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VECTORIAL GENERALIZED g–FRACTIONAL DIRECT AND ITERATED APPROXIMATIONS BY LINEAR OPERATORS

Abstract. In this work we consider quantitatively with rates the convergence of sequences of linear operators applied on Banach space valued functions. The results are pointwise estimates with rates. To prove our main results we use an elegant and natural boundedness property of our linear operators by their companion positive linear operators. Our inequalities are generalized g-direct and iterated fractional involving the right and left vector Caputo type generalized g-direct and iterated fractional derivatives, built in vector moduli of continuity. We treat wide and general classes of Banach space valued functions. We give applications to vectorial Bernstein operators.

Key words: vector generalized g-direct and iterated fractional derivatives, Bochner integral, Vector generalized g-direct and iterated Fractional Taylor formulae, Vector modulus of continuity, vector linear operators, positive linear operators, quantitative approximations.

AMS Mathematics Subject Classification: 26A33, 41A17, 41A25, 41A36, 41A80, 46B25.

1. Motivation

Let $(X, \|\cdot\|)$ be a Banach space, $N \in \mathbb{N}$. Consider $g \in C([0,1])$ and the classic Bernstein polynomials

$$\left(\widetilde{B}_N g\right)(t) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{1}$$

Let also $f \in C([0,1], X)$ and define the vector valued in X Bernsein linear operators

$$(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{2}$$

That is $(B_N f)(t) \in X$. Clearly here $\|f\| \in C([0,1])$.

We notice that

$$\|(B_N f)(t)\| \leq \sum_{k=0}^{N} \left\| f\left(\frac{k}{N}\right)\right\| \binom{N}{k} t^k (1-t)^{N-k} = \left(\widetilde{B}_N(\|f\|)\right)(t), \quad \forall\, t \in [0,1]\,. \tag{3}$$

The property

$$\|(B_N f)(t)\| \leq \left(\widetilde{B}_N(\|f\|)\right)(t), \quad \forall t \in [0,1], \tag{4}$$

is shared by almost all summation/integration similar operators and motivates our work here.

If $f(x) = c \in X$ the constant function, then

$$(B_N c) = c. \tag{5}$$

If $g \in C([0,1])$ and $c \in X$, then $cg \in C([0,1], X)$ and

$$(B_N(cg)) = c\widetilde{B}_N(g)\,. \tag{6}$$

Again (5), (6) are fulfilled by many summation/integration operators.

In fact here (6) implies (5), when $g \equiv 1$.

The above can be generalized from $[0,1]$ to any interval $[a,b] \subset \mathbb{R}$. All this discussion motivates us to consider the following situation.

Let $L_N : C([a,b], X) \hookrightarrow C([a,b], X)$, $(X, \|\cdot\|)$ a Banach space, L_N is a linear operator, $\forall\, N \in \mathbb{N}$, $x_0 \in [a,b]$. Let also $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, a sequence of positive linear operators, $\forall\, N \in \mathbb{N}$.

We assume that

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{7}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x_0 \in [a,b]$, $\forall\, f \in C([a,b], X)$.

When $g \in C([a,b])$, $c \in X$, we assume that

$$(L_N(cg)) = c\widetilde{L}_N(g)\,. \tag{8}$$

The special case of

$$\widetilde{L}_N(1) = 1, \tag{9}$$

implies

$$L_N(c) = c, \quad \forall\, c \in X. \tag{10}$$

We call $\widetilde{L}_N$ the companion operator of L_N.

Based on the above fundamental properties we study the fractional approximation properties of the sequence of linear operators $\{L_N\}_{N\in\mathbb{N}}$, i.e. their fractional convergence to the unit operator. No kind of positivity property of $\{L_N\}_{N\in\mathbb{N}}$ is assumed.

2. Background

We need

Definition 1 ([3]). *Let $[a,b]\subset\mathbb{R}$, $(X,\|\cdot\|)$ a Banach space, $g\in C^1([a,b])$ and increasing, $f\in C([a,b],X)$, $\nu>0$.*

We define the left Riemann-Liouville generalized fractional Bochner integral operator

$$\left(I^{\nu}_{a+;g}f\right)(x):=\frac{1}{\Gamma(\nu)}\int_a^x (g(x)-g(z))^{\nu-1}g'(z)f(z)\,dz, \tag{11}$$

$\forall\ x\in[a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f\in C([a,b],X)$, then $f\in L_\infty([a,b],X)$. By [3] we get that $I^{\nu}_{a+;g}f\in C([a,b],X)$. Above we set $I^{0}_{a+;g}f:=f$ and see that $\left(I^{\nu}_{a+;g}f\right)(a)=0$.

When g is the identity function id, we get that $I^{\nu}_{a+;id}=I^{\nu}_{a+}$, the ordinary left Riemann-Liouville fractional integral

$$\left(I^{\nu}_{a+}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}f(t)\,dt, \tag{12}$$

$\forall\ x\in[a,b]$, $\left(I^{\nu}_{a+}f\right)(a)=0$.

We mention

Theorem 1 ([3]). *Let $\mu,\nu>0$ and $f\in C([a,b],X)$. Then*

$$I^{\mu}_{a+;g}I^{\nu}_{a+;g}f=I^{\mu+\nu}_{a+;g}f=I^{\nu}_{a+;g}I^{\mu}_{a+;g}f. \tag{13}$$

We need

Definition 2 ([3]). *Let $[a,b]\subset\mathbb{R}$, $(X,\|\cdot\|)$ a Banach space, $g\in C^1([a,b])$ and increasing, $f\in C([a,b],X)$, $\nu>0$.*

We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(I^{\nu}_{b-;g}f\right)(x):=\frac{1}{\Gamma(\nu)}\int_x^b (g(z)-g(x))^{\nu-1}g'(z)f(z)\,dz, \tag{14}$$

$\forall\ x\in[a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f \in C([a,b], X)$, then $f \in L_\infty([a,b], X)$. By [3] we get that $I_{b-;g}^{\nu} f \in C([a,b], X)$. Above we set $I_{b-;g}^{0} f := f$ and see that $\left(I_{b-;g}^{\nu} f\right)(b) = 0$.

When g is the identity function id, we get that $I_{b-;id}^{\nu} = I_{b-}^{\nu}$, the ordinary right Riemann-Liouville fractional integral

$$\left(I_{b-}^{\nu} f\right)(x) = \frac{1}{\Gamma(\nu)} \int_x^b (t-x)^{\nu-1} f(t)\, dt, \tag{15}$$

$\forall\, x \in [a,b]$, with $\left(I_{b-}^{\nu} f\right)(b) = 0$.

We mention

Theorem 2 ([3]). *Let $\mu, \nu > 0$ and $f \in C([a,b], X)$. Then*

$$I_{b-;g}^{\mu} I_{b-;g}^{\nu} f = I_{b-;g}^{\mu+\nu} f = I_{b-;g}^{\nu} I_{b-;g}^{\mu} f. \tag{16}$$

We will use

Definition 3 ([3]). *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{17}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [3], we have that $\left(D_{a+;g}^{\alpha} f\right) \in C([a,b], X)$.

