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UNITARILY EQUIVALENT BILATERAL WEIGHTED SHIFTS WITH OPERATOR WEIGHTS

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Abstract. The aim of this paper is to study unitarily equivalent bilateral weighted shifts with operator weights. Our purpose is to establish a general characterization of unitary equivalence of such shifts under the assumption that the weights are quasi-invertible. Under further assumptions on weights it was proved that unitary equivalence of bilateral weigthed shifts with operator weights defined on $\mathbb{C}^2$ can always be given by a unitary operator with at most two non-zero diagonals. The paper contains also examples of unitarily equivalent shifts with weights defined on $\mathbb{C}^k$ such that every unitary operator, which intertwines them has at least k non-zero diagonals.

Keywords: weighted shifts, operator weights, unitary equivalence.

Mathematics Subject Classification: 47B37, 47B02.

1. INTRODUCTION

Classical weighted shifts (both unilateral and bilateral) have been studied extensively for many years (see [15] for comprehensive work on weighted shifts). In [11] the weighted shift operators were studied from the point of view of linear dynamics. One possible generalization of classical weighted shifts is to replace scalar weights with operator weights. We consider unilateral weighted shifts:

$$\ell^2(\mathbb{N},\mathcal{H}) \ni (x_i)_{i\in\mathbb{N}} \mapsto (0, S_1x_0, S_2x_1, \dots) \in \ell^2(\mathbb{N},\mathcal{H})$$

and bilateral weighted shifts:

$$\ell^2(\mathbb{Z},H) \ni (x_i)_{i\in\mathbb{Z}} \mapsto (S_ix_{i-1})_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},H),$$

where all S_i's are bounded operators on a Hilbert space $\mathcal{H}$. The matrix representation of bilateral weighted shift takes the form

$$\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & 0 & 0 & \ddots \\ \ddots & S_0 & \boxed{0} & 0 & \ddots \\ \ddots & 0 & S_1 & 0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1.1}$$

where $\boxed{\cdot}$ denotes the entry in zeroth column and zeroth row of this matrix. In [10] Lambert studied unitary equivalence of unilateral weighted shifts with invertible weights. In [12] Orovčanec characterized unitarily equivalent unilateral weighted shifts with quasi-invertible weights (see also [1] for the similar result for shifts with weights having dense ranges). The similarity and quasi-similarity of both unilateral and bilateral weighted shifts was studied by Ivanovski and Orovčanec (see [8] and [7]). Guyker showed (see [4]) that if the weights can be divided into two sequences of normal and commuting operators, then the bilateral shift with such weights is unitarily equivalent to the shift with non-negative weights. Recently, in [9] Kośmider characterized bilateral shifts with quasi-invertible weights, which are unitarily equivalent by a unitary operator of diagonal form.

The aim of this paper is to generalize the results of [9] to the case of unitary equivalence by any unitary operator (not necessarily of diagonal form). In Section 2 we review some of the standard facts on bilateral weighted shifts, which are crucial in the subsequent part of this paper. Section 3 contains the main results of this paper. In Theorem 3.1 we present the counterpart of [10, Corollary 3.3] for bilateral weighted shifts. Next, we show how to deduce the condition given by [9, Corollary 2.4] from our result (see Corollary 3.5). We also give some more convenient characterizations of unitary equivalence under further assumptions on weights. Theorem 3.8 provides the characterization of unitary equivalence in terms of factors in polar decomposition of the bilateral weighted shift. The remaining part of Section 3 is devoted to study certain shifts with positive weights. In Theorem 3.10 it is proved that when $\mathcal{H}$ is two dimensional, then under certain assumptions on weights the unitary equivalence is always given by a unitary operator with at most two non-zero diagonals. At the end of this section we provide a class of examples of two unitarily equivalent shifts with weights on $\mathbb{C}^k$ such that every unitary operator intertwining these two shifts has at least k non-zero diagonals.

2. PRELIMINARIES

Denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of non-negative integers and integers, respectively and by $\mathbb{R}$ and $\mathbb{C}$ the field of real and complex numbers, respectively. For $p \in \mathbb{R}$ and $A \subset \mathbb{R}$

we set

$$A_p = \{x \in A : x \geq p\}.$$

If $\mathcal{H}$, $\mathcal{K}$ are complex Hilbert spaces, then $\mathbf{B}(\mathcal{H},\mathcal{K})$ stands for the space of all linear and bounded operators from $\mathcal{H}$ to $\mathcal{K}$; if $\mathcal{H}=\mathcal{K}$, then we simply write $\mathbf{B}(\mathcal{H})$. For $T \in \mathbf{B}(\mathcal{H})$ we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range of T and the kernel of T, respectively. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *normal* if T commutes with its adjoint; T is *positive* if $\langle Th, h\rangle > 0$ for every $h \in \mathcal{H}$, $h \neq 0$. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *quasi-invertible* if $\mathcal{N}(T) = \{0\}$ and $\mathcal{N}(T^*) = \{0\}$. It can be easily seen that

$$T, S - \text{quasi-invertible} \implies TS - \text{quasi-invertible}, \quad T, S \in \mathbf{B}(\mathcal{H}), \tag{2.1}$$

and that

$$T - \text{positive} \implies T - \text{quasi-invertible}, \quad T \in \mathbf{B}(\mathcal{H}).$$

An operator $U \in \mathbf{B}(\mathcal{H})$ is called a *partial isometry* if $U|_{\mathcal{N}(U)^\perp}$ is isometric. If $T \in \mathbf{B}(\mathcal{H})$, then there exists the unique partial isometry $U \in \mathbf{B}(\mathcal{H})$ such that $\mathcal{N}(T) = \mathcal{N}(U)$ and $T = U|T|$, where $|T| = (T^*T)^{1/2}$ (see [16, Theorem 7.20]).

> The polar decomposition of T is the decomposition $T = U|T|$, where U is the unique partial isometry satisfying $\mathcal{N}(U) = \mathcal{N}(T)$. (2.2)

We say that the operators $T, S \in \mathbf{B}(\mathcal{H})$ are *unitarily equivalent* if there exists a unitary operator $U \in \mathbf{B}(\mathcal{H})$ such that $US = TU$. Observe that if $T, S \in \mathbf{B}(\mathcal{H})$ are unitarily equivalent, then so are S^* and T^* (by the same unitary operator). A family $\mathcal{F} \subset \mathbf{B}(\mathcal{H})$ of operators is *doubly commuting* if for every $A, B \in \mathcal{F}$, $AB = BA$ and $A^*B = BA^*$. If $(M_n)_{n\in\mathbb{Z}}$ is the sequence of subspaces of $\mathcal{H}$, then we denote by $\bigvee_{n\in\mathbb{Z}} M_n$ the closed linear span of $\bigcup_{n\in\mathbb{Z}} M_n$. If $(\mathcal{H}_i)_{i\in\mathbb{Z}}$ is the sequence of Hilbert spaces, then we define its orthogonal sum as follows:

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{H}_i = \left\{ h = (h_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \mathcal{H}_i \colon \sum_{i\in\mathbb{Z}} \|h_n\|^2 < \infty \right\};$$

this is a Hilbert space with the inner product given by the formula:

$$\langle (h_i)_{i\in\mathbb{Z}}, (h_i')_{i\in\mathbb{Z}}\rangle = \sum_{i\in\mathbb{Z}} \langle h_i, h_i'\rangle, \quad (h_i)_{i\in\mathbb{Z}}, (h_i')_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathcal{H}_i.$$

If $\mathcal{H}_i = \mathcal{H}$ for every $i \in \mathbb{Z}$, then the above orthogonal sum will be denoted by $\ell^2(\mathbb{Z},\mathcal{H})$. For $x \in \mathcal{H}$ and $k \in \mathbb{Z}$ denote by $x^{(k)} \in \ell^2(\mathbb{Z},\mathcal{H})$ the vector given as follows: $x_k^{(k)} = x$ and $x_i^{(k)} = 0$ for $i \in \mathbb{Z} \setminus \{k\}$. Note that every operator $T \in \mathbf{B}(\ell^2(\mathbb{Z},\mathcal{H}))$ has a matrix representation $[T_{i,j}]_{i,j\in\mathbb{Z}}$, where $T_{i,j} \in \mathbf{B}(\mathcal{H})$, satisfying the following formula (see [5, Chapter 8]):

$$T(x_i)_{i\in\mathbb{Z}} = \left(\sum_{j\in\mathbb{Z}} T_{i,j} x_j \right)_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathcal{H}).$$

Observe that

$$T = [T_{i,j}]_{i,j\in\mathbb{Z}} \in \mathbf{B}(\ell^2(\mathbb{Z},\mathcal{H})) \implies T^* = [T^*_{j,i}]_{i,j\in\mathbb{Z}}, \tag{2.3}$$

and

$$T = [T_{i,j}]_{i,j\in\mathbb{Z}}, S = [S_{i,j}]_{i,j\in\mathbb{Z}} \in \mathbf{B}(\ell^2(\mathbb{Z},\mathcal{H})) \implies TS = \left[\sum_{k\in\mathbb{Z}} T_{i,k}S_{k,j}\right]_{i,j\in\mathbb{Z}}. \tag{2.4}$$

For the convenience let us introduce the following notation: if $(A_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ is a uniformly bounded sequence of operators, then by $D[(A_i)_{i\in\mathbb{Z}}] \in \mathbf{B}\left(\ell^2(\mathbb{Z},\mathcal{H})\right)$ we denote the diagonal operator with operators A_i ($i \in \mathbb{Z}$) on the diagonal, that is, $D[(A_i)_{i\in\mathbb{Z}}]\colon \ell^2(\mathbb{Z},\mathcal{H}) \to \ell^2(\mathbb{Z},\mathcal{H})$ is defined as

$$D[(A_i)_{i\in\mathbb{Z}}](x_i)_{i\in\mathbb{Z}} = (A_i x_i)_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathcal{H}). \tag{2.5}$$

For a uniformly bounded sequence $(S_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ of operators we define the bilateral weighted shift $S \in \mathbf{B}\left(\ell^2(\mathbb{Z},\mathcal{H})\right)$ with weights $(S_i)_{i\in\mathbb{Z}}$ by the formula:

$$S(x_i)_{i\in\mathbb{Z}} = (S_i x_{i-1})_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathcal{H}).$$

Let F be the bilateral weighted shift with all weights equal to the identity operator on $\mathcal{H}$. It is easy to see that F is a unitary operator. An operator $T \in \mathbf{B}\left(\ell^2(\mathbb{Z},\mathcal{H})\right)$ is *of diagonal form* if there exists a uniformly bounded sequence $(A_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ such that $T = F^k D[(A_i)_{i\in\mathbb{Z}}]$ for some $k \in \mathbb{Z}$. In the following lemma we gather basic properties of bilateral weighted shifts with operator weights; the proof (which is a straightforward application of (1.1), (2.3) and (2.4)) is left to the reader.

