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C^n-PSEUDO ALMOST AUTOMORPHIC SOLUTIONS OF CLASS r FOR NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS UNDER THE LIGHT OF MEASURE THEORY

Abstract. The aim of this work is to present new approach to study C^n-(μ,ν)-pseudo almost automorphic solutions of class r for some neutral partial functional differential equations in a Banach space when the delay is distributed. We use the variation of constants formula and the spectral decomposition of the phase space.

1. Introduction

In this work, we study the existence and uniqueness of C^n-(μ,ν)-pseudo almost automorphic solutions of class r for the following functional differential equation

$$\frac{d}{dt}u_t = Au_t + L(u_t) + f(t) \text{ for } t \in \mathbb{R}, \tag{1}$$

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where A is a linear operator on a Banach space X satisfying the Hille-Yosida condition, that is, there exist $M_0 \geq 1$ and $\beta \in \mathbb{R}$ such that $(\beta, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{M_0}{\lambda - \beta} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \beta,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. In sequel, without lost of generality, we assume that $M_0 = 1$. We denote by $C = C([-r, 0]; X)$ the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. If $u \in C([-r, +\infty); X)$ and $t \geq 0$, then $u_t \in C([-r, 0]; X)$ denotes the history function of u defined by

$$u_t(\theta) = u(t + \theta), \quad \theta \in [-r, 0].$$

The operator L is a bounded linear operator from C into X and $f : \mathbb{R} \to X$ is a continuous function.

Some recent contributions concerning pseudo almost automorphic solutions for abstract differential equations similar to equation (1) have been made. For example, in [9], the authors studied the existence of C^n-almost periodic solutions and C^n-almost automorphic solutions ($n \geq 1$), for partial neutral functional differential equations. They proved that the existence of a bounded integral solution on $\mathbb{R}^+$ implies the existence of C^n-almost periodic and C^n-almost automorphic strict solutions. When the exponential dichotomy holds for the homogeneous linear equation, they show the uniqueness of C^n-almost periodic and C^n-almost automorphic strict solutions.

In [12], the authors studied a new concept of C^n-almost automorphic and asymptotically C^n-almost automorphic functions with values in a Banach space. They gave a new result related to the existence and uniqueness of an asymptotically almost automorphic solution of a semilinear fractional differential equation of the form $D^\alpha x(t) = Ax(t) + F(t, x(t), Bx(t))$ with $t \in \mathbb{R}$, $0 < \alpha < 1$, where A generates a family of α-resolvent family $S_\alpha(t)$ and f satisfies some Lipschitz conditions.

In [4], the authors present new approach to study weighted pseudo almost automorphic functions using the measure theory. They present a new concept of weighted ergodic functions, which is more general than the classical one. Then they establish many interesting results on the functional space of such functions like completeness and composition theorems. The theory of their work generalizes the classical results on weighted pseudo almost automorphic functions.

The aim of this work is to prove the existence of C^n-(μ,ν)-pseudo almost automorphicc solutions of the equation (1) when the delay is distributed on $[-r,0]$. Our approach is based on the variation of constants formula and the spectral decomposition of the phase space developed in [3] and a new approach developped in [4].

This work is organised as follows. In Section 2, we recall some prelimary results on variation of constants formula and spectral decomposition. In Section 3, we recall some preliminary results on C^n-(μ,ν)-pseudo almost automorphic functions and neutral partial functional differential equations that will be used in this work. In Section 4, we give some properties of C^n-(μ,ν)-pseudo almost automorphic functions of class r. In Section 5, we discuss the main result of this paper. By using the strict contraction principle, we show the existence and uniqueness of C^n-(μ,ν)-pseudo almost automorphic solution of class r for the equation (1). Finally, for illustration, we propose to study the existence and uniqueness of C^n-(μ,ν)-pseudo almost automorphic solution for some model arising in the population dynamics.

2. Variation of constants formula and spectral decomposition

We associate to the equation (1) the following initial value problem

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \geq 0, \\ u_0 = \varphi \in C = C([-r,0];X), \end{cases} \tag{2}$$

where $f\colon \mathbb{R}^+ \to X$ is a continuous function.

Definition 1. *We say that a continuous function u from $[-r,+\infty)$ into X is an integral solution of the equation* (2)*, if the following conditions hold:*

(i) $\displaystyle\int_0^t u(s)ds \in D(A)$ *for* $t \geq 0$,

(ii) $\displaystyle u(t) = \varphi(0) + A\int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds$ *for* $t \geq 0$,

(iii) $u_0 = \varphi$.