We see that

$$\left(I_{a+;g}^{n-\alpha} \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha} f\right)(x), \quad \forall\, x \in [a,b]. \tag{18}$$

We set

$$D_{a+;g}^{n} f(x) := \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) \in C([a,b], X), \quad n \in \mathbb{N}, \tag{19}$$

$$D_{a+;g}^{0} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha} f = D_{a+;id}^{\alpha} f = D_{*a}^{\alpha} f, \tag{20}$$

the usual left X-valued Caputo fractional derivative, see [5].

We will use

Definition 4 ([3]). *Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$.*

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$(21)\quad \left(D^{\alpha}_{b-;g} f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt,$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [3], we have that $\left(D^{\alpha}_{b-;g} f\right) \in C([a,b], X)$.

We see that

$$(22)\quad I^{n-\alpha}_{b-;g}\left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D^{\alpha}_{b-;g} f\right)(x), \quad a \le x \le b.$$

We set

$$(23)\quad D^{n}_{b-;g} f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b], X), \quad n \in \mathbb{N},$$

$$D^{0}_{b-;g} f(x) := f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$(24)\quad D^{\alpha}_{b-;g} f(x) = D^{\alpha}_{b-;id} f(x) = D^{\alpha}_{b-} f,$$

the usual right X-valued Caputo fractional derivative, see [5].

We make

Remark 1. All as in Definition 3. We have (by Theorem 2.5, p. 7, [7] and [9])

$$(25)\quad \left\|\left(D^{\alpha}_{a+;g} f\right)(x)\right\| \le \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left\|\left(f \circ g^{-1}\right)^{(n)} (g(t))\right\| dt$$
$$\le \frac{\left\|\left(f \circ g^{-1}\right)^{(n)} \circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)} \int_{g(a)}^{g(x)} (g(x) - g(t))^{n-\alpha-1} dg(t)$$
$$= \frac{\left\|\left(f \circ g^{-1}\right)^{(n)} \circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(x) - g(a))^{n-\alpha}.$$

That is

$$\left\| \left(D_{a+;g}^{\alpha} f \right)(x) \right\| \leq \frac{\left\| \left(f \circ g^{-1} \right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} \left(g(x) - g(a) \right)^{n-\alpha}, \tag{26}$$

$\forall\, x \in [a,b]$.

If $\alpha \notin \mathbb{N}$, then $\left(D_{a+;g}^{\alpha} f \right)(a) = 0$.

Similarly, by Definition 4 we derive

$$\begin{aligned} \left\| \left(D_{b-;g}^{\alpha} f \right)(x) \right\| &\leq \frac{1}{\Gamma(n-\alpha)} \\ &\int_x^b \left(g(t) - g(x) \right)^{n-\alpha-1} g'(t) \left\| \left(f \circ g^{-1} \right)^{(n)} (g(t)) \right\| dt \\ &\leq \frac{\left\| \left(f \circ g^{-1} \right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)} \int_{g(x)}^{g(b)} \left(g(t) - g(x) \right)^{n-\alpha-1} dg(t) \\ &= \frac{\left\| \left(f \circ g^{-1} \right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} \left(g(b) - g(x) \right)^{n-\alpha}. \end{aligned} \tag{27}$$

That is

$$\left\| \left(D_{b-;g}^{\alpha} f \right)(x) \right\| \leq \frac{\left\| \left(f \circ g^{-1} \right)^{(n)} \circ g \right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} \left(g(b) - g(x) \right)^{n-\alpha}, \tag{28}$$

$\forall\, x \in [a,b]$.

If $\alpha \notin \mathbb{N}$, then $\left(D_{b-;g}^{\alpha} f \right)(b) = 0$.

Notation 1. We denote by

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha} D_{a+;g}^{\alpha} \ldots D_{a+;g}^{\alpha} \quad (n \text{ times}),\ n \in \mathbb{N}, \tag{29}$$

$$I_{a+;g}^{n\alpha} := I_{a+;g}^{\alpha} I_{a+;g}^{\alpha} \ldots I_{a+;g}^{\alpha}, \tag{30}$$

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha} D_{b-;g}^{\alpha} \ldots D_{b-;g}^{\alpha}, \tag{31}$$

and

$$I_{b-;g}^{n\alpha} := I_{b-;g}^{\alpha} I_{b-;g}^{\alpha} \ldots I_{b-;g}^{\alpha}, \tag{32}$$

(n times), $n \in \mathbb{N}$.

We mention the following g-left generalized modified X-valued Taylor's formula.

Theorem 3 ([3])**.** *Let* $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Let* $F_k := D^{k\alpha}_{a+;g} f$, $k = 1, ..., n$, *that fulfill* $F_k \in C^1([a,b], X)$.
Then

$$f(x) = \sum_{i=0}^{n} \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{a+;g} f\right)(a) + \frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{a+;g} f\right)(t)\, dt, \tag{33}$$

$\forall\, x \in [a,b]$.
Clearly here $D^{(n+1)\alpha}_{a+;g} f \in C([a,b], X)$.

We also mention the following g-right generalized modified X-valued Taylor's formula.

Theorem 4 ([3])**.** *Let* $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Suppose that* $F_k := D^{k\alpha}_{b-;g} f$, $k = 1, ..., n$, *fulfill* $F_k \in C^1([a,b], X)$, *where* $0 < \alpha \leq 1$, $n \in \mathbb{N}$.
Then

$$f(x) = \sum_{i=0}^{n} \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D^{i\alpha}_{b-;g} f\right)(b) + \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{b-;g} f\right)(t)\, dt, \tag{34}$$

$\forall\, x \in [a,b]$.
Clearly here $D^{(n+1)\alpha}_{b-;g} f \in C([a,b], X)$.

For differentiation of functions from real numbers to normed linear spaces the definition is the same as for the real valued functions, however the limit and convergence is in the norm of linear space $(X, \|\cdot\|)$.
We state

Corollary 1 (to Theorem 3)**.** *Let* $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^1([g(a), g(b)])$. *Assume* $D^{k\alpha}_{a+;g} f \in C^1([a,b], X)$, $k = 1, ..., n$, *and* $\left(D^{i\alpha}_{a+;g} f\right)(a) = 0$, $i = 0, 2, 3, ..., n$.
Then

$$f(x) = \frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left(D^{(n+1)\alpha}_{a+;g} f\right)(t)\, dt, \tag{35}$$

$\forall\ x \in [a, b]$.

We state

Corollary 2 (to Theorem 4). *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$. Suppose that $D_{b-;g}^{k\alpha} f \in C^1([a,b], X)$, for $k = 1, ..., n$; where $0 < \alpha < 1$, $n \in \mathbb{N}$. We further assume that $\left(D_{b-;g}^{i\alpha} f\right)(b) = 0$, $i = 0, 2, 3, ..., n$.*

Then

$$f(x) = \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t)\, dt, \tag{36}$$

$\forall\ x \in [a, b]$.