Lemma 2.1. *Let $\mathcal{H}$ be a complex Hilbert space. Let $(S_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ be a uniformly bounded sequence of operators and let S be the bilateral weighted shift $S \in \mathbf{B}\left(\ell^2(\mathbb{Z},\mathcal{H})\right)$ with weights $(S_i)_{i\in\mathbb{Z}}$. Then:*

(i) *for every $n \in \mathbb{N}_1$,*

$$(S^n)_{i,j} = \begin{cases} 0, & \text{for } i \neq j+n, \\ S_{j+n}\cdots S_{j+1}, & \text{for } i = j+n, \end{cases} \quad i,j \in \mathbb{Z},$$

are the entries of the matrix representation of S^n,

(ii) *for every $n \in \mathbb{N}_1$,*

$$(S^{*n})_{i,j} = \begin{cases} 0, & \text{for } j \neq i+n, \\ S^*_{i+1}\cdots S^*_{i+n}, & \text{for } j = i+n, \end{cases} \quad i,j \in \mathbb{Z},$$

are the entries of the matrix representation of S^{*n},

(iii) *for every $n \in \mathbb{N}_1$,*

$$(S^{*n}S^n)_{i,j} = \begin{cases} 0, & \text{for } i \neq j, \\ |S_{j+n}\cdots S_{j+1}|^2, & \text{for } i = j, \end{cases} \quad i,j \in \mathbb{Z},$$

are the entries of the matrix representation of $S^{*n}S^n$,

(iv) *for every* $n \in \mathbb{N}_1$,

$$(S^n S^{*n})_{i,j} = \begin{cases} 0, & \text{for } j \neq i, \\ |S^*_{i-n+1} \cdots S^*_i|^2, & \text{for } j = i, \end{cases} \quad i, j \in \mathbb{Z},$$

are the entries of the matrix representation of $S^n S^{*n}$.

The next lemma characterizes the operators, which intertwines two bilateral weighted shifts (see also [13, Lemma 4]).

Lemma 2.2. *Let* $\mathcal{H}$ *be a complex Hilbert space. Let* $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ *be two uniformly bounded sequences of operators. Denote by* S, T *the bilateral weighted shifts with weights* $(S_i)_{i\in\mathbb{Z}}$ *and* $(T_i)_{i\in\mathbb{Z}}$*, respectively. Suppose* $A = [A_{i,j}]_{i,j\in\mathbb{Z}} \in \mathbf{B}(\ell^2(\mathbb{Z}, \mathcal{H}))$*. Let* $n \in \mathbb{N}_1$*. Then:*

(i) $AS^n = T^n A$ *if and only if*

$$A_{i+n,j+n}(S_{j+n} \cdots S_{j+1}) = T_{i+n} \cdots T_{i+1} A_{i,j}, \quad i, j \in \mathbb{Z},$$

(ii) $AS^{*n} = T^{*n} A$ *if and only if*

$$A_{i,j}(S^*_{j+1} \cdots S^*_{j+n}) = T^*_{i+1} \cdots T^*_{i+n} A_{i+n,j+n}, \quad i, j \in \mathbb{Z}.$$

Proof. Observe that, by Lemma 2.1,

$$(AS^n)_{i,j} = \sum_{k\in\mathbb{Z}} A_{i,k}(S^n)_{k,j} = A_{i,j+n} S_{j+n} \cdots S_{j+1}, \quad i, j \in \mathbb{Z},$$
$$(T^n A)_{i,j} = \sum_{k\in\mathbb{Z}} (T^n)_{i,k} A_{k,j} = T_i \cdots T_{i-n+1} A_{i-n,j}, \quad i, j \in \mathbb{Z},$$

are the entries of matrix representations of AS^n and $T^n A$, respectively. Replacing i with $i+n$ in the above equalities, we obtain (i). In turn, again by Lemma 2.1,

$$(AS^{*n})_{i,j} = \sum_{k\in\mathbb{Z}} A_{i,k}(S^{*n})_{k,j} = A_{i,j-n} S^*_{j-n+1} \cdots S^*_j, \quad i, j \in \mathbb{Z},$$
$$(T^{*n} A)_{i,j} = \sum_{k\in\mathbb{Z}} (T^{*n})_{i,k} A_{k,j} = T^*_{i+1} \cdots T^*_{i+n} A_{i+n,j}, \quad i, j \in \mathbb{Z},$$

are the entries of matrix representations of AS^{*n} and $T^{*n} A$, respectively. Replacing j with $j+n$ in the above equalities, we obtain (ii). □

In the following lemma we describe the polar decomposition of a bilateral weighted shift.

Lemma 2.3. *Let* $\mathcal{H}$ *be a complex Hilbert space. Let* $(S_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ *be a uniformly bounded sequence of operators and let* S *be the bilateral weighted shift with weights* $(S_i)_{i\in\mathbb{Z}}$*. For* $i \in \mathbb{Z}$ *let* $S_i = V_i|S_i|$ *be the polar decomposition of* S_i*. Denote by* V *the bilateral weighted shift with weights* $(V_i)_{i\in\mathbb{Z}}$*. Then* $S = V|S|$ *is the polar decomposition of* S*.*

Proof. For every $(x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathcal{H})$ we have

$$\begin{aligned} S(x_i)_{i\in\mathbb{Z}} &= (S_i x_{i-1})_{i\in\mathbb{Z}} = (V_i|S_i|x_{i-1})_{i\in\mathbb{Z}} \\ &= FD[(V_{i+1})_{i\in\mathbb{Z}}]D[(|S_{i+1}|)_{i\in\mathbb{Z}}](x_i)_{i\in\mathbb{Z}}. \end{aligned}$$

By Lemma 2.1, $D[(|S_{i+1}|)_{i\in\mathbb{Z}}] = |S|$. It can be easily verified that $V = FD[(V_{i+1})_{i\in\mathbb{Z}}]$. Next,

$$\begin{aligned} \mathcal{N}(S) &= \big\{(x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathcal{H})\colon S_i x_{i-1} = 0 \text{ for all } i\in\mathbb{Z}\big\} \\ &\overset{(2.2)}{=} \big\{(x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z},\mathcal{H})\colon V_i x_{i-1} = 0 \text{ for all } i\in\mathbb{Z}\big\} = \mathcal{N}(V). \end{aligned}$$

Thus, V is a partial isometry satisfying $\mathcal{N}(S) = \mathcal{N}(V)$, so $S = V|S|$ is the polar decomposition of S as in (2.2). □

3. UNITARY EQUIVALENCE OF WEIGHTED SHIFTS

In this section we provide a general characterization of unitarily equivalent weighted shifts with operator weights. The result presented below is the counterpart of the theorem of Lambert (cf. [10, Corollary 3.3]).

Theorem 3.1. *Let $\mathcal{H}$ be a Hilbert space. Let $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ be two uniformly bounded sequences of quasi-invertible operators and let S, T be the bilateral weighted shifts with weights $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$, respectively.*

(i) *If $US = TU$ for a unitary operator $U \in \mathbf{B}(\ell^2(\mathbb{Z},\mathcal{H}))$, then the isometry $U_0 \in \mathbf{B}(\mathcal{H}, \ell^2(\mathbb{Z},\mathcal{H}))$ defined by $U_0 x = Ux^{(0)}$, $x\in\mathcal{H}$, satisfies the following conditions:*

$$U_0|S_n\cdots S_1| = D[(|T_{i+n}\cdots T_{i+1}|)_{i\in\mathbb{Z}}]U_0, \quad n\in\mathbb{N}_1, \tag{3.1}$$

$$U_0|S^*_{-n+1}\cdots S^*_0| = D[(|T^*_{i-n+1}\cdots T^*_i|)_{i\in\mathbb{Z}}]U_0, \quad n\in\mathbb{N}_1, \tag{3.2}$$

$$T^{[k]}(\mathcal{R}(U_0)) \perp T^{[m]}(\mathcal{R}(U_0)), \quad k,m\in\mathbb{Z},\ k\neq m, \tag{3.3}$$

$$\bigvee_{i\in\mathbb{Z}} T^{[i]}(\mathcal{R}(U_0)) = \ell^2(\mathbb{Z},\mathcal{H}), \tag{3.4}$$

where

$$T^{[k]} = \begin{cases} T^k, & \text{if } k\in\mathbb{N}_1, \\ I, & \text{if } k = 0, \\ T^{*-k}, & \text{if } -k\in\mathbb{N}_1. \end{cases}$$

(ii) *If there exists an isometry $U_0 \in \mathbf{B}(\mathcal{H}, \ell^2(\mathbb{Z},\mathcal{H}))$ satisfying* (3.1)–(3.4)*, then S and T are unitarily equivalent.*