If $\overline{D(A)} = X$, the integral solutions coincide with the known mild solutions. One can see that if $u(t)$ is an integral solution of the equation (2), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$, which is defined by

$$\begin{cases} D(A_0) = \left\{ x \in D(A) \colon \; Ax \in \overline{D(A)} \right\}, \\ A_0 x = Ax \;\text{ for }\; x \in D(A_0). \end{cases}$$

Lemma 2. *([1]) The part A_0 of the operator A generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

In the sequel, we consider the following condition:

$(\mathbf{H_0})$ the operator A satisfies the Hille-Yosida condition.

Proposition 3. *([2]) If $(\boldsymbol{H_0})$ holds, then for all $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, the equation (2) has a unique integral solution u on $[-r, +\infty)$. Moreover, u is given by*

$$u(t) = T_0(t)\varphi(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-s) B_\lambda (L(u_s) + f(s)) ds, \;\; t \geq 0,$$

where $B_\lambda = \lambda R(\lambda, A)$ for $\lambda > \omega$.

The phase space C_0 of the equation (2) is defined by

$$C_0 = \left\{ \varphi \in C \colon \varphi(0) \in \overline{D(A)} \right\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t)\varphi = v_t(., \varphi),$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt} v(t) = Av(t) + L(v_t), \;\; t \geq 0, \\ v_0 = \varphi \in C. \end{cases}$$

Proposition 4. *([3]) $(\mathcal{U}(t))_{t\geq 0}$ is a strongly continuous semigroup of linear operators on C_0. Moreover, $(\mathcal{U}(t))_{t\geq 0}$ satisfies for $t \geq 0$ and $\theta \in [-r, 0]$ the following* *translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) & \text{for } t+\theta \geq 0, \\ \varphi(t+\theta) & \text{for } t+\theta \leq 0. \end{cases}$$

Proposition 5. *([3]) Let $\mathcal{A}_{\mathcal{U}}$ be defined on C_0 by the condition that $D(\mathcal{A}_{\mathcal{U}})$ consists of all functions $\varphi \in C^1([-r,0];X)$ such that $\varphi(0) \in D(A)$, $\varphi(0)' = A\varphi(0) + L(\varphi) \in \overline{D(A)}$ and $\mathcal{A}_{\mathcal{U}}\varphi = \varphi'$ for all $\varphi \in D(\mathcal{A}_{\mathcal{U}})$. Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on C_0.*

Let $\langle X_0 \rangle$ be the space defined as follows

$$\langle X_0 \rangle = \{X_0 c \colon c \in X\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r,0), \\ c & \text{for } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0 c| = |\phi|_C + |c|$, $(\phi, c) \in C_0 \times X$, is a Banach space. Consider the extension of $\mathcal{A}_{\mathcal{U}}$ defined on $C_0 \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \left\{\varphi \in C^1([-r,0];X) \colon \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)}\right\}, \\ \widetilde{\mathcal{A}_{\mathcal{U}}}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 6. *([3]) Assume that $(\boldsymbol{H_0})$ holds. Then $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hille-Yosida condition on $C_0 \oplus \langle X_0 \rangle$, that is, there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}_{\mathcal{U}}}$ on $D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = C_0$ is exactly the operator $\mathcal{A}_{\mathcal{U}}$.

Now, we can state the variation of constants formula associated to the equation (2).

Proposition 7. *([3]) Assume that ($\boldsymbol{H_0}$) holds. Then for all $\varphi \in C_0$, the solution u of the equation* (2) *is given by the following formula*

$$u_t = \mathcal{U}(t)\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\widetilde{B}_\lambda(X_0 f(s))ds, \quad t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_\mathcal{U})^{-1}$ *for* $\lambda > \widetilde{\omega}$.

Definition 8. *We say that a semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A}_\mathcal{U}) \cap i\mathbb{R} = \emptyset.$$

In the sequel, we make the following assumption:

($\mathbf{H_1}$) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

Proposition 9. *([3]) Assume that ($\boldsymbol{H_0}$) and ($\boldsymbol{H_1}$) hold. Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is compact for $t > r$.*

From the compactness of the semigroup $(\mathcal{U}(t))_{t\geq 0}$, we get the following result on the spectral decomposition of the phase space C_0.

