Advances in Science and Technology Research Journal

Advances in Science and Technology Research Journal, 18(6), 1–12
https://doi.org/10.12913/22998624/191044
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Received: 2024.07.06
Accepted: 2024.08.15
Published: 2024.09.08

New Approaches to Generalized Logistic Equation with Bifurcation Graph Generation Tool

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ABSTRACT

This paper proposed two new generalizations of the logistic function, each drawing on non-extensive thermodynamics, the q-logistic Equation and the logistic Equation of arbitrary order, respectively. It demonstrated the impact of chaos theory by integrating it with logistics Equations and revealed how minor parameter variations will change system behavior from deterministic to non-deterministic behavior. Moreover, this work presented BifDraw – a Python program for drawing bifurcation diagrams using classical logistic function and its generalizations illustrating the diversity of the system's response to the changes in the conditions. The research gave a pivotal role to the place of the logistic Equation in chaos theory by looking at its complicated dynamics and offering new generalizations that may be new in terms of thermodynamic basic states and entropy. Also, the paper investigated dynamics nature of the Equations and bifurcation diagrams in it which present complexity and the surprising dynamic systems features. The development of the BifDraw tool exemplifies the practical application of theoretical concepts, facilitating further exploration and understanding of logistic Equations within chaos theory. This study not only deepens the comprehension of logistic Equations and chaos theory, but also introduces practical tools for visualizing and analyzing their behaviors.

Keywords: generalization, chaos theory, bifurcation, logistic Equation, non-extensive thermodynamics.

INTRODUCTION

The concept of chaos has been recognized since ancient times. Ancient civilizations, such as the Babylonians [1], Chinese [2], Greeks [3], and Jewish [4] tribes, shared a common vision of the world's genesis – chaos transforming into order. Historically, order was associated with good, while disorder was seen as evil [5]. By the 18th century, significant strides in uncovering nature's laws led to a belief that almost everything about the universe was understood [6]. This era, influenced by Newton's laws, viewed the world as a clockwork mechanism [7].

However, the 19th century brought challenges to this deterministic view [8]. Henri Poincaré, while studying the stability of planetary motion in the Solar System, discovered signs of chaos [9]. Nearly a century after Poincaré, a work by Edward Lorenz laid the foundation for chaos theory [10]. Although the concept of deterministic chaos emerged in the 1960s [11], it was not until the proliferation of computers and their computational power that the theory gained widespread acceptance and interest [12]. This theory offers a novel and effective approach to scientifically describe complex physical and social systems that were previously deemed too intricate [13]. Almost all empirical sciences, from astronomy to sociology, have encountered chaos theory [14].

A notable example of understanding chaos is the analysis of the logistic Equation and its representation using a bifurcation diagram. This simple Equation exhibits remarkable properties, where minor parameter changes can shift its behavior from deterministic to random. The primary objective of this publication was to present the concept of the logistic Equation, its significance in chaos theory, and propose two potential generalizations, including one based on non-extensive thermodynamics. Furthermore, an application that generates bifurcation diagrams was introduced.

Over many years, a lot of papers and books have been published and interesting visualizations have been done in the area of logistic Equation. Its simplicity is a confirmation that in many cases simple systems can lead to very complex behaviors. Also, it has been proven that many systems can behave similarly to logistic Equation, especially creating different bifurcation diagrams. In this paper, two new proposals are presented, but this time they refer to non-extensive entropy and the concept of fractional generalizations. To the best knowledge of authors, such approaches have not been done so far and it is believed that can expand the point of logistic Equation perception.

LOGISTIC EQUATION

The logistic Equation is often associated with deterministic chaos due to its significant intersection with experimental chaos theory [15]. This chapter will cover the origins of the logistic Equation, its role in chaos theory, and two generalization ideas. The concept of bifurcation and a sample bifurcation plot of the logistic mapping will be discussed.

Origin of the logistic Equation

In the 1970s, Australian scholar Robert May, working at the University of Oxford, published a breakthrough paper [16]. Although not initially about chaos theory, the analysis of the logistic Equation made it one of the most cited texts in theoretical ecology [17]. May's work focused on the theoretical aspects of population dynamics. The simplest mathematical model for such ecosystems, where the lifespan of one generation lasts one season, is the linear model, often referred to as the Malthus [18] model:

$$N_{i+1} = \alpha N_i \tag{1}$$

When analyzing Equation (1), three population development scenarios are possible. In nature, however, unlimited growth never occurs due to environmental constraints [8]. The Equation had to be adjusted to account for resource limits. Belgian mathematician Pierre François Verhulst proposed a model in 1838 [19] that considered food access limitations:

$$\alpha = \alpha(N_i) \tag{2}$$

$$\alpha = \alpha\left(1 - \frac{N_i}{K}\right) \tag{3}$$

where: K stands for the amount of available resources or habitat capacity limit.

Combining models (1) and (3) results in:

$$N_{i+1} = \alpha\left(1 - \frac{N_i}{K}\right) N_i \tag{4}$$

This can be further transformed into the logistic Equation:

$$x_{i+1} = \alpha x_i (1 - x_i) \tag{5}$$

Despite its simplicity, May [16] demonstrated that the Equation (5) has complex dynamics.

The logistic Equation in chaos theory

At first glance, the logistic Equation seems simple and predictable. However, for certain values of α, the model exhibits periodic and even chaotic seasonal population changes, highly sensitive to initial values x_0. For instance, with $\alpha = 4$, the behavior of the logistic Equation 5 appears random and is extremely sensitive to initial conditions, as illustrated in Figures 1, 2, 3 and 4 which show changes in linear plots. The logistic Equation showcases unique mathematical properties typical of deterministic chaos. By slightly modifying the α parameter, the nature of the Equation changes from deterministic to random, highly sensitive to initial conditions. This is why May's paper [16] gained popularity. The presented form of the logistic Equation (5) is its most known, but it can undergo certain modifications.

GENERALIZATION OF THE LOGISTIC EQUATION

Various approaches exist to generalize the logistic mapping. This chapter presents two such ideas. The first is based on the generalized q-logistic Equation proposed in [20], and the second is based on the logistic model of any order presented in [21].