Proof. (i) Let $U = [U_{i,j}]_{i,j\in\mathbb{Z}} \in \mathbf{B}(\ell^2(\mathbb{Z},\mathcal{H}))$ be a unitary operator satisfying $US = TU$. Then, we also have $US^* = T^*U$. This implies that for every $n\in\mathbb{N}_1$, $U|S^n|^2 = |T^n|^2U$ and $U|S^{*n}|^2 = |T^{*n}|^2U$. Using the square root theorem (see [14, p.265]) and Berberian's trick (see [2]) we obtain that $U|S^n| = |T^n|U$ and $U|S^{*n}| = |T^{*n}|U$. By Lemma 2.1, the above equalities take the form

$$U_{i,j}|S_{j+n}\cdots S_{j+1}| = |T_{i+n}\cdots T_{i+1}|U_{i,j}, \quad i,j\in\mathbb{Z},\ n\in\mathbb{N}_1, \tag{3.5}$$

$$U_{i,j}|S^*_{j-n+1}\cdots S^*_j| = |T^*_{i-n+1}\cdots T^*_i|U_{i,j}, \quad i,j\in\mathbb{Z},\ n\in\mathbb{N}_1. \tag{3.6}$$

We will prove that the isometry U_0 satisfies (3.1)–(3.4). By (3.5) and (3.6), (3.1) and (3.2) hold. For $n\in\mathbb{Z}\setminus\{0\}$ define $U_n\colon \mathcal{H}\to\ell^2(\mathbb{Z},\mathcal{H})$ as follows:

$$U_n x = Ux^{(n)}, \quad x\in\mathcal{H}.$$

Since U is unitary, U_n is isometric for all $n\in\mathbb{Z}$. Next, by Lemma 2.2,

$$U_{i+n,n}S_n\cdots S_1 = T_{i+n}\cdots T_{i+1}U_{i,0}, \quad i\in\mathbb{Z},\ n\in\mathbb{N}_1. \tag{3.7}$$

For $n\in\mathbb{N}_1$ and $i\in\mathbb{Z}$, let

$$S_n\cdots S_1 = V_n|S_n\cdots S_1|, \tag{3.8}$$

$$T_{i+n}\cdots T_{i+1} = W_{n,i}|T_{i+n}\cdots T_{i+1}| \tag{3.9}$$

be the polar decompositions. Since, by (2.1), $S_n\cdots S_1$ and $T_{i+n}\cdots T_{i+1}$ are quasi-invertible, it follows from (2.2) that partial isometries V_n and $W_{n,i}$ ($n\in\mathbb{N}_1$, $i\in\mathbb{Z}$) are unitary operators. We have

$$\begin{aligned} W_{n,i}|T_{i+n}\cdots T_{i+1}|U_{i,0} &= T_{i+n}\cdots T_{i+1}U_{i,0} \\ &\overset{(3.7)}{=} U_{i+n,n}S_n\cdots S_1 \\ &= U_{i+n,n}V_n|S_n\cdots S_1|, \quad n\in\mathbb{N}_1,\ i\in\mathbb{Z}. \end{aligned}$$

This implies that for every $n\in\mathbb{N}_1,\ i\in\mathbb{Z}$,

$$W^*_{n,i}U_{i+n,n}V_n|S_n\cdots S_1| = |T_{i+n}\cdots T_{i+1}|U_{i,0} \overset{(3.5)}{=} U_{i,0}|S_n\cdots S_1|.$$

Since $S_n\cdots S_1$ is quasi-invertible, $\mathcal{N}(|S_n\cdots S_1|) = \{0\}$, which implies that $\overline{\mathcal{R}(|S_n\cdots S_1|)} = \mathcal{H}$. It follows from (3.5) that $W^*_{n,i}U_{i+n,n}V_n = U_{i,0}$; using the fact that $W_{n,i}$ is unitary, it can be written equivalently as

$$U_{i+n,n} = W_{n,i}U_{i,0}V^*_n, \quad n\in\mathbb{N}_1,\ i\in\mathbb{Z}. \tag{3.10}$$

From (3.10) and (2.5) we derive that

$$U_n = F^n D[(W_{n,i})_{i\in\mathbb{Z}}]U_0V^*_n, \quad n\in\mathbb{N}_1. \tag{3.11}$$

Now we prove similar formula for U_{-n} ($n\in\mathbb{N}_1$); we provide the whole reasoning, because it differs from the above in some details. Again, by Lemma 2.2 we have

$$U_{i-n,-n}(S^*_{-n+1}\cdots S^*_0) = T^*_{i-n+1}\cdots T^*_i U_{i,0}, \quad n\in\mathbb{N}_1,\ i\in\mathbb{Z}. \tag{3.12}$$

For $n \in \mathbb{N}_1$ and $i \in \mathbb{Z}$, let

$$S_0 \cdots S_{-n+1} = V_{-n}|S_0 \cdots S_{-n+1}|, \tag{3.13}$$
$$T_i \cdots T_{i-n+1} = W_{-n,i}|T_i \cdots T_{i-n+1}| \tag{3.14}$$

be the polar decompositions. As before, since $S_0 \cdots S_{-n+1}$ and $T_i \cdots T_{i-n+1}$ are quasi-invertible, partial isometries V_{-n} and $W_{-n,i}$ ($n \in \mathbb{N}$, $i \in \mathbb{Z}$) are unitary operators. We infer from [16, Exercise 7.26(c)] that for every $n \in \mathbb{N}_1$ and $i \in \mathbb{Z}$,

$$(S_0 \cdots S_{-n+1})^* = S^*_{-n+1} \cdots S^*_0 = V^*_{-n}|S^*_{-n+1} \cdots S^*_0| \tag{3.15}$$
$$(T_i \cdots T_{i-n+1})^* = T^*_{i-n+1} \cdots T^*_i = W^*_{-n,i}|T^*_{i-n+1} \cdots T^*_i| \tag{3.16}$$

are the polar decompositions of $(S_0 \cdots S_{-n+1})^*$ and $(T_i \cdots T_{i-n+1})^*$, respectively. We have

$$\begin{aligned} W^*_{-n,i}|T^*_{i-n+1} \cdots T^*_i|U_{i,0} &\overset{(3.16)}{=} T^*_{i-n+1} \cdots T^*_i U_{i,0} \\ &\overset{(3.12)}{=} U_{i-n,-n} S^*_{-n+1} \cdots S^*_0 \\ &\overset{(3.15)}{=} U_{i-n,-n} V^*_{-n}|S^*_{-n+1} \cdots S^*_0|, \quad n \in \mathbb{N}_1,\ i \in \mathbb{Z}. \end{aligned}$$

Thus, for every $n \in \mathbb{N}_1,\ i \in \mathbb{Z}$,

$$W_{-n,i} U_{i-n,-n} V^*_{-n}|S^*_{-n+1} \cdots S^*_0| = |T^*_{i-n+1} \cdots T^*_i|U_{i,0} \overset{(3.6)}{=} U_{i,0}|S^*_{0-n+1} \cdots S^*_0|.$$

Since $\overline{\mathcal{R}(|S^*_{-n+1} \cdots S^*_0|)} = \mathcal{H}$, it follows from (3.6) that $W_{-n,i} U_{i-n,-n} V^*_{-n} = U_{i,0}$. Equivalently,

$$U_{i-n,-n} = W^*_{-n,i} U_{i,0} V_{-n}, \quad n \in \mathbb{N}_1,\ i \in \mathbb{Z}.$$

Hence,

$$U_{-n} = F^{-n} D[(W^*_{-n,i})_{i \in \mathbb{Z}}] U_0 V_{-n}, \quad n \in \mathbb{N}_1. \tag{3.17}$$

Now we will prove (3.3). First, observe that if $k \in \mathbb{N}_1$, then

$$\begin{aligned} U_k S_k \cdots S_1 &\overset{(3.11)}{=} F^k D[(W_{k,i})_{i \in \mathbb{Z}}] U_0 V^*_k S_k \cdots S_1 \\ &= F^k D[(W_{k,i})_{i \in \mathbb{Z}}] U_0 |S_k \cdots S_1| \\ &\overset{(3.5)}{=} F^k D[(W_{k,i})_{i \in \mathbb{Z}}] D[(|T_{i+k} \cdots T_{i+1}|)_{i \in \mathbb{Z}}] U_0 \\ &\overset{(3.9)}{=} F^k D[(T_{i+k} \cdots T_{i+1})_{i \in \mathbb{Z}}] U_0 \\ &\overset{\text{Lemma 2.1}}{=} T^k U_0. \end{aligned} \tag{3.18}$$

In turn, if $-k \in \mathbb{N}_1$, then

$$\begin{aligned} U_k S_{k+1}^* \cdots S_0^* &\overset{(3.17)}{=} F^k D[(W_{k,i}^*)_{i\in\mathbb{Z}}] U_0 V_k S_{k+1}^* \cdots S_0^* \\ &= F^k D[(W_{k,i}^*)_{i\in\mathbb{Z}}] U_0 |S_{k+1}^* \cdots S_0^*| \\ &\overset{(3.6)}{=} F^k D[(W_{k,i}^*)_{i\in\mathbb{Z}}] D[(|T_{i+k+1}^* \cdots T_i^*|)_{i\in\mathbb{Z}}] U_0 \\ &\overset{(3.16)}{=} F^k D[(T_{i+k+1}^* \cdots T_i^*)_{i\in\mathbb{Z}}] U_0 \\ &\overset{\text{Lemma 2.1}}{=} T^{*-k} U_0. \end{aligned} \tag{3.19}$$

Since $Ux^{(k)} = U_k x$ for all $k \in \mathbb{Z}$ and $x \in \mathcal{H}$, we obtain that $\mathcal{R}(U_k) \perp \mathcal{R}(U_n)$ for all $k, n \in \mathbb{Z}$, $n \neq k$. In particular, for all $k, m \in \mathbb{Z}$, $k \neq m$, $\langle U_k \widehat{x_k}, U_m \widehat{x_m} \rangle = 0$, where

$$\widehat{x_j} = \begin{cases} S_j \cdots S_1 x_j, & \text{if } j > 0, \\ x_0, & \text{if } j = 0, \\ S_{j+1}^* \cdots S_0^* x_j, & \text{if } j < 0, \end{cases} \quad j = k, m,$$

and $x_k, x_m \in \mathcal{H}$. But from the above computation it follows that $U_j \widehat{x_j} = T^{[j]} U_0 x_j$, $j = k, m$. Hence, (3.3) holds.

