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Degradation model for system incorporating heterogeneities

Keywords

condition-based maintenance, heterogeneities, random effects model

Abstract

A system subject to several deterioration processes is studied. These processes arrive to the system following a Cox process and they grow according to a homogeneous gamma process. The system is failed when a degradation process exceeds a certain failure threshold. The maintenance strategy implemented on the system is condition-based maintenance, the deterioration state of the system is checked, and replacements are performed if necessary. A random effects model is considered to deal with the heterogeneities between processes, in particular, a uniform distribution is used to model the inverse of the scale parameter of the gamma process. Finally, the analytic cost model is obtained and analysed through some numerical examples.

1. Introduction

It is increasingly necessary to model the deterioration of a system subject to multiple degradation processes (Wu & Castro, 2020; Jia & Gardoni, 2018). This can take two different approaches, the first of which assumes that all deterioration processes affecting the system start at the same instant in time. However, this is unlikely or difficult to occur, so a second approach will be adopted which assumes that deterioration processes start at random times and deterioration increases depending on the system and its environment. These processes may or may not be considered independent of each other.

Reliability depends heavily on the failure data that determine the suitable lifetime models. Due to monitoring techniques, degradation models have become an important tool instead of the traditional reliability study. In general, aging-related failures are associated to stochastic deterioration of the components. Degradation-threshold models (Caballé, 2015; Caballé et al., 2015) have been included in order to facilitate the reliability analysis of complex systems. The system is considered to be failed when the degradation processes reach a certain predefined threshold.

The failure of a system relates to multiple degradation processes, each of them with different characteristics. These degradation processes could also be affected by the environment and external working conditions.

The degradation processes of a complex system could be dependent because usually the failure of a component is reflected in others (Castro & Landesa, 2019). There are also covariates that influence the model, such as the environmental conditions. That provides us external information that affect the system. For example: temperature, humidity, or climate conditions. The study of covariates was carried out by Lawless and Crowder in

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(Lawless et al., 2004), for Wiener and gamma processes. The environmental conditions are represented as explanatory variables, whose effects are modeled by a covariate-effect function used to modify the model parameters.

Recent literature reviews on degradation and maintenance can be found in van Noortwijk, de Jonge and Scarf, among others (Noortwijk, 2009). Time-based maintenance was first studied by Barlow and Hunter (Barlow et al., 1966). Preventive maintenance policies have had an increasing popularity in the last years. It may be due to the advances in sensor technology. Condition monitoring is impossible in some devices because costs are high. On the other hand, condition-based maintenance is difficult to plan (Alaswad, 2017; Huynh et al., 2017). Instead of that, aged-based maintenance is sometimes used. The age of the system determines when preventive maintenance takes place. It has other problems, early preventive maintenance results in a high cost and late maintenance results in a higher probability of failure.

In this chapter, we study the modelling and maintenance strategy of a system subject to various degradation processes. For the sake of simplicity, we consider perfect repairs on the system. It means that the system is replaced by a completely new one when replacements at periodical inspection times are performed. However, realistic maintenance is mostly imperfect and the system state after performing a maintenance is between the as-good-as-new state (AGAN) and the as-bad-as-old state (ABAO).

The purpose of many research works on reliability is to determine the lifetime distribution or, at least, adjust an appropriate distribution. Our approach is to assume a gamma process (suitable for representing non-decreasing degradation) and to consider the shape and scale parameter as random parameters (Abdel-Hamed, 1975). This assumption provides a good description of failure data and heterogeneities between the different components of a system.

Heterogeneity of the different components of a system can be represented through two main sources of variability.

1) *Heterogeneous components*. Maintenance in multi-component systems composed of many heterogeneous components with different behaviours is a key challenge in industry. These components may require different maintenance actions that should be studied separately. Electrical components, for example, fail without warning. On the contrary, degrading components may show a certain deterioration that is a precursor of failure. The heterogeneities between components can be modelled in many ways: using different distributions for representing their degradation paths, or using the same distribution, but with random parameters to represent that variability between components.
2) *Lead time*. In practice, there is usually a time between the failure and the arrival of the repair team for performing the corresponding maintenance action. This lead time is due to multiple factors.

A probability distribution is used on the scale parameter of the gamma process to represent this variability. The uniform distribution has been chosen in this chapter. This distribution is flexible since is used to model uncertainty based on previously available data; it is commonly used as a prior distribution in a Bayesian approach.

