BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 63, No. 1, 2015

PARTIAL DIFFERENTIAL EQUATIONS

Existence of Solutions for the Keller–Segel Model of Chemotaxis with Measures as Initial Data

by

Piotr BILER and Jacek ZIENKIEWICZ

Presented by Bogdan BOJARSKI

Summary. A simple proof of the existence of solutions for the two-dimensional Keller-Segel model with measures with all the atoms less than 8π as the initial data is given. This result was obtained by Senba and Suzuki (2002) and Bedrossian and Masmoudi (2014) using different arguments. Moreover, we show a uniform bound for the existence time of solutions as well as an optimal hypercontractivity estimate.

1. Introduction. We consider the classical parabolic-elliptic Keller–Segel model of chemotaxis in two space dimensions,

$$u_t - \Delta u + \nabla \cdot (u\nabla v) = 0, \tag{1.1}$$

$$\Delta v + u = 0, \tag{1.2}$$

supplemented with a nonnegative initial condition

$$u(x,0) = u_0(x) \geq 0. \tag{1.3}$$

Here for $(x,t) \in \mathbb{R}^2 \times [0,T)$, the function $u = u(x,t) \geq 0$ denotes the density of the population of microorganisms, and $v = v(x,t)$ the density of the chemical altractant secreted by them that makes them aggregate. System (1.1)–(1.2) is also used in modelling the gravitational attraction of particles in the mean field astrophysical models (see [2]).

As is well known (cf. e.g. [7]), the total mass of the initial condition

$$M = \int u_0(x)\,dx \tag{1.4}$$

is the critical quantity for the global-in-time existence of nonnegative solutions. Namely, if $M \leq 8\pi$, then solutions of (1.1)–(1.3) (with u_0 a finite

2010 *Mathematics Subject Classification*: 35Q92, 35B44, 35K55.
Key words and phrases: chemotaxis, blowup of solutions.

DOI: 10.4064/ba63-1-6

nonnegative measure) exist for all $t \geq 0$. For local-in-time existence, it should be assumed that all the atoms of the measure u_0 are of mass less than 8π (see [1, Th. 2]). When $M > 8\pi$, nonnegative solutions blow up in finite time, and for radially symmetric solutions, mass equal to 8π concentrates at the origin at the blowup time (see e.g. [6]).

Our goal in this note is to give an alternative proof of the local-in-time existence of solutions to (1.1)–(1.3) when $u_0 \in \mathcal{M}(\mathbb{R}^2)$ is a nonnegative finite measure with all its atoms of mass less than 8π. We believe that this approach is conceptually simpler than that in the recent paper [1] (which used elaborate arguments for interactions of solutions emanating from localized pieces of initial data), and those in previous papers [10], [11]. The latter approaches used heavily the free energy functional for system (1.1)–(1.2) considered in bounded domains. Moreover, our condition (1.5) seems to be clearer and shows that measures with small atoms which are not well separated (as was assumed in [1]) are also admissible as initial data for (1.1)–(1.2). Compared to [1], however, we obtain neither the uniqueness of solutions nor the Lipschitz property of the solution map.

The main result of this paper is

THEOREM 1.1. *Let* $0 \leq u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *be a smooth initial density for* (1.1)–(1.2) *such that*

$$\|u_0 * \mathbb{1}_{B(1)}\|_\infty \leq 8\pi - \varepsilon_0 \tag{1.5}$$

for some fixed $\varepsilon_0 > 0$ *and the unit ball* $B(1)$ *centered at the origin of* $\mathbb{R}^2$. *Then there exists a solution of problem* (1.1)–(1.3) *on the interval* $[0, t_0]$ *with* $t_0 = t_0(\varepsilon_0, M)$ *such that*

$$\sup_{0<t\leq t_0} t^{1-1/p}\|u(t)\|_p \leq B, \tag{1.6}$$

where the constant B *depends on* M *and* ε_0 (*in particular,* B *does not depend on* $\|u_0\|_\infty$).

Note that condition (1.5) reads

$$\int_{B(x,1)} u_0(y)\,dy = \int_{B(1)} u_0(x-y)\,dy \leq 8\pi - \varepsilon_0$$

for all balls $B(x,1)$ of radius 1 centered at arbitrary $x \in \mathbb{R}^2$, and this in particular means that if, more generally, u_0 were a nonnegative measure then its atoms would be strictly less than 8π. The key property of our estimate of t_0 is that it depends only on M and ε_0 for all u_0 satisfying (1.5) with a given M in (1.4).