We mention the following g-left generalized X-valued Taylor's formula:

Theorem 5 ([3]). *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Then*

$$\begin{aligned} f(x) &= f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha - 1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt \\ &= f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha - 1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \end{aligned} \tag{37}$$

$\forall\ x \in [a, b]$.

We also mention the following g-right generalized X-valued Taylor's formula:

Theorem 6 ([3]). *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Then*

$$\begin{aligned} f(x) &= f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha - 1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt \end{aligned} \tag{38}$$

$$= f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b))$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} \left(\left(D_{b-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\, x \in [a,b].$$

Integrals in (33)-(38) are of Bochner type. For the Bochner integral excellent resources are [6], [8], [9] and [1], pp. 422-428.

We need

Definition 5 ([4]). *Let $f \in C([a,b], X)$, $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ a Banach space. We define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]:\\ |x-y|\le\delta}} \|f(x) - f(y)\|, \quad 0 < \delta \le b - a. \tag{39}$$

If $\delta > b - a$, then $\omega_1(f,\delta) = \omega_1(f, b-a)$.

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$.

Clealry f is uniformly continuous and $\omega_1(f,\delta) < \infty$. For $f \in B([a,b], X)$ (bounded functions) $\omega_1(f,\delta)$ is defined the same way.

Lemma 1 ([4]). *We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$ iff $f \in C([a,b], X)$.*

We need

Theorem 7 ([4]). *Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x,b]), \quad \delta > 0, \ \ x \in [a,b]. \tag{40}$$

Then G is continuous on $[a,b]$.

Theorem 8 ([4]). *Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a,x]), \tag{41}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We need

Lemma 2 ([2], p. 208, Lemma 7.1.1). *Let $f \in B([a,b], X)$, $(X, \|\cdot\|)$ is a Banach space. Then*

$$\|f(x) - f(x_0)\| \le \omega_1(f,h) \left\lceil \frac{|x - x_0|}{h} \right\rceil \le \omega_1(f,h) \left(1 + \frac{|x - x_0|}{h}\right), \tag{42}$$

$\forall\, x, x_0 \in [a,b]$, $h > 0$.

We make

Remark 2. All here as in (7)-(10). Let $f, g \in C([a,b], X)$. We have that

$$\|L_N(f)(x_0) - L_N(g)(x_0)\| = \|(L_N(f-g))(x_0)\| \overset{(7)}{\leq} \left(\widetilde{L}_N(\|f-g\|)\right)(x_0), \tag{43}$$

$\forall\, N \in \mathbb{N}, \forall\, x_0 \in [a,b]$.

In this work, for simplicity, from now on we will assume that

$$\widetilde{L}_N(1) = 1, \ \ \forall\, N \in \mathbb{N}. \tag{44}$$

We observe that

$$\|(L_N(f))(x_0) - f(x_0)\| \overset{(10)}{=} \|(L_N(f))(x_0) - (L_N(f(x_0)))(x_0)\| = \|(L_N(f - f(x_0)))(x_0)\| \overset{(43)}{\leq} \widetilde{L}_N(\|f(\cdot) - f(x_0)\|)(x_0), \tag{45}$$

$\forall\, x_0 \in [a,b]$, $\forall\, N \in \mathbb{N}$.

We have proved that

$$\|(L_N(f))(x_0) - f(x_0)\| \leq \widetilde{L}_N(\|f(\cdot) - f(x_0)\|)(x_0) \tag{46}$$

$\forall\, x_0 \in [a,b]$, $\forall\, N \in \mathbb{N}$.

3. Main results

We need

Definition 6. *Let $D_{x_0;g}^{(n+1)\alpha} f$ denote any of $D_{x_0-;g}^{(n+1)\alpha} f$, $D_{x_0+;g}^{(n+1)\alpha} f$, $n \in \mathbb{N}$ and $\delta > 0$. We set*

$$\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right) \circ g^{-1}, \delta\right) := \max\left\{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha} f\right) \circ g^{-1}, \delta\right)_{[g(a),g(x_0)]}, \omega_1\left(\left(D_{x_0+;g}^{(n+1)\alpha} f\right) \circ g^{-1}, \delta\right)_{[g(x_0),g(b)]}\right\}, \tag{47}$$

where $x_0 \in [a,b]$. Here the moduli of continuity are considered over $[g(a), g(x_0)]$ and $[g(x_0), g(b)]$, respectively.

We will use the following:

Theorem 9. *Let* $0 < \alpha < 1$, $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, g *is strictly increasing and* $g^{-1} \in C^1([g(a), g(b)])$. *Assume that* $D^{k\alpha}_{x_0-;g} f \in C^1([a,x_0], X)$ *and* $D^{k\alpha}_{x_0+;g} f \in C^1([x_0,b], X)$, *for* $k = 1, ..., n$; *where* $x_0 \in [a,b]$ *is fixed. Further assume that* $\left(D^{i\alpha}_{x_0\pm;g} f\right)(x_0) = 0$, $i = 2, ..., n+1$, $n \in \mathbb{N}$. *Then*

$$\|f(x) - f(x_0)\| \leq \frac{\omega_1\left(\left(D^{(n+1)\alpha}_{x_0;g} f\right) \circ g^{-1}, \delta\right)}{\Gamma((n+1)\alpha + 1)} \left[|g(x) - g(x_0)|^{(n+1)\alpha} + \frac{|g(x) - g(x_0)|^{(n+1)\alpha+1}}{\delta((n+1)\alpha+1)}\right], \tag{48}$$

$\forall\, x \in [a,b]$, $\delta > 0$.

Proof. By $\left(D^{i\alpha}_{x_0+;g} f\right)(x_0) = 0$, for $i = 2, ..., n+1$, and (33) we have

$$f(x) - f(x_0) = \frac{1}{\Gamma((n+1)\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{(n+1)\alpha - 1} g'(t) \left[\left(D^{(n+1)\alpha}_{x_0+;g} f\right)(t) - \left(D^{(n+1)\alpha}_{x_0+;g} f\right)(x_0)\right] dt, \tag{49}$$

$\forall\, x \in [x_0, b]$.

Here we apply the change of variables method for Bochner integrals, see Theorem 2.5, p. 7, [7] and [9].

Hence $(z := g(t))$

$$f(x) - f(x_0) = \frac{1}{\Gamma((n+1)\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1} \left[\left(\left(D^{(n+1)\alpha}_{x_0+;g} f\right) \circ g^{-1}\right)(z) - \left(\left(D^{(n+1)\alpha}_{x_0+;g} f\right) \circ g^{-1}\right)(g(x_0))\right] dz, \tag{50}$$

$\forall\, x \in [x_0, b]$.