An important advantage of the uniform distribution it is simplicity.

1.1. Objectives

The objective of this chapter is to analyse a condition-based maintenance (CBM) strategy and maintenance planning for a system subject to internal degradation.

The arrival process intensity is considered not deterministic but stochastic. That notion was introduced by Cox through the doubly stochastic Poisson process, or Cox process, which consists of a Poisson process with stochastic intensity modulated by an outside process that influences the evolution of the counting process (Pinsky & Karlin, 2011). The motivation for using this process is that external shocks can accelerate the arrival of new processes or shocks to the system, hence the deterministic intensity is not suitable. It is useful for modelling the system lifetime of a dynamic environment (Straub, 2009), apart from different applications in reliability and maintenance, when the failure rate suffers an abrupt increment due to external factors.

In our case, this external process is a Poisson process. The internal degradation is the result of the stochastic arrival of multiple degradation processes which grow according to a homogeneous

gamma process.
The main objectives covered in the work are:
1) develop a condition-based maintenance model,
2) introduce preventive replacements and periodic inspections,
3) analyse the expected cost rate of this maintenance policy,
4) optimise the expected cost rate by the optimization of the preventive threshold and the time between inspections,
5) consider heterogeneities in the model,
6) study the influence of the gamma process parameters in the model.

2. Gamma process models

Deterioration models play an important role in improving the durability and reliability and planning an appropriate maintenance strategy for the system. The gamma process is usually employed in modelling degradation processes that involve independent non-negative increments. However, in most systems there is substantial heterogeneity between the degradation paths of the components. To model this variability, random effects are introduced in the degradation models, for example, by using random coefficients.

Definition 1. The homogeneous gamma process $\{X(t),\ t \geq 0\}$ with shape function $\alpha(t) > 0$ and scale parameter $\beta > 0$ is a continuous-time stochastic process with independent gamma-distributed increment and the following properties:
1. $X(0) = 0$ with probability 1.
2. $X(t_2) - X(t_1)$ follows a gamma process with shape parameter $\alpha(t_2 - t_1)$ and scale parameter β.
3. $X(t)$ has independent increments.

The gamma process is widely employed in modelling continuous degradation or damage in materials and engineering structures.

Definition 2. The probability density function of a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ is:

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-\beta x\},\quad x > 0, \quad (1)$$

with

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \quad (2)$$

Remark 1. The system fails when the deterioration level of one of the degradation processes exceeds the failure threshold. Supposing that the degradation grows following a gamma process, the following random variable is defined:

$$\sigma_L = inf\,\{t : X(t) > L\}. \quad (3)$$

This variable represents the first time at which the gamma degradation process exceeds the threshold L.

Definition 3. The distribution function of σ_L is given by

$$F_{\sigma_L}(t) = \mathrm{P}(X(t) \geq L)$$

$$= \int_L^\infty f_{\alpha t,\beta}(x)dx = \frac{\Gamma(\alpha t, \beta L)}{\Gamma(\alpha t)} \quad (4)$$

Remark 2. Another important survival function is the lifetime distribution of $\sigma_L - \sigma_M$

$$\overline{F_{\sigma_L - \sigma_M}}(t) =$$

$$\int_{x=0}^\infty \int_{y=M}^\infty f_{\sigma_M, X(\sigma_M)}(x,y) F_{\alpha t,\beta}(L-y)\,dy\,dx \quad (5)$$

where $F_{\alpha t,\beta}$ is the gamma distribution function with parameters αt and β. In this formula the overshoot effect of gamma processes has not been taken into account.

2.1. Random effects

To deal with the variability between processes in this section, we consider that the degradation paths can be described by a gamma process with random effects, that is, a model in which one or both parameters of the gamma process are random (Lawless et al., 2004; Pulcini et al., 2013) and (Rodríguez-Picón et al., 2018). Here the heterogeneities between the different degradation processes are quantified by randomizing the scale parameter of the gamma process. This approach is assumed instead of the analysis of explicit dependences between degradation paths since the formulation would be more complex and copulas should be used (Liu et al., 2014; Wang et al., 2012).
In this case, we assume that the scale parameter is random. With that, both the mean and the variance of the process are influenced by this choice.