We recall that for each $\lambda > 0$ and each solution u of (1.1)–(1.2) of mass M the function

$$u_\lambda(x,t) = \lambda^2 u(\lambda x, \lambda^2 t) \tag{1.7}$$

is also a solution, with mass again equal to M.

Of course, by a suitable scaling (1.7) of initial data we may satisfy the assumptions of the local existence result in Theorem 1.1 for any nonnegative $u_0 \in \mathcal{M}(\mathbb{R}^2)$ with atoms strictly less than 8π. Thus we arrive at the following corollary (cf. [1, Theorem 2]).

COROLLARY 1.2. *System* (1.1)–(1.2) *has a local-in-time solution for each initial nonnegative finite measure* u_0 *with all atoms strictly less than* 8π.

Indeed, it is sufficient to approximate u_0 (in the sense of weak convergence of measures) by a sequence of initial data satisfying (after the rescaling (1.7) with a single $\lambda > 0$) all the assumptions of Theorem 1.1. This approximation is possible by taking, e.g., $e^{\delta_n \Delta} u_0$ for any sequence $\delta_n \searrow 0$. Then the existence time t_0 is bounded from below by a positive quantity (since t_0 depends on M and λ only). Next, we infer from the hypercontractivity estimate (1.6) and from the standard regularity theory for parabolic equations that for every multiindex α,

$$\|D^\alpha u(t)\|_p \le C_\alpha B\, t^{1/p-1-|\alpha|/2},$$

which permits us to pass to the limit with (a subsequence of) the approximating solutions which are, in fact, smooth on $\mathbb{R}^2 \times (0, t_0)$. We thus obtain a solution to (1.1)–(1.2) with the measure u_0, and this solution is also smooth on $\mathbb{R}^2 \times (0, t_0)$.

The proof of Theorem 1.1 will be a consequence of a well-known estimate [2, 3] of the existence time by the mass of the initial condition only (see (2.7)), by using a rather delicate localization argument repeatedly.

The existence results are proved (e.g. in [2]) for the integral formulation of system (1.1)–(1.3),

$$u(t) = e^{t\Delta} u_0 + B(u, u)(t), \tag{1.8}$$

whose solutions are called *mild* solutions of the original Cauchy problem. Here, the bilinear term B is defined as

$$B(u, z)(t) = -\int_0^t (\nabla e^{(t-s)\Delta}) \cdot (u(s)\, \nabla(-\Delta)^{-1} z(s))\, ds. \tag{1.9}$$

It is well known that the heat semigroup $e^{t\Delta}$ satisfies the L^q-L^p estimates

$$\|e^{t\Delta} z\|_p \le C t^{1/p-1/q} \|z\|_q \tag{1.10}$$

and

$$\|\nabla e^{t\Delta} z\|_p \le C t^{-1/2+1/p-1/q} \|z\|_q \tag{1.11}$$

for all $1 \le q \le p \le \infty$. Moreover, for each $p > 1$ and $z \in L^1(\mathbb{R}^2)$,

$$\lim_{t \to 0} t^{1-1/p} \|e^{t\Delta} z\|_p = 0. \tag{1.12}$$

This is (for example) a consequence of a much more general inequality valid for every finite measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ and every $p > 1$,

$$\limsup_{t\to 0} t^{1-1/p}\|e^{t\Delta}\mu\|_p \le C_p\|\mu_{\mathrm{at}}\|_{\mathcal{M}(\mathbb{R}^2)} \equiv C_p \sum_{\{x:\,\mu(\{x\})\neq 0\}} |\mu(\{x\})|, \tag{1.13}$$

where μ_{at} denotes the purely atomic part of the measure μ. The proof of (1.13) is contained in [9, Lemma 4.4]. This fact together with the condition (1.5) rescaled to other balls of a fixed radius (see (1.14) below) is crucial in the analysis of applicability of the Banach contraction argument to (1.8).

We recall that the formulation of our existence results in [2] used in fact condition (1.13) in the definition of the function space where the solutions were looked for:

$$\Big\{u : (0,T) \to L^p(\mathbb{R}^2) : |||u||| \equiv \sup_{0<t<T} t^{1-1/p}\|u(t)\|_p < \infty\Big\},$$

and then a smallness condition was assumed on the quantity $|||e^{t\Delta}u_0|||$.