By $\left(D^{i\alpha}_{x_0-;g} f\right)(x_0) = 0$, for $i = 2, ..., n+1 \in \mathbb{N}$, and (34) we have

$$f(x) - f(x_0) = \frac{1}{\Gamma((n+1)\alpha)} \int_{x}^{x_0} (g(t) - g(x))^{(n+1)\alpha - 1} g'(t) \left[\left(D^{(n+1)\alpha}_{x_0-;g} f\right)(t) - \left(D^{(n+1)\alpha}_{x_0-;g} f\right)(x_0)\right] dt, \tag{51}$$

$\forall\, x \in [a, x_0]$.

Hence $(z := g(t))$

$$f(x) - f(x_0) = \frac{1}{\Gamma((n+1)\alpha)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(n+1)\alpha - 1} \left[\left(\left(D^{(n+1)\alpha}_{x_0-;g} f\right) \circ g^{-1}\right)(z) - \left(\left(D^{(n+1)\alpha}_{x_0-;g} f\right) \circ g^{-1}\right)(g(x_0))\right] dz, \tag{52}$$

$\forall\ x \in [a, x_0]$.

We have that $(x_0 \leq x \leq b)$

$$\|f(x) - f(x_0)\| \leq \frac{1}{\Gamma((n+1)\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1}$$

$$\left\| \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right)(z) - \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1} \right)(g(x_0)) \right\| dz \underset{(\delta_1 > 0)}{\leq}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1} \tag{53}$$

$$\omega_1 \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1}, \frac{\delta_1 |z - g(x_0)|}{\delta_1} \right)_{[g(x_0), g(b)]} dz \overset{(42)}{\leq}$$

$$\frac{\omega_1 \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1}, \delta_1 \right)_{[g(x_0), g(b)]}}{\Gamma((n+1)\alpha)}$$

$$\int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1} \left(1 + \frac{(z - g(x_0))}{\delta_1} \right) dz =$$

$$\frac{\omega_1 \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1}, \delta_1 \right)_{[g(x_0), g(b)]}}{\Gamma((n+1)\alpha)} \left[\frac{(g(x) - g(x_0))^{(n+1)\alpha}}{(n+1)\alpha} + \right.$$

$$\left. \frac{1}{\delta_1} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(n+1)\alpha - 1} (z - g(x_0))^{2-1} dz \right] = \tag{54}$$

$$\frac{\omega_1 \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1}, \delta_1 \right)_{[g(x_0), g(b)]}}{\Gamma((n+1)\alpha)} \left[\frac{(g(x) - g(x_0))^{(n+1)\alpha}}{(n+1)\alpha} \right.$$

$$\left. + \frac{1}{\delta_1} \frac{\Gamma((n+1)\alpha)\Gamma(2)}{\Gamma((n+1)\alpha + 2)} (g(x) - g(x_0))^{(n+1)\alpha + 1} \right] =$$

$$\frac{\omega_1 \left(\left(D_{x_0+;g}^{(n+1)\alpha} f \right) \circ g^{-1}, \delta_1 \right)_{[g(x_0), g(b)]}}{\Gamma((n+1)\alpha)} \tag{55}$$

$$\left[\frac{(g(x) - g(x_0))^{(n+1)\alpha}}{(n+1)\alpha} + \frac{1}{\delta_1} \frac{(g(x) - g(x_0))^{(n+1)\alpha + 1}}{(n+1)\alpha((n+1)\alpha + 1)} \right].$$

We have proved that

$$\|f(x)-f(x_0)\| \leq \frac{\omega_1\left(\left(D_{x_0+;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_1\right)_{[g(x_0),g(b)]}}{\Gamma((n+1)\alpha+1)} \left[(g(x)-g(x_0))^{(n+1)\alpha}+\frac{(g(x)-g(x_0))^{(n+1)\alpha+1}}{\delta_1((n+1)\alpha+1)}\right], \tag{56}$$

$\forall\ x\in[x_0,b]$, $\delta_1>0$.

We have that $(a\leq x\leq x_0)$

$$\|f(x)-f(x_0)\| \overset{(52)}{\leq} \frac{1}{\Gamma((n+1)\alpha)}\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}$$

$$\left\|\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right)(z)-\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1}\right)(g(x_0))\right\|dz \underset{(\delta_2>0)}{\leq}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}$$

$$\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\frac{\delta_2|z-g(x_0)|}{\delta_2}\right)_{[g(a),g(x_0)]}dz \overset{(42)}{\leq} \tag{57}$$

$$\frac{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma((n+1)\alpha)}$$

$$\int_{g(x)}^{g(x_0)}(z-g(x))^{(n+1)\alpha-1}\left(1+\frac{g(x_0)-z}{\delta_2}\right)dz=$$

$$\frac{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma((n+1)\alpha)}\left[\frac{(g(x_0)-g(x))^{(n+1)\alpha}}{(n+1)\alpha}+\right.$$

$$\left.\frac{1}{\delta_2}\int_{g(x)}^{g(x_0)}(g(x_0)-z)^{2-1}(z-g(x))^{(n+1)\alpha-1}dz\right]= \tag{58}$$

$$\frac{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma((n+1)\alpha)}\left[\frac{(g(x_0)-g(x))^{(n+1)\alpha}}{(n+1)\alpha}+\right.$$

$$\left.\frac{1}{\delta_2}\frac{\Gamma(2)\Gamma((n+1)\alpha)}{\Gamma((n+1)\alpha+2)}(g(x_0)-g(x))^{(n+1)\alpha+1}\right]=$$

$$\frac{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma\left((n+1)\alpha\right)}$$

$$\left[\frac{(g(x_0)-g(x))^{(n+1)\alpha}}{(n+1)\alpha}+\frac{1}{\delta_2}\frac{(g(x_0)-g(x))^{(n+1)\alpha+1}}{(n+1)\alpha((n+1)\alpha+1)}\right]= \tag{59}$$

$$\frac{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma\left((n+1)\alpha+1\right)}$$

$$\left[(g(x_0)-g(x))^{(n+1)\alpha}+\frac{(g(x_0)-g(x))^{(n+1)\alpha+1}}{\delta_2((n+1)\alpha+1)}\right].$$

We have proved that

$$\|f(x)-f(x_0)\|\leq\frac{\omega_1\left(\left(D_{x_0-;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma\left((n+1)\alpha+1\right)} \tag{60}$$

$$\left[(g(x_0)-g(x))^{(n+1)\alpha}+\frac{(g(x_0)-g(x))^{(n+1)\alpha+1}}{\delta_2((n+1)\alpha+1)}\right],$$

$\forall\ x\in[a,x_0]$, $\delta_2>0$.

By (56) and (60), setting $\delta=\delta_1=\delta_2>0$, we derive (48). ■

We state

Corollary 3 (to Theorem 9)**.** *All as in Theorem 9. Then*

$$\|f(\cdot)-f(x_0)\|\leq\frac{\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha}f\right)\circ g^{-1},\delta\right)}{\Gamma\left((n+1)\alpha+1\right)} \tag{61}$$

$$\left[|g(\cdot)-g(x_0)|^{(n+1)\alpha}+\frac{|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}}{\delta((n+1)\alpha+1)}\right],$$

$\delta>0$, *true over* $[a,b]$.

