{\infty}\frac{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}{(1-\eta)}\,a_n \leqslant 1.$$

Thus, (16) will be true if

$$\frac{(n+2-\delta)}{(1-\delta)}\,|z|^{n+1} < \frac{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}{(1-\eta)}$$

or equivalently

$$|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)}\,\frac{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}{(1-\eta)}$$

which yields the starlikeness of the family. □

The proof of the following theorem is analogous to that of Theorem 9, and so we omit the proof.

Theorem 10. *Let $f\in\sigma_p^*(\eta,k,\upsilon)$. Then f is meromorphically convex of order δ $(0\leqslant\delta\leqslant 1)$ in the unit disc $|z|<r_2$, where*

$$r_2=\inf_n\left[\frac{(1-\delta)}{n\,(n+2-\delta)}\,\frac{\big[n\,(1+k)+(\eta+k)\big]\,\phi_n(\upsilon)}{(1-\eta)}\right]^{\frac{1}{n+1}}, \qquad n\geqslant 1.$$

The result is sharp for the extremal function $f(z)$ given by (14).

5. Partial sums

Let $f \in \sum$ be a function of the form (1). Motivated by Silverman [7] and Silvia [8], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n\, z^n, \qquad m \in \mathbb{N}. \tag{17}$$

In this section we consider partial sums of function from the class $\sigma_p^*(\eta, k, \upsilon)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 11. *Let $f \in \sigma_p^*(\eta, k, \upsilon)$ be given by* (1) *and define the partial sums $f_1(z)$ and $f_m(z)$ by*

$$f_1(z) = \frac{1}{z} \quad \textit{and} \quad f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n|\, z^n, \qquad m \in \mathbb{N} \setminus \{1\}. \tag{18}$$

Suppose also that $\sum\limits_{n=1}^{\infty} d_n\,|a_n| \leqslant 1$, where

$$d_n \geqslant \begin{cases} 1, & \textit{if } n = 1, 2, \cdots, m, \\ \frac{[n\,(1+k)+(\eta+k)]\,\phi_n(\upsilon)}{(1-\eta)}, & \textit{if } n = m+1, m+2, \ldots \end{cases} \tag{19}$$

Then $f \in \sigma_p^*(\eta, k, \upsilon)$. Furthermore

$$\operatorname{Re}\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{20}$$

and

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{21}$$

Proof. For the coefficient d_n given by (19) it is not difficult to verify that

$$d_{m+1} > d_m > 1. \tag{22}$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leqslant \sum_{n=1}^{\infty} |a_n| \, d_m \leqslant 1 \tag{23}$$

by using the hypothesis (19). By setting

$$g_1(z) = d_{m+1} \left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}} \right) \right) = 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n \, z^{n-1}}{1 + \sum\limits_{n=1}^{\infty} |a_n| \, z^{n-1}}$$

then it sufficient to show that

$$\operatorname{Re}\big(g_1(z)\big) \geqslant 0, \quad z \in U, \qquad \text{or} \qquad \left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leqslant 1, \quad z \in U,$$

and applying (23), we find that

$$\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leqslant \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum\limits_{n=1}^{m} |a_n| - d_{m+1} \sum\limits_{n=m+1}^{\infty} |a_n|} \leqslant 1, \qquad z \in U,$$

which ready yields the assertion (20) of Theorem 11. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{24}$$

gives sharp result, we observe that for

$$z = r e^{\frac{i\pi}{m}} \quad \text{that} \quad \frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}} \qquad \text{as} \quad r \to 1^-.$$

Similarly, if we takes $g_2(z) = (1 + d_{m+1}) \left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}} \right)$ and making use of (23), we denote that

$$\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| < \frac{(1 + d_{m+1}) \sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2 \sum\limits_{n=1}^{m} |a_n| - (1 - d_{m+1}) \sum\limits_{n=m+1}^{\infty} |a_n|}$$

which leads us immediately to the assertion (21) of Theorem 11.

The bound in (21) is sharp for each $m \in \mathbb{N}$ with extremal function $f(z)$ given by (24). $\square$

The proof of the following theorem is analogous to that of Theorem 11, so we omit the proof.

Theorem 12. *If $f \in \sigma_p^*(\eta, k, \upsilon)$ be given by* (1) *and satisfies the condition* (8) *then*

$$\operatorname{Re}\left(\frac{f'(z)}{f'_m(z)}\right) > 1 - \frac{m+1}{d_{m+1}}$$

and

$$\operatorname{Re}\left(\frac{f'_m(z)}{f'(z)}\right) > \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geqslant \begin{cases} n, & \text{if } n = 2, 3, \cdots, m, \\ \frac{[n(1+k)+(\eta+k)]\phi_n(\upsilon)}{(1-\eta)}, & \text{if } n = m+1, m+2, \ldots \end{cases}$$

The bounds are sharp with the extremal function $f(z)$ of the form (12).

6. Neighborhoods for the class $\sigma_p^{*\xi}(\eta, k, \upsilon)$

In this section, we determine the neighborhood for the class $\sigma_p^{*\xi}(\eta, k, \upsilon)$ which we define as follows

Definition 13. *A function $f \in \sum$ is said to be in the class $\sigma_p^{*\xi}(\eta, k, \upsilon)$ if there exits a function $g \in \sigma_p^*(\eta, k, \upsilon)$ such that*

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \xi, \qquad z \in E, \quad 0 \leqslant \xi < 1. \tag{25}$$

Following the earlier works on neighbourhoods of analytic functions by Goodman [5] and Ruscheweyh [6], we define the $\delta-$neighbourhoods of function $f \in \sum$ by

$$N_\delta(f) = \left\{g \in \sum : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \sum_{n=1}^{\infty} n\,|a_n - b_n| \leqslant \delta\right\}. \tag{26}$$

Theorem 14. *If* $g \in \sigma_p^*(\eta, k, \upsilon)$ *and*

$$\xi = 1 - \frac{\delta\,(2\,k+\eta+1)\,\phi_1(\upsilon)}{(2\,k+\eta+1)\,\phi_1(\upsilon)-(1-\eta)} \tag{27}$$

then $N_\delta(g) \subset \sigma_p^{*\xi}(\eta, k, \upsilon)$.

Proof. Let $f \in N_\delta(g)$. Then we find from (26) that

$$\sum_{n=1}^{\infty} n\,|a_n - b_n| \leqslant \delta \tag{28}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leqslant \delta, \qquad n \in \mathbb{N}. \tag{29}$$

Since $g \in \sigma_p^*(\eta, k, \upsilon)$, we have

$$\sum_{n=1}^{\infty} b_n \leqslant \frac{(1-\eta)}{(2\,k+\eta+1)\,\phi_1(\upsilon)}. \tag{30}$$

So that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum\limits_{n=1}^{\infty} |a_n - b_n|}{1 - \sum\limits_{n=1}^{\infty} b_n} = \frac{\delta\,(2\,k+\eta+1)\,\phi_1(\upsilon)}{(2\,k+\eta+1)\,\phi_1(\upsilon)-(1-\eta)} = 1 - \xi$$

provided ξ is given by (27). Hence by definition, $f \in \sigma_p^{*\xi}(\eta, k, \upsilon)$ for ξ given by which completes the proof. □

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