The heuristics behind the argument leading to the proof of Theorem 1.1 is the following: the initial data diffuse into a domain whose size grows as $t^{1/2}$ in time, as in Corollary 2.8. Thus, we need to find a time $\tau \ge 0$ when a counterpart of (1.5),

$$\|u(\tau) * \mathbb{1}_{B(\varrho)}\| \le m_0, \tag{1.14}$$

holds with a sufficiently small m_0 given in (2.10) and $\varrho > 0$ suitably small in order to apply the local existence result of Theorem 2.7.

Remark. When (1.2) is replaced by the nonhomogeneous heat equation $\tau v_t = \Delta v + u$ (and thus we consider the parabolic-parabolic version of the Keller–Segel model), the situation seems to be more complicated. For instance, if $\tau \gg 1$, then there exist global-in-time solutions with $M > 8\pi$ which emanate from $M\delta_0$ as (purely atomic) initial data. These are self-similar solutions which are regular and *nonunique* for sufficiently large M (cf. e.g. [4] and comments in [5]).

In the proof of Theorem 1.1 we will apply simple (but subtle) techniques of weight functions and scalings. The core of our analysis is in the uniform (with respect to the initial distributions) estimates on the maximal existence time, expressed in terms of dispersion of the initial data.

Notation. Integrals with no integration limits are taken over the whole space $\mathbb{R}^2$: $\int = \int_{\mathbb{R}^2}$. The letter C denotes various constants which may vary from line to line but they are independent of solutions. The norm in $L^p(\mathbb{R}^2)$ is denoted by $\|\cdot\|_p$. The kernel of the heat semigroup on $\mathbb{R}^2$, denoted by $e^{t\Delta}$, is given by $G(x,t) = (4\pi t)^{-1}\exp(-|x|^2/(4t))$.

2. Proof of Theorem 1.1. The proof is split into several lemmata.

For any fixed $x_0 \in \mathbb{R}^2$ we define the *local moment* of a solution u by

$$\Lambda(t) \equiv \int \psi(x - x_0) u(x,t)\, dx. \tag{2.1}$$

Here the weight function

$$\psi(x) = (1-|x|^2)_+^2 \ \text{ with } \ \nabla\psi(x) = -4x(1-|x|^2)_+,\ \Delta\psi(x) = 16|x|^2 - 8 \geq -8, \tag{2.2}$$

is a fixed radial, piecewise $\mathcal{C}^2$, nonnegative function ψ, supported on the unit ball such that $\psi(0) = 1$. Our particular choice of ψ is not critical.

LEMMA 2.1. *Suppose that $w = w(x)$ is a nonnegative function locally in $L^1 \cap L^\infty$, $\int_{B(1)} w(x)\, dx \leq m$, and $\varrho, \delta \in (0,1)$. Then:*

(i) *There exists $H_0 \in (0,1)$ such that*

$$\int_{B(\varrho)} w(x)\, dx \leq (1-\delta)m \ \Rightarrow\ \int \psi(x) w(x)\, dx \leq (1 - H_0)m.$$

(ii) *Similarly, there exists $H_1 \in (0,1)$ such that if*

$$\int_{B(1)} w(x)\, dx \leq m \text{ and } \int \psi(x) w(x)\, dx \geq (1-H_1)m$$
$$\Rightarrow \int_{B(\varrho)} w(x)\, dx \geq (1 - \delta/2)m.$$

(iii) *Suppose that*

$$\int \psi(x) w(x)\, dx \leq (1-H)m \quad \text{with some } H \in (0,1).$$

Then

$$\int_{B(\beta)} w(x)\, dx \leq (1 - H/2)m \quad \text{for } \beta^2 \leq H/4 \leq 1/4.$$

Proof. Properties (i)–(iii) are simple consequences of (2.2). Indeed,

$$\begin{aligned}
\int \psi(x) w(x)\, dx &\leq \int_{B(\varrho)} w(x)\, dx + \sup_{B(1)\setminus B(\varrho)} \psi(x) \cdot \int_{B(1)\setminus B(\varrho)} w(x)\, dx \\
&\leq \int_{B(\varrho)} w(x)\, dx + (1-\varrho^2)^2 \int_{B(1)\setminus B(\varrho)} w(x)\, dx \\
&= (1-\varrho^2)^2 \int_{B(1)} w(x)\, dx + (1 - (1-\varrho^2)^2) \int_{B(\varrho)} w(x)\, dx \\
&\leq (1-\varrho^2)^2 m + (1-(1-\varrho^2)^2)(1-\delta)m \\
&= (1-H_0)m,
\end{aligned}$$

where $1 - H_0 = (1-\varrho^2)^2 + (1-(1-\varrho^2)^2)(1-\delta) = 1 - \delta(1-(1-\varrho^2)^2)$.