We need

Definition 7. *Let* $D_{x_0;g}^{\alpha}f$ *denote any of* $D_{x_0\pm;g}^{\alpha}f$ *and* $\delta>0$. *We set*

$$\omega_1\left(\left(D_{x_0;g}^{\alpha}f\right)\circ g^{-1},\delta\right):=\max\left\{\omega_1\left(\left(D_{x+;g}^{\alpha}f\right)\circ g^{-1},\delta\right)_{[g(x_0),g(b)]},\right. \tag{62}$$
$$\left.\omega_1\left(\left(D_{x_0-;g}^{\alpha}f\right)\circ g^{-1},\delta\right)_{[g(a),g(x_0)]}\right\},$$

where $x_0\in[a,b]$. *Here the moduli of continuity are considered over* $[g(x_0),g(b)]$ *and* $[g(a),g(x_0)]$, *respectively.*

We will use:

Theorem 10. *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing such that $g^{-1} \in C^n([g(a), g(b)])$. Let $x_0 \in [a,b]$, be fixed such that $\left(f \circ g^{-1}\right)^{(i)}(g(x_0)) = 0$, for, $i = 1, ..., n-1$. Then*

$$(63) \quad \|f(x) - f(x_0)\| \leq \frac{\omega_1\left(\left(D^{\alpha}_{x_0;g} f\right) \circ g^{-1}, \delta\right)}{\Gamma(\alpha+1)} \left[|g(x) - g(x_0)|^{\alpha} + \frac{|g(x) - g(x_0)|^{\alpha+1}}{\delta(\alpha+1)}\right],$$

$\forall\ \delta > 0$, $\forall\ x \in [a,b]$.

If $0 < \alpha < 1$, then we do not need any initial conditions.

Proof. By Remark 1 we get that $\left(D^{\alpha}_{x_0\pm;g} f\right)(x_0) = 0$, for $\alpha \notin \mathbb{N}$. By (37) we obtain

$$(64) \quad f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{\alpha-1} g'(t) \left(D^{\alpha}_{x_0+;g} f\right)(t)\, dt,$$

$\forall\ x \in [x_0, b]$.

And, by (38) we get

$$(65) \quad f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (g(t) - g(x))^{\alpha-1} g'(t) \left(D^{\alpha}_{x_0-;g} f\right)(t)\, dt,$$

$\forall\ x \in [a, x_0]$.

When $0 < \alpha < 1$, i.e. $n = 1$, then (64), (65) are valid without any initial conditions.

We can write

$$(66) \quad f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (g(x) - g(t))^{\alpha-1} g'(t) \left[\left(D^{\alpha}_{x_0+;g} f\right)(t) - \left(D^{\alpha}_{x_0+;g} f\right)(x_0)\right] dt,$$

$\forall\ x \in [x_0, b]$,

$$(67) \quad f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (g(t) - g(x))^{\alpha-1} g'(t) \left[\left(D^{\alpha}_{x_0-;g} f\right)(t) - \left(D^{\alpha}_{x_0-;g} f\right)(x_0)\right] dt,$$

$\forall\ x \in [a, x_0]$.

We can rewrite (66) and (67) as follows (by $z := g(t)$):

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left[\left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}\right)(z) - \left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}\right)(g(x_0))\right] dz, \tag{68}$$

$\forall\, x \in [x_0, b]$, and

$$f(x) - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left[\left(\left(D_{x_0-;g}^{\alpha} f\right) \circ g^{-1}\right)(z) - \left(\left(D_{x_0-;g}^{\alpha} f\right) \circ g^{-1}\right)(g(x_0))\right] dz, \tag{69}$$

$\forall\, x \in [a, x_0]$.

We have that $(x_0 \leq x \leq b)$

$$\|f(x) - f(x_0)\| = \frac{1}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1}$$

$$\left\|\left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}\right)(z) - \left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}\right)(g(x_0))\right\| dz \leq$$

$(\delta_1 > 0)$

$$\frac{1}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1}$$

$$\omega_1\left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}, \frac{\delta_1 |z - g(x_0)|}{\delta_1}\right)_{[g(x_0), g(b)]} dz \overset{(42)}{\leq}$$

$$\frac{\omega_1\left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}, \delta_1\right)_{[g(x_0), g(b)]}}{\Gamma(\alpha)} \tag{70}$$

$$\int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left(1 + \frac{(z - g(x_0))}{\delta_1}\right) dz =$$

$$\frac{\omega_1\left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}, \delta_1\right)_{[g(x_0), g(b)]}}{\Gamma(\alpha)}$$

$$\left[\frac{(g(x) - g(x_0))^{\alpha}}{\alpha} + \frac{1}{\delta_1} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} (z - g(x_0))^{2-1} dz\right] =$$

$$\frac{\omega_1\left(\left(D_{x_0+;g}^{\alpha} f\right) \circ g^{-1}, \delta_1\right)_{[g(x_0), g(b)]}}{\Gamma(\alpha)}$$

$$\left[\frac{(g(x) - g(x_0))^{\alpha}}{\alpha} + \frac{1}{\delta_1} \frac{\Gamma(\alpha)\Gamma(2)}{\Gamma(\alpha + 2)} (g(x) - g(x_0))^{\alpha+1}\right] =$$

$$\frac{\omega_1\left(\left(D^{\alpha}_{x_0+;g}f\right)\circ g^{-1},\delta_1\right)_{[g(x_0),g(b)]}}{\Gamma(\alpha)}$$

$$\left[\frac{(g(x)-g(x_0))^{\alpha}}{\alpha}+\frac{1}{\delta_1}\frac{1}{\alpha(\alpha+1)}(g(x)-g(x_0))^{\alpha+1}\right], \tag{71}$$

$\forall\ x\in[x_0,b]$.

We have proved that

$$\|f(x)-f(x_0)\| \leq \frac{\omega_1\left(\left(D^{\alpha}_{x_0+;g}f\right)\circ g^{-1},\delta_1\right)_{[g(x_0),g(b)]}}{\Gamma(\alpha+1)} \left[(g(x)-g(x_0))^{\alpha}+\frac{(g(x)-g(x_0))^{\alpha+1}}{\delta_1(\alpha+1)}\right], \tag{72}$$

$\forall\ x\in[x_0,b]$, $\delta_1>0$.