The density function of a homogeneous gamma process considering that the scale parameter $\beta^{-1} \sim U(a,b)$ follows a uniform distribution on the interval (a,b) is expressed as follows:

$$f_{X_h(t)}(u) = \int_a^b f_{\alpha t,\beta}(u) \frac{1}{b-a} d\beta$$

$$= \frac{u^{\alpha t-1}}{(b-a)\Gamma(\alpha t)} \int_a^b \exp(-\beta u)\, \beta^{\alpha t} d\beta. \quad (6)$$

With that, we obtain the expectation of the process as:

$$E[X_h(t)] = \frac{1}{b-a}\int_a^b \frac{\alpha t}{\beta} d\beta = \frac{\alpha t}{b-a}\log\left(\frac{b}{a}\right). \quad (7)$$

We can see that the expectation is a linear function with respect to the time *t*.

And finally, the variance of the process considering random effects on the scale parameter is the following expression:

$$Var\big(X_h(t)\big) = \frac{\alpha t+(\alpha t)^2}{ab} - \left(\frac{\alpha t}{b-a}\log(b/a)\right)^2. \quad (8)$$

The study of the monotony of the ratio between the variance and the expectation of the homogeneous gamma process with random effects shows that this ratio is increasing with respect to time *t*.

$$\frac{d}{dt}\frac{Var\big(X_h(t)\big)}{E\big(X_h(t)\big)} = \frac{\alpha(t-1)}{\alpha t\log(t)}\left(1 - \frac{t\log(t)^2}{(t-1)^2}\right) \geq 0. \quad (9)$$

3. Preventive maintenance policy

The deterioration levels of the different processes cannot be directly observed, so that the system is periodically monitored. These inspections check the system state and reveal its exact degradation.

We assume a preventive maintenance policy in this work. For this purpose, two degradation thresholds are considered: the preventive threshold (called M) and the corrective threshold (denoted by L), and the inequality $M < L$ holds. If the degradation level of a process exceeds the corrective threshold, this means that the system is highly degraded and can no longer perform its function. This failure results in corrective maintenance. However, if only the value of the preventive threshold is exceeded, the system is deteriorated but can continue to perform its function. Placing two different thresholds is very useful for cost optimization and to avoid unnecessary replacements. Preventive and corrective replacements both consists of the total replacements of the system and the system is said to be failed if one of the degradation processes reaches the corrective or failure threshold.

The main assumptions and important features of the model are explained below.

3.1. Assumptions of model

- The system is subject to continuous internal degradation initiated at random times by several processes.
- The failures arrive at the system following a shot-noise Cox process and they grow according to a homogeneous gamma process.
- Failures of the system are not self-announcing, that is, they are only detected at inspection times.
- The system is inspected periodically each T time units to check its deterioration state. The following actions are performed:
  1) If the degradation levels of the processes do not exceed the preventive threshold, the system is in a good condition, and it is left as it is.
  2) If the deterioration level of one of the multiple degradation processes exceeds the preventive maintenance threshold but not the corrective one, a preventive replacement is performed, which consists of the replacement of the system by a completely new one.
  3) If the system is failed at an inspection time, a corrective replacement is performed, and the system is also replaced by a new one.

A realization of the deterioration process is shown in Figure 1. The times $S_1,\ S_2,\ S_3$ represent the arrivals of new events. These three processes grow according to a homogeneous gamma process. The variability between them is due to different shape parameters. Preventive and corrective thresholds are shown, and also the first time at which the processes exceed them, which is given by (3).

- The time duration of the replacements is considered negligible.
- A sequence of costs associated with the maintenance tasks is assumed:
  1) cost due to corrective replacements: C_c monetary units,

2) cost due to preventive replacements: C_p monetary units,
3) cost of inspection times: C_I monetary units,
4) cost due to downtimes of the system: C_d monetary units per unit time.

- The cost due to corrective replacements is always higher than the cost due to preventive replacements, that is why is better to performed preventive maintenance on the system before it cannot work anymore.
- When the system has a corrective failure, in the time between the corrective maintenance action and the previous inspection time, the system is not working, which involves a cost.

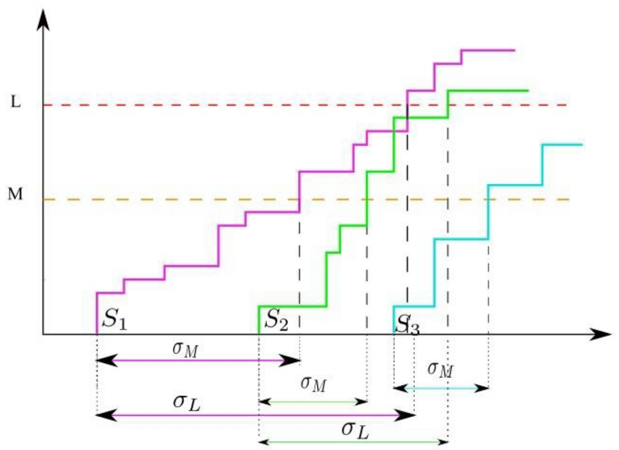

Figure 1. A realization of the deterioration process.