(ii) is equivalent to (i) with δ replaced by $\frac{1}{2}\delta$.

(iii) For $|x| \le \beta$, $\beta^2 \le H/4$ and $H \le 1$ we have
$$\psi(x) \ge \frac{1-H}{1-H/2}. \ \blacksquare$$

Next, we establish a result on the dispersion of the initial data evolving according to (1.1)–(1.2).

LEMMA 2.2. *Let u be a solution to* (1.1)–(1.2) *such that for $t \in [0, A]$,*
$$\|u_0 * \mathbb{1}_{B(R_0)}\|_\infty \le m \tag{2.3}$$
for some $A > 0$, $R_0 = 6 \cdot 128\pi M/\varepsilon_0 > 0$ and $m_0 \le m \le 8\pi - \varepsilon_0$. Then there exist numbers $A_1 = A_1(M, \varepsilon_0)$, $\delta = \delta(M, \varepsilon_0, m_0)$ and $\varrho = \varrho(M, \varepsilon_0, m_0)$ such that if
$$\int\limits_{|y-x_0|\le\varrho} u(y,t)\,dy \ge (1-\delta)m \quad \textit{for some } t \in [0, A],$$
then
$$\Lambda'(t) \le -\vartheta \quad \textit{with some } \vartheta = \vartheta(M, \varepsilon_0, m_0) > 0.$$

Proof. First we give a uniform estimate of the time derivative of the moment $\Lambda(t)$:
$$|\Lambda'(t)| \le C_M.$$
Let us compute the time derivative of Λ using (1.1)–(1.2) and (2.2). Symmetrizing the bilinear integral $\int u(x,t)\nabla v(x,t) \cdot \nabla\psi(x)\,dx$ with the solution v of (1.2) given by
$$v(x,t) = -\frac{1}{2\pi}\int u(y,t)\log|x-y|\,dy$$
we obtain
$$\Lambda'(t) = \int u(x,t)\Delta\psi(x)\,dx + \frac{1}{4\pi}\iint \frac{\nabla\psi(x) - \nabla\psi(y)}{|x-y|^2} \cdot (x-y)u(x,t)u(y,t)\,dx\,dy. \tag{2.4}$$
From (2.4) and (2.2) we immediately get $\Lambda'(t) \le 8M + 4M^2$.

Using (2.2), the bound $|\nabla\psi(x) - \nabla\psi(y)| \le 4$ and the relation
$$\int\limits_{B(\varrho)} u(x,t)\,dx \le 8\pi \quad \text{with} \quad \varrho \le 1 < 2 \le R_0 \text{ and } \frac{1}{R_0 - \varrho} \le \frac{2}{R_0},$$
we arrive at
$$\left| \int\limits_{|x|<\varrho} \int\limits_{2\le R_0<|y|} \frac{4x(1-|x|^2)_+ - 4y(1-|y|^2)_+}{|x-y|^2} \cdot (x-y)u(x,t)u(y,t)\,dx\,dy \right| \le \frac{16\pi M}{R_0} \cdot 4. \tag{2.5}$$

Next, in the annulus $\varrho < |y| \le R_0$ we have

$$\left| \int\limits_{|x|<\varrho} \int\limits_{\varrho<|y|\le R_0} \frac{\nabla\psi(x) - \nabla\psi(y)}{|x-y|^2} \cdot (x-y) u(x,t) u(y,t)\, dx\, dy \right| \le B \cdot 8\pi\delta, \tag{2.6}$$

where we have applied the bound

$$|\nabla\psi(x) - \nabla\psi(y)| \le B|x-y| \tag{2.7}$$

for some constant B, as well as $\int_{\varrho<|y|\le R_0} u(y,t)\, dy \le \delta m < \delta \cdot 8\pi$.