Finally, since $U(x_i)_{i\in\mathbb{Z}} = \sum_{i\in\mathbb{Z}} U_i x_i$,

$$\ell^2(\mathbb{Z}, \mathcal{H}) = \bigvee_{i\in\mathbb{Z}} \mathcal{R}(U_i). \tag{3.20}$$

By the fact that $\overline{\mathcal{R}(S_n \cdots S_1)} = \mathcal{H}$ and $\overline{\mathcal{R}(S_{-n+1}^* \cdots S_0^*)} = \mathcal{H}$ for $n \in \mathbb{N}_1$, we obtain from (3.11) and (3.17) that

$$\mathcal{R}(U_n) = \overline{\mathcal{R}(U_n S_n \cdots S_1)} \quad \text{and} \quad \mathcal{R}(U_{-n}) = \overline{\mathcal{R}(U_{-n} S_{-n+1}^* \cdots S_0^*)}$$

for all $n \in \mathbb{N}_1$. Combining the above and the equalities (3.18) and (3.19) we get that $\mathcal{R}(U_n) = \overline{\mathcal{R}(T^{[n]} U_0)}$ for all $n \in \mathbb{Z}$. Hence, (3.20) takes the form

$$\ell^2(\mathbb{Z}, \mathcal{H}) = \bigvee_{i\in\mathbb{Z}} \overline{\mathcal{R}(T^{[i]} U_0)} = \bigvee_{i\in\mathbb{Z}} \mathcal{R}(T^{[i]} U_0).$$

Therefore, (3.4) holds.

(ii) For $n \in \mathbb{N}_1$ let U_n be defined by (3.11) and let U_{-n} be defined by (3.17), where U_0 is as in (ii). Since U_0 is an isometry, so is U_n for every $n \in \mathbb{Z}$. By the fact that $\overline{\mathcal{R}(S_n \cdots S_1)} = \mathcal{H}$ and $\overline{\mathcal{R}(S_{-n+1}^* \cdots S_0^*)} = \mathcal{H}$, using (3.18) and (3.19), we obtain from (3.3) and (3.4) that $\mathcal{R}(U_k) \perp \mathcal{R}(U_n)$ for all $n, k \in \mathbb{Z}$, $n \neq k$, and that

$$\ell^2(\mathbb{Z}, \mathcal{H}) = \bigvee_{i\in\mathbb{Z}} \mathcal{R}(U_i). \tag{3.21}$$

Define

$$U \colon \ell^2(\mathbb{Z}, \mathcal{H}) \ni (x_i)_{i\in\mathbb{Z}} \longmapsto \sum_{i\in\mathbb{Z}} U_i x_i \in \ell^2(\mathbb{Z}, \mathcal{H}).$$

Since all U_n's ($n \in \mathbb{Z}$) are isometries with mutually orthogonal ranges, U is also isometric. By (3.21), U is unitary. It remains to show that $US = TU$. It is sufficient to prove that

$$USx^{(k)} = TUx^{(k)}, \quad x \in \mathcal{H},\ k \in \mathbb{Z}. \tag{3.22}$$

First, let us check (3.22) for $k = 0$. In this case we have

$$\begin{aligned} USx^{(0)} &= U(S_1x)^{(1)} = U_1S_1x \overset{(3.11)}{=} FD[(W_{1,i})_{i\in\mathbb{Z}}]U_0V_1^*S_1x \\ &\overset{(3.8)}{=} FD[(W_{1,i})_{i\in\mathbb{Z}}]U_0|S_1|x \\ &\overset{(3.1)}{=} F[(W_{1,i})_{i\in\mathbb{Z}}]D[(|T_{i+1}|)_{i\in\mathbb{Z}}]U_0x \\ &\overset{(3.9)}{=} FD[(T_{i+1})_{i\in\mathbb{Z}}]U_0x = TU_0x = TUx^{(0)}. \end{aligned}$$

Next, assume $k \in \mathbb{N}_1$. Since $\overline{\mathcal{R}(S_k \cdots S_1)} = \mathcal{H}$, it is enough to check (3.22) for vectors of the form $S_k \cdots S_1x$, where $x \in \mathcal{H}$. We have

$$\begin{aligned} &US(S_k \cdots S_1x)^{(k)} = U(S_{k+1} \cdots S_1x)^{(k+1)} = U_{k+1}S_{k+1} \cdots S_1x \\ &\overset{(3.18)}{=} T^{k+1}U_0x = TT^kU_0x \overset{(3.18)}{=} TU_kS_k \cdots S_1x \\ &= TU(S_k \cdots S_1x)^{(k)}. \end{aligned}$$

Now suppose that $-k \in \mathbb{N}_1$. Note that it suffices to verify that

$$S_{k+1}^*U_{k+1}^* = U_k^*T^*. \tag{3.23}$$

Indeed, if (3.23) holds, then by taking adjoints we obtain $U_{k+1}S_{k+1} = TU_k$. But

$$U_{k+1}S_{k+1}x = U(S_{k+1}x)^{(k+1)} = USx^{(k)} \quad \text{and} \quad TUx^{(k)} = TU_kx, \quad x \in \mathcal{H}.$$

Again, since $\overline{\mathcal{R}(T_{i+1}^* \cdots T_{i-k-1}^*)} = \mathcal{H}$ for all $i \in \mathbb{Z}$, it suffices to verify (3.23) for vectors of the form $(T_{i+1}^* \cdots T_{i-k-1}^* x_{i-k-1})_{i\in\mathbb{Z}}$, where $(x_i)_{i\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathcal{H})$. For such a vector we have

$$\begin{aligned} &U_k^*T^*(T_{i+1}^* \cdots T_{i-k-1}^* x_{i-k-1})_{i\in\mathbb{Z}} \overset{\text{Lemma 2.1}}{=} U_k^*(T_{i+1}^* \cdots T_{i-k}^* x_{i-k})_{i\in\mathbb{Z}} \\ &\overset{(3.17)}{=} V_k^*U_0^*D[(W_{k,i})_{i\in\mathbb{Z}}]F^{-k}(T_{i+1}^* \cdots T_{i-k}^* x_{i-k})_{i\in\mathbb{Z}} \\ &= V_k^*U_0^*D[(W_{k,i})_{i\in\mathbb{Z}}](T_{i+k+1}^* \cdots T_i^* x_i)_{i\in\mathbb{Z}} \\ &\overset{(3.16)}{=} V_k^*U_0^*(|T_{i+k+1}^* \cdots T_i^*|x_i)_{i\in\mathbb{Z}} \\ &\overset{(3.2)}{=} V_k^*|S_{k+1}^* \cdots S_0^*|U_0^*(x_i)_{i\in\mathbb{Z}} \\ &= (S_{k+1}^* \cdots S_0^*)U_0^*(x_i)_{i\in\mathbb{Z}} \end{aligned}$$

and

$$\begin{aligned}
&S_{k+1}^* U_{k+1}^* (T_{i+1}^* \cdots T_{i-k-1}^* x_{i-k-1})_{i\in\mathbb{Z}}\\
&\overset{(3.17)}{=} S_{k+1}^* V_{k+1}^* U_0^* D[(W_{k+1,i})_{i\in\mathbb{Z}}] F^{-(k+1)} (T_{i+1}^* \cdots T_{i-k-1}^* x_{i-k-1})_{i\in\mathbb{Z}}\\
&= S_{k+1}^* V_{k+1}^* U_0^* D[(W_{k+1,i})_{i\in\mathbb{Z}}] (T_{i+k+2}^* \cdots T_i^* x_i)_{i\in\mathbb{Z}}\\
&\overset{(3.16)}{=} S_{k+1}^* V_{k+1}^* U_0^* (|T_{i+k+2}^* \cdots T_i^*| x_i)_{i\in\mathbb{Z}}\\
&\overset{(3.2)}{=} S_{k+1}^* V_{k+1}^* |S_{k+2}^* \cdots S_0^*| U_0^* (x_i)_{i\in\mathbb{Z}}\\
&= S_{k+1}^* S_{k+2}^* \cdots S_0^* U_0^* (x_i)_{i\in\mathbb{Z}}.
\end{aligned}$$

Hence, (3.23) holds, which completes the proof. □

Remark 3.2. The careful look on the proof of Theorem 3.1 reveals that all entries in the matrix representation of the unitary operator U making S and T unitarily equivalent are uniquely determined by U_0 and that the isometry U_0 in (ii) is exactly the zeroth column of the unitary operator we construct. The reader can easily verify that the choice of the zeroth column is arbitrary; we may prove that all entries of U are uniquely determined by any other fixed column as well.

As a corollary we obtain the counterpart of [10, Corollary 3.2] for bilateral shifts.

Corollary 3.3. *Let $\mathcal{H}$ be a Hilbert space. Let $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ be two sequences of unitary operators on $\mathcal{H}$ and let S, T be the bilateral weighted shifts with weights $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$, respectively. Then S and T are unitarily equivalent.*

Proof. Define $U_0 \colon \mathcal{H} \to \ell^2(\mathbb{Z}, \mathcal{H})$ by

$$U_0 x = x^{(0)}, \quad x \in \mathcal{H}.$$

It is a matter of routine to verify that (3.1)–(3.4) hold. The application of Theorem 3.1(ii) finishes the proof. □

It turns out that under some additional assumptions on weights, we can simplify the conditions in Theorem 3.1.