4. Optimization problem

The purpose of this work is to obtain the optimal expected cost of the system maintenance process, considering the shape and scale parameters of the gamma model and the associated costs. For this purpose, we will try to minimize the cost by optimizing two parameters: the preventive threshold and the time between inspections. In general, if the preventive threshold is lower, the system will be replaced more frequently; and if the inspections are more frequent in time, the system will not fail as much, since it will be repaired soon after the failure has occurred. By knowing these optimal parameters, an optimal maintenance cost can be obtained.

The renewal-reward theorem provides the following expression of the asymptotic cost rate, reducing the process to the first renewal:

$$C(T, M) = \frac{E[C]}{E[R]}$$

where $E[C]$ is the expected cost in a replacement cycle (that is, the time between two inspections) and $E[R]$ is the expected time to a replacement.

Then, the expected cost rate can be developed as:

$$C(T, M) = C_c \sum_{k=1}^{\infty} P_c(kT) + C_p \sum_{k=1}^{\infty} P_p(kT)$$

$$+ C_I E[N_I] + \sum_{k=0}^{\infty} C_d E_d(kT, (k+1)T) \qquad (10)$$

where $E[N_I]$ is the expected number of inspections, $E_d(kT, (k+1)T)$ is the expected number of downs of the system during the interval $(kT, (k+1)T)$ and $P_c(kT), P_p(kT)$ are the probabilities of performing a preventive or a corrective replacement at inspection time kT, respectively.

5. Numerical examples

Some numerical examples obtained by simulation of the stochastic processes involved in the model The expression for the objective cost function is difficult to manage it since it evolves infinity sums. A classical way to evaluate the objective cost function is to perform simulations on the points of the mesh and to find the optimal combination by visualization or using optimization metaheuristic methods such as genetic algorithms (Marseguerra & Zio, 2022).

The optimal maintenance strategy considering a model without heterogeneity and a model with heterogeneities is next analysed.

5.1. Model without heterogeneities

We assume that the system is working in a dynamic environment (Cha & Finkelstein, 2017) and is subject to external shocks that arrive to the system according to a homogeneous Poisson process with rate $\mu = 2$.

The stochastic intensity considered for this model has the following expression, as it appears in (Lemoine, 1986):

$$\lambda(t) = 1 + \sum_{i=1}^{N(t)} e^{-0.5(t-T_i)} \qquad (11)$$

with $N(t)$ being the number of Poisson processes in the system at time *t*. These Poisson processes underlying the main process are the ones that determines the arrival times T_i which allow to calculate the intensity $\lambda(t)$ of the main process.

The degradation processes grow according to a gamma process with shape parameter $\alpha = 1$ and scale parameter $\beta = 1.5$. The system fails when a degradation process exceeds the failure or corrective threshold $L = 10$.

The associated costs assumed in this numerical example are the following:

- $C_p = 150$ monetary units,
- $C_c = 300$ monetary units,
- $C_I = 50$ monetary units.
- $C_d = 75$ monetary units per unit time.

The optimal maintenance policy consists of finding the pair of values (T_{opt}, M_{opt}) that optimize the cost $C(T, M)$ given in (10), subject to the restriction $M \leq L$.

5.1.1. Optimization procedure

The procedure for optimizing the preventive threshold and the time between inspections, together with the minimization of the expected cost, is carried out as follows.

- A grid of size 10 is obtained by discretizing the interval [0, 25] into 10 equally spaced points for T. We denoted them T_i, $i = 1, \dots, 10$.
- Similarly, a grid of size 8 is obtained by discretizing the interval [0, 10] into 8 equal parts for M, denoted by M_j, $j = 1, \dots 8$.
- For each combination of the pair (T_i, M_j), $i = 1, \dots 10$, $j = 1, \dots, 8$, we have performed 10,000 simulations applying the Monte-Carlo method to obtain the expected cost rate.
- The optimal values for T and M are obtained by direct visualization. They correspond to the values which minimise the expected cost rate shown in (10).