By (69) we obtain

$$\|f(x)-f(x_0)\|\leq\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}$$

$$\left\|\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1}\right)(z)-\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1}\right)(g(x_0))\right\|dz\leq$$

$(\delta_2>0)$

$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}$$

$$\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\frac{\delta_2(g(x_0)-z)}{\delta_2}\right)_{[g(a),g(x_0)]}dz\overset{(42)}{\leq}$$

$$\frac{\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma(\alpha)}$$

$$\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(1+\frac{g(x_0)-z}{\delta_2}\right)dz= \tag{73}$$

$$\frac{\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma(\alpha)}$$

$$\left[\frac{(g(x_0)-g(x))^{\alpha}}{\alpha}+\frac{1}{\delta_2}\int_{g(x)}^{g(x_0)}(g(x_0)-z)^{2-1}(z-g(x))^{\alpha-1}dz\right]=$$

$$\frac{\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma(\alpha)}$$

$$\left[\frac{(g(x_0)-g(x))^{\alpha}}{\alpha}+\frac{1}{\delta_2}\frac{\Gamma(2)\Gamma(\alpha)}{\Gamma(\alpha+2)}(g(x_0)-g(x))^{\alpha+1}\right]=$$

$$\frac{\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma(\alpha)}$$

$$\left[\frac{(g(x_0)-g(x))^{\alpha}}{\alpha}+\frac{1}{\delta_2}\frac{1}{\alpha(\alpha+1)}(g(x_0)-g(x))^{\alpha+1}\right]= \tag{74}$$

$$\frac{\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma(\alpha+1)}$$

$$\left[(g(x_0)-g(x))^{\alpha}+\frac{1}{\delta_2(\alpha+1)}(g(x_0)-g(x))^{\alpha+1}\right], \tag{75}$$

$\forall\ x\in[a,x_0]$.

In conclusion we derive

$$\|f(x)-f(x_0)\|\leq\frac{\omega_1\left(\left(D^{\alpha}_{x_0-;g}f\right)\circ g^{-1},\delta_2\right)_{[g(a),g(x_0)]}}{\Gamma(\alpha+1)}\left[(g(x_0)-g(x))^{\alpha}+\frac{(g(x_0)-g(x))^{\alpha+1}}{\delta_2(\alpha+1)}\right], \tag{76}$$

$\forall\ x\in[a,x_0]$, $\delta_2>0$.

Choosing $\delta=\delta_1=\delta_2>0$ by (72) and (76), we get (63). ■

We state

Corollary 4. *All as in Theorem 10. Then*

$$\|f(\cdot)-f(x_0)\|\leq\frac{\omega_1\left(\left(D^{\alpha}_{x_0;g}f\right)\circ g^{-1},\delta\right)}{\Gamma(\alpha+1)}\left[|g(\cdot)-g(x_0)|^{\alpha}+\frac{|g(\cdot)-g(x_0)|^{\alpha+1}}{\delta(\alpha+1)}\right], \tag{77}$$

true over $[a,b]$, $\delta>0$.

If $0<\alpha<1$, *no initial conditions are needed.*

We make

Assumption 1. *Let* $L_N: C([a,b],X)\hookrightarrow C([a,b],X)$, $(X,\|\cdot\|)$ *a Banach space,* L_N *is a linear operator,* $\forall\ N\in\mathbb{N}$, $x_0\in[a,b]$. *Let also* $\widetilde{L}_N: C([a,b])\hookrightarrow C([a,b])$, *a sequence of positive linear operators,* $\forall\ N\in\mathbb{N}$.

We assume that

$$\left\| \left(L_N (f) \right) (x_0) \right\| \leq \left(\widetilde{L}_N \left(\| f \| \right) \right) (x_0) , \tag{78}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x_0 \in [a,b]$, $\forall\, f \in C\left([a,b], X\right)$.

When $g \in C([a,b])$, $c \in X$, *we assume that*

$$\left(L_N (cg) \right) = c\widetilde{L}_N (g) , \ \ \forall\, N \in \mathbb{N}. \tag{79}$$

We treat the special case of

$$\widetilde{L}_N (1) = 1, \ \ \forall\, N \in \mathbb{N}, \tag{80}$$

which implies

$$L_N (c) = c, \ \ \forall\, c \in X, \ \ \forall\, N \in \mathbb{N}. \tag{81}$$

We call $\widetilde{L}_N$ *the companion operator of* L_N.

Operators $\widetilde{B}_N$ see (1), and B_N see (2), fulfill Assumption 1.

We make

Remark 3. Let $\widetilde{L}_N : C([a,b]) \to C([a,b])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. By Riesz representation theorem (see [10], p. 304) we have

$$\widetilde{L}_N (f, x_0) = \int_{[a,b]} f(t)\, d\mu_{N_{x_0}} (t) , \tag{82}$$

$\forall\, x_0 \in [a,b]$, where $\mu_{N_{x_0}}$ is a unique positive finite measure on σ-Borel algebra of $[a,b]$. Call

$$\widetilde{L}_N (1, x_0) = \mu_{N_{x_0}} ([a,b]) = M_{N_{x_0}}. \tag{83}$$

In our case of $\widetilde{L}_N (1, x_0) = 1$, we have that $M_{N_{x_0}} = 1$, and $\mu_{N_{x_0}}$ is a probability measure.

By Hölder's inequality we obtain

$$\begin{aligned} \widetilde{L}_N \left(|g(\cdot) - g(x_0)|^{\alpha} \right) (x_0) &= \int_{[a,b]} |g(t) - g(x_0)|^{\alpha}\, d\mu_{N_{x_0}} (t) \\ &\leq \left(\int_{[a,b]} |g(t) - g(x_0)|^{\alpha+1}\, d\mu_{N_{x_0}} (t) \right)^{\frac{\alpha}{\alpha+1}} \\ &= \left(\widetilde{L}_N \left(|g(\cdot) - g(x_0)|^{\alpha+1} \right) (x_0) \right)^{\frac{\alpha}{\alpha+1}} . \end{aligned} \tag{84}$$

Similarly, it holds

$$\begin{aligned} &\widetilde{L}_N \left(|g(\cdot) - g(x_0)|^{(n+1)\alpha} \right) (x_0) \\ &\quad \leq \left(\widetilde{L}_N \left(|g(\cdot) - g(x_0)|^{(n+1)\alpha+1} \right) (x_0) \right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}} . \end{aligned} \tag{85}$$

We present

Theorem 11. *Under Assumption 1 and all as in Theorem 9, we derive*

$$\left\| \left(L_N\left(f\right)\right)\left(x_0\right) - f\left(x_0\right)\right\| \leq \frac{\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right) \circ g^{-1}, \delta\right)}{\Gamma\left(\left(n+1\right)\alpha+1\right)} \tag{86}$$

$$\left[\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{(n+1)\alpha}\right)\left(x_0\right) + \frac{\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{(n+1)\alpha+1}\right)\left(x_0\right)}{\delta\left(\left(n+1\right)\alpha+1\right)}\right],$$

$\forall\ N \in \mathbb{N}$, $\delta > 0$.