Corollary 3.4. *Suppose that S and T are as in Theorem 3.1 with additional assumption that the sequences $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$ are doubly commuting.*

(i) *if $US = TU$ for a unitary operator $U \in \mathbf{B}(\ell^2(\mathbb{Z}, \mathcal{H}))$, then the isometry $U_0 \in \mathbf{B}(\mathcal{H}, \ell^2(\mathbb{Z}, \mathcal{H}))$ defined in Theorem 3.1*(i) *satisfies*

$$U_0 |S_n| = D[(|T_{i+n}|)_{i\in\mathbb{Z}}] U_0, \quad n \in \mathbb{N}_1, \tag{3.24}$$

$$U_0 |S_{-n+1}^*| = D[(|T_{i-n+1}^*|)_{i\in\mathbb{Z}}] U_0, \quad n \in \mathbb{N}_1. \tag{3.25}$$

(ii) *If there exists an isometry $U_0 \in \mathbf{B}(\mathcal{H}, \ell^2(\mathbb{Z}, \mathcal{H}))$ satisfying* (3.24), (3.25), (3.3) *and* (3.4)*, then S and T are unitarily equivalent.*

Proof. Since $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$ are sequences of doubly commuting operators, by [1, Lemma 2.1] we have

$$|S_n|\cdots|S_1| = |S_n\cdots S_1|, \quad n\in\mathbb{N}_1, \tag{3.26}$$

$$|T_{i+n}\cdots T_{i+1}| = |T_{i+n}|\cdots|T_{i+1}|, \quad n\in\mathbb{N}_1,\ i\in\mathbb{Z} \tag{3.27}$$

(i) By Theorem 3.1, the isometry $U_0\colon \mathcal{H}\to\ell^2(\mathbb{Z},\mathcal{H})$ satisfies (3.1)–(3.4). We check that (3.24) and (3.25) hold. By (3.26) and (3.27), (3.1) takes the form

$$U_0|S_n|\cdots|S_1| = D[(|T_{i+n}|)_{i\in\mathbb{Z}}]\cdots D[(|T_{i+1}|)]U_0, \quad n\in\mathbb{N}_1. \tag{3.28}$$

If $n=1$, then (3.28) coincides with (3.24). If $n\in\mathbb{N}_2$, then, using (3.28) with n replaced with $n-1$, we obtain

$$U_0|S_n||S_{n-1}|\cdots|S_1| = D[(|T_{i+n}|)_{i\in\mathbb{Z}}]U_0|S_{n-1}|\cdots|S_1|.$$

Since $\overline{\mathcal{R}(|S_{n-1}|\cdots|S_1|)} = \mathcal{H}$, the above equality implies (3.24). Similar reasoning shows that (3.2) implies (3.25).

(ii) Arguing as in (i) we derive the equalities (3.1) and (3.2) from (3.24) and (3.25). □

Corollary 3.4 gives (in particular) the characterization of unitary equivalence for classical bilateral weighted shifts (with scalar weights). However, in this case the proof is much simpler than the proof of Theorem 3.1; in particular, the unitary operator making two bilateral weighted shifts unitarily equivalent is always of diagonal form (see [15, Theorem 1, p. 53]).

Now, let us show how to deduce [9, Corollary 2.4] from Theorem 3.1.

Corollary 3.5. *Suppose that S and T satisfy the assumptions of Theorem 3.1. The following conditions are equivalent:*

(i) *there exists a unitary operator $U\in\mathbf{B}\left(\ell^2(\mathbb{Z},\mathcal{H})\right)$ of diagonal form such that $US=TU$,*
(ii) *there exist $p\in\mathbb{Z}$ and a unitary operator $U_{p,0}\in\mathbf{B}(\mathcal{H})$ such that*

$$\|S_n\cdots S_1v\| = \|T_{p+n}\cdots T_{p+1}U_{p,0}v\|, \quad v\in\mathcal{H},\ n\in\mathbb{N}_1, \tag{3.29}$$

$$\|S^*_{-n+1}\cdots S^*_0v\| = \|T^*_{p-n+1}\cdots T^*_pU_{p,0}v\|, \quad v\in\mathcal{H},\ n\in\mathbb{N}_1. \tag{3.30}$$

Proof. For $p\in\mathbb{Z}$ and $n\in\mathbb{N}_1$ let V_n, V_{-n}, $W_{n,p}$ and $W_{-n,p}$ be as in (3.8), (3.13), (3.9) and (3.14), respectively. Since S_n and T_n are quasi-invertible, we infer from (2.1) and (2.2) that V_n and $W_{n,p}$ are unitary for every $n\in\mathbb{Z}$.

(i)⇒(ii) By Theorem 3.1(i), the isometry $U_0\colon\mathcal{H}\to\ell^2(\mathbb{Z},\mathcal{H})$ satisfies (3.1) and (3.2). Since U is of diagonal form, U_0 is of the form $U_0x=(U_{p,0}x)^{(p)}$, where $U_{p,0}\in\mathbf{B}(\mathcal{H})$ is a unitary operator and $p\in\mathbb{Z}$. Then (3.1) takes the form

$$U_{p,0}V_n^*S_n\cdots S_1 = W^*_{n,p}T_{p+n}\cdots T_{p+1}U_{p,0}, \quad n\in\mathbb{N}_1.$$

Thus, (3.29) easily follows from the above equality. In turn, using (3.15) and (3.16), (3.2) takes the form

$$U_{p,0}V_{-n}S^*_{-n+1}\cdots S^*_0 = W_{-n,p}T^*_{p-n+1}\cdots T^*_pU_{p,0}, \quad n\in\mathbb{N}_1.$$

From the above we deduce (3.30).

(ii)⇒(i) Observe that for every $x\in\mathcal{H}$,

$$\begin{aligned}\||S_n\cdots S_1|x\|^2 &= \|V_n|S_n\cdots S_1|x\|^2 = \|S_n\cdots S_1x\|^2\\ &\overset{(3.29)}{=} \|T_{p+n}\cdots T_{p+1}U_{p,0}x\|^2 = \|W_{n,p}|T_{p+n}\cdots T_{p+1}|U_{p,0}x\|^2\\ &= \||T_{p+n}\cdots T_{p+1}|U_{p,0}x\|^2 = \|U^*_{p,0}|T_{p+n}\cdots T_{p+1}|U_{p,0}x\|^2.\end{aligned}$$

Hence,

$$\langle|S_n\cdots S_1|^2x,x\rangle = \langle(U^*_{p,0}|T_{p+n}\cdots T_{p+1}|U_{p,0})^2x,x\rangle, \quad x\in\mathcal{H},$$

which implies that

$$|S_n\cdots S_1|^2 = (U^*_{p,0}|T_{p+n}\cdots T_{p+1}|U_{p,0})^2, \quad n\in\mathbb{N}_1.$$

By the uniqueness of square root,

$$|S_n\cdots S_1| = U^*_{p,0}|T_{p+n}\cdots T_{p+1}|U_{p,0}, \quad n\in\mathbb{N}_1.$$

It follows that

$$U_{p,0}|S_n\cdots S_1| = |T_{p+n}\cdots T_{p+1}|U_{p,0}, \quad n\in\mathbb{N}_1. \tag{3.31}$$

In a similar manner we prove that

$$U_{p,0}|S^*_{-n+1}\cdots S^*_0| = |T^*_{p-n+1}\cdots T^*_p|U_{p,0}, \quad n\in\mathbb{N}. \tag{3.32}$$

Define $U_0\colon\mathcal{H}\to\ell^2(\mathbb{Z},\mathcal{H})$ by the formula: $U_0x=(U_{p,0}x)^{(p)}$, $x\in\mathcal{H}$. Clearly, U_0 is isometric. In view of (3.31) and (3.32), (3.1) and (3.2) hold; (3.3) and (3.4) are trivially satisfied. The application of Theorem 3.1(ii) completes the proof. □

Next, we show a convenient necessary condition for bilateral weighted shifts to be unitarily equivalent by a unitary operator of diagonal form.

Lemma 3.6. *Suppose that S and T satisfy the assumptions of Theorem 3.1. Assume that there exists a unitary operator $U=[U_{i,j}]_{i,j\in\mathbb{Z}}\in\mathbf{B}(\ell^2(\mathbb{Z},\mathcal{H}))$ of diagonal form such that $US=TU$. Then there exists $p\in\mathbb{Z}$ such that $|S_i|$ and $|T_{i+p}|$ are unitarily equivalent for all $i\in\mathbb{Z}$; in particular, $\sigma(|S_i|)=\sigma(|T_{i+p}|)$ for every $i\in\mathbb{Z}$.*

Proof. Since U is of diagonal form, there exists $p\in\mathbb{Z}$ such that $U_{i+p,i}$ ($i\in\mathbb{Z}$) are the only non-zero entries of the matrix representation of U. This implies that $U_{i+p,i}$ has to be unitary for every $i\in\mathbb{Z}$. The equalities $US=TU$ and $US^*=T^*U$ implies that $U|S|=|T|U$. Now, it follows from Lemma 2.1 that

$$U_{i+p,i}|S_{i+1}| = |T_{i+p+1}|U_{i+p,i}, \quad i\in\mathbb{Z}.$$

Hence, $|S_i|$ and $|T_{i+p}|$ are unitarily equivalent for all $i\in\mathbb{Z}$. □

It can be easily deduced from the Schur decomposition (see [6, Theorem 2.3.3]) that if $A \in \mathbf{B}(\mathbb{C}^m)$ is normal, then $\sigma(|A|) = \{|\lambda| : \lambda \in \sigma(A)\}$. Hence, in case $\mathcal{H} = \mathbb{C}^m$ ($m \in \mathbb{N}_1$) the necessary condition $\sigma(|S_i|) = \sigma(|T_{i+p}|)$ in Lemma 3.6 actually says that if the two bilateral weighted shifts S and T with normal and invertible weights are unitarily equivalent by a unitary operator of diagonal form, then the moduli of eigenvalues of S_i and T_{i+p} are equal for some $p \in \mathbb{Z}$ (see [9, Proposition 2.7] for the proof in the case $\mathcal{H} = \mathbb{C}^2$). However, in [9, Example 2.8] the author showed that this condition does not guarantee that the shifts are unitarily equivalent by an operator of diagonal form, even if the weights are normal and commuting. The next result shed a new light on the aforementioned example.