Proposition 10. *([10]) Assume that ($\boldsymbol{H_1}$) holds. If the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic, then the space C_0 decomposes as a direct sum $C_0 = S \oplus U$ of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on U is a group, and there exist positive constants $\overline{M}$ and ω such that*

$$\begin{aligned} |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{-\omega t}|\varphi| \ \ \text{for } t \geq 0 \text{ and } \varphi \in S, \\ |\mathcal{U}(t)\varphi| &\leq \overline{M}e^{\omega t}|\varphi| \ \ \text{for } t \leq 0 \text{ and } \varphi \in U. \end{aligned}$$

The spaces S and U are called, respectively, the stable and unstable space. By Π^s and Π^u, we denote the projection operators on S and U, respectively.

3. Almost automorphic functions and (μ, ν)-ergodic functions

In this section, we recall some properties about (μ, ν)-pseudo almost automorphic functions. The notion of μ-pseudo almost periodicity is a generalization of the pseudo almost periodicity introduced by Zhang [15–17]. It is also a generalization of weighted pseudo almost automorphicity given by Diagana [6].

Let $BC(\mathbb{R};X)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to X equipped with the uniform norm topology. We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R})=+\infty$ and $\mu([a,b])<\infty$ for all $a,b\in\mathbb{R}$ $(a\le b)$.

Definition 11. *A continuous function $\phi\colon \mathbb{R}\to X$ is called almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that the limit*

$$g(t)=\lim_{n\to+\infty}\phi(t+s_n)$$

is well defined for each $t\in\mathbb{R}$ and

$$\lim_{n\to+\infty}g(t-s_n)=\phi(t)$$

for each $t\in\mathbb{R}$. We denote by $AA(\mathbb{R},X)$ the space of all such functions.

Definition 12. *Let X_1 and X_2 be two Banach spaces. A continuous function $\phi\colon \mathbb{R}\times X_1\to X_2$ is called almost automorphic in $t\in\mathbb{R}$ uniformly for each $x\in X_1$ if for every real sequence (s_m), there exists a subsequence (s_n) such that*

$$g(t,x)=\lim_{n\to+\infty}\phi(t+s_n,x)$$

is well defined for each $t\in\mathbb{R}$ and each $x\in X_1$ and

$$\lim_{n\to+\infty}g(t-s_n,x)=\phi(t,x)$$

for each $t\in\mathbb{R}$. We denote by $AA(\mathbb{R}\times X_1;X_2)$ the space of all such functions.

Definition 13. *A continuous function $\phi\colon \mathbb{R}\to X$ is called compact almost automorphic if for each real sequence (s_m), there exists a subsequence (s_n) such that*

$$g(t)=\lim_{n\to+\infty}\phi(t+s_n)\ \ \textit{and}\ \ \lim_{n\to+\infty}g(t-s_n)=\phi(t)$$

uniformly on compact subsets of $\mathbb{R}$. We denote by $AA_c(\mathbb{R};X)$ the space of all such functions.

Definition 14. *Let X_1 and X_2 be two Banach spaces. A continuous function $\phi\colon \mathbb{R} \times X_1 \to X_2$ is called compact almost automorphic in $t \in \mathbb{R}$ if for every real sequence (s_m), there exists a subsequence (s_n) such that*

$$g(t,x) = \lim_{n\to+\infty} \phi(t+s_n,x) \quad \text{and} \quad \lim_{n\to+\infty} g(t-s_n,x) = \phi(t,x),$$

where the limits are uniform on compact subsets of $\mathbb{R}$ for each $x \in X_1$. We denote by $AA_c(\mathbb{R} \times X_1; X_2)$ the space of all such functions.

Let $C^n(\mathbb{R}; X)$ be the space of all continuous functions $h\colon \mathbb{R} \to X$ which have a continuous n-th derivative on $\mathbb{R}$, and let $C_b^n(\mathbb{R}; X)$ be the subspace of $C^n(\mathbb{R}; X)$ which consists of all functions $h\colon \mathbb{R} \to X$ satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |h^{(i)}(t)| < \infty,$$

where $h^{(i)}$ denotes the i-th derivative of h. Then $C_b^n(\mathbb{R}; X)$ is a Banach space provided with the following norm

$$|h|_n = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |h^{(i)}(t)|.$$

Now, we state a new concept of the C^n-almost automorphy, which generalizes the one of the C^n-almost periodicity.

Definition 15. *A continuous function $h\colon \mathbb{R} \to X$ is said to be C^n-almost automorphic for $n \geq 1$ if for $i = 1, \dots, n$, the i-th derivative $h^{(i)}$ of h is almost automorphic. We denote by $AA^{(n)}(\mathbb{R}; X)$ the space of C^n-almost automorphic functions.*

Definition 16. *A continuous function $h\colon \mathbb{R} \to X$ is said to be C^n-compact almost automorphic for $n \geq 1$ if for $i = 1, \dots, n$, the i-th derivative $h^{(i)}$ of h is compact almost automorphic. We denote by $AA_c^{(n)}(\mathbb{R}; X)$ the space of C^n-compact almost automorphic functions.*

By [13], since $AA(\mathbb{R}; X)$ and $AA_c(\mathbb{R}; X)$ are Banach spaces, we get also the following result.

Proposition 17. *([8]) The spaces $AA^{(n)}(\mathbb{R};X)$ and $AA_c^{(n)}(\mathbb{R};X)$ equipped with the norm $|\,.\,|_n$ are Banach spaces.*

In the sequel, we use some preliminary results concerning the (μ,ν)-pseudo almost automorphic functions, which have been established recently in [4]. The symbol $\mathcal{E}(\mathbb{R};X,\mu,\nu)$ stands for the following space of functions

$$\left\{u \in BC(\mathbb{R};X)\colon \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|u(t)|d\mu(t)=0\right\}.$$

To study delayed differential equations for which the history belongs to the space $C([-r,0];X)$, we also consider the space

$$\left\{u \in BC(\mathbb{R};X)\colon \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|u(\theta)|\Big)d\mu(t)=0\right\},$$

which we will denote by $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$. In addition to the above-mentioned spaces, we also consider the space

$$\left\{u \in BC(\mathbb{R}\times X_1;X_2)\colon \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|u(t,x)|_{X_2}d\mu(t)=0\right\}$$

and denote it by $\mathcal{E}(\mathbb{R}\times X_1,X_2,\mu,\nu)$, as well as the space

$$\left\{u \in BC(\mathbb{R}\times X_1;X_2)\colon \lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|u(\theta,x)|_{X_2}\Big)d\mu(t)=0\right\}$$

and denote it by $\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$, where in both cases the limit (as $\tau\to+\infty$) is uniform in compact subset of X_1. In view of previous definitions, it is clear that the spaces $\mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ and $\mathcal{E}(\mathbb{R}\times X_1;X_2,\mu,\nu,r)$ are continuously embedded in $\mathcal{E}(\mathbb{R};X,\mu,\nu)$ and $\mathcal{E}(\mathbb{R}\times X_1,X_2,\mu,\nu)$, respectively. On the other hand, one can observe that a ρ-weighted pseudo almost automorphic function is μ-pseudo almost automorphic, where the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon-Nikodym derivative is ρ:

$$d\mu(t)=\rho(t)dt.$$

Example 18. (see also [4]) Let ρ be a nonnegative $\mathcal{B}$-measurable function. Denote by μ the positive measure defined by

$$\mu(A) = \int_A \rho(t)dt \quad \text{for} \quad A \in \mathcal{B}, \tag{3}$$

where dt denotes the Lebesgue measure on $\mathbb{R}$. The function ρ in the equation (3) is called the Radon-Nikodym derivative of μ with respect to the Lebesgue measure on $\mathbb{R}$.

Definition 19. *A function $h \in C_b^n(\mathbb{R}; X)$ is said to be a C^n-ergodic function if $h^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$ for $i = 1, \ldots, n$. We denote by $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$ the space of C^n-ergodic functions.*

Definition 20. *A function $h \in C_b^n(\mathbb{R}; X)$ is said to be a C^n-ergodic function of class r if $h^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$ for $i = 1, \ldots, n$. We denote by $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ the space of C^n-ergodic functions.*

For any $\mu, \nu \in \mathcal{M}$, we formulate the following condition:

$(\mathbf{H_2})$ $\limsup_{\tau \to +\infty} \dfrac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \alpha < \infty.$

We have the following result.

Lemma 21. *Assume $(\boldsymbol{H_2})$ holds and let $f \in C_b^n(\mathbb{R}; X)$. Then $f \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$ if and only if for any $\varepsilon > 0$ and for $i = 1, \ldots, n$ the following equality holds*

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau, \tau]} = 0,$$

where

$$M_{\tau,\varepsilon}(f^{(i)}) = \left\{ t \in [-\tau, \tau] \colon |f^{(i)}(t)| \geq \varepsilon \right\}.$$

Proof. Suppose that $f \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$. Then, by Definition 19, we can write $f^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$ for $i = 1, \ldots, n$. We also have

$$\begin{aligned} & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f^{(i)}(t)|d\mu(t) = \\ = \quad & \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})}|f^{(i)}(t)|d\mu(t) + \\ + \quad & \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f^{(i)})}|f^{(i)}(t)|d\mu(t) \geq \\ \geq \quad & \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})}|f^{(i)}(t)|d\mu(t) \geq \frac{\varepsilon}{\nu([-\tau,\tau])}M_{\tau,\varepsilon}(f^{(i)}). \end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty}\frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau,\tau]} = 0.$$

Conversely, suppose that $f \in C_b^n(\mathbb{R}; X)$ is such that for any $\varepsilon > 0$ and for $i = 1, \ldots, n$, the following equality holds

$$\lim_{\tau\to+\infty}\frac{\mu(M_{\tau,\varepsilon}(f^{(i)}))}{\nu([-\tau,\tau]} = 0.$$

We can assume $|f^{(i)}(t)| \leq N$ for all $t \in \mathbb{R}$. By using $(\mathbf{H_2})$, we have

$$\begin{aligned} & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f^{(i)}(t)|d\mu(t) = \\ = \quad & \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})}|f^{(i)}(t)|d\mu(t) + \\ + \quad & \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f^{(i)})}|f^{(i)}(t)|d\mu(t) \leq \\ \leq \quad & \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})}d\mu(t) + \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f^{(i)})}|f^{(i)}(t)|d\mu(t) \leq \\ \leq \quad & \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f^{(i)})}d\mu(t) + \frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}d\mu(t) \leq \\ \leq \quad & \frac{N}{\nu([-\tau,\tau])}M_{\tau,\varepsilon}(f^{(i)}) + \frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}, \end{aligned}$$

which implies for any $\varepsilon > 0$ the following inequality:

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}|f^{(i)}(t)|d\mu(t) \leq \alpha\varepsilon.$$

Therefore $f^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$ for $i = 1, \ldots, n$, and hence $f \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$. $\square$

Definition 22. *Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi \in C_b^n(\mathbb{R}; X)$ is called C^n-(μ,ν)-pseudo almost automorphic (respectively C^n-(μ,ν)-pseudo compact almost automorphic) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}, X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$ (resp. $\phi_1 \in AA_c^{(n)}(\mathbb{R}, X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu)$). We denote by $PAA^{(n)}(\mathbb{R}; X, \mu, \nu)$ (resp. $PAA_c^{(n)}(\mathbb{R}; X, \mu, \nu)$) the space of all such functions.*

Definition 23. *Let $\mu, \nu \in \mathcal{M}$ and X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi \in C_b^n(\mathbb{R} \times X_1, X_2)$ is called uniformly C^n-(μ,ν)-pseudo almost automorphic (resp. uniformly C^n-(μ,ν)-pseudo compact almost automorphic) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R} \times X_1, X_2, \mu, \nu)$ (resp. $\phi_1 \in AA_c^{(n)}(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R} \times X_1, X_2, \mu, \nu)$). We denote by $PAA^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu)$ (resp. $PAA_c^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu)$) the space of all such functions.*

Definition 24. *Let $\mu, \nu \in \mathcal{M}$. A bounded continuous function $\phi \in C_b^n(\mathbb{R}; X)$ is called C^n-(μ,ν)-pseudo almost automorphic of class r (resp. C^n-(μ,ν)-pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ (resp. $\phi_1 \in AA_c^{(n)}(\mathbb{R}, X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$). We denote by $PAA^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ (resp. $PAA_c^{(n)}(\mathbb{R}; X, \mu, \nu, r)$) the space of all such functions.*

Definition 25. *Let $\mu, \nu \in \mathcal{M}$ and let X_1 and X_2 be two Banach spaces. A bounded continuous function $\phi \in C_b^n(\mathbb{R} \times X_1, X_2)$ is called uniformly C^n-(μ,ν)-pseudo almost automorphic of class r (resp. uniformly C^n-(μ,ν)-pseudo compact almost automorphic of class r) if $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu, r)$ (resp. $\phi_1 \in AA_c^{(n)}(\mathbb{R} \times X_1; X_2)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R} \times X_1, X_2, \mu, \nu, r)$). We denote by $PAA^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu, r)$ (resp. $PAA_c^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu, r)$) the space of all such functions.*

4. Properties of $C^{(n)}$-(μ,ν)-pseudo almost automorphic functions of class r

Lemma 26. *Let $\mu, \nu \in \mathcal{M}$. The space $PAA^{(n)}(\mathbb{R}; X, \mu, \nu, r)$ endowed with the norm $|.|_n$ is a Banach space.*

Proof. Let (x_m) be a sequence in $PAA^{(n)}(\mathbb{R};X,\mu,\nu,r)$ such that the limit $x=\lim_{m\to\infty} x_m$ belongs to $BC^n(\mathbb{R};X)$. For each m, let $x_m = y_n + z_m$, where $y_m \in AA^{(n)}(\mathbb{R};X)$ and $z_m \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Since $y_m \in AA^{(n)}(\mathbb{R};X)$, we have by Definition 15: $y_m^{(i)} \in AA(\mathbb{R};X)$ for $i=0,1,\ldots,n$. By [11, Lemma 1.2], the sequence $(y_m^{(i)})_m$ converges to some $y^{(i)} \in AA(\mathbb{R};X)$. Consequently, $y \in AA^{(n)}(\mathbb{R};X)$ by Definition 15. Since $z_m \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$, Definition 20 implies that $z_m^{(i)} \in \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ and $(z_m^{(i)})_m$ converges to some $z^{(i)} \in BC(\mathbb{R};X)$. We also have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z^{(i)}(\theta)|\Big)d\mu(t) \le\\
\le\ &\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)-z^{(i)}(\theta)|\Big)d\mu(t)+\\
+\ &\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)|\Big)d\mu(t) \le\\
\le\ &\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}|z_m^{(i)}(t)-z^{(i)}(t)|\Big)d\mu(t)+\\
+\ &\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)|\Big)d\mu(t) \le\\
\le\ &\|z_m^{(i)}-z^{(i)}\|\cdot\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|z_m^{(i)}(\theta)|\Big)d\mu(t),
\end{aligned}$$

where

$$\|z_m^{(i)}-z^{(i)}\| = \int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}|z_m^{(i)}(t)-z^{(i)}(t)|\Big)d\mu(t).$$

Then we get $z^{(i)} \in \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ for $i=0,1,\ldots,n$, so $z\in\mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$. It follows that $x \in PAA^{(n)}(\mathbb{R};X,\mu,\nu,r)$. ☐

The next result is a characterization of $C^{(n)}$-(μ,ν)-ergodic functions of class r.

Theorem 27. *Let $\mu,\nu\in\mathcal{M}$ be such that ($\boldsymbol{H_2}$) holds and let I be a bounded interval (we do not exclude the case $I=\emptyset$). Then for every $f\in C_b^n(\mathbb{R};X)$, the following assertions are equivalent:*

(i) $f\in\mathcal{E}^{(n)}(\mathbb{R},X,\mu,\nu,r)$.

(ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)=0$ *for* $i=0,1,\ldots,n$.

(iii) $\lim\limits_{\tau\to+\infty}\mu\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in[t-r,t]}|f^{(i)}(\theta)|>\varepsilon\Big\}\Big)/\nu([-\tau,\tau]\setminus I)=0$ *for any* $\varepsilon>0$ *and* $i=0,1,\dots,n$.

Proof. To show (i)⇔(ii), let us denote:

$$A=\nu(I),\quad B=\int_I\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t).$$

Since the interval I is bounded and the function $f^{(i)}$ is bounded and continuous for $i=0,1,\dots,n$, we have: $A,B\in\mathbb{R}$. For $\tau>0$ such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)=\\
=\;&\frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)-B\Big]=\\
=\;&\frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].
\end{aligned}$$

From the above equalities and the fact that $\nu(\mathbb{R})=+\infty$, we deduce that (ii) is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)=0,\quad i=0,1,\dots,n,$$

which, by Definition 19, is equivalent to (i).

To show (iii)⇒(ii), let us denote the following sets

$$\begin{aligned}
A_\tau^\varepsilon&=\Big\{t\in[-\tau,\tau]\setminus I\colon\ \sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|>\varepsilon\Big\},\\
B_\tau^\varepsilon&=\Big\{t\in[-\tau,\tau]\setminus I\colon\ \sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\le\varepsilon\Big\}.
\end{aligned}$$

Assume that (iii) holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}=0. \tag{4}$$

From the equality

$$\begin{aligned}&\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)=\\ =\;&\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)+\int_{B_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t),\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)\le\\ \le\;&\|f^{(i)}\|_\infty\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}\end{aligned}$$

for τ sufficiently large, where $\|f^{(i)}\|_\infty=\sup\limits_{t\in\mathbb{R}}|f^{(i)}(t)|$. By using $(\mathbf{H_2})$, it follows that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)\le\alpha\varepsilon$$

for any $\varepsilon>0$ and $i=0,1,\ldots,n$. Consequently, (ii) holds.

To show (ii)⇒(iii), assume that (ii) holds. For every $i=0,1,\ldots n$, if τ is sufficiently large, then the integral

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)$$

is not smaller than

$$\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t).$$

In consequence, we obtain:

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)\quad\ge\quad\varepsilon\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},$$

and hence:

$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}|f^{(i)}(\theta)|\Big)d\mu(t)\quad\ge\quad\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}.$$

The last inequality implies (4), and hence (iii) holds. □

In what follows, we prove some preliminary results concerning the composition of (μ,ν)-pseudo almost periodic functions of class r.

Theorem 28. *Let* $\mu, \nu \in \mathcal{M}$ *and*

$$\phi \in PAA^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu, r), \quad h \in PAA^{(n)}(\mathbb{R}; X_1, \mu, \nu, r).$$

Assume that there exists a function $L_\phi : \mathbb{R} \to [0, +\infty)$ *satisfying*

$$|\phi(t, x_1) - \phi(t, x_2)| \leq L_\phi(t)|x_1 - x_2|, \ \ t \in \mathbb{R}, \ \ x_1, x_2 \in X_1. \tag{5}$$

If $\dfrac{1}{\nu([-\tau, \tau])} \displaystyle\int_{-\tau}^{\tau} \Big(\sup_{\theta \in [t-r, t]} L_\phi(\theta) \Big) d\mu(t) < \infty$ *and for each* $\xi \in \mathcal{E}(\mathbb{R}, \mu, \nu)$*, we have*

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta \in [t-r, t]} L_\phi(\theta) \Big) \xi(t) d\mu(t) = 0, \tag{6}$$

then the function $t \mapsto \phi(t, h(t))$ *belongs to* $PAA^{(n)}(\mathbb{R}; X_2, \mu, \nu, r)$.

Proof. Assume that $\phi = \phi_1 + \phi_2$, $h = h_1 + h_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R} \times X_1; X_2)$, $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R} \times X_1; X_2, \mu, \nu, r)$ and $h_1 \in AA^{(n)}(\mathbb{R}; X_1)$, $h_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X_1, \mu, \nu, r)$. Then $\phi_1^{(i)} \in AA(\mathbb{R} \times X_1; X_2)$, $\phi_2^{(i)} \in \mathcal{E}(\mathbb{R} \times X_1; X_2, \mu, \nu, r)$ and $h_1^{(i)} \in AA(\mathbb{R}; X_1)$, $h_2^{(i)} \in \mathcal{E}(\mathbb{R}; X_1, \mu, \nu, r)$ for $i = 0, 1, \ldots, n$. Consider the following decomposition

$$\phi^{(i)}(t, h(t)) = \phi_1^{(i)}(t, h_1^{(i)}(t)) + [\phi^{(i)}(t, h^{(i)}(t)) - \phi^{(i)}(t, h_1^{(i)}(t))] + \phi_2^{(i)}(t, h_1^{(i)}(t)).$$

From [4], we know that $\phi_1^{(i)}(., h_1^{(i)}(.)) \in AA(\mathbb{R}; X_2)$ for $i = 0, 1, \ldots, n$. It remains to prove that for $i = 0, 1, \ldots, n$ both $\phi^{(i)}(., h^{(i)}(.)) - \phi^{(i)}(., h_1^{(i)}(.))$ and $\phi_2^{(i)}(., h_1^{(i)}(.))$ belong to $\mathcal{E}(\mathbb{R}; X_2, \mu, \nu, r)$. By using the equation (5), it follows that the quotient

$$\frac{\mu\Big(\Big\{t \in [-\tau, \tau] \colon \sup\limits_{\theta \in [t-r, t]} |\phi^{(i)}(\theta, h^{(i)}(\theta)) - \phi^{(i)}(\theta, h_1^{(i)}(\theta))| > \varepsilon\Big\}\Big)}{\nu([-\tau, \tau])}$$

is not greater than

$$\frac{\mu\Big(\Big\{t \in [-\tau, \tau] \colon \sup\limits_{\theta \in [t-r, t]} (L_\phi(\theta)|h_2^{(i)}(\theta)|) > \varepsilon\Big\}\Big)}{\nu([-\tau, \tau])},$$

which in turn is not greater than

$$\frac{\mu\Big(\Big\{t \in [-\tau, \tau] \colon \Big(\sup\limits_{\theta \in [t-r, t]} L_\phi(\theta)\Big)\Big(\sup\limits_{\theta \in [t-r, t]} |h_2^{(i)}(\theta)|\Big) > \varepsilon\Big\}\Big)}{\nu([-\tau, \tau])}.$$

Since $h_2^{(i)}$ is (μ,ν)-ergodic of class r, Theorem 27 and the equation (6) yield that for the above-mentioned ε, we have

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]\colon \Big(\sup\limits_{\theta\in[t-r,t]}L_\phi(\theta)\Big)\Big(\sup\limits_{\theta\in[t-r,t]}|h_2^{(i)}(\theta)|\Big)>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0.$$

Then we obtain

$$\lim_{\tau\to+\infty}\frac{\mu\Big(\Big\{t\in[-\tau,\tau]\colon \sup\limits_{\theta\in[t-r,t]}|\phi^{(i)}(\theta,h^{(i)}(\theta))-\phi(\theta,h_1(\theta))|>\varepsilon\Big\}\Big)}{\nu([-\tau,\tau])}=0. \tag{7}$$

By Theorem 27, the equation (7) shows that for $i=0,1,\dots,n$ the function $t\mapsto\phi^{(i)}(t,h^{(i)}(t))-\phi^{(i)}(t,h_1^{(i)}(t))$ is (μ,ν)-ergodic of class r. Now, to complete the proof, it is enough to prove that $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of class r. Since $\phi_2^{(i)}$ is uniformly continuous on the compact set $K_i=\overline{\{h_1^{(i)}(t)\colon t\in\mathbb{R}\}}$ with respect to the second variable x, we deduce that for a given $\varepsilon>0$, there exists $\delta>0$ such that $|\xi_1-\xi_2|\le\delta$ implies $|\phi_2^{(i)}(t,\xi_1^{(i)}(t))-\phi_2^{(i)}(t,\xi_2^{(i)}(t))|\le\varepsilon$ for all $t\in\mathbb{R}$, $\xi_1,\xi_2\in K_i$. Therefore, there exist $m(\varepsilon)\ge1$ and $z_k^{(i)}\in K_i$ for $k=1,\dots,m(\varepsilon)$ such that

$$K_i\subset\bigcup_{k=1}^{m(\varepsilon)}B_\delta(z_k^{(i)},\delta).$$

Then, we have

$$|\phi_2^{(i)}(t,h_1^{(i)}(t))|\le\varepsilon+\sum_{k=1}^{m(\varepsilon)}|\phi_2^{(i)}(t,z_i)|.$$

Since

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,z_k^{(i)})|\Big)d\mu(t)=0,\quad k\in\{1,\dots,m(\varepsilon)\},$$

we deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,h_1^{(i)}(t))|\Big)d\mu(t)\le\varepsilon$$

for every $\varepsilon>0$. This implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\phi_2^{(i)}(\theta,h_1^{(i)}(t))|\Big)d\mu(t)=0.$$

Consequently, for $i = 0, 1, \ldots, n$, the function $t \mapsto \phi_2^{(i)}(t, h^{(i)}(t))$ is (μ, ν)-ergodic of class r. By using Definition 15 and Definition 20, it follows that the function $t \mapsto \phi(t, h(t))$ belongs to $PAA^{(n)}(\mathbb{R}; X_2, \mu, \nu, r)$. $\square$

For $\mu \in \mathcal{M}$, we formulate the following two conditions:

($\mathbf{H_3}$) for all $a, b, c \in \mathbb{R}$ such that $0 \leq a < b \leq c$, there exist $\delta_0, \alpha_0 > 0$ such that $|\delta| \geq \delta_0$ implies $\mu(a+\delta, b+\delta) \leq \alpha_0 \mu(\delta, c+\delta)$,

($\mathbf{H_4}$