Proof. By (61) we get

$$\widetilde{L}_N\left(\left\|f\left(\cdot\right) - f\left(x_0\right)\right\|\right)\left(x_0\right) \leq \frac{\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right) \circ g^{-1}, \delta\right)}{\Gamma\left(\left(n+1\right)\alpha+1\right)} \tag{87}$$

$$\left[\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{(n+1)\alpha}\right)\left(x_0\right) + \frac{\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{(n+1)\alpha+1}\right)\left(x_0\right)}{\delta\left(\left(n+1\right)\alpha+1\right)}\right],$$

along with (46) proving the claim. ■

We also give

Theorem 12. *Under Assumption 1 and all as in Theorem 10, we derive*

$$\left\| \left(L_N\left(f\right)\right)\left(x_0\right) - f\left(x_0\right)\right\| \leq \frac{\omega_1\left(\left(D_{x_0;g}^{\alpha} f\right) \circ g^{-1}, \delta\right)}{\Gamma\left(\alpha+1\right)} \tag{88}$$

$$\left[\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{\alpha}\right)\left(x_0\right) + \frac{\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{\alpha+1}\right)\left(x_0\right)}{\delta\left(\alpha+1\right)}\right],$$

$\forall\ N \in \mathbb{N}$, $\delta > 0$.

Proof. By (77) we get

$$\widetilde{L}_N\left(\left\|f\left(\cdot\right) - f\left(x_0\right)\right\|\right)\left(x_0\right) \leq \frac{\omega_1\left(\left(D_{x_0;g}^{\alpha} f\right) \circ g^{-1}, \delta\right)}{\Gamma\left(\alpha+1\right)} \tag{89}$$

$$\left[\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{\alpha}\right)\left(x_0\right) + \frac{\widetilde{L}_N\left(\left|g\left(\cdot\right) - g\left(x_0\right)\right|^{\alpha+1}\right)\left(x_0\right)}{\delta\left(\alpha+1\right)}\right],$$

along with (46) proving the claim. ■

We give

Theorem 13. *Under Assumption 1 and all as in Theorem 9, we obtain*

$$\|(L_N(f))(x_0) - f(x_0)\| \leq \frac{((n+1)\alpha+2)}{\Gamma((n+1)\alpha+2)} \tag{90}$$

$$\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right)\circ g^{-1}, \left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}}\right)$$

$$\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}}, \quad \forall\ N\in\mathbb{N}.$$

If $\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0) \to 0$, *then* $L_N(f)(x_0) \to f(x_0)$, *as* $N\to\infty$.

Proof. By (87) and (85) we obtain that

$$\text{Right hand side (87)} \leq \frac{\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right)\circ g^{-1}, \delta\right)}{\Gamma((n+1)\alpha+1)}$$

$$\left[\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}}\right. \tag{91}$$

$$\left.+\frac{\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)}{\delta((n+1)\alpha+1)}\right] =: (*).$$

Momentarily we assume that $\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0) > 0$, and choose

$$\delta := \left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}} > 0. \tag{92}$$

Thus, we have

$$(*) = \frac{\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right)\circ g^{-1}, \left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}}\right)}{\Gamma((n+1)\alpha+1)}$$

$$\left[\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}} + \frac{\delta^{(n+1)\alpha+1}}{\delta((n+1)\alpha+1)}\right] = \tag{93}$$

$$\frac{\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right)\circ g^{-1}, \left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}}\right)}{\Gamma((n+1)\alpha+1)}$$

$$\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}}\left[1+\frac{1}{(n+1)\alpha+1}\right]=$$

$$\frac{((n+1)\alpha+2)}{\Gamma((n+1)\alpha+2)}\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha}f\right)\circ g^{-1},\right. \tag{94}$$

$$\left.\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}}\right)$$

$$\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}}.$$

So far we have proved that

$$\widetilde{L}_N(\|f(\cdot)-f(x_0)\|)(x_0)\leq\frac{((n+1)\alpha+2)}{\Gamma((n+1)\alpha+2)} \tag{95}$$

$$\omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha}f\right)\circ g^{-1},\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}}\right)$$

$$\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}},$$

when $\delta>0$.

If $\delta=0$, we get R.H.S.(95)$=0$, and

$$\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{(n+1)\alpha+1}\right)(x_0) \tag{96}$$
$$=\int_{[a,b]}|g(t)-g(x_0)|^{(n+1)\alpha+1}\,d\mu_{N_{x_0}}(t)=0.$$

Hence $|g(t)-g(x_0)|=0$, a.e. on $[a,b]$. That is $g(t)=g(x_0)$, a.e. on $[a,b]$ and $t=x_0$, a.e. on $[a,b]$, because g is strictly increasing. More precisely $\mu_{N_{x_0}}\{t\in[a,b]:t\neq x_0\}=0$. Therefore $\mu_{N_{x_0}}$ concentrates on x_0, that is $\mu_{N_{x_0}}=\delta_{x_0}$, the unit Dirac measure.

Consequently, it holds

$$\widetilde{L}_N(\|f(\cdot)-f(x_0)\|)(x_0)=\int_{[a,b]}\|f(t)-f(x_0)\|\,d\delta_{x_0}(t) \tag{97}$$
$$=\|f(x_0)-f(x_0)\|=0.$$

Therefore (95) is true always, in both cases, completing the proof of the theorem, by the use of (46). ∎

We also give

Theorem 14. *Under Assumption 1 and all as in Theorem 10, we obtain*

$$\left\|\left(L_N(f)\right)(x_0)-f(x_0)\right\|\leq\frac{(\alpha+2)}{\Gamma(\alpha+2)}\omega_1\left(\left(D^{\alpha}_{x_0;g}f\right)\circ g^{-1},\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right)\left(\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}},\quad\forall\ N\in\mathbb{N}.\tag{98}$$

If $\widetilde{L}_N\left(|g(\cdot)-g(x_0)|^{\alpha+1}\right)(x_0)\to0$*, then* $L_N(f)(x_0)\to f(x_0)$*, as* $N\to\infty$*.*

Proof. Similar to the proof of Theorem 13. It is based on (89), (84) and (46). We omit the details. ■

We give

Corollary 5 (to Theorem 13)**.** *All as in Theorem 13. Then*

$$\left\|\left(L_N(f)\right)(x_0)-f(x_0)\right\|\leq\frac{((n+1)\alpha+2)}{\Gamma((n+1)\alpha+2)}\omega_1\left(\left(D^{(n+1)\alpha}_{x_0;g}f\right)\circ g^{-1},\left\|g'\right\|_{\infty,[a,b]}\left(\widetilde{L}_N\left(|\cdot-x_0|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{1}{(n+1)\alpha+1}}\right)\left(\left\|g'\right\|_{\infty,[a,b]}\right)^{(n+1)\alpha}\left(\widetilde{L}_N\left(|\cdot-x_0|^{(n+1)\alpha+1}\right)(x_0)\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}},\quad\forall\ N\in\mathbb{N}.\tag{99}$$

If $\widetilde{L}_N\left(|\cdot-x_0|^{(n+1)\alpha+1}\right)(x_0)\to0$*, then* $L_N(f)(x_0)\to f(x_0)$*, as* $N\to\infty$*.*