Finally, by (2.4) we have simply

$$\left| \int u(x,t) \Delta\psi(x)\, dx + 8 \int\limits_{B(\varrho)} u(x,t)\, dx \right| \le 64\pi\delta + 16\varrho^2\, 8\pi.$$

Now, the crucial estimate for the bilinear integral in (2.4) is

$$\left| \int\limits_{|x|<\varrho} \int\limits_{|y|<\varrho} \frac{4x(1-|x|^2)_+ - 4y(1-|y|^2)_+}{|x-y|^2} \cdot (x-y) u(x,t) u(y,t)\, dx\, dy \right. \\ \left. - 4\Big(\int\limits_{B(\varrho)} u(x,t)\, dx \Big)^2 \right| \le B\delta M^2. \tag{2.8}$$

Here we have used the following properties of the weight function ψ:
$|\psi(x)-1| = \big|(1-|x|^2)_+^2 - 1\big| = \big|2|x|^2 - |x|^4\big|_+ \le 2|x|^2$, $\quad |\Delta\psi(x)+8| \le 16|x|^2$,
and an improvement of (2.7):

$$|\nabla\psi(x) - \nabla\psi(y) + 4(x-y)| \le B\varrho|x-y|,$$

valid for all $|x|, |y| \le \varrho$. Therefore, we get

$$\Lambda'(t) \le -8 \int\limits_{B(\varrho)} u(x,t)\, dx + \frac{1}{\pi}\Big(\int\limits_{B(\varrho)} u(x,t)\, dx \Big)^2 + \frac{128\pi M}{R_0} \\ + 16\pi B\delta + 64\pi\delta + 16\varrho^2 8\pi.$$

Since

$$\frac{1}{\pi} \int\limits_{B(\varrho)} u(x,t)\, dx \Big(-8\pi + \int\limits_{B(\varrho)} u(x,t)\, dx \Big) \le -\varepsilon_0,$$

it suffices to choose

$$R_0 = \frac{6C_0}{\varepsilon_0}, \qquad \delta \le \frac{\varepsilon_0}{6C_1}, \qquad \varrho^2 \le \frac{\varepsilon_0}{6C_2},$$

therefore $R_0 = 6 \cdot 128\pi M/\varepsilon_0$. ∎

Note that a variant of Lemma 2.3 below has been obtained in [11] by a (rather elaborate) radial rearrangement argument of [8]. The proof we present uses only weight functions and localized moments defined by them.

LEMMA 2.3. *Suppose that $u = u(x,t)$ is a solution of* (1.1)–(1.2) *satisfying all the assumptions of Lemma 2.2. Then for all $t \ge A_0 = A_0(M, m_0, \varepsilon_0)$ and $H_1 = H_1(M, m_0, \varepsilon_0)$ we have*

$$\int_{B(\varrho)} u(x,t)\,dx \le (1 - H_1)m.$$

Proof. Let $\tau_0 = 0$. If $\int_{B(\varrho)} u(x,\tau_0)\,dx \le (1-\delta)m$, then set $\tau_1 = \tau_0$. Otherwise, let

$$\tau_1 = \inf\Big\{\tau < A : \int_{B(\varrho)} u(x,\tau)\,dx = (1-\delta)m\Big\}.$$

In order to obtain necessary estimates for τ_1, observe that by Lemma 2.2,

$$\frac{d}{dt}\Lambda(t) \le -\vartheta \quad \text{ for all } t \in [\tau_0, \tau_1].$$

Since $\Lambda(0) \le m$ and $\Lambda(t) \ge 0$, we have $\tau_1 - \tau_0 \le m/\vartheta$. By Lemma 2.1(i) we arrive at $\Lambda(\tau_1) \le (1 - H_0)m$. Next, we define

$$\tau_2 = \inf\{\tau_1 < \tau < A : \Lambda(\tau) = (1 - H_1)m\}$$

if this exists, otherwise $\tau_2 = A$. Then for every $t \in [\tau_1, \tau_2]$ we obtain $\Lambda \le (1 - H_1)m$. If $\tau_2 = A$, we are done. If not, by Lemma 2.1(ii) we infer that

$$\int_{B(\varrho)} u(x,\tau_2)\,dx \ge (1 - \delta/2)m > (1-\delta)m.$$

Therefore, $\Lambda(\tau_2) \le -\vartheta$ implies that $\Lambda(\tau_2 - h) > (1 - H_1)m$ for a sufficiently small $h > 0$, contrary to the definition of τ_2. Thus we get $\Lambda(t) \le (1 - H_1)m$ for $t \in [\tau_1, A]$. Consequently, by Lemma 2.1, we have