Corollary 3.7. *Let $\mathcal{H} = \mathbb{C}^m$, $m \in \mathbb{N}_1$. Let $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ be two uniformly bounded sequences of normal, invertible and commuting operators on $\mathcal{H}$ and let S, T be the bilateral weighted shifts with weights $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$, respectively. The following conditions are equivalent:*

(i) *S and T are unitarily equivalent by a unitary operator $U \in \mathbf{B}(\ell^2(\mathbb{Z}, \mathcal{H}))$ of diagonal form,*
(ii) *there exist $p \in \mathbb{Z}$ and two orthonormal basis $(v_n)_{n=1}^m, (w_n)_{n=1}^m$ of $\mathcal{H}$ such that*
  (a) *v_n is an eigenvector of $|S_i|$ for all $i \in \mathbb{Z}$ and $n = 1, \ldots, m$,*
  (b) *w_n is an eigenvector of $|T_i|$ for all $i \in \mathbb{Z}$ and $n = 1, \ldots, m$,*
  (c) *for every $\lambda \in \mathbb{C}$ and every $n = 1, \ldots, m$*

$$|S_i|v_n = \lambda v_n \iff |T_{i+p}|w_n = \lambda w_n, \quad i \in \mathbb{Z}.$$

Proof. (i)⇒(ii) Since S_i's ($i \in \mathbb{Z}$) are normal and commuting, we infer from Fuglede-Putnam theorem (see [3, Theorem 6.7]), that they are doubly commuting; the same holds for T_i's ($i \in \mathbb{Z}$). It follows from Corollary 3.4(i) that the isometry $U_0 \colon \mathcal{H} \to \ell^2(\mathbb{Z}, \mathcal{H})$ satisfies (3.24), (3.25), (3.3) and (3.4). By (i), U_0 is such that $U_{i,0} = 0$ for all $i \in \mathbb{Z} \setminus \{p\}$ and $U_{p,0}$ is unitary with some $p \in \mathbb{Z}$. Since $|S_i|$'s ($i \in \mathbb{Z}$) are normal and commuting, we deduce from [6, Theorem 2.5.5] that there exists an orthonormal basis $(v_n)_{n=1}^m$ such that

$$|S_i|v_n = \lambda_{i,n} v_n, \quad i \in \mathbb{Z},\ n = 1, \ldots, m;$$

in particular, (a) holds. Using (3.24) and (3.25), we get

$$\lambda_{i,n} U_{p,0} v_n = U_{p,0}|S_i|v_n = |T_{p+i}|U_{p,0}v_n, \quad i \in \mathbb{Z},\ n = 1, \ldots, m.$$

By the above, $(U_{p,0}v_n)_{n=1}^m$ is the orthonormal basis of $\mathcal{H}$ satisfying (b) and (c).

(ii)⇒(i) Define $U_{p,0} \in \mathbf{B}(\mathcal{H})$ as follows:

$$U_{p,0} v_n = w_n, \quad n = 1, \ldots, m.$$

Clearly, $U_{p,0}$ is unitary. For every $n = 1, \ldots, m$, if $S_i v_n = \lambda_{i,n} v_n$, then

$$U_{p,0}|S_i|v_n = \lambda_{i,n} U_{p,0} v_n = \lambda_{i,n} w_n \overset{(c)}{=} |T_{i+p}|w_n = |T_{i+p}|U_{p,0}v_n.$$

Define $U_0\colon \mathcal{H} \to \ell^2(\mathbb{Z}, \mathcal{H})$ by the formula

$$U_0 x = (U_{p,0}x)^{(p)}, \quad x \in \mathcal{H}.$$

Then U_0 is an isometry satisfying (3.24) and (3.25). Clearly, the conditions (3.3) and (3.4) also hold. Applying Corollary 3.4(ii) we get (i). □

Now let us present another characterization of unitary equivalence of bilateral shifts with operator weights in terms of factors in their polar decompositions.

Theorem 3.8. *Suppose that S and T satisfy the assumptions of Theorem 3.1. Let $S = V_S|S|$ and $T = V_T|T|$ be the polar decompositions of S and T, respectively. For a unitary operator $U\colon \ell^2(\mathbb{Z}, \mathcal{H}) \to \ell^2(\mathbb{Z}, \mathcal{H})$ the following conditions are equivalent:*

(i) $US = TU$,
(ii) $U|S| = |T|U$ *and* $UV_S = V_T U$.

Proof. (i)⇒(ii) Repeating an argument as in the proof of Theorem 3.1, we have $U|S| = |T|U$. Moreover,

$$UV_S|S| = US = TU = V_T|T|U = V_T U|S|.$$

Since, by Lemma 2.1(iii),

$$\overline{\mathcal{R}(|S|)} = \overline{\mathcal{R}(D[(|S_{i+1}|)_{i\in\mathbb{Z}}])} = \ell^2(\mathbb{Z}, \mathcal{H}),$$

it follows that $UV_S = V_T U$.

(ii)⇒(i) We have

$$US = UV_S|S| = V_T U|S| = V_T|T|U = TU.$$

This completes the proof. □

By Lemma 2.3, the operators V_S and V_T in the above theorem are weighted shifts with unitary weights. By Corollary 3.3, the equality $UV_S = V_T U$ always holds with some unitary operator U; however, this operator U does not have to satisfy $U|S| = |T|U$. Since the unitary operator U making S and T unitarily equivalent can be found among all unitary operators intertwining V_s and V_T, the possible further research is to describe all unitary operators, which intertwine two shifts with unitary weights.

The rest of this section is devoted to study unitary equivalence of shifts with positive commuting weights. First, we prove that unitary operator intertwining bilateral shifts with positive weights has to be constant on diagonals (see [13, Lemma 5] for similar result for self-adjoint operator intertwining bilateral shifts).

Lemma 3.9. *Let $\mathcal{H}$ be a Hilbert space. Let $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ be two uniformly bounded sequences of positive operators on $\mathcal{H}$ and denote by S and T the bilateral weighted shifts with weights $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$, respectively. Suppose $U = [U_{i,j}]_{i,j\in\mathbb{Z}} \in \mathbf{B}(\ell^2(\mathbb{Z}, \mathcal{H}))$ is a unitary operator such that $US = TU$. Then U is constant on diagonals, that is, $U_{i+1,j+1} = U_{i,j}$ for all $i, j \in \mathbb{Z}$.*

Proof. From Theorem 3.8 we obtain that $U|S| = |T|U$. Since the weights of S and T are positive, Lemma 2.1(iii) shows that $U_{i,j}S_{j+1} = T_{i+1}U_{i,j}$ for all $i, j \in \mathbb{Z}$. On the other hand, from Lemma 2.2 it follows that $U_{i+1,j+1}S_{j+1} = T_{i+1}U_{i,j}$ for all $i, j \in \mathbb{Z}$. Since $\overline{\mathcal{R}(S_j)} = \mathcal{H}$ for every $j \in \mathbb{Z}$, we conclude that $U_{i+1,j+1} = U_{i,j}$ for all $i, j \in \mathbb{Z}$. □

The next result states that if $\mathcal{H}$ is two dimensional, then under certain assumptions on weights the unitary equivalence is always given by the operator having at most two non-zero diagonal.

Theorem 3.10. *Suppose $\mathcal{H} = \mathbb{C}^2$. Let $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}} \subset \mathbf{B}(\mathcal{H})$ be two uniformly bounded sequences of positive and commuting operators on $\mathcal{H}$ and assume additionally that every S_j and every T_j ($j \in \mathbb{Z}$) has two distinct eigenvalues. Denote by S, T the bilateral weighted shifts with weights $(S_i)_{i\in\mathbb{Z}}$ and $(T_i)_{i\in\mathbb{Z}}$, respectively. The following conditions are equivalent:*

- (i) *S and T are unitarily equivalent,*
- (ii) *S and T are unitarily equivalent by a unitary operator with at most two non-zero diagonals.*

Before we state the proof we need two lemmas.

Lemma 3.11. *Let $\mathcal{H}$, $(S_i)_{i\in\mathbb{Z}}, (T_i)_{i\in\mathbb{Z}}$ satisfy the assumptions of Theorem 3.10. Suppose $U = [U_{i,j}]_{i,j\in\mathbb{Z}} \in \mathbf{B}(\ell^2(\mathbb{Z}, \mathcal{H}))$ is a unitary operator such that $US = TU$. Let $\{v_1, v_2\} \subset \mathcal{H}$ be the common orthonormal basis of eigenvectors for the sequence $(S_i)_{i\in\mathbb{Z}}$ and let $\{w_1, w_2\} \subset \mathcal{H}$ be the common orthonormal basis of eigenvectors for the sequence $(T_i)_{i\in\mathbb{Z}}$. Then:*

- (i) *for every $j \in \{1, 2\}$ there exist $k_j \in \{1, 2\}$ and $i_j \in \mathbb{Z}$ such that*

$$\langle U_{i_j,0}v_j, w_{k_j}\rangle \neq 0,$$

- (ii) *for every $k \in \{1, 2\}$ there exist $j_k \in \{1, 2\}$ and $i_k \in \mathbb{Z}$ such that*

$$\langle U_{i_k,0}v_j, w_k\rangle \neq 0.$$

Proof. (i) For $j \in \{1, 2\}$, we have

$$1 = \|Uv_j^{(0)}\|^2 = \sum_{i\in\mathbb{Z}} \|U_{i,0}v_j\|^2.$$

This implies that there exists $i_j \in \mathbb{Z}$ such that $U_{i_j,0}v_j \neq 0$. Since $\{w_1, w_2\}$ is the orthonormal basis of $\mathcal{H}$, there exists $k_j \in \{1, 2\}$ such that $\langle U_{i_j,0}v_j, w_{k_j}\rangle \neq 0$.