By simulation, the optimal values obtained are $T_{opt} = 5.500$ and $M_{opt} = 6.333$, which implies an expected cost rate of 29.4070 monetary units per unit time.

In more complex systems with many components, the optimisation procedure can be very difficult. In these cases, metaheuristics are used, for instance, the genetic algorithm (GA). The algorithms used will depend on the case under study.

5.2. Model with heterogeneities

To develop the model with heterogeneities, we assume in this section that the scale parameter β is random. The stochastic intensity given by (11) defines the degradation rate of the process.

The shape parameter $\alpha = 1$ of the gamma process is the same as in the previous example. The scale parameter is random and follows a uniform distribution in the interval $(1, 2.5)$. The failure of the system occurs when the deterioration level reaches the corrective threshold $L = 10$. The sequence of costs given in the model without heterogeneities are imposed in this model.

To obtain the expected cost rate represented in Figure 2, we have performed 10,000 simulations on a grid of size 10 for variable time between inspection T in the interval $(0, 25)$ and another grid of size 8 for the preventive threshold M in the interval $(0, 10)$. By visual inspection, the optimal values are $T_{opt} = 8.53$ and $M_{opt} = 2.86$, with an expected cost rate of 24.8601 monetary units per unit time.

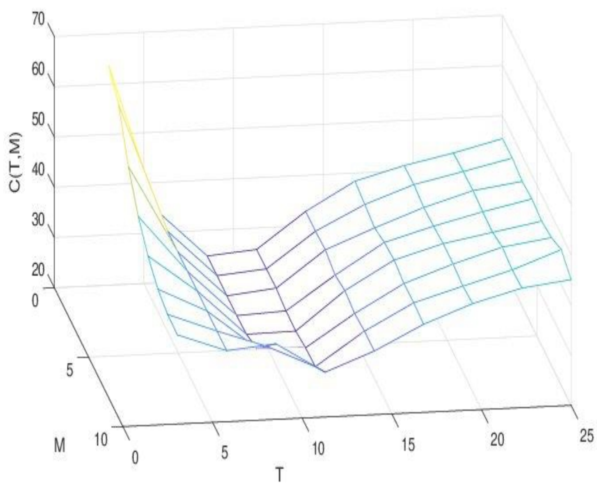

Figure 2. Expected cost rate of the model with heterogeneities.

5.2.1. Sensitivity analysis of model including heterogeneities

We focus now on the influence of the main model parameters on the expected cost rate through a sensitivity analysis of the gamma process parameters. Since α is fixed in each model and β varies in the interval $(1, b)$, different values for parameters α and b are studied.

The values for α and b are modified according to the following formulas

$$\alpha_{(r_i\%)} = \alpha \left[1 + \frac{r_i}{100}\right]$$

$$b_{(r_j\%)} = b \left[1 + \frac{r_j}{100}\right]$$

where $\alpha_{(r_i\%)}$ and $b_{(r_j\%)}$ represent the variation percentages of α and b.

Let us quickly look at the influence of the parameters in the model without heterogeneities. Studying computationally how the stochastic intensity and the variability of the parameters affect the proposed model through a sensitivity analysis, it can be observed and deduced that the shape and scale parameter have both effect on the optimal expected cost. More differences are found when the scale parameter varies. Similarly, both parameters have effect on the optimal value for T. However, any clear influence is shown in the case of the optimal value for the threshold M. The expected cost rate decreases as the shape and scale parameters increase.

On the other hand, we will analyse the case with heterogeneities, which turns out to be much more interesting, since the scale parameter influences the cost of the model to a greater extent.

The sensitivity analysis of the expected cost rate and the optimal values for the time between inspections T and preventive threshold M is summarized in Tables 1, 2 and 3, with α varying between 1 and 1.9, and b varying between 1 and 1.75 in increments of 0.15 units.

The optimal expected cost rate shown in Table 1 increases as the values of α and b increase, which implies that the degradation of the system is accelerated. In this numerical example, the random choice of parameter β as a uniform distribution on $(0, b)$ has little influence on the resulting expected cost.