$$\int_{B(\beta\varrho)} u(x,t)\,dx \le (1 - H_1/2)m \quad \text{ for } t \ge \frac{m}{\vartheta} \equiv A_0(M, \varepsilon_0) \text{ and } \beta < \frac{1}{2}H^{1/2}$$

as in Lemma 2.1(iii). In what follows we denote $\beta\varrho$ again by ϱ. ∎

COROLLARY 2.4. *If u solves system* (1.1)–(1.2) *on the time interval $[0, AT]$ with $A \ge A_0$, and satisfies the estimate*

$$\int_{B(T^{1/2})} u(x,t)\,dx \le m < 8\pi - \varepsilon_0, \tag{2.9}$$

then

$$\int_{B(\varrho T^{1/2})} u(x,t)\,dx \le (1 - H_1)m \quad \textit{for } t \in [A_0 T, AT].$$

Proof. This follows from Lemma 2.3 applied to the rescaled function $u_T(x,t) = Tu(T^{1/2}x, Tt)$, which, evidently, is also a solution of (1.1)–(1.2). ∎

Corollary 2.5. *If a solution u of* (1.1)–(1.2) *exists on the time interval* $[0, A]$ *then for all* $t \in [A_0(1 + \varrho_1 + \cdots + \varrho_1^{j-1}), A)$ *with* $\varrho_1 = \varrho/R_0$,

$$\int_{B(\varrho_1^j)} u(x,t)\,dx \le (1 - H_1)^j m \quad \textit{as long as} \quad (1 - H_1)^j m \ge m_0.$$

Proof. It suffices to apply Corollary 2.4 to the functions $u(x,t)$, $u(x, A_0 + t)$, $u(x, A_0 + \varrho_1 A_0 + t)$, …, rescaled with $T_0 = 1$, $T_1 = \varrho_1^2$, $T_3 = \varrho_1^4$, $T_3 = \varrho_1^6$, …, consecutively. ■

Corollary 2.6. *Suppose that a solution u of* (1.1)–(1.2) *exists for* $t \in [0, A]$ *and satisfies*

$$\|u(t) * \mathbb{1}_{B(1)}\|_\infty \le 8\pi - \varepsilon_0.$$

Then

$$\|u(t) * \mathbb{1}_{B(sr)}\|_\infty \le m_0 \quad \textit{with} \quad r = \varrho_1^{[|\log 8\pi/\log(1-H_1)|]+1}$$

for any $s \in [0, 1]$ *and* $t \in \big[\frac{s^2}{1-\varrho_1} A_0, A\big]$.

Proof. Apply Corollary 2.5 to the rescaled solution. ■

Now we recall the existence result of [2, 3].

Theorem 2.7. *There exists a* (*small*) $m_0 > 0$ *such that the condition*

$$\|u_0 * \mathbb{1}_{B(1)}\|_\infty \le 2m_0 \tag{2.10}$$

guarantees the existence of a local-in-time solution (*on a time interval* $[0, T]$ *with* $T = T(M)$) *satisfying* $\|u(t)\|_p \le C t^{1/p-1}$ *for* $p = 4/3$.

In fact, we will use the following immediate consequence of Theorem 2.7 which takes into account the scale invariance (1.7).

Corollary 2.8. *For each $\sigma > 0$ there exists $\alpha > 0$ such that the condition*

$$\|u_0 * \mathbb{1}_{B(\sigma\tau^{1/2})}\|_\infty \le 2m_0$$

implies the existence of a solution of (1.1)–(1.3) *on the time interval* $[0, \alpha\tau]$. *Here $\tau > 0$ is any small positive number and α can be chosen as $\alpha = \alpha_0\sigma^2$ for some $\alpha_0 > 0$.*

Theorem 2.7 has been proved (even for sign-changing measures) in [2, Theorem 2] (cf. also [3, proof of Theorem 2.2]) using a standard contraction argument applied to the formulation (1.8).

Lemma 2.9. *Suppose that $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a smooth nonnegative function satisfying the condition*

$$\|u_0 * \psi\|_\infty \le 8\pi - \varepsilon_0.$$

Then the solution u with the initial condition (1.3) *exists at least on the time interval* $[0, \varepsilon_0/(2C_M)]$.