(ii) For $k \in \{1, 2\}$, we have

$$\begin{aligned} 1 = \|U^*w_k^{(0)}\|^2 &= \sum_{i\in\mathbb{Z}} \|(U^*)_{i,0}w_k\|^2 \\ &= \sum_{i\in\mathbb{Z}} \|U^*_{0,i}w_k\|^2 \overset{\text{Lemma 3.9}}{=} \sum_{i\in\mathbb{Z}} \|U^*_{-i,0}w_k\|^2. \end{aligned}$$

Arguing as in (i) we find $i_k \in \mathbb{Z}$ and $j_k \in \{1,2\}$ such that

$$\langle v_{j_k}, U_{i_k,0}^* w_k\rangle = \langle U_{i_k,0} v_{j_k}, w_k\rangle \neq 0,$$

which proves (ii). □

Lemma 3.12. *Under the assumptions of Lemma 3.11 there exist a bijection* $\sigma\colon \{1,2\} \to \{1,2\}$ *and a function* $\tau\colon \{1,2\} \to \mathbb{Z}$ *such that*

$$\langle U_{\tau(j),0} v_j, w_{\sigma(j)}\rangle \neq 0, \quad j \in \{1,2\}.$$

Proof. For $j \in \{1,2\}$ set

$$B_j = \{k \in \{1,2\} \colon \langle U_{i,0} v_j, w_k\rangle \neq 0 \text{ for some } i \in \mathbb{Z}\}.$$

By Lemma 3.11(i), $B_j \neq \varnothing$ for $j \in \{1,2\}$. From Lemma 3.11(ii) we deduce that it is not possible that $B_1 = B_2 = \{j\}$ for any $j \in \{1,2\}$. The only possibilities for the sets B_1, B_2 are listed in Table 1.

Table 1
Possibilities for the sets B_1 and B_2

B_1	B_2
$\{1\}$	$\{2\}$
$\{1\}$	$\{1,2\}$
$\{2\}$	$\{1\}$
$\{2\}$	$\{1,2\}$
$\{1,2\}$	$\{1\}$
$\{1,2\}$	$\{2\}$
$\{1,2\}$	$\{1,2\}$

It is a matter of routine to verify that in every case there exists a bijection $\sigma\colon \{1,2\} \to \{1,2\}$ such that $\sigma(j) \in B_j$ for $j \in \{1,2\}$. The existence of the required function $\tau\colon \{1,2\} \to \mathbb{Z}$ easily follows. □

Now we are in the position to prove Theorem 3.10.

Proof of Theorem 3.10. It is enough to prove that (i) implies (ii). Let $\sigma\colon \{1,2\} \to \{1,2\}$ and $\tau\colon \{1,2\} \to \mathbb{Z}$ be as in Lemma 3.12. For $k \in \mathbb{Z}$ let $\sigma(S_k) = \{\lambda_{k,1}, \lambda_{k,2}\}$. By Lemma 2.2,

$$\lambda_{k,j} U_{\tau(j),0} v_j = U_{\tau(j),0} S_k v_j = T_{\tau(j)+k} U_{\tau(j),0} v_j, \quad j \in \{1,2\}, k \in \mathbb{Z}. \tag{3.33}$$

Since $\langle U_{\tau(j),0} v_j, w_{\sigma(j)}\rangle \neq 0$, $U_{\tau(j),0} v_j$ is an eigenvector of $T_{\tau(j)+k}$ for every $k \in \mathbb{Z}$. By the assumption, this implies that for $j \in \{1,2\}$ there exists $\mu_j \in \mathbb{C} \setminus \{0\}$ satisfying $U_{\tau(j),0} v_j = \mu_j w_{\sigma(j)}$; since U is unitary, we have $|\mu_j| \leq 1$. For $i \in \tau(\{1,2\})$ define $V_{i,0} \in \mathbf{B}(\mathcal{H})$ as follows:

$$V_{i,0} x = \sum_{\substack{\ell \in \{1,2\} \\ i = \tau(\ell)}} \langle x, v_\ell\rangle w_{\sigma(\ell)}, \quad x \in \mathcal{H};$$

for $i \notin \tau(\{1,2\})$ we set $V_{i,0} = 0$. It is evident that $V_{i,0}$ is a partial isometry for every $i \in \mathbb{Z}$ and that

$$V^*_{\tau(j),0}x = \sum_{\substack{\ell\in\{1,2\}\\ \tau(j)=\tau(\ell)}} \langle x, w_{\sigma(\ell)}\rangle v_\ell, \quad x \in \mathcal{H},\ j \in \{1,2\}.$$

From the above we deduce that for every $j \in \{1,2\}$, $V^*_{\tau(j),0}V_{\tau(j),0}$ is the orthogonal projection of $\mathcal{H}$ onto the space $\operatorname{Lin}\{v_\ell : \tau(j) = \tau(\ell)\}$ and that $V_{\tau(j),0}V^*_{\tau(j),0}$ is the orthogonal projection of $\mathcal{H}$ onto the space $\operatorname{Lin}\{w_\ell : \tau(j) = \tau(\ell)\}$; in particular, if $\tau(1) = \tau(2)$, then $V_{\tau(1),0}$ is unitary. Define $V_0 \in \mathbf{B}(\mathcal{H}, \ell^2(\mathbb{Z},\mathcal{H}))$ by the formula

$$V_0x = (V_{i,0}x)_{i\in\mathbb{Z}}, \quad x \in \mathcal{H}.$$

If $x \in \mathcal{H}$, then

$$\|V_0x\|^2 = \sum_{i\in\mathbb{Z}}\|V_{i,0}x\|^2 = \sum_{j=1}^{2}|\langle x, v_j\rangle|^2 = \|x\|^2.$$

Thus, V_0 is the isometry. We will show that V_0 satisfies the conditions in Corollary 3.4. First, we will prove (3.24) and (3.25). For every $j \in \{1,2\}$ and $k \in \mathbb{Z}$ we have

$$\begin{aligned}\lambda_{k,j}\mu_j w_{\sigma(j)} &= \lambda_{k,j}U_{\tau(j),0}v_j = U_{\tau(j),0}S_kv_j\\ &\overset{(3.33)}{=} T_{\tau(j)+k}U_{\tau(j),0}v_j = \mu_jT_{\tau(j)+k}w_{\sigma(j)},\end{aligned}$$

which implies that

$$\lambda_{k,j}w_{\sigma(j)} = T_{\tau(j)+k}w_{\sigma(j)}.$$

By the definition of $V_{\tau(j)}$, this equality takes the form

$$V_{\tau(j),0}S_kv_j = T_{\tau(j)+k}V_{\tau(j),0}v_j, \quad j \in \{1,2\},\ k \in \mathbb{Z}$$

If $j, \ell \in \{1,2\}$ and $\tau(j) \neq \tau(\ell)$, then for every $k \in \mathbb{Z}$ we have

$$V_{\tau(j),0}S_kv_\ell = 0 = T_{\tau(j)+k}V_{\tau(j),0}v_\ell.$$

In turn, if $i \notin \tau(\{1,2\})$, then $V_{i,0} = 0$. Therefore,

$$V_0S_k = D[(T_{i+k})_{i\in\mathbb{Z}}]V_0, \quad k \in \mathbb{Z}.$$

From this we conclude that (3.24) and (3.25) hold. Next, we will show (3.3). If $k, m \in \mathbb{N}_1$, $k \neq m$, then, by Lemma 2.1, for $j_1, j_2 \in \{1,2\}$,

$$\begin{aligned}&\langle T^{[k]}V_0v_{j_1}, T^{[m]}U_0v_{j_2}\rangle = \langle T^{[k]}(V_{\tau(j_1),0}v_{j_1})^{\tau(j_1)}, T^{[m]}(V_{\tau(j_2),0}v_{j_2})^{\tau(j_2)}\rangle \\ &= \langle (T_{\tau(j_1)+k}\cdots T_{\tau(j_1)+1}w_{\sigma(j_1)})^{\tau(j_1)+k}, (T_{\tau(j_2)+m}\cdots T_{\tau(j_2)+1}w_{\sigma(j_2)})^{\tau(j_2)+m}\rangle.\end{aligned} \tag{3.34}$$

If $\tau(j_1) + k \neq \tau(j_2) + m$, then the right hand side of (3.34) is equal to zero. If $\tau(j_1) + k = \tau(j_2) + m$, then $\tau(j_1) \neq \tau(j_2)$, because $k \neq m$. This implies that $j_1 \neq j_2$

and $\sigma(j_1) \neq \sigma(j_2)$, which implies that $\langle w_{\sigma(j_1)}, w_{\sigma(j_2)} \rangle = 0$. From the fact that $w_{\sigma(j)}$ is the eigenvector of T_k for every $j \in \{1, 2\}$ and $k \in \mathbb{Z}$ we deduce that the right hand side of (3.34) is equal to zero also in this case. Since $\{v_1, v_2\}$ is the orthonormal basis of $\mathcal{H}$, it follows that

$$\langle T^{[k]} V_0 x, T^{[m]} V_0 y \rangle = 0, \quad x, y \in \mathcal{H}, k, m \in \mathbb{N}_1, k \neq m.$$