Proof. By Theorem 13 and

$$|g(x)-g(x_0)|\leq\left\|g'\right\|_{\infty,[a,b]}|x-x_0|,\quad\forall\ x,\ x_0\in[a,b].\tag{100}$$

■

We give

Corollary 6 (to Theorem 14))**.** *All as in Theorem 14. Then*

$$\left\|\left(L_N(f)\right)(x_0)-f(x_0)\right\|\leq\frac{(\alpha+2)}{\Gamma(\alpha+2)}\omega_1\left(\left(D^{\alpha}_{x_0;g}f\right)\circ g^{-1},\left\|g'\right\|_{\infty,[a,b]}\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{1}{\alpha+1}}\right)\left(\left\|g'\right\|_{\infty,[a,b]}\right)^{\alpha}\left(\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)(x_0)\right)^{\frac{\alpha}{\alpha+1}},\quad\forall\ N\in\mathbb{N}.\tag{101}$$

If $\widetilde{L}_N\left(|\cdot-x_0|^{\alpha+1}\right)(x_0)\to0$*, then* $L_N(f)(x_0)\to f(x_0)$*, as* $N\to\infty$*.*

Proof. By Theorem 14 and

$$|g(x) - g(x_0)| \leq \|g'\|_{\infty,[a,b]} |x - x_0|, \quad \forall\, x,\ x_0 \in [a,b]. \tag{102}$$

∎

We make

Remark 4. Theorems 13, 14 and Corollaries 5, 6 have direct applications to B_N, $\widetilde{B}_N$ Bernstein operators over $[0,1]$. We obtain there $B_N(f)(x_0) \to f(x_0)$, as $N \to \infty$.

We notice that

$$|\cdot - x_0|^{\alpha+1} = |\cdot - x_0|^{\alpha} |\cdot - x_0| \leq |\cdot - x_0|, \text{ over } [0,1].$$

Hence

$$\widetilde{B}_N\left(|\cdot - x_0|^{\alpha+1}\right)(x_0) \leq \widetilde{B}_N(|\cdot - x_0|)(x_0), \quad x_0 \in [0,1].$$

Similarly, it holds

$$|\cdot - x_0|^{(n+1)\alpha+1} = |\cdot - x_0|^{(n+1)\alpha} |\cdot - x_0| \leq |\cdot - x_0|, \text{ over } [0,1].$$

Hence

$$\widetilde{B}_N\left(|\cdot - x_0|^{(n+1)\alpha+1}\right)(x_0) \leq \widetilde{B}_N(|\cdot - x_0|)(x_0), \quad x_0 \in [0,1].$$

We have that

$$\widetilde{B}_N(|\cdot - x_0|)(x_0) \leq \left(\widetilde{B}_N\left((\cdot - x_0)^2\right)(x_0)\right)^{\frac{1}{2}} = \sqrt{\frac{x_0(1-x_0)}{N}} \leq \frac{1}{2\sqrt{N}}, \quad \forall\, N \in \mathbb{N}. \tag{103}$$

We give

Corollary 7. *Here $[a,b] = [0,1]$ and all as in Theorem 9. Then*

$$\|(B_N(f))(x_0) - f(x_0)\| \leq \frac{((n+1)\alpha+2)}{\Gamma((n+1)\alpha+2)} \omega_1\left(\left(D_{x_0;g}^{(n+1)\alpha} f\right) \circ g^{-1}, \|g'\|_{\infty,[a,b]} \left(\frac{1}{2\sqrt{N}}\right)^{\frac{1}{(n+1)\alpha+1}}\right) \left(\|g'\|_{\infty,[a,b]}\right)^{(n+1)\alpha} \left(\frac{1}{2\sqrt{N}}\right)^{\frac{(n+1)\alpha}{(n+1)\alpha+1}}, \quad \forall\, N \in \mathbb{N},\ x_0 \in [0,1]. \tag{104}$$

As $N \to \infty$, we get $B_N(f)(x_0) \to f(x_0)$.

Proof. By Corollary 5 and (103). ∎

We give

Corollary 8. *Here* $[a,b]=[0,1]$ *and all as in Theorem 10. Then*

$$\|(B_N(f))(x_0)-f(x_0)\| \leq \frac{(\alpha+2)}{\Gamma(\alpha+2)} \omega_1\left(\left(D^{\alpha}_{x_0;g}f\right)\circ g^{-1}, \left\|g'\right\|_{\infty,[a,b]}\left(\frac{1}{2\sqrt{N}}\right)^{\frac{1}{\alpha+1}}\right) \left(\left\|g'\right\|_{\infty,[a,b]}\right)^{\alpha}\left(\frac{1}{2\sqrt{N}}\right)^{\frac{\alpha}{\alpha+1}}, \quad \forall\, N\in\mathbb{N}. \tag{105}$$

As $N\to\infty$*, we get* $B_N(f)(x_0)\to f(x_0)$*,* $x_0\in[0,1]$.

Proof. By Corollary 6 and (103). ∎

We continue with the following

Corollary 9 (to Corollary 8)**.** *Here* $[a,b]=[0,1]$, *all as in Theorem 10, and* $g(x)=e^x$, $x\in[0,1]$. *Then*

$$\|(B_N(f))(x_0)-f(x_0)\| \leq \frac{(\alpha+2)}{\Gamma(\alpha+2)} \omega_1\left(\left(D^{\alpha}_{x_0;e^x}f\right)\circ \ln x, e\left(\frac{1}{2\sqrt{N}}\right)^{\frac{1}{\alpha+1}}\right) e^{\alpha}\left(\frac{1}{2\sqrt{N}}\right)^{\frac{\alpha}{\alpha+1}}, \quad \forall\, N\in\mathbb{N}. \tag{106}$$

As $N\to\infty$*, we get* $B_N(f)(x_0)\to f(x_0)$*,* $x_0\in[0,1]$.

Proof. By Corollary 8. ∎

We finish with

Corollary 10 (to Corollary 9)**.** *Here* $[a,b]=[0,1]$, *all as in Theorem 10,* $g(x)=e^x$, $x\in[0,1]$ *and* $\alpha=\frac{1}{2}$ *(no initial conditions needed). Then*

$$\|(B_N(f))(x_0)-f(x_0)\| \leq \frac{10}{3}\sqrt{\frac{e}{\pi}} \omega_1\left(\left(D^{\frac{1}{2}}_{x_0;e^x}f\right)\circ \ln x, e\left(\frac{1}{2\sqrt{N}}\right)^{\frac{2}{3}}\right)\left(\frac{1}{2\sqrt{N}}\right)^{\frac{1}{3}}, \tag{107}$$

$\forall\, N\in\mathbb{N}$, $\forall\, x_0\in[0,1]$. *As* $N\to\infty$*, we get* $B_N(f)(x_0)\to f(x_0)$*, any* $x_0\in[0,1]$..

Proof. By Corollary 9. ∎

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