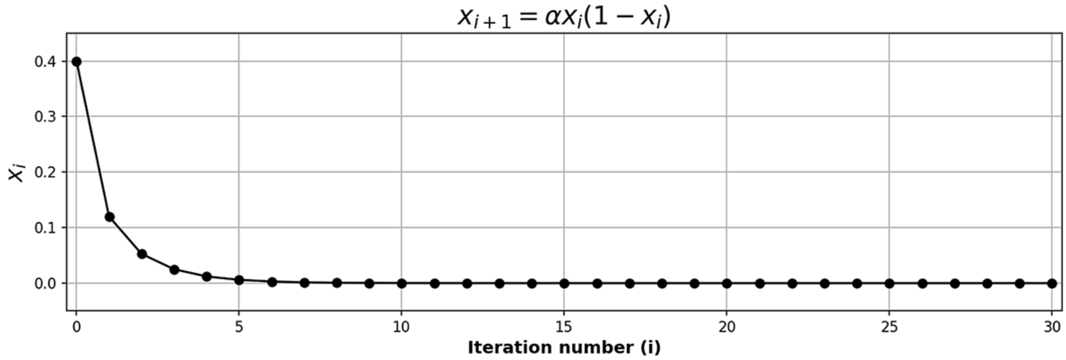

Figure 1. Graph of the value of the logistic equation for $x_0 = 0.4$ and $\alpha = 0.5$

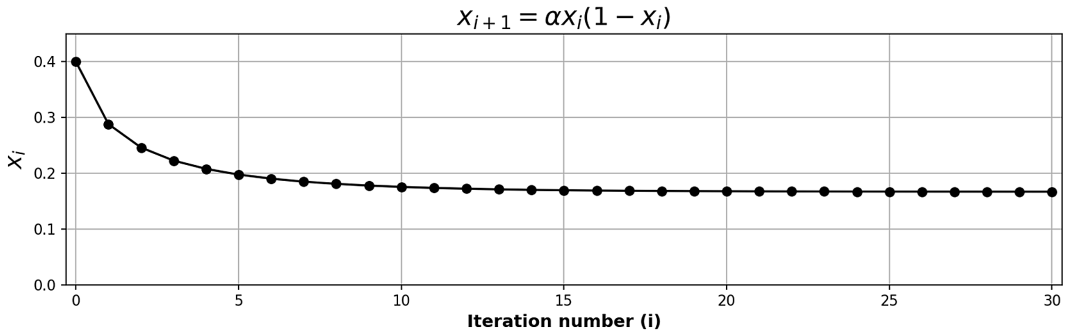

Figure 2. Graph of logistic equation values for $x_0 = 0.4$ and $\alpha = 1.2$

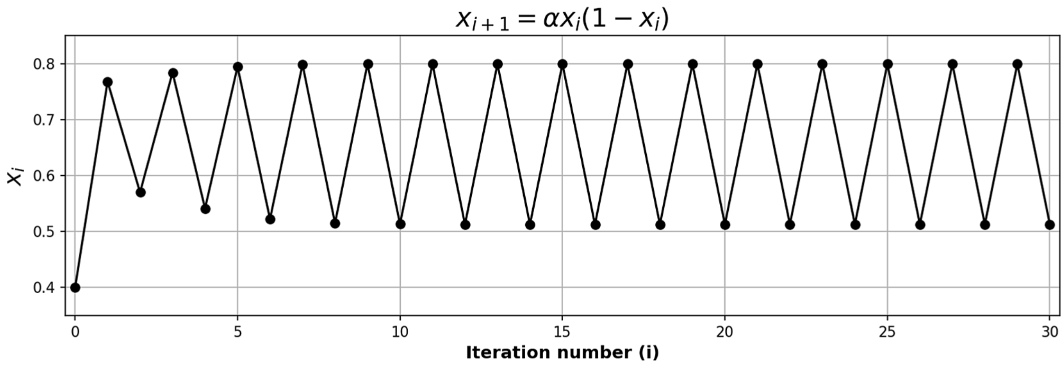

Figure 3. Graph of the value of the logistic equation for $x_0 = 0.4$ and $\alpha = 3.2$

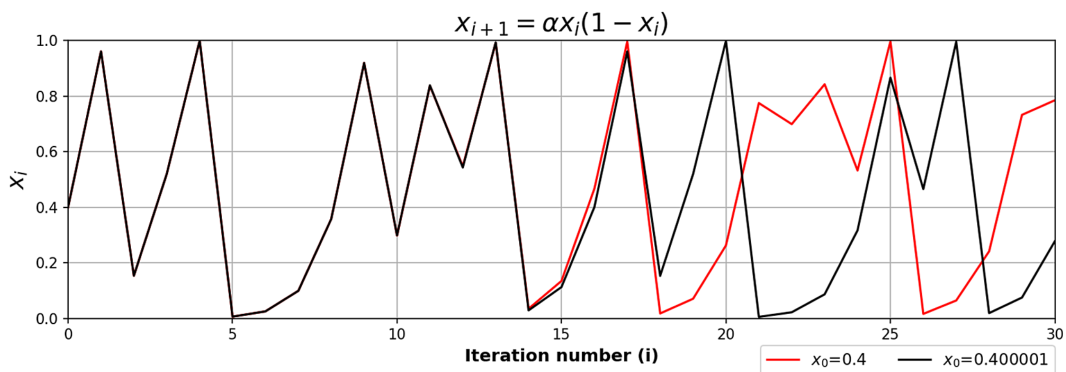

Figure 4. Graph of the value of the logistic equation for two starting values $x_0 = 0.4$ and $x_0 = 0.400001$ with dynamics $\alpha = 4$

Generalized q-logistic Equation

A crucial concept related to chaos theory is entropy, a fundamental thermodynamics concept. It quantitatively characterizes the irreversibility of physical and chemical processes. Rudolf Clausius, a German physicist, laid the foundation for entropy theory in 1850 with his statement: "It is impossible to transfer heat from a body at a lower temperature to one at a higher temperature without introducing other changes in both bodies and their surroundings" [22]. This discovery quickly gained popularity in the scientific world.

In systems far from equilibrium, classical (extensive) thermodynamics can fail. Constantino Tsallis proposed an extension in 1988, based on Equation (6), which was presented as a way to construct generalized thermodynamic foundations of statistical physics [23].

$$S_q = \frac{1-\sum_i p_i^q}{q-1} \tag{6}$$

In the simplest population model, the growth rate w of any organism population X is somewhat dependent on X itself. Given that X_i is the current population size at time i = 1, 2, 3,… and ignoring migration effects, w is:

$$w = \frac{X_{i+1}-X_i}{X_i} \tag{7}$$

where: w can also be treated as the difference between birth rate b and mortality rate d. Assuming w = const. and transforming formula (7), we get:

$$X_{i+1} = X_i + wX_i = (1+w)X_i. \tag{8}$$

From this, it is easy to infer that with $w > 0$, the population size X will tend towards infinity over time. Further transforming Equation (8) gives:

$$X_{i+1} - X_i = wX_i \tag{9}$$

Considering the Verhulst model, there is an expected maximum population size N that a given habitat can accommodate. If the population size at a given time X_i is less than N, further growth is expected. However, if $X_i > N$, the population should decrease. This leads to the Equation:

$$b_0 - k_b X_i = d_0 + k_d X_i \tag{10}$$

Transformation of formula (10) gives:

$$X_i = \frac{b_0 - d_0}{k_b + k_d} \tag{11}$$

Further transformations yield:

$$X_{i+1} - X_i = wX_i = [(b_0 - k_b X_i) - (d_0 + k_d X_i)]X_i = [b_0 - d_0 - (k_b + k_d)X_i] = \left(w - \frac{wX_i}{N}\right)X_i = \\ = wX_i\left(1 - \frac{X_i}{N}\right) \tag{12}$$

Assuming $X_i/N = x_i$ and $x_i \in \langle 0, 1\rangle$, Equation (12) becomes:

$$x_{i+1} - x_i = wx_i(1 - x_i) \tag{13}$$

To obtain the continuous form of the logistic Equation, let us transform formula (13) to:

$$\frac{dx(t)}{dt} = wx(t)[1 - x(t)] \tag{14}$$

Solving formula (14) gives:

$$x(t) = \frac{1}{1+\left(\frac{1}{x(0)}-1\right)e^{-wT}} \tag{15}$$

There's also a simplified version:

$$x(t) = \frac{1}{1+e^{-wt}} \tag{16}$$

Considering Tsallis's definition of the q-exponential function [24], Equation (16) can be generalized. Replacing the exponential function with the q-exponential function gives:

$$x_{(q)}(t) = \frac{1}{1+e_q^{-w_q t}} \tag{17}$$

Further transformations yield [20]:

$$x_{(q)^{i+1}} = w_q x^{q}_{(q)^i}\left(1 - x_{(q)^i}\right)^{-q+2} \quad (18)$$

Figures 5 and 6 illustrate the behavior of the generalized q-logistic Equation for various values of w. Figure 7 and 8 show the behavior for different values of q.
In conclusion, Tsallis's q-generalization offers a new approach to well-known models like the Verhulst model with the logistic Equation. The generalized q-logistic Equation, despite its relative simplicity, can lead to interesting results.

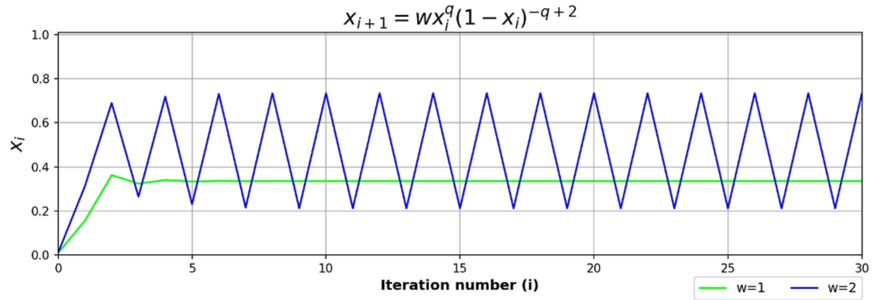

Figure 5. Generalized q-logistic equation for two example values of *w*

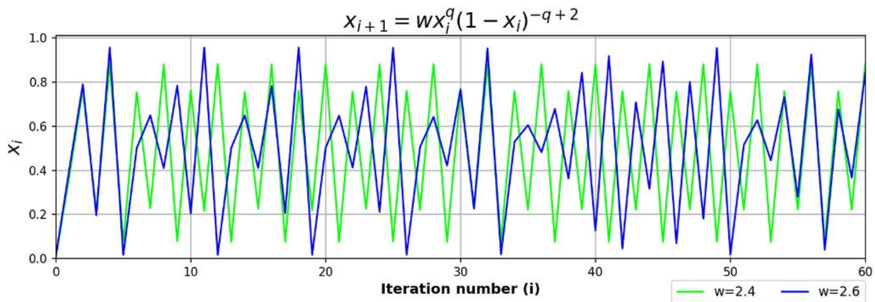

Figure 6. Generalized q-logistic equation for two example values of *w*

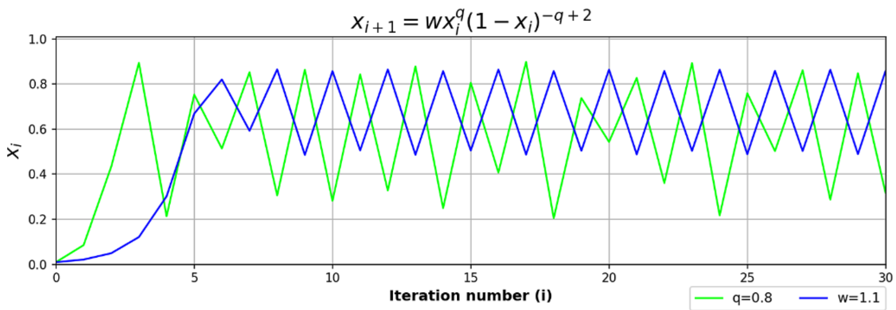

Figure 7. Generalized q-logistic equation for two example values of *q*

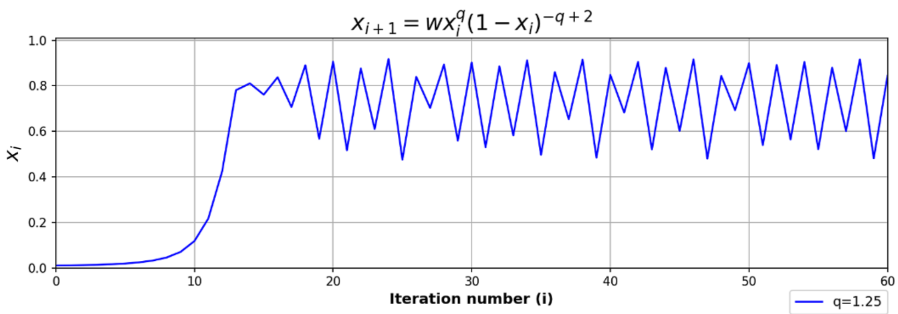

Figure 8. Generalized q-logistic equation for q = 1.25 and w = 3.45

Generalized logistic Equation of any order

Natural and artificial systems, such as humans, animals, plants, markets, and wireless computer networks, can exist in one of two distinct thermodynamic states based on internal and external conditions: sub-extensive or super-extensive [25]. The former represents reversible self-organization [26–32]. However, when such a system reaches a certain percolation threshold [33], it inevitably tends towards irreversible super-extensive self-organization [34–45].

Self-organization is defined as a system that initially consists of independent parts that, due to various processes, become a unified whole based on different types of connections [46–48].
Physical phenomena of self-organization in different systems are tied to a specific temporal-spatial context. Space is associated with system resources, ideally unlimited but limited in reality. Time is related to the processes occurring within the system. In an ideal scenario, these processes are homogeneous and stable, without self-organizing effects. However, in reality, processes are often heterogeneous and unstable, depending on the sensitivity to the initial conditions of the processes in the system [21].

When analyzing the given context, the Malthusian Equation refers to a simple system with unlimited resources and homogeneous processes, which is far from reality. The logistic Equation is slightly better, considering limited resources but still referring to homogeneous representatives of a species [18]. However, in reality, representatives of a species can differ to some extent. There is a lack of a general Equation of any order that would describe a system with limited resources governed by heterogeneous self-organizing processes. A solution to this problem has been proposed as a generalization of the logistic Equation formula with limited external system resources and internal heterogeneous processes.

The logistic Equation was created in response to the imperfections of the Malthus model, based on Verhulst's concept formula (4). Considering Equation (4), the logistic Equation refers only to homogeneous and sub-extensive processes in the system. However, in reality, every system can have both sub-extensive and super-extensive features with homogeneous and heterogeneous modes of self-organization. The traditional logistic Equation shows only one path, from a zero population state ($N = 0$) to reaching the habitat capacity limit ($N = K$). In general, a system can follow different self-organization paths, depending on both external (macroscopic) and internal (microscopic) conditions. Due to this, the logistic Equation is criticized for its lack of universality by some scientists [49–52].

Considering the taxonomy in the context, Equation (4), and other factors, the logistic Equation (5) can be transformed to match the Malthus model – formula (1):

$$x_{i+1} = \alpha x_i (1 - x_i)^0 \tag{19}$$

Similarly, the same Equation can be rewritten in relation to the Verhulst model, resulting in the classic logistic Equation:

$$x_{i+1} = \alpha x_i (1 - x_i)^1 \tag{20}$$

From the context, a complex system model will be complete if the processes occurring in the system are of any order, meaning they can occur in both sub-extensive and super-extensive forms. A solution to this problem was presented as a generalized logistic Equation of any order, referring to the entire family of systems:

$$x_{i+1} = \alpha x_i (1 - x_i)^f \tag{21}$$

The parameter f in Equation (21) describes the self-organization of homogeneous and heterogeneous processes and refers to the local (microscopic) part of the system and can be of any order, where $-\infty < f < +\infty$.

Comparing the generalized logistic Equation while maintaining the classic logistic mapping, a graph was presented in Figure 9.

From Figure 9, the behavior of the generalized logistic Equation of any order is identical to the classic mapping. This state remains regardless of the value of f. In another test, the value of α is increased to 1.2, as shown in Figure 10 where the behavior of the function is again similar to the classic Equation. However, a relationship can be observed: the function monotonicity is inversely proportional to the value of f.

However, the similarities between the Equations end here. The last meaningful result was obtained for $\alpha = 1.7$, as shown in Figure 11 where the behavior is atypical for the classic logistic Equation. For values of $f \in (0.3, 1)$, the function behaves as before. However, unusual things happen at $f \in \langle 0.2, 0.3 \rangle$.

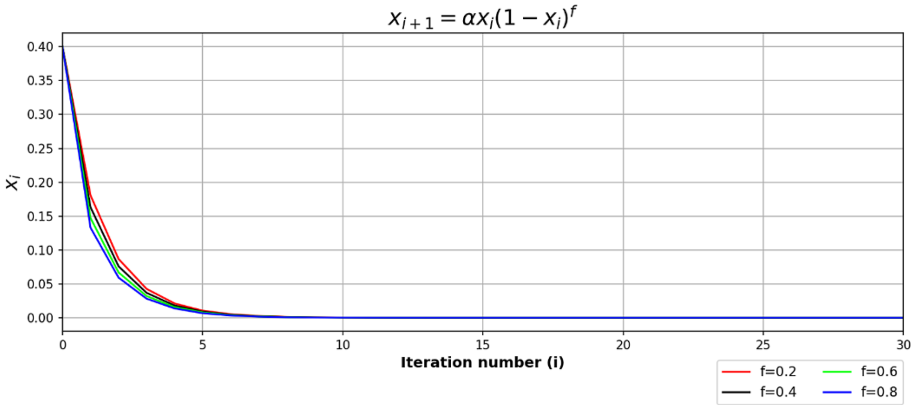

Figure 9. Value graph of logistic equation of any order for $x_0 = 0.4$ and $\alpha = 0.5$

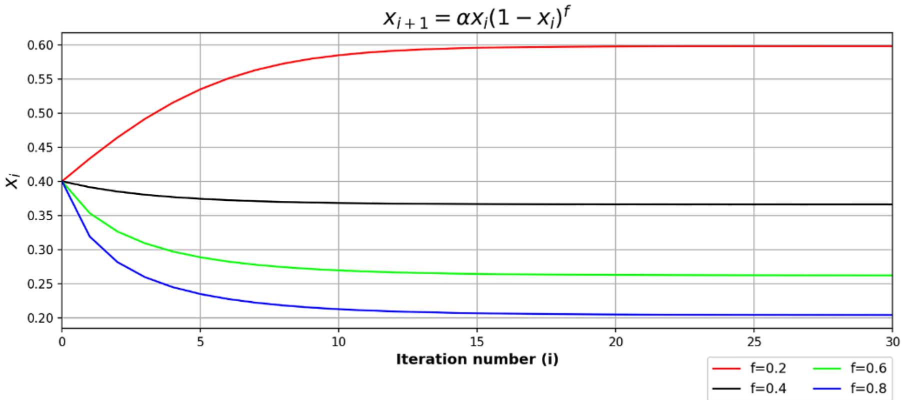

Figure 10. Value graph of logistic equation of any order for $x_0 = 0.4$ and $\alpha = 1.2$

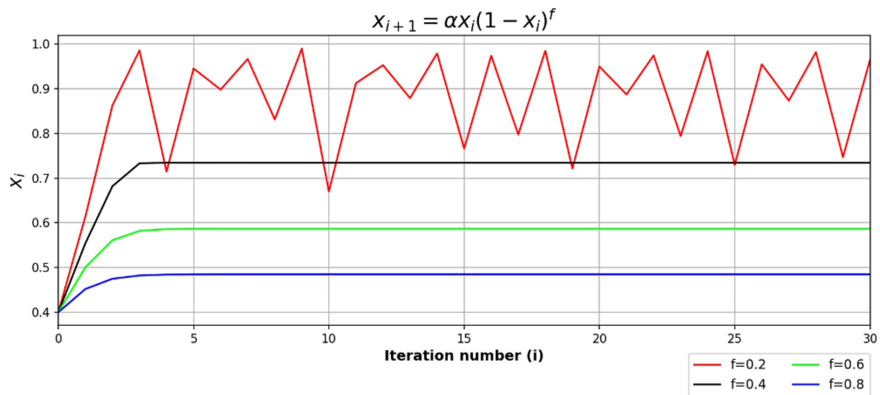

Figure 11. Value graph of logistic equation of any order for $x_0 = 0.4$ and $\alpha = 1.7$

BIFURACTION DIAGRAM

As mentioned before, the classic logistic Equation has a unique property of changing its nature depending on the value of the parameter α. For a parameter $\alpha < 3$, the population tends towards a specific number. However, for a value in the range (3, 4), such as $\alpha = 3.2$, the population does not reach one value but two, alternating every other season. Certain conclusions can be drawn from the analysis of this phenomenon. If the sequence of subsequent values x_i approaches a limit, the condition for the limit value x_g can be determined. This is the so-called fixed point condition of the function: $x_g = f(x_g)$. This relationship can be used to determine the fixed points of the logistic mapping depending on the parameter α. After appropriate calculations, two solutions emerge: $x_g = 0$ and $x_g = 1 - \frac{1}{\alpha}$. Figure 12 shows the graph of the fixed points of logistic mapping depending on the parameter α.

When analyzing Figure 12, it is essential to note that while $x_g = 0$ is a fixed point for any value of the parameter α, the meaningful values of the logistic Equation are in the range $x_i \in (0.1)$. Thus, the second fixed point exists for values $\alpha \in (1.4)$. When the Equation depends on a parameter and the number of solutions changes with this parameter, this phenomenon is called bifurcation. A bifurcation point is $\alpha = 1$, where the system starts to have two solutions instead of one. Regardless of the initial condition, for $\alpha < 1$, the population always tends towards extinction. Such a point to which the system is attracted fulfills the definition of an attractor. If $\alpha > 1$, then two fixed points exist. Starting in this area from any starting point (except $x_0 = 0$), the logistic Equation will always tend towards the second solution x_g, its attractor. This means that if the solution $x = 0$ is disturbed by any small number, e.g., $x = 0.00001$, the population will inevitably start to tend towards $x_g = 1 - \frac{1}{\alpha}$. In other words, the stable point at $\alpha < 1$ becomes unstable at $\alpha > 1$.

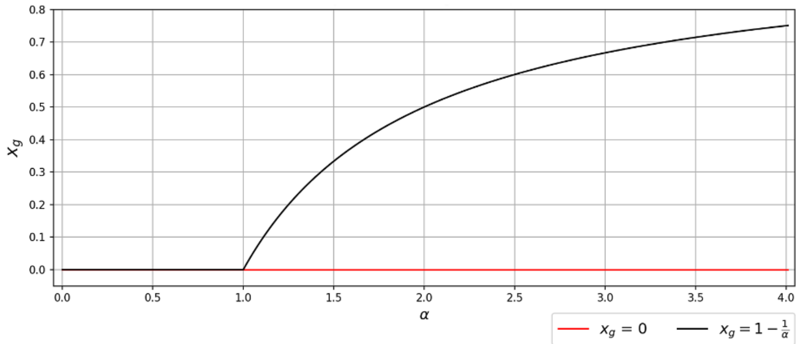

Figure 12. Graph of logistic mapping fixed points depending on the parameter α

According to the above assumptions, at $\alpha = 3.2$, a fixed point should occur for the Equation: $x_g = 1 - \frac{1}{\alpha}$. This is indeed the case. Figure 13 shows the stability graph of the fixed points of logistic mapping for $\alpha = 3.2$ and the initial condition $x_0 = 1 - \frac{1}{\alpha}$. From the analysis of Figure 13, it appears that for $x_0 = 1 - \frac{1}{\alpha}$, there are stable fixed points for the logistic Equation. However, this situation changes even with a slight modification of the value x_0. As it can be seen in Figure 13, another bifurcation occurs, resulting in the solution $x_g = 1 - \frac{1}{\alpha}$ losing stability in favor of oscillations. Since they occur between two values, it can be assumed that at $\alpha = 3.2$, the system has a periodic point with a period of 2. Assuming that the population takes the same number every other season, the mapping $g(x) = f(f(x))$, which advances the system by two seasons, can be considered. Fixed points for this Equation can be calculated similarly as before. After composition, the following polynomial is obtained (22):

$$x = \alpha^2 x - \alpha^2 x^2 - \alpha^3 x^2 + 2\alpha^3 x^3 - \alpha^3 x^4 \tag{22}$$

After performing appropriate calculations on Equation (22), it turns out that four fixed point Equations can be obtained [53]:

$$x = \frac{\alpha - \sqrt{\alpha^2 - 2\alpha - 3} + 1}{2\alpha} \tag{23}$$

$$x = \frac{\alpha + \sqrt{\alpha^2 - 2\alpha - 3} + 1}{2\alpha} \tag{24}$$

$$x = \frac{\alpha - 1}{\alpha} \tag{25}$$

$$x = 0 \tag{26}$$

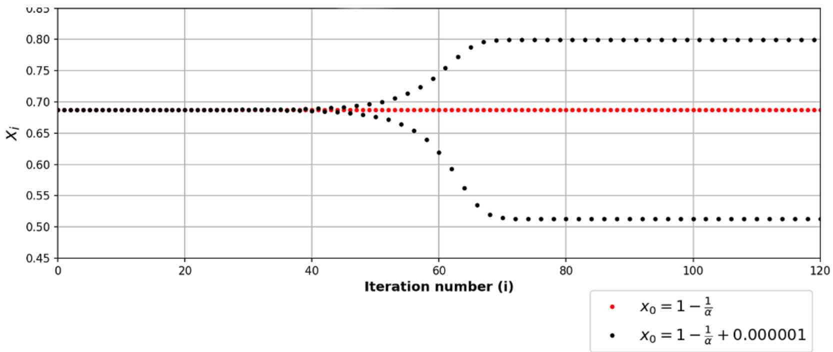

Figure 13. Stability graph of the fixed points of the logistic function

Figure 14 shows the graph of fixed points (23–26) of the logistic mapping depending on the parameter α. When analyzing it, the emergence of another bifurcation is confirmed, this time at α = 3. This is consistent with the results presented in Figure 13. Such a graph as in Figure 14 is called a bifurcation diagram. However, it is not complete in this case. Presumably, since two bifurcations have appeared, there is no reason to assume that there will not be more. Further analysis, however, poses significant obstacles. Analytical examination of fixed points of further compositions of the mapping $g(x) = f(f(f(x)))$ is impossible, as in the previous case, after composition, a fourth-degree polynomial is obtained. According to Galois' theory, analytical formulas for polynomial roots generally end at the fourth degree [54]. The analysis of chaotic systems seems extremely difficult. A solution to this problem might be the use of computer simulations. It can be assumed that the x-axis will be the values of the parameter α, while the y-axis – the stable fixed points of the logistic Equation. These points will be obtained by simulating a large number of iterations of the logistic Equation, from which several of the last results will be selected for each α. These values will then be plotted on the graph. In this way, a full-fledged, classic bifurcation diagram is obtained. Figure 15 shows such a graph.

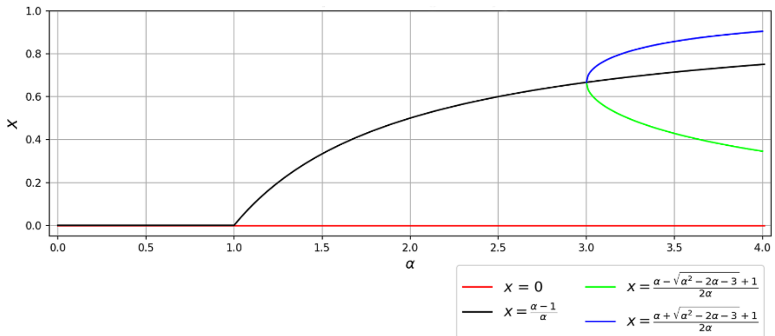

Figure 14. Graph of logistic mapping fixed points depending on the parameter α

When analyzing Figure 15, several conclusions can be drawn, e.g., for the parameter value $\alpha = 3.2$, there are 2 stable fixed points (every other season the population has the same number), for $\alpha = 3.5$, there are already 4 such points, for $\alpha = 3.56$: 8 points, etc. As the parameter α increases, all branches of the existing cycle split. This process continues until α reaches the limit value $\alpha = 3.5699456$ [54–57]. From this moment on, the cycle has an infinite number of elements, spread along the entire segment $[0, 1]$. In other words, the population's number changes unpredictably and never repeats. The classic bifurcation diagram thus confirms the relationships discussed in this chapter. Figure 16 presents a comparison of the fixed point graph (Fig. 14) and the bifurcation diagram (Fig. 15).

Using a relatively simple algorithm, the structure of the fixed points of the logistic mapping was explored. Sometimes there is just one point, and sometimes there are infinitely many. The bifurcation diagram is one of the most impressive tools to showcase the extraordinary, chaotic nature of the logistic Equation.

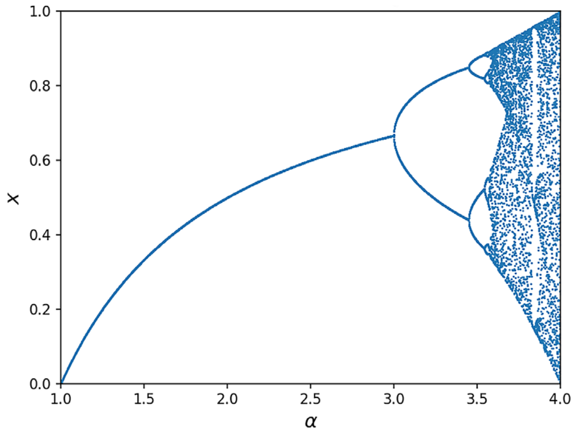

Figure 15. Classic bifurcation diagram

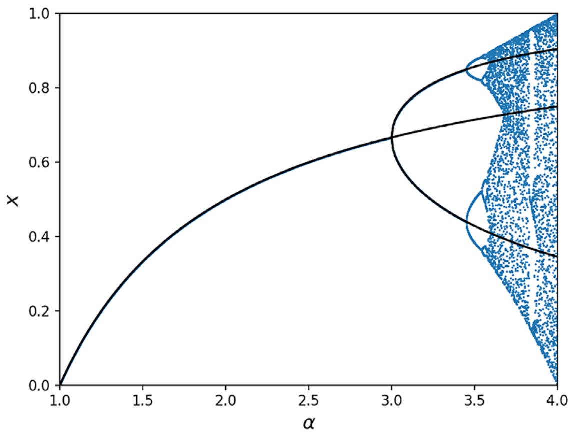

Figure 16. Composition of bifurcation diagram and fixed point diagram

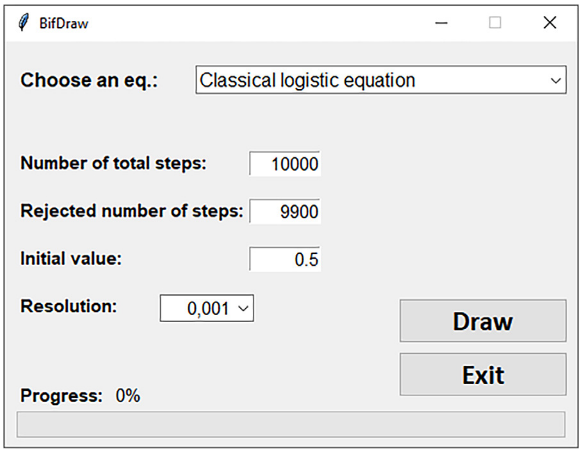

Figure 17. Interface of BifDraw application

Bifurcation graph generation tool

The section describes the BifDraw application, a tool written in Python designed to generate bifurcation diagrams for the classic logistic Equation and its q-generalization. The application was developed using the Microsoft Visual Studio Code environment and utilizes Python 3.7. Key libraries employed include NumPy, Matplotlib, and TkInter. The interface of the application is straightforward, allowing users to modify various parameters to influence the generated diagrams. Notably, at the time of publication, BifDraw is the only known program to the authors that can produce such charts. The application's functionality and its graphical user interface are illustrated in Figure 17. The complete documentation and source code are available on GitHub at [58].

CONCLUSIONS

This research aimed to elucidate the logistic Equation, emphasizing its significance within chaos theory. Initially, the logistic Equation, despite its seeming simplicity, was demonstrated to exhibit complex dynamics. The investigation uncovered two generalizations: a generalized q-logistic Equation and a logistic Equation of arbitrary order. The former ties into the concept of entropy and the q-exponential function, exhibiting unique characteristics. The latter, representing both sub-extensive and super-extensive thermodynamic states, has attracted criticism due to its perceived lack of universality. As main paper results, the following can be indicated:

- introduction of two innovative generalizations rooted in non-extensive thermodynamics and fractional generalizations,
- set of new bifurcation diagrams with graphical representation of the fixed points of logistic Equation as a function of the parameter α
- a practical tool, BifDraw, developed to generate bifurcation diagrams for both the traditional logistic Equation and its q-generalized version.

In conclusion, it is believed that this research offers a deeper understanding of the logistic Equation within chaos theory, shedding light on its behavior and generalizations. The development of BifDraw serves as a testament to the blend of theory and application, facilitating further exploration and comprehension of the subject.

REFERENCES

1. Eoyang, E. Chaos Misread: Or, There's Wonton in My Soup! Comparative Literature Studies, 1989, 271–284.
2. Lee, A.C. The Chinese Creation Myth of Nu Kua and the Biblical Narrative in Genesis 1–11. Biblical interpretation, 1994, 2(3), 312–324.
3. Höfele, A., Levin, C., Müller, R., Quiring, B. (Eds.). Chaos from the Ancient world to Early modernity: formations of the formless. Walter de Gruyter GmbH & Co KG. 2020.
4. Gunkel, H. Creation and chaos in the primeval era and the eschaton: A religio-historical study of Genesis 1 and Revelation 12. Wm. B. Eerdmans Publishing. 2006.
5. Roest, B. Order and disorder: the Poor Clares between foundation and reform, 2013, 8. Brill.
6. Prigogine, I., Stengers, I. The end of certainty. Simon and Schuster. 1997.
7. Smith, C. W. Empiricist devotions: Science, religion, and poetry in early eighteenth-century England. University of Virginia Press. 2016.
8. Hacking, I. Nineteenth century cracks in the concept of determinism. Journal of the History of Ideas, 1983, 44(3), 455–475.
9. Poincaré, H. Analysis situs, 1895, 1–121. Paris, France: Gauthier-Villars.
10. Tucker, W. The Lorenz attractor exists. Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, 1999, 328(12), 1197–202.
11. Dove, W.F. A Long Walk with René: Order out of Chaos via Avenue Lambda. VIROLOGY, 1964, 24, 241–253.
12. Schuster, H.G., Just, W. Deterministic chaos: an introduction. John Wiley & Sons. 2006.
13. Cilliers, P. Complexity and postmodernism: Understanding complex systems. routledge. 2002.
14. Mann, S.R. Chaos theory and strategic thought. Parameters, 1992, 22(3), 54–68.
15. Kumar, A. Chaos theory: impact on and applications in medicine. Journal of Health and Allied Sciences NU, 2012, 2(04), 93–99.
16. May, R.M. Stability and complexity in model ecosystems (Vol. 1). Princeton university press. 2019.
17. Fox, J. And the most-cited ecology papers from the 70s, 80s, and 90s are... (UPDATEDx6). Dynamic Ecology. 2013, June 15. https://dynamicecology.wordpress.com/ 2013/06/10/and-the-most-cited-ecology-papers-from-the-70s-80s-and-90s-are/
18. Malthus, T.R. An essay on the principle of population (1798). The Works of Thomas Robert Malthus, London, Pickering & Chatto Publishers, 1986, 1, 1–139.
19. Verhulst, P.F. Notice sur la loi que la population suit dans son accroissement. Correspondence mathematique et physique, 1838, 10, 113–129.
20. Strzałka, D., Grabowski, F. Towards possible q-generalizations of the Malthus and Verhulst growth models. Physica A: Statistical Mechanics and its Applications, 2008, 387(11), 2511–2518.
21. Grabowski, F. Logistic Equation of arbitrary order. Physica A: Statistical Mechanics and its Applications, 2010, 389(16), 3081–3093.
22. Clausius, R. Ueber die bewegende Kraft der Wärme und die Gesetze, welche sich daraus für die Wärmelehre selbst ableiten lassen. Annalen der Physik, 1850, 155(3), 368–397.
23. Tsallis, C. Possible generalization of Boltzmann-Gibbs statistics. Journal of statistical physics, 1988, 52, 479–487.
24. Yamano, T. Some properties of q-logarithm and q-exponential functions in Tsallis statistics. Physica A: Statistical Mechanics and its Applications, 2002, 305(3–4), 486–496.
25. Tsallis, C. What should a statistical mechanics satisfy to reflect nature?. Physica D: Nonlinear Phenomena, 2004, 193(1–4), 3–34.
26. Ashby, W.R. Principles of the self-organizing dynamic system. The Journal of general psychology, 1947, 37(2), 125–128.
27. Prehofer, C., Bettstetter, C. Self-organization in communication networks: principles and design paradigms. IEEE Communications magazine, 2005, 43(7), 78–85.
28. Dressler, F. A study of self-organization mechanisms in ad hoc and sensor networks. Computer Communications, 2008, 31(13), 3018–3029.
29. Hinchey, M.G., Sterritt, R. Self-managing software. Computer, 2006, 39(2), 107–109.
30. Mamei, M., Menezes, R., Tolksdorf, R., Zambonelli, F. Case studies for self-organization in computer science. Journal of Systems Architecture, 2006, 52(8–9), 443–460.
31. Mills, K.L. A brief survey of self-organization in wireless sensor networks. Wireless Communications and Mobile Computing, 2007, 7(7), 823–834.
32. Bantz, D.F., Bisdikian, C., Challener, D., Karidis, J.P., Mastrianni, S., Mohindra, A., Shea D.G., Vanover, M. Autonomic personal computing. IBM Systems Journal, 2003, 42(1), 165–176.
33. Alencar, A.M., Andrade, J.S., Lucena, L.S. Self-organized percolation. Physical Review E, 1997, 56(3), R2379.
34. Bak, P., Tang, C., Wiesenfeld, K. Self-organized criticality: An explanation of the 1/f noise. Physical review letters, 1987, 59(4), 381.
35. Bak, P., Tang, C., Wiesenfeld, K. Self-organized criticality. Physical review A, 1988, 38(1), 364.
36. Bak, P. How nature works: the science of self-organized criticality. Springer Science & Business

Media. 2013.

37. Abe, S., Suzuki, N. Statistical similarities between internetquakes and earthquakes. Physica D: Nonlinear Phenomena, 2004, 193(1–4), 310–314.
38. Polyakov, A.A. Process of self-organization in metal. Materials Science, 1993, 29(2), 121–127.
39. Abe, S., Suzuki, N. Omori's law in the Internet traffic. Europhysics Letters, 2003, 61(6), 852.
40. Jørgensen, S.E., Mejer, H., Nielsen, S.N. Ecosystem as self-organizing critical systems. Ecological Modelling, 1998, 111(2–3), 261–268.
41. Scheinkman, J.A., Woodford, M. Self-organized criticality and economic fluctuations. The American Economic Review, 1994, 84(2), 417–421.
42. Guerin, S., Kunkle, D. Emergence of constraint in self-organizing systems. Nonlinear Dynamics, Psychology, and Life Sciences, 2004, 8(2), 131–146.
43. Turcotte, D.L., Malamud, B.D., Guzzetti, F., Reichenbach, P. Self-organization, the cascade model, and natural hazards. Proceedings of the National Academy of Sciences, 99(1), 2002, 2530–2537.
44. Grabowski, F. Difference between the 1/f noise spectral density before and after stress as a measure of the submicron MOS transistors degradation. Microelectronics Reliability, 1995, 35(3), 511–528.
45. Sornette, D. Predictability of catastrophic events: Material rupture, earthquakes, turbulence, financial crashes, and human birth. Proceedings of the National Academy of Sciences, 2002, 99(1), 2522–2529.
46. Prigogine, I., Nicolis, G. Self-organization. Non-Equilibrium System. 1977.
47. Heylighen, F. The science of self-organization and adaptivity. The encyclopedia of life support systems, 2001, 5(3), 253–280.
48. Haken, H. Self-organization. Scholarpedia, 2008, 3(8), 1401.
49. Richerson, P.J., Boyd, R. Homage to Malthus, Ricardo, and Boserup: Toward a general theory of population, economic growth, environmental deterioration, wealth, and poverty. Human Ecology Review, 1997, 85–90.
50. Nelder, J.A. 182. note: An alternative form of a generalized logistic Equation. Biometrics, 1962, 18(4), 614–616.
51. Rosen, G. Characterizing conditions for generalized Verhulst logistic growth of a biological population. Bulletin of mathematical biology, 1984, 46, 963–965.
52. Gabriel, J.P., Saucy, F., Bersier, L.F. Paradoxes in the logistic Equation?. Ecological Modelling, 2005, 185(1), 147–151.
53. El-Sayed, A.M.A., El-Mesiry, A.E.M., El-Saka, H.A.A. On the fractional-order logistic Equation. Applied Mathematics Letters, 2007, 20(7), 817–823.
54. Funkhouser, H.G. A short account of the history of symmetric functions of roots of Equations. The American mathematical monthly, 1930, 37(7), 357–365.
55. Lee, W.K., Phan, R.C. W., Yap, W.S., Goi, B.M. Spring: a novel parallel chaos-based image encryption scheme. Nonlinear Dynamics, 2018, 92(2), 575–593.
56. Han, Z., Feng, W.X., Hui, L.Z., Da Hai, L., Chou, L.Y. A new image encryption algorithm based on chaos system. In IEEE International Conference on Robotics, Intelligent Systems and Signal Processing, 2003. Proceedings. 2003, 2, 778–782.
57. Crane, J. Chaos Theory and Military Applications. 2007.
58. github.com/zszprz/bifdraw