Table 1. Sensitivity cost analysis on the values of α and b for the expected cost rate in a model with heterogeneities

$\alpha \setminus b$	1.15	1.3	1.45	1.6	1.75
1	20.54	20.67	24.94	20.43	24.31
1.15	24.83	22.99	27.32	22.28	26.56
1.3	26.85	24.78	32.32	24.26	28.75
1.45	29.59	26.95	33.59	25.84	29.44
1.6	31.37	30.83	37.33	28.37	34.09
1.75	33.09	32.55	40.22	31.76	35.94
1.9	34.91	32.50	41.10	33.15	38.11

On the other hand, Table 2 shows that the optimal value of the time between inspections T decreases as the values of α and b increase. More deterioration implies more frequent inspections, so it makes sense that the periodic inspection time would be decreased as the degradation parameters of the gamma process increase. However, the value of b has almost no influence on the model. Finally, no trend seems to be found when varying parameters α and b for the optimal values of the preventive threshold M represented in Table 3.

Table 2. Sensitivity cost analysis on the values of α and b for the optimal value of T in a model with heterogeneities

$\alpha \setminus b$	1.15	1.3	1.45	1.6	1.75
1	8.40	8.07	6.75	8.24	7.28
1.15	6.84	7.26	6.59	7.67	6.84
1.3	6.77	6.98	5.42	6.91	6.31
1.45	5.67	6.20	5.01	6.18	5.66
1.6	5.29	5.71	4.78	5.79	5.01
1.75	5.01	5.07	4.41	5.10	4.91
1.9	4.63	5.25	4.46	5.09	5.39

Table 3. Sensitivity cost analysis on the values of α and b for the optimal values of M in a model with heterogeneities

$\alpha \setminus b$	1.15	1.3	1.45	1.6	1.75
1	3.86	3.59	1.60	3.38	2.01
1.15	2.88	1.73	2.07	1.08	2.00
1.3	1.58	1.53	2.27	1.52	2.97
1.45	1.82	1.66	2.15	1.97	2.22
1.6	2.09	1.63	1.93	3.36	2.90
1.75	2.65	1.32	2.81	1.77	1.29
1.9	1.58	2.87	2.82	2.72	2.85

In general, the decision to implement a model with heterogeneities or a model without them depends largely on the parameters chosen for the model. In our model, with the chosen parameters, it is observed that a model with heterogeneities is much better. The cost rate of such a model is 24.8601 monetary units per unit time, while in the initial model, without heterogeneities, it is 29.4070 monetary units per unit time. Similarly, in the model with heterogeneities, the optimal inspection parameter T increases with respect to the model without heterogeneities, and the preventive threshold M decreases, resulting in the system being replaced before it is too deteriorated. Through the sensitivity analysis it has been possible to determine which parameters influence the model and to what percentage they do so.

6. Conclusion

We have studied a combined model of initiating and effective events in this chapter. It is well known that, if the initiating events (degradation processes) arrive to the system according to a Cox process and the degradation processes evolve following gamma processes, the resulting combined model follows again a Cox process (Bautista et al., 2021).

Without any maintenance, the failure time is increasing failure rate, hence a preventive maintenance policy is potentially worth implementing to improve the system reliability.

A maintenance strategy using random effects in the gamma distribution is presented. Although these heterogeneities are presented in practice, there is not enough research on that. Studying the effects of the parameters could be worthy to improve the maintenance and reliability of the system. We study the common gamma distribution, which is appropriate for modelling continuous and non-decreasing deterioration. The shape and scale parameters can be determined based on the failure data of the system. However, we have considered that the scale parameter is random, following a uniform distribution.

Random coefficients often implies that maintenance is performed more frequently, and failures are less probably to occur. The analysis is completed using a random effects model and performing several sensitivity analyses.

Although in this chapter we have assume that the initiation times follow a shot-noise Cox process, the result can be extended considering a different Cox process such as a Weibull renewal process.

The developed approach assumes that the degradation processes start following a shot-noise pattern. Further works can consider that degradation processes start according to a non-Cox process such as a Hawkes process (Chevalier, 2017; Cui et al., 2020). A Hawkes process is a point process whose main characteristic is that it is a self-exciting process meaning that each arrival of a degradation process increases the rate of future arrivals for some period of time (Grandell, 1976). Another crucial assumption of our work is that the degradation processes evolve independently and according to the same degradation pattern. It could be an important limitation, because in real systems, components are not independent, and we can establish dependencies between them or with external processes. For future research, different degradation patters for the degradation processes and dependence between the processes can be considered, as well as new structures of the systems (series, parallel or *k*-out-of-*n* structure). Another interesting research would be to study the model with uncertainty in the scale and shape parameter, that is, without knowing the probability distribution and using a Bayesian approach.

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