Similar argument can be used to prove that the above equality holds also when $k, m \in \mathbb{Z}$ are both non-positive or are of different sign. Therefore, (3.3) is satisfied. Finally, we will prove (3.4). It suffices to show that

$$x^{(n)} \in \bigvee_{k \in \mathbb{Z}} T^{[k]} \mathcal{R}(V_0), \quad x \in \mathcal{H},\ n \in \mathbb{Z}.$$

Observe that if $i - \tau(j) < 0$, then

$$\begin{aligned} T^{[i-\tau(j)]} (V_{\tau(j),0} v_j)^{\tau(j)} &= T^{*(\tau(j)-i)} (V_{\tau(j),0} v_j)^{(\tau(j))} \\ &\overset{\text{Lemma 2.1}}{=} (T_{i+1} \cdots T_{\tau(j)} w_{\sigma(j)})^{(i)} \\ &\overset{(3.33)}{=} (\lambda_{i-\tau(j)+1,j} \cdots \lambda_{0,j} w_{\sigma(j)})^{(i)}, \end{aligned}$$

so

$$\|T^{[i-\tau(j)]} V_0 v_j\| = \lambda_{i-\tau(j)+1,j} \cdots \lambda_{0,j}.$$

Similarly, if $i - \tau(j) > 0$, then

$$\begin{aligned} T^{[i-\tau(j)]} (V_{\tau(j),0} v_j)^{(\tau(j))} &= (T_i \cdots T_{\tau(j)+1} w_{\sigma(j)})^{(i)} \\ &= (\lambda_{i-\tau(j),j} \cdots \lambda_{1,j} w_{\sigma(j)})^{(i)}, \end{aligned}$$

so

$$\|T^{[i-\tau(j)]} V_0 v_j\| = \lambda_{i-\tau(j),j} \cdots \lambda_{1,j}.$$

Therefore, if $x \in \mathcal{H}$ and $n \in \mathbb{Z}$, then

$$x^{(n)} = \sum_{j=1}^{2} \frac{1}{\|T^{[n-\tau(j)]} V_0 v_j\|} \langle x, w_{\sigma(j)} \rangle T^{[n-\tau(j)]} V_0 v_j \in \bigvee_{k \in \mathbb{Z}} T^{[k]} \mathcal{R}(U_0).$$

Thus, we get (3.4). The application of Corollary 3.4(ii) and Remark 3.2 gives (ii). □

In [9, Example 3.1] the author presented an example of two unitarily equivalent bilateral shifts with operator weights defined on $\mathbb{C}^2$, for which the unitary equivalence cannot be given by a unitary operator of diagonal form. Below we present a general construction of two unitarily equivalent shifts with weights defined on $\mathbb{C}^k$ such that the every unitary operator making them unitarily equivalent has at least k non-zero diagonals.

Example 3.13. Assume $k \in \mathbb{N}_2$. For every $n \in \mathbb{Z}$ let $(x_{n,i})_{i=1}^k \subset (0,\infty)$ be the sequence of positive numbers such that $x_{k,i} \neq x_{\ell,j}$ for $(k,i) \neq (\ell,j)$ and that

$$\sup_{n\in\mathbb{Z}} \max_{i=1,\ldots,k} x_{n,i} < \infty \tag{3.35}$$

Define

$$S_n = \begin{bmatrix} x_{n,1} & 0 & \ldots & 0 \\ 0 & x_{n,2} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & x_{n,k} \end{bmatrix}, \quad n \in \mathbb{Z},$$

and

$$T_n = \begin{bmatrix} x_{n-1,1} & 0 & \ldots & 0 \\ 0 & x_{n-2,2} & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & x_{n-k,k} \end{bmatrix}, \quad n \in \mathbb{Z}.$$

Obviously, both S_n and T_n are positive for every $n \in \mathbb{Z}$. By (3.35), the sequences $(S_n)_{n\in\mathbb{Z}}, (T_n)_{n\in\mathbb{Z}} \subset \mathbf{B}(\mathbb{C}^k)$ are uniformly bounded. Let $S, T \in \mathbf{B}(\ell^2(\mathbb{Z},\mathbb{C}^k))$ be the bilateral weighted shifts with weights $(S_n)_{n\in\mathbb{Z}}$ and $(T_n)_{n\in\mathbb{Z}}$, respectively. We show that S and T are unitarily equivalent by a unitary operator with at least k non-zero diagonals. For $i = 1,\ldots,k$, let $U_{i,0}$ be the orthogonal projection of $\mathbb{C}^k$ onto $\operatorname{Lin}\{e_i\}$, where $(e_j)_{j=1}^k \subset \mathbb{C}^k$ stands for the standard orthonormal basis of $\mathbb{C}^k$; for $i \in \mathbb{Z} \setminus \{1,\ldots,k\}$ set $U_{i,0} = 0$. Define $U_0 \in \mathbf{B}(\mathbb{C}^k, \ell^2(\mathbb{Z},\mathbb{C}^k))$ by the formula

$$U_0 x = (U_{i,0}x)_{i\in\mathbb{Z}}, \quad x \in \mathbb{C}^k.$$

We check that U_0 satisfies Corollary 3.4(ii). Obviously, U_0 is an isometry. We show that (3.24) and (3.25) hold. Since S_n and T_n are positive for $n \in \mathbb{Z}$ it is enough to verify that

$$U_0 S_n = D[(T_{i+n})_{i\in\mathbb{Z}}]U_0, \quad n \in \mathbb{Z}. \tag{3.36}$$

For every $n \in \mathbb{Z}$ and $j = 1,\ldots,k$,

$$U_0 S_n e_j = U_0 x_{n,j} e_j = (x_{n,j}e_j)^{(j)}$$

and

$$D[(T_{i+n})_{i\in\mathbb{Z}}]U_0 e_j = D[(T_{i+n})_{i\in\mathbb{Z}}]e_j^{(j)} = (T_{j+n}e_j)^{(j)} = (x_{n,j}e_j)^{(j)}.$$

Hence, (3.36) holds. Next, we check (3.3). Suppose $n, m \in \mathbb{N}_1$ and $n \neq m$. We show that

$$\langle T^{[n]}U_0 x, T^{[m]}U_0 y\rangle = 0, \quad x, y \in \mathbb{C}^k. \tag{3.37}$$

It is sufficient to verify (3.37) for $x = e_{j_n}$, $y = e_{j_m}$, where $j_n, j_m \in \{1,\ldots,k\}$. By Lemma 2.1, for every $\ell \in \mathbb{N}_1$ and $j = 1,\ldots,k$,

$$T^\ell U_0 e_j = T^\ell e_j^{(j)} = (T_{j+n}\cdots T_{j+1}e_j)^{j+\ell} = (x_{\ell,j}\cdots x_{1,j}e_j)^{j+\ell}.$$

Hence,

$$\langle T^n U_0 e_{j_n}, T^m U_0 e_{j_m}\rangle =, \quad j_n + n \neq j_m + m.$$

If $j_n + n = j_m + m$, then $j_n \neq j_m$, because $n \neq m$. Thus,

$$\langle T^n U_0 e_{j_n}, T^m U_0 e_{j_m}\rangle = 0, \qquad j_n + n = j_m + m.$$

Similarly, we can show that (3.37) holds for m, n being both non-positive or of different sign. It remains to verify (3.4). It is enough to check that

$$x^{(n)} \in \bigvee_{\ell\in\mathbb{Z}} T^{[\ell]}\mathcal{R}(U_0), \quad x \in \mathbb{C}^k, n \in \mathbb{Z}.$$

It is a matter of routine to verify that for $x \in \mathbb{C}^k$ and $n \in \mathbb{Z}$,

$$x^{(n)} = \sum_{j=1}^{k} \langle x, e_j\rangle e_j = \sum_{j=1}^{k} \langle x, e_j\rangle \frac{1}{\|T^{[n-j]}U_0 e_j\|^2} T^{[n-j]}U_0 e_j \in \bigvee_{\ell\in\mathbb{Z}} T^{[\ell]}\mathcal{R}(U_0)$$

(see the proof of Theorem 3.10). By Corollary 3.4 and Remark 3.2, S and T are unitarily equivalent by a unitary operator with k non-zero diagonals. Now we will show that there is no unitary operator with at most $k-1$ non-zero diagonals making S and T unitarily equivalent. Suppose to the contrary U is such an operator. By Corollary 3.4(i), $U_0 \in \mathbf{B}(\mathbb{C}^k, \ell^2(\mathbb{Z}, \mathbb{C}^k))$ has $k-1$ non-zero entries and satisfies (3.36); suppose $U_{n_1,0}, \ldots, U_{n_\ell,0} \in \mathbf{B}(\mathbb{C}^k)$ are the only non-zero entries of U_0, where $\ell \leq k-1$, $n_j \in \mathbb{Z}$ for $j = 1, \ldots, \ell$ and $n_1 < \ldots < n_\ell$. By (3.36),

$$x_{n,j} U_{n_i,0} e_j = U_{n_i,0} S_n e_j = T_{n_i+n} U_{n_i,0} e_j, \quad i = 1, \ldots, \ell,\ j = 1, \ldots, k.$$

Since, $x_{n,j}$ is the eigenvalue only of the weight T_{n+j}, it follows that

$$U_{n_i,0} e_j \neq 0 \implies n_i = j, \quad j = 1, \ldots, k,\ i = 1, \ldots, \ell.$$

From the fact that U_0 is isometric, we deduce that for every $j = 1, \ldots, k$ there exists n_i such that $U_{n_i,0} e_j \neq 0$. Hence, $\{1, \ldots, k\} = \{n_1, \ldots, n_\ell\}$, which is impossible (we asssumed $\ell \leq k-1$).

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