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A Discrete SIS Model of Epidemic for a Heterogeneous Population without Discretization of its Continuous Counterpart

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ABSTRACT

In this paper we propose a model of an infectious disease transmission in a heterogeneous population consisting of two different subpopulations: individuals with accordingly low and high susceptibility to an infection. This is a discrete model which was built without discretization of its continuous counterpart. It is not a typical approach. We assume that parameters describing particular processes in each subpopulation have different values. This assumption makes model analysis more complicated comparing to models without this assumption. We investigate conditions for existence and local stability of stationary states. The novelty of this paper lies in presenting the explicit condition concerning stationary states, including stability. We compute the basic reproduction number R_0 of the given system, which determines the local stability of the disease-free stationary state. Additionally, we consider a situation when there is no illness transmission in the subpopulation with the low susceptibility. Theoretical result are complemented with numerical simulations in which we fit the model to epidemic data from the Warmian-Masurian province of Poland. These data reflect the case of the tuberculosis epidemic for which the homeless people were treated as a group with the high susceptibility.

Keywords: SIS epidemic model, local stability, discrete modeling.

INTRODUCTION

In order to investigate the epidemic dynamics from the mathematical point, continuous models are commonly used. However, discrete models are also gaining attention. In most cases, these discrete systems arise from discretization of their continuous counterparts. We introduced and analyzed models of this type in [1, 2] and [3]. There are various methods of discretization, which are described and analyzed in the literature. In our previous papers we used the explicit Euler method and non–standard discretization methods, including the strictly positive scheme. In this paper we show an unusual concept of building a discrete model of epidemic dynamics without discretization of its continuous counterpart. This concept is presented in [4] and [5]. In the constructed model we will include demographic and epidemiological processes which were considered in models from our previous papers.

Here as a homogeneous population we understand a population in which we cannot distinguish individuals concerning the risk of being infected. A heterogeneous population is a population in which this distinction can be done. Here we assume that the heterogeneous population consists of two homogeneous subpopulations. The first subpopulation is formed by individuals with low susceptibility to an infection and the second subpopulation is composed of the people with high one. We will call these subpopulations the low subpopulation (LS) and high subpopulation (HS), respectively. The infection can be transmitted among each subpopulation separately and from HS to LS. We will reasonably assume that there is no transmission from LS to HS.

The proposed model is an example of SIS (susceptible-infected-susceptible) models – an infected individual after recovery does not gain immunity and can be infected again. In this paper we assume that every process for each subpopulation is independent. This assumption results in different values of coefficients describing specific processes in every subpopulation. According to our knowledge, there are no papers in which this case is analyzed in such a type of SIS models which is presented here, particularly in the context of presenting explicit results for stability analysis. The reason of neglecting it is obtaining complicated computations. In this paper transmission of the infection is described with a general form of functions which values can be interpreted as probability of remaining susceptible by an individual. These functions will be defined later.

Although discrete SIS models built with the discretization are not widely used, one find some examples of them in literature. Authors in [4] analyze two-dimension SIS model for a homogeneous population. This model was extended in [6], where environment seasonality is considered. To do so, the authors introduced a periodic function reflecting the birth and recruitment process. They emphasized formerly omitted an impact of demographic processes on population density. Martcheva in her book [5] applied the approach from [4] to a model for a heterogeneous population. However, she did not conduct stability analysis of stationary states. An interesting approach to discrete modelling is depicted in [7]. The authors indicated in the subpopulation four compartments: susceptible, exposed (infected but not infectious), infective, and recovered one. Each compartment is divided into n patches. The author introduced two time scales: a slow and fast one, reflecting accordingly disease dynamics and movement between patches. The authors focused on analysis of the disease–free stationary state stability. The concept from [7] is continued in the very recent paper [8] – here the author analyzed the additional case when disease dynamics is faster than movement between patches.

In epidemiological modeling the concept of the basic reproduction number, customarily denoted with R_0, is an important issue. Considering an epidemic in a heterogeneous population, we define R_0 as a number of new infections produced by an infective individual in a population at the disease-free stationary state. In this paper we compute R_0 for the given system and formulate condition for local stability of the disease-free stationary state, defined later, in the context of R_0.

Motivation of presented research arose from the case of tuberculosis (TB) spread in the Warmian-Masurian province of Poland in years 2001–2018. In the population of this province two subpopulations were indicated – the non-homeless people, being LS, and the homeless people, which comprise HS. In the community of homeless people programs of Active Case Finding (ACF) were conducted. As a result, the TB incidence dropped not only among homeless individuals, but in the whole population in the province. This observation emphasized usefulness of indicating LS and HS in the population so TB preventive actions can be conducted only for specified subpopulation. It is therefore reasonable to investigate TB dynamics in a population where its heterogeneity is considered. A description of the ACF programmes held in the Warmian-Masurian province and their impacts are presented in [10]. In the cited paper authors introduced a continuous model of TB dynamics for the given heterogeneous population. The heuristic assumptions of this continuous model are considered in our proposed system. We should stress that both models (mentioned in [10] and proposed by us) can be used for other diseases and other subpopulations of a heterogeneous population.

In this paper we assume, if it is not written otherwise, that $n \in N \cup \{0\}$.

DESCRIPTION OF THE MODEL

Let us now describe elements of the analyzed model. Each variable and parameter corresponding to LS and HS has a lower subscript equal to 1 and 2. respectively. If the lower subscript is denoted by i, then $i \in \{1,2\}$. Following this notation, we will denote by S_1 and S_2 a size of the group of healthy people in HS and LS, analogically. The sizes of the infected individuals in each subpopulation are expressed with I1 and I_2. With N_1 and N_2 we indicate the sizes of the whole subpopulations. Naturally we have $N_i = S_i + I_i$. We consider that there are constant inflows C_i into both subpopulations. The probabilities of individuals' survival are equal to r_i. The ratios of recovery and mortality related to the illness are denoted by γ_i and α_i.

Our proposed model has a form

$$S_{n+1}^{(1)} = C_1 + r_1 S_n^{(1)} G\left(\beta_1 \frac{I_n^{(1)}}{N_n^{(1)}} + \beta \frac{I_n^{(2)}}{N_n^{(2)}}\right) + (r_1 - \alpha_1)\gamma_1 I_n^{(1)},$$

$$I_{n+1}^{(1)} = r_1 S_n^{(1)} \left(1 - G\left(\beta_1 \frac{I_n^{(1)}}{N_n^{(1)}} + \beta \frac{I_n^{(2)}}{N_n^{(2)}}\right)\right) + (r_1 - \alpha_1)(1 - \gamma_1) I_n^{(1)},$$

$$S_{n+1}^{(2)} = C_2 + r_2 S_n^{(2)} H\left(\beta_2 \frac{I_n^{(2)}}{N_n^{(2)}}\right) + (r_2 - \alpha_2)\gamma_2 I_n^{(2)},$$

$$I_{n+1}^{(2)} = r_2 S_n^{(2)} \left(1 - H\left(\beta_2 \frac{I_n^{(2)}}{N_n^{(2)}}\right)\right) + (r_2 - \alpha_2)(1 - \gamma_2) I_n^{(2)},$$

where $S_n^{(i)}$ and $I_n^{(i)}$ mean the sizes of the groups from the i-th population at the n-th node of the discrete time scale. The G and H functions reflect probability of staying healthy in LS and HS, accordingly. These functions are based on the standard incidence function commonly used in epidemiological modeling [11]. With β_i we express efficiency of the illness transmission in the i-th subpopulation and β corresponds to efficiency of the transmission from HS to LS. Every parameter is fixed and positive. More specifically, we assume that $r_i, \gamma_i, \beta_i, \beta \in (0,1)$ and $\alpha_i \in (0, r_i)$. Using notations $S_i^+ := S_{n+1}^{(i)}$, $I_i^+ := I_{n+1}^{(i)}$, $N_i^+ := N_{n+1}^{(i)}$, $S_i := S_n^{(i)}$, we rewrite the proposed system as

$$S_1^+ = C_1 + r_1 S_1 G\left(\beta_1 \frac{I_1}{N_1} + \beta \frac{I_2}{N_2}\right) + (r_1 - \alpha_1)\gamma_1 I_1, \tag{1a}$$

$$I_1^+ = r_1 S_1 \left(1 - G\left(\beta_1 \frac{I_1}{N_1} + \beta \frac{I_2}{N_2}\right)\right) + (r_1 - \alpha_1)(1 - \gamma_1) I_1, \tag{1b}$$

$$S_2^+ = C_2 + r_2 S_2 H\left(\beta_2 \frac{I_2}{N_2}\right) + (r_2 - \alpha_2)\gamma_2 I_2, \tag{1c}$$

$$I_2^+ = r_2 S_2 \left(1 - H\left(\beta_2 \frac{I_2}{N_2}\right)\right) + (r_2 - \alpha_2)(1 - \gamma_2). \tag{1d}$$

Somewhere, if it does not provide to ambiguity, we will write

$$G = G\left(\beta_1 \frac{I_1}{N_1} + \beta \frac{I_2}{N_2}\right), H = H\left(\beta_2 \frac{I_2}{N_2}\right).$$

Determining the domain of these functions, we follow the obvious fact that $I_i \leq N_i$. We assume the following properties of the function G: G(x): [0,2) → [0,1), G(0)=1, G(2)=0, G'(x)<0, G''(x)>0. Analogically, we determine the function H with properties: H(x): [0,1) → [0,1), H(0)=1, H(1)=0, H'(x)<0, H''(x)>0.

Observe that for an initial condition $(S_0^{(1)}, I_0^{(1)}, S_0^{(2)}, I_0^{(2)}) \geq 0$ we have $S_n^{(1)}, S_n^{(2)} > 0$ and $I_n^{(1)}, I_n^{(2)} \geq 0$.

As an example of G and H one can point $G(x) = \frac{4}{x+2} - 1, H(x) = \frac{2}{x+1} - 1$.

Later we will relate to the values of G'(0) and H'(0) for the general form of functions G and H. If G(x) is a surjection, we formulate the lemma:

Lemma 1. *If G(x) is a surjection, then*

$$G'(0) < -\frac{1}{2}. \tag{2}$$

Proof. Let us analyze a linear function G*(x)=ax+b, a, b ∈ R, G*(x): [0,2] → [0,1] such that G*(0)=1 and G*(2)=0. Immediately we get b=1. From the equation 0=2a+1 we get a=-1/2, so G*(x) = -x/2 +1. See that the functions G(x) and G*(x) intersect in x=0 and x=2. Because G'(x)<0 and G''(x)>0, we have G(x)<G*(x) in x ∈ [0,2]. Since (G*)'(0) = -1/2 we get Ineq. (2).

Analogically, we can conclude that

Corollary 1. *If H(x) is a surjection, then*

$$H'(0) \leftarrow 1. \tag{3}$$

Existence of stationary states

Let us investigate forms and existence of stationary states (S_1, I_1, S_2 , I_2) of System (1). For every stationary state adding Eqs. (1a)–(1b) or (1c)–(1d) by sides yields

$$N_i = C_i + r_i S_i + (r_i - \alpha_i) I_i \tag{4}$$

or equivalently $N_i = C_i + r_i N_i - \alpha_i I_i$, what provides to

$$N_i = \frac{C_i - \alpha_i I_i}{1 - r_i}. \tag{5}$$

Eq. (4) can be written also as $S_i + I_i = C_i + r_i S_i + (r_i - \alpha_i) I_i$, giving

$$S_i = \frac{C_i - (1 - r_i + \alpha_i) I_i}{1 - r_i}. \tag{6}$$

Let us define

$$\sigma_i := 1 - r_i + \alpha_i > 0 \tag{7}$$

and express Eq. (6) as

$$S_i = \frac{C_i - \sigma_i I_i}{1 - r_i}. \tag{8}$$

For $I_i = 0$ we immediately get the disease–free stationary state

$$E_{df} := \left(\frac{C_1}{1 - r_1}, 0, \frac{C_2}{1 - r_2}, 0\right),$$

which always exists. From Eq. (8) for $I_i, S_i \neq 0$ we obtain that $S_i > 0$ if

$$I_i < \frac{C_i}{\sigma_i} \tag{9}$$

and $I_i > 0$ if

$$S_i < \frac{C_i}{1 - r_i}. \tag{10}$$

In a further analysis we will use a notation

$$\kappa_i := 1 - (r_i - \alpha_i)(1 - \gamma_i). \tag{11}$$

Naturally we have $\kappa_i \in (0,1)$. See that dynamics of Eqs. (1c)–(1d) is independent on Eqs. (1a)–(1b), hence Eqs. (1c)–(1d) can be analyzed solely.

Before investigating the existence of the stationary states, let us introduce auxiliary functions and describe their properties.

Auxiliary functions

Let us define a function

$$F(I_2) := \kappa_2 \frac{I_2}{C_2 - \sigma_2 I_2}, \kappa_2 := \frac{\kappa_2 (1 - r_2)}{r_2}. \tag{12}$$

This function is continuous in $\left[0, \frac{C_2}{\sigma_2}\right]$, where $F(0) = 0$,

$$\lim_{I_2 \to \frac{C_2}{\sigma_2}} F(I_2) = \infty$$

$$F'(I_2) = \frac{C_2 \kappa_2}{(C_2 - \sigma_2 I_2)^2} > 0, \tag{13}$$

and

$$F''(I_2) = \frac{2 C_2 \sigma_2 \kappa_2}{(C_2 - \sigma_2 I_2)^3} > 0.$$

Let us introduce an another function

$$F_a(I_2) := \beta_2 (1 - r_2) \frac{I_2}{C_2 - \alpha_2 I_2}, \tag{14}$$

defined on $\left[0, \frac{C_2}{\alpha_2}\right]$. The functions F and F_a, because of their forms, have similar properties. Hence, F_a is continuous on $\left[0, \frac{C_2}{\alpha_2}\right]$ and

$$F_a(0) = 0, \quad \lim_{I_2 \to \frac{C_2}{\alpha_2}} F_a(I_2) = \infty, F_a'(I_2) > 0, F_a''(I_2) > 0. \tag{15}$$

From Ineq. (9) we have $I_2 < \frac{C_2}{\sigma_2}$ The definition of σ_2 in Eq. (7) yields $\frac{C_2}{\sigma_2} < \frac{C_2}{\alpha_2}$. Hence, we restrict the domain of $F_a(I_2)$ to $\left[0, \frac{C_2}{\sigma_2}\right]$. The supremum of the values' set of F_a for the narrowed domain equals

$$F_a\left(\frac{C_2}{\sigma_2}\right) = \beta_2(1-r_2)\frac{\frac{C_2}{\sigma_2}}{C_2 - \alpha_2\frac{C_2}{\sigma_2}} = \beta_2.$$

Monotonicity of F_a gives

$$F_a(I_2) \in [0,\beta_2] \in [0,1]. \tag{16}$$

Now let us investigate a function

$$F_a(I_2) := 1 - H\big(F_a(I_2)\big). \tag{17}$$

Looking at the properties of H and F_a, we state that the composition $H(F_a)$ is decreasing. Remind that the domain of H is [0, 1]. The first and the third conditions from Eqs. (15) and the dependence (16) yield that we do not have additional conditions because of the composition.

Case $I_2>0$

Let us analyze a case when $I_2>0$, meaning that in the population there is at least one infected individual from HS. We formulate the theorem:

Theorem 1. *If $I_2>0$ for any stationary state of System (1), then there is a unique pair of the coordinates: $(S_2, I_2) = (\overline{S}_2, \overline{I}_2)$, where $\overline{I}_2$ is a solution of an equation*

$$\frac{\kappa_2(1-r_2)}{r_2}\frac{I_2}{C_2-\sigma_2 I_2} = 1 - H\left(\beta_2(1-r_2)\frac{I_2}{C_2-\alpha_2 I_2}\right), \tag{18}$$

The pair $(\overline{S}_2, \overline{I}_2)$ exists if

$$-H'(0) \geq \frac{\kappa_2}{\beta_2 r_2}. \tag{19}$$

Proof. Considering Eqs. (5), (8) and (11) in Eq. (1d) for any stationary state gives Eq. (18). This equation can be written, using the definitions (12) and (17), as $F(I_2) = F_b(I_2)$. Let us investigate the intersection point of F and F_b for the domain $I_2 \in \left(0, \frac{C_2}{\sigma_2}\right)$ determined in Subsection 3.1. Observe that H is decreasing and concave up, so F_b is increasing and concave down. Both F_a and F_b intersect at $I_2=0$. See that we have F_a', F_b', $F_a''>0$ and $F_b''<0$. They have one unique intersection point $\overline{I}_2 > 0$ if and only if

$$F_a'(0) \leq F_b'(0). \tag{20}$$

Eq. (13) gives

$$F_a'(0) = \frac{\kappa_2(1-r_2)}{r_2}\frac{C_2}{C_2^2} = \frac{\kappa_2(1-r_2)}{C_2 r_2}. \tag{21}$$

From Eq. (17) we obtain

$$F_a'(I_2) = -H'\big(\beta_2(1-r_2)F(I_2)\big)\beta_2(1-r_2)F'(I_2).$$

Substituting $I_2 = 0$ yields

$$F_b'(0) = -\frac{\beta_2(1-r_2)}{C_2}H'(0). \tag{22}$$

Considering Eqs. (21) and (22) in Ineq. (20) leads to

$$\frac{\kappa_2(1-r_2)}{C_2 r_2} \leq -\frac{\beta_2(1-r_2)}{C_2}H'(0),$$

giving Ineq. (19).

Observe that combining Theorem 1 and Corollary 1, we get that

Corollary 2. *If H(x) is a surjection, then Ineq. (19) in Theorem 1 holds if $\kappa_2 > \beta_2 r_2$.*

Case $I_2=0$

Now we investigate the case when $I_2 = 0$. Let us consider the existence of a proposed stationary state

$$E_1 := \left(S_1, I_1, \frac{C_2}{1-r_2}, 0\right), S_1, I_1 > 0. \tag{23}$$

See that considering the pair $(S_2, I_2) = \left(\frac{C_2}{1-r_2}, 0\right)$ in Eqs. (1) for E_1 leads to the analogical reasoning as in the proof of Theorem 1. We get an equation which is analogical to Eq. (18) and has the form

$$\frac{\kappa_1(1-r_1)}{r_1}\frac{I_1}{C_1-\sigma_1 I_1} = 1 - G\left(\beta_1(1-r_1)\frac{I_1}{C_1-\sigma_1 I_1}\right). \tag{24}$$

Eq. (24) has one positive unique solution $I_1 = I_1$. On the grounds of Theorem 1 we conclude that

Corollary 3. *In System (1) there is a stationary state E_1 defined in (23) existing if*

$$-G'(0) \geq \frac{\kappa_1}{\beta_1 r_1}. \tag{25}$$

Analogically to Corollary 2, we state that

Corollary 4. *If G(x) is a surjection, then Ineq. (25) from Corollary 3 holds if* $\kappa_1 > \beta_1 r_1$.

Existence of an endemic state

Now let us investigate existence of a postulated positive (endemic) stationary state. For this state we have $I_2>0$, what is the assumption of Theorem 1. Hence, for the positive state we have $(S_2, I_2) = \left(\overline{S_2}, \overline{I_2}\right)$. We formulate the theorem:

Theorem 2. *In System (1) there is a positive (endemic) stationary state* $E_e := \left(\overline{S_1}, \overline{I_1}, \overline{S_2}, \overline{I_2}\right)$ *existing if (19) and*

$$-G'\left(F_a\left(\overline{I_2}\right)\right) \geq \frac{\kappa_1}{\beta_1 r_1}. \tag{26}$$

Proof. Including the pair $\left(\overline{S_2}, \overline{I_2}\right)$ in Eq. (1b) for the postulated stationary state leads to the equation similar to Eq. (18). The obtained equation has a form

$$\frac{\kappa_1(1-r_1)}{r_1}\frac{I_1}{C_1-\sigma_1 I_1} = 1 - G\left(\beta_1(1-r_1)\frac{I_1}{C_1-\alpha_1 I_1} + F_a\left(\overline{I_2}\right)\right). \tag{27}$$

Let us repeat the approach from the proof of Theorem 1. We define functions:

$$\Phi(I_1) := \frac{\kappa_1(1-r_1)}{r_1}\frac{I_1}{C_1-\sigma_1 I_1}.$$

$$\Phi_a(I_1) := 1 - G\left(\beta_1(1-r_1)\frac{I_1}{C_1-\alpha_1 I_1} + F_a\left(\overline{I_2}\right)\right).$$

They are continuous for $I_1 \in \left[0, \frac{C_1}{\sigma_1}\right]$ and fulfill Φ', Φ'', $\Phi_a' > 0$ and $\Phi_a'' < 0$. We state that there is the unique positive point $\overline{I_1}$ being an intersection of Φ_a and Φ if and only $\Phi'(0) \leq \Phi_a'(0)$, what can be written as

$$\frac{\kappa_1(1-r_1)}{C_1 r_1} \leq -\frac{\beta_1(1-r_1)}{C_1} G'\left(F_a\left(\overline{I_2}\right)\right)$$

and (26). Ineq. (19) comes from Theorem 1.

Let us compare the values of I_1 and $\overline{I_1}$. Remind that they are the positive unique solutions of Eqs. accordingly (24) and (27). Since $G'<0$, we have

$$1 - G\left(\beta_1(1-r_1)\frac{I_1}{C_1-\sigma_1 I_1}\right) < 1 - G\left(\beta_1(1-r_1)\frac{I_1}{C_1-\sigma_1 I_1} + F_a\left(\overline{I_2}\right)\right).$$

Hence, we state that $I_1 < \overline{I_1}$.

Now we remind Ineqs. (25) and (26). They appear in accordingly Corollary 3 and Theorem 2 stating about the existence of the states E_1 and E_e, respectively. From the properties of G we have $G'(0) < G'\left(F_a\left(\overline{I_2}\right)\right) < 0$. Hence, Ineq. (26) is stricter than Ineq. (25). Considering this fact and the forms of the E_1 and E_e stationary states, we state the state E_e exists for smaller ranges of parameter values comparing to the state E_1. This situation is desirable from the epidemiological point.

Local stability of the stationary states

Let us investigate local stability of the obtained stationary states. The Jacobian matrix for System (1) can be written as a block matrix $J(S_1, I_1, S_2, I_2) = \begin{pmatrix} J_1 & J_a \\ 0 & J_2 \end{pmatrix}$, where J_1 , J_a and J_2 have the forms

$$J_1 = \begin{pmatrix} r_1 G + r_1\beta_1 \frac{S_1 I_1}{N_1^2} G' & r_1\beta_1 \frac{S_1^2}{N_1^2} G' + (r_1 - \alpha_1)\gamma_1 \\ r_1 - r_1 G - r_1\beta_1 \frac{S_1 I_1}{N_1^2} G' & -r_1\beta_1 \frac{S_1^2}{N_1^2} G' + 1 - \kappa_1 \end{pmatrix},$$

$$J_a = \begin{pmatrix} -\beta r_1 S_1 \frac{I_2}{N_2^2} G' & -\beta r_1 S_1 \frac{S_2}{N_2^2} G' \\ \beta r_1 S_1 \frac{I_2}{N_2^2} G' & \beta r_1 S_1 \frac{S_2}{N_2^2} G' \end{pmatrix},$$

$$J_2 = \begin{pmatrix} r_2 H + r_2\beta_2 \frac{S_2 I_2}{N_2^2} H' & r_2\beta_2 \frac{S_2^2}{N_2^2} H' + (r_2 - \alpha_2)\gamma_2 \\ r_2 - r_2 H - r_2\beta_2 \frac{S_2 I_2}{N_2^2} H' & -r_2\beta_2 \frac{S_2^2}{N_2^2} H' + 1 - \kappa_2 \end{pmatrix}.$$

We will investigate the local stability of the states, computing eigenvalues of J. Hence, it is sufficient to consider eigenvalues of J_1 and J_2.

Local stability of E_{df}

We start from the local stability of E_{df}. We will prove the theorem:

Theorem 3. *E_{df} is locally stable if*

$$-G'(0) < \frac{\kappa_1}{r_1\beta_1} \tag{28}$$

and

$$-H'(0) < \frac{\kappa_2}{r_2\beta_2}. \tag{29}$$

Proof. Remind that for E_{df} we have G(0)=1, H(0)=1 and $N_i=S_i$. The matrices $J_1(E_{df})$ and $J_2(E_{df})$ read

$$J_1(E_{df}) = \begin{pmatrix} r_1 & r_1\beta_1 G'(0) + (r_1 - \alpha_1)\gamma_1 \\ 0 & -r_1\beta_1 G'(0) + 1 - \kappa_1 \end{pmatrix}, J_2(E_{df}) = \begin{pmatrix} r_2 & r_2\beta_2 H'(0) + (r_2 - \alpha_2)\gamma_2 \\ 0 & -r_2\beta_2 H'(0) + 1 - \kappa_2 \end{pmatrix}.$$

We obtain the eigenvalues

$$\lambda_1 = r_1, \lambda_2 = -r_1\beta_1 G'(0) + 1 - \kappa_1, \lambda_3 = r_2, \lambda_4 = -r_2\beta_2 H'(0) + 1 - \kappa_2.$$

From the definition of r_i we get $|\lambda_{1,3}|<1$. A condition $|\lambda_2|<1$ is equivalent to $-1 < -r_1\beta_1 G'(0) + 1 - \kappa_1 < 1$, what can be written as $-2 < -r_1\beta_1 G'(0) - \kappa_1 < 0$. This compound inequality can be expressed as two separated ones:

$$-2 < -r_1\beta_1 G'(0) - \kappa_1 \tag{30}$$

and

$$-r_1\beta_1 G'(0) - \kappa_1 < 0. \tag{31}$$

From Ineq. (30) we have $\kappa_1 - 2 < -r_1\beta_1 G'(0)$. Since $\kappa_1 \in (0,1)$, the left–hand side of the above inequality is negative, whereas its right–hand side is positive. Hence, we state that this inequality always holds. The condition $|\lambda_2|<1$ is true if (31), what can be written as Ineq. (28). For λ_4 we conduct analogical reasoning like for λ_4. We state that $|\lambda_4|<1$ if (29).

Recall Ineq. (25), what is the condition for the E_1 existence. This inequality stays on the contrary to Ineq. (28), what is one of the conditions for the E_{df} local stability. The similar reasoning can be done for Ineq. (19) being the condition for the E_e existence. We conclude that

Corollary 5. *If E_1 or* E_e *exists, then* E_{df} *loses stability.*

Local stability of E_1

Now we determine conditions for E_1 local stability. In the next theorem and its proof we will use notations:

$$G = G\left(\beta_1 \frac{I_1}{N_1}\right), \tag{32}$$

$$\vartheta_1 := r_1\beta_1 \frac{S_1 I_1}{N_1^2} \in (0,1), \tag{33}$$

$$\theta_1 := r_1\beta_1 \frac{S_1^2}{N_1^2} \in (0,1). \tag{34}$$

and

$$\eta_1 := r_1 - \alpha_1 \in (0,1). \tag{35}$$

The variable's value in the above definitions relate to E_1. We will indicate the derivative of G at $\beta_1 \frac{I_1}{N_1}$ by G' and the identity 2x2 matrix by I.

Theorem 4. E*xisting E_1 is locally stable if (29) and if one of sets of conditions:*

$$2 > (1-\gamma_1)\eta_1 + r_1 G - (\theta_1 - \vartheta_1)G', \tag{36}$$

$$-\big(\theta_1 - \vartheta_1(1-\eta_1)\big)G' + (1-\eta_1)r_1 G + \big(1-\gamma_1(1-r_1)\big)\eta_1 < 1 - r_1\theta_1 G' \tag{37}$$

or

$$r_1(\eta_1 G - \theta_1 G') < 1 + \eta_1(r_1\gamma_1 - \vartheta_1 G') \tag{38}$$

holds.

Proof. We assume that E_1 exists. For this state we have $J_2(E_1)=J_2(E_{df})$ and hence we obtain Ineq. (29). The determinant of $J_1(E_1)$- λI reads

$$det\begin{pmatrix} r_1 G + \vartheta_1 G' - \lambda & \theta_1 G' + \eta_1\gamma_1 \\ r_1 - r_1 G - \vartheta_1 G' & -\theta_1 G' + \eta_1(1-\gamma_1) - \lambda \end{pmatrix}.$$

Adding the first row to the second one gives

$$det(J_1(E_1) - \lambda I) = det\begin{pmatrix} r_1 G + \vartheta_1 G' - \lambda & \theta_1 G' + \eta_1\gamma_1 \\ r_1 - \lambda & \eta_1 - \lambda \end{pmatrix}.$$

We write the characteristic polynomial of $J_1(E_1)$ as $P(\lambda) := \lambda^2$-bλ+c, where

$$\begin{aligned} b &:= (\vartheta_1 - \theta_1)G' + r_1 G + (1-\gamma_1)\eta_1, \\ c &:= \eta_1(r_1 G + \vartheta_1 G') - r_1(\theta_1 G' + \eta_1\gamma_1). \end{aligned} \tag{39}$$

Character of eigenvalues of $J_1(E_1)$ depends on the sign of the discriminant of $P(\lambda)$. We will denote this discriminant with Δ . The non–generic case when Δ=0 will be omitted.

1. Firstly we assume that Δ>0. The eigenvalues are real and equal to

$$\lambda_{1,2} = \frac{-b \mp \sqrt{b^2 - 4c}}{2}.$$

It is sufficient to check λ_1>-1 and λ_2<1. From the first inequality we have

$$\sqrt{b^2 - 4c} < 2 - b. \tag{40}$$

If 2-b>0, what can be written as Ineq. (36), then Ineq. (40) is reasonable. Hence, we raise its both sides to a square and get

$$b^2 - 4c < 4 - 4b + b^2 \Rightarrow b < 1 + c \tag{41}$$

Using the definitions from (39), we rewrite the last inequality from (41) as Ineq. (37).
The condition λ_2<1 can be written as

$$\sqrt{b^2 - 4c} < 2 + b. \tag{42}$$

See that Ineq. (42) is weaker than Ineq. (40), so the case λ_2<1 does not have to be analyzed.

2. Now let Δ<0. The eigenvalues are complex with non-zero imaginary part and equal to

$$\lambda_{1,2} = \frac{-b \pm i\sqrt{4c - b^2}}{2},$$

where i is an imaginary unit. The dependence $\lambda_1\,\lambda_2 = |\lambda|$<1 guaranties the local stability of E_1. See that

$$|\lambda| = \left(\frac{b}{2}\right)^2 + \frac{4c - b^2}{4} = c.$$

Hence, only the condition c<1 has to hold. This inequality can be written as Ineq. (38).

Local stability of E_e

Here we determine local stability of E_e. We will use notations

$$G = G\left(\beta_1 \frac{\bar{I}_1}{\bar{N}_1} + \beta \frac{\bar{I}_2}{\bar{N}_2}\right), H = H\left(\beta_2 \frac{\bar{I}_2}{\bar{N}_2}\right),$$
$$G' = G'\left(\beta_1 \bar{I}_1/\bar{N}_1 + \beta \bar{I}_2/\bar{N}_2\right), H' = H'\left(\beta_2 \frac{\bar{I}_2}{\bar{N}_2}\right), \tag{43}$$
$$\zeta_1 = r_1\beta_1 \frac{\bar{S}_1\bar{I}_1}{\bar{N}_1^2}, \zeta_2 = r_2\beta_2 \frac{\bar{S}_2\bar{I}_2}{\bar{N}_2^2}, \xi_1 = r_1\beta_1 \frac{\bar{S}_1^2}{\bar{N}_1^2}, \xi_2 = r_2\beta_2 \frac{\bar{S}_2^2}{\bar{N}_2^2}.$$

The determinants of $J_1(E_e)$ and $J_2(E_e)$ read accordingly

$$det\begin{pmatrix} r_1G + \zeta_1G' & \xi_1G' + (r_1 - \alpha_1)\gamma_1 \\ r_1 - r_1G - \zeta_1G' & -\xi_1G' + 1 - \kappa_1 \end{pmatrix}, det\begin{pmatrix} r_2H + \zeta_2H' & \xi_2H' + (r_2 - \alpha_2)\gamma_2 \\ r_2 - r_2H - \zeta_2H' & -\xi_2H' + 1 - \kappa_2 \end{pmatrix}.$$

We conduct similar approach as for the determinant of $J_1(E_1)$- λI. Hence, we state that

Corollary 6. E*xisting state* E_e *is locally stable if one of sets of conditions:*

$$2 > (1 - \gamma_1)\eta_1 + r_1G - (\xi_1 - \zeta_1)G',$$
$$-\left(\xi_1 - \zeta_1(1 - \eta_1)\right)G' + (1 - \eta_1)r_1G + \left(1 - \gamma_1(1 - r_1)\right)\eta_1 < 1 - r_1\xi_1G',$$
$$2 > (1 - \gamma_2)\eta_2 + r_2H - (\xi_2 - \zeta_2)H',$$
$$-\left(\xi_2 - \zeta_2(1 - \eta_2)\right)H' + (1 - \eta_2)r_2H + \left(1 - \gamma_2(1 - r_2)\right)\eta_2 < 1 - r_2\xi_2H'$$

or

$$r_1(\eta_1G - \xi_1G') < 1 + \eta_1(r_1\gamma_1 - \zeta_1G'), \tag{44}$$
$$r_2(\eta_2H - \xi_2H') < 1 + \eta_2(r_2\gamma_2 - \zeta_2H') \tag{45}$$

holds.

The basic reproduction number

Now we compute R_0 of System (1) with the use of the next-generation approach. This approach was introduced and described in [9]. Firstly we rearrange System (1) so that two first equations correspond to variables describing the groups of infected individuals. We obtain

$$I_1^+ = r_1S_1(1 - G) + (r_1 - \alpha_1)(1 - \gamma_1)I_1,$$
$$I_2^+ = r_2S_2(1 - H) + (r_2 - \alpha_2)(1 - \gamma_2)I_2,$$
$$S_1^+ = C_1 + r_1S_1G + (r_1 - \alpha_1)\gamma_1I_1,$$
$$S_2^+ = C_2 + r_2S_2H + (r_2 - \alpha_2)\gamma_2I_2,. \tag{46}$$

The disease–free state in System (46) has a form $\overline{E}_{df} := \left(0,0,\frac{C_1}{1-r_1},\frac{C_2}{1-r_2}\right)$.

In a further analysis we will use a definition $\zeta := \beta r_1 \frac{C_1}{C_2}\frac{1-r_2}{1-r_1}$.

The Jacobian matrix at $\overline{E}_{df}$ for System (46) can be written as a block matrix $\begin{pmatrix} J_a & 0 \\ J_b & J_c \end{pmatrix}$, where

$$J_a := \begin{pmatrix} -r_1\beta_1G'(0) + 1 - \kappa_1 & -\zeta G'(0) \\ 0 & -r_2\beta_2H'(0) + 1 - \kappa_2 \end{pmatrix},$$
$$J_b := \begin{pmatrix} r_1\beta_1G'(0) + (r_1 - \alpha_1)\gamma_1 & \zeta G'(0) \\ 0 & r_2\beta_2H'(0) + \eta_2r_2 \end{pmatrix}, J_c := \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

We express J_a as F+T, where F reflects new infections and T corresponds to the other process for infective states. The matrices F and T read

$$F = \begin{pmatrix} -r_1\beta_1G'(0) & -\zeta G'(0) \\ 0 & -r_2\beta_2H'(0) \end{pmatrix}, T = \begin{pmatrix} 1 - \kappa_1 & 0 \\ 0 & 1 - \kappa_2 \end{pmatrix}.$$

We write I-T and $(I-T)^{-1}$ and $F(I-T)^{-1}$ as

$$I-T=\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix},(I-T)^{-1}=\begin{pmatrix}\frac{1}{\kappa_1} & 0\\ 0 & \frac{1}{\kappa_2}.\end{pmatrix},F(I-T)^{-1}=\begin{pmatrix}\frac{-r_1\beta_1}{\kappa_1}G'(0) & \frac{-\zeta}{\kappa_2}G'(0)\\ 0 & \frac{-r_2\beta_2}{\kappa_2}H'(0)\end{pmatrix}.$$

We define ρ(M) as a spectral radius of a matrix M. We require that ρ (J_c)<1 and ρ (T)<1. These conditions can be written as accordingly max(r_1, r_2)<1 and max(1-κ_1,1-κ_2)<1. Both inequalities always hold. We define R_0 as ρ(F(I-T)$^{-1}$), what has the form

$$R_0=max\left(\frac{-r_1\beta_1}{\kappa_1}G'(0),\frac{-r_2\beta_2}{\kappa_2}H'(0)\right).$$

According to Theorem 3, we conclude that

Corollary 7. *The state $\overline{E}_{df}$ of System (1) is locally stable if R_0<1.*

Case β_1=0

Let us analyze the cases when the values of transmission coefficients equal 0. Firstly notice that the form and existence of E_{df} is independent on the values of these coefficients. Now see that if β=0, then System (1) describes the epidemic in separated subpopulations. This case does not reflect dynamics of epidemic in a heterogeneous population. Now assume that β_2=0 This equality corresponds to an unrealistic situation in which any individual from HS cannot get infected. The remaining case β_1=0 means that there is no illness transmission in LS. This situation is nearly impossible in reality, but can happen when the individuals from LS do not interact. It can occur during general isolation, like it happened during the COVID–19 pandemic. Hence, in this paragraph we assume that β_1=0.

Under this assumption the domain of the function G is redefined so that G: [0,1] → [0,1]. Thanks to this redefinition, as G we can choose the function F. System (1) becomes

$$\begin{aligned}S_1^+&=C_1+r_1S_1H\left(\beta\frac{I_2}{N_2}\right)+(r_1-\alpha_1)\gamma_1I_1,\\ I_1^+&=r_1S_1\left(1-H\left(\beta\frac{I_2}{N_2}\right)\right)+(r_1-\alpha_1)(1-\gamma_1)I_1,\\ S_2^+&=C_2+r_2S_2H\left(\beta_2\frac{I_2}{N_2}\right)+(r_2-\alpha_2)\gamma_2I_2,\\ I_2^+&=r_2S_2\left(1-H\left(\beta_2\frac{I_2}{N_2}\right)\right)+(r_2-\alpha_2)(1-\gamma_2)I_2.\end{aligned}\tag{47}$$

Let us investigate existence of stationary state of System (47). Similarly as previously, we consider two cases: I_2=0 and I_2>0.

Firstly let us assume that I_2=0. Hence, we get $S_2=\frac{C_2}{1-r_2}$. If$(I_2,S_2)=\left(0,\frac{C_2}{1-r_2}\right)$, then two first equations of System (47) become an independent system having a form

$$\begin{aligned}S_1^+&=C_1+r_1S_1H(0)+(r_1-\alpha_1)\gamma_1I_1,\\ I_1^+&=r_1S_1\big(1-H(0)\big)+(r_1-\alpha_1)(1-\gamma_1)I_1.\end{aligned}\tag{48}$$

Since H(0)=1, we rewrite System (48) as

$$S_1^+=C_1+r_1S_1+(r_1-\alpha_1)\gamma_1I_1,(49a)$$
$$I_1^+=(r_1-\alpha_1)(1-\gamma_1)I_1(49b).$$

The unique stationary state of System (49) is $(S_1,I_1)=\left(\frac{C_1}{1-r_1},0\right)$. We conclude that for I_2=0 the unique stationary state of System (47) is E_{df}. Now we assume that $(S_2,I_2)=(\overline{S}_2,\overline{I}_2)$. In the following theorem and its proof we will notations

$$\sigma:=\big(1-r_1\overline{H}\big)\kappa_1-(r_1-\alpha_1)r_1\gamma_1\big(1-\overline{H}\big)\tag{50}$$

and

$$\overline{H} := H\left(\beta \frac{\overline{I}_2}{\overline{N}_2}\right). \tag{51}$$

From the properties of H we have

$$0 < \overline{H} < 1. \tag{52}$$

Let us formulate the theorem:

Theorem 5. *System (47) has a positive stationary state* $\left(S_1^{\$}, I_1^{\$}, \overline{S}_2, \overline{I}_2\right)$*, where*

$$S_1^{\$} := \frac{\kappa_1 C_1}{r_1 \sigma}, \tag{53}$$

$$I_1^{\$} := \frac{C_1(1-\overline{H})}{\sigma}. \tag{54}$$

This state exists if

$$\gamma_1 > \frac{1}{2}. \tag{55}$$

Proof. Including $(S_2, I_2) = \left(\overline{S}_2, \overline{I}_2\right)$ in two first equations of System (47) for a stationary state yields

$$\begin{aligned} S_1 &= C_1 + r_1 S_1 \overline{H} + (r_1 - \alpha_1)\gamma_1 I_1, \\ I_1 &= r_1 S_1 \left(1 - \overline{H}\right) + (1 - \kappa_1) I_1. \end{aligned} \tag{56}$$

From the above equations we get accordingly

$$S_1 = \frac{C_1 + (r_1 - \alpha_1)\gamma_1 I_1}{1 - r_1 \overline{H}}, S_1 = \frac{\kappa_1 I_1}{r_1(1-\overline{H})}. \tag{57}$$

Solving System (57) gives (54). From the second equation of (57) we have Eq. (53). See that Ineq. (52) yields $1 - \overline{H} \in (0,1)$. Thanks to this property we obtain the positivity of $S_i^{\$}$ and $I_i^{\$}$, under the condition $\sigma > 0$. Using Eq. (50), this inequality can be written as

$$\left(1 - r_1 \overline{H}\right)\kappa_1 - (r_1 - \alpha_1) r_1 \gamma_1 \left(1 - \overline{H}\right) > 0,$$

Applying Eq. (11), from the above inequality we get

$$1 - r_1 \overline{H} + (r_1 - \alpha_1)\gamma_1 (1 - r_1) > r_1 \overline{H}(r_1 - \alpha_1)(1 - 2\gamma_1). \tag{58}$$

The condition $r_1 \overline{H} < 1$ is always true. Let us consider the inequality

$$(r_1 - \alpha_1)\gamma_1 (1 - r_1) > r_1 \overline{H}(r_1 - \alpha_1)(1 - 2\gamma_1), \tag{59}$$

which is stronger than Ineq. (58). Ineq. (59) holds if

$$\gamma_1 (1 - r_1) + 2\gamma_1 r_1 \overline{H} > r_1 \overline{H}. \tag{60}$$

If Ineq. (55) holds, what is probable from the epidemiological point, then Ineq. (60) and consequently Ineq. (58) are true. Hence, we obtain the fulfillment of $I_i^* > 0$ and $S_i^* > 0$ under Ineq. (55).

NUMERICAL SIMULATIONS

Finally we illustrate the dynamics of System (1) for values of parameters fitted to real data. These data reflect the case of the TB epidemic mentioned in Section 1. Simulations were performed with Matlab software. Numerical results of the simulations and actual epidemic data were compared so that the best-fitted values of the parameters were obtained. These values are shown in Table 1.

Numbers of the homeless people were obtained from the Regional Center for Social Policy, Office of the Marshall of the Warmian-Masurian province in Poland. Demographic figures were taken from statistical yearbooks [12]. Epidemic data were anonymized by the Independent Public Tuberculosis and Lung Diseases Unit in Olsztyn, Poland. Only numerical details were provided. All data are fully available without any restrictions. The numbers of healthy and infected individuals for each subpopulation for year 2001 were chosen as the initial condition for System (1). The values

Table 1. The values of the parameters for the model described by System (1)

Name	Described process	Value (no units)
C_1	Constant inflow into LS	11000
C_2	Constant inflow into HS	60
r_1, r_2	Natural death	0.991
γ_1, γ_2	Recovery	0.9
α_1, α_2	Disease-related death	0.09
β_1	Transmission among LS	0.4286
β_2	Transmission among HS	0.4445
β	Transmission from HS to LS	0.0150

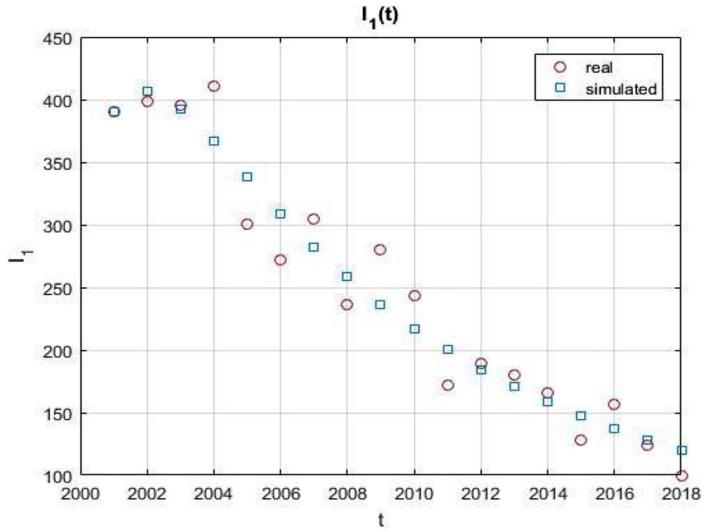

Figure 1. Tuberculosis in the Warmian-Mazurian province over the years 2001–2018 (number of the infected non-homeless individuals). Comparison between the model and the actual data

of β1, β2 and β were estimated with the use of the built-in lsqcurvefit function in Matlab. This function is based on the Gauss-Newton algorithm [13]. The values of the remaining parameters were obtained from the Central Statistical Office of Poland. In Figure 1 comparison between the simulated values and the actual data is presented.

CONCLUSIONS

In this paper we introduced and analyzed the discrete model of the illness transmission in the heterogeneous population. The model was constructed without using discretization of the continuous model, what is not a usual approach. We defined stationary states of the proposed system – we determined the conditions for their existence. Later we investigated the conditions for their local stability and computed the basic reproduction number R_0 of the system. Finally we discussed the case of the lack of the illness transmission in the subpopulation with low susceptibility to infection.

We provided the mathematical analysis of the proposed system. We indicated appearing stationary states and expressed the explicit conditions of their existence and local stability. What is important, we assumed that the parameters describing each subpopulation have different values. Furthermore, we did not define specific functions describing the illness transmission. These assumptions make our model general.

The results obtained in this paper are in line with those concerning similar continuous models. In System (1) there are both disease–free (E_{df}) and endemic (E_e) stationary states, what is typical in epidemiological models. The conditions for local stability of E_{df} depend on the values of parameters and derivatives of transmission functions for E_{df}. These values constitute the expression for R_0 which determines the local stability of E_{df}. Crossing the critical value $R_0=1$ provides instability of this state, what should be expected. For E_e we obtained explicit conditions for its local stability, what is not obvious for a four–dimensional system. Besides E_{df} and E_e, there exists also the stationary state E_1 for which the infection appears only in LS, what is desirable from the epidemiological point.

System (1), because of the mentioned properties appearing also in analogical continuous systems, can be exploited in cases when it is reasonable to investigate the discrete nature of epidemic spread. What is important, System (1) does not include a step size of the discretization method, which appears in discrete systems built with discretization of their continuous counterpart. Thanks to that, there are no cases where some conditions, for example for stability of stationary states, depend on the step size.

To conclude, the model proposed and analyzed in this paper can be treated as an exemplary one for researchers investigating epidemic dynamics in heterogeneous populations. From the theoretical point, the obtained outcomes can be used for analyzing the systems of four discrete equations which are built from coupling two–dimensional discrete systems.

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