Proof. The inequality $\Lambda'(t) \le C_M$ implies that $\Lambda(t) \le 8\pi - \varepsilon_0/2$ for $t \le \tau_1 \equiv \min\{\varepsilon_0/(2C_M), \tau\}$, where τ is the maximal existence time of u. By Lemma 2.1(iii) we obtain the bound

$$\|u(t) * \mathbb{1}_{B(\varrho_1)}\|_\infty \le 8\pi - \varepsilon_0/4 \quad \text{for all } t \le \tau_1.$$

From Corollary 2.6 we infer that there exists $\sigma_0 = \eta\tau_1^{1/2}$ such that

$$\|u(\tau_2) * \mathbb{1}_{B(\sigma_0)}\|_\infty \le m_0 \quad \text{for } \tau_1/2 < \tau_2 < \tau_1.$$

By Theorem 2.7 the solution with $u_0 = u(x, \tau_2)$ as the initial condition exists for $t \in [0, \alpha\tau_2]$ with some $\alpha > 0$ independent of τ_2. Therefore, by Theorem 2.7, this solution can be continued onto the interval $[0, \tau_1 + \alpha\tau_2]$. This solution satisfies

$$\|u(\tau_1 + \alpha\tau_2)\|_p \le C\tau_2^{1/p-1} \quad \text{for each } p \in [4/3, 2].$$

Finally, a recurrence argument permits us to obtain a classical solution $u = u(x,t)$ on the whole interval $[0, T_0]$ with $T_0 = T_0(\varepsilon_0, M)$, and applying once more Corollaries 2.5 and 2.8 gives the hypercontractive estimate

$$\|u(t)\|_p \le Ct^{1/p-1} \quad \text{for } p \in [4/3, 2].$$

The extrapolation of that estimate to the whole range of $p \in (1, \infty)$ is standard (see e.g. [3]). ■

Acknowledgments. The preparation of this paper was supported by the NCN grants 2013/09/B/ST1/04412, and DEC-2012/05/B/ST1/00692.

References

[1] J. Bedrossian and N. Masmoudi, *Existence, uniqueness and Lipschitz dependence for Patlak–Keller–Segel and Navier–Stokes in* $\mathbb{R}^2$ *with measure-valued initial data*, Arch. Ration. Mech. Anal. 214 (2014), 717–801.

[2] P. Biler, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation*, Studia Math. 114 (1995), 181–205.

[3] P. Biler, T. Cieślak, G. Karch and J. Zienkiewicz, *Local criteria for blowup of solutions in two-dimensional chemotaxis models*, arXiv:1410.7807v2 (2014).

[4] P. Biler, L. Corrias and J. Dolbeault, *Large mass self-similar solutions of the parabolic-parabolic Keller–Segel model*, J. Math. Biology 63 (2011), 1–32.

[5] P. Biler, I. Guerra and G. Karch, *Large global-in-time solutions of the parabolic-parabolic Keller–Segel system on the plane*, Comm. Pure Appl. Anal., to appear; arXiv:1401.7650 (2014).

[6] P. Biler, G. Karch and J. Zienkiewicz, *Optimal criteria for blowup of radial solutions of chemotaxis systems*, arXiv:1407.4501 (2014).

[7] A. Blanchet, J. Dolbeault and B. Perthame, *Two-dimensional Keller–Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations 2006, no. 44, 32 pp.

[8] J. I. Díaz, T. Nagai and J.-M. Rakotoson, *Symmetrization techniques on unbounded domains: application to a chemotaxis system on* $\mathbb{R}^N$, J. Differential Equations 145 (1998), 156–183.

[9] Y. Giga, T. Miyakawa and H. Osada, *Two-dimensional Navier–Stokes flow with measures as initial vorticity*, Arch. Ration. Mech. Anal. 104 (1988), 223–250.

[10] T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations 6 (2001), 21–50.

[11] T. Senba and T. Suzuki, *Weak solutions to a parabolic-elliptic system of chemotaxis*, J. Funct. Anal. 191 (2002), 17–51.

Piotr Biler, Jacek Zienkiewicz
Instytut Matematyczny
Uniwersytet Wrocławski
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: Piotr.Biler@math.uni.wroc.pl
Jacek.Zienkiewicz@math.uni.wroc.pl

Received February 17, 2015;
received in final form July 2, 2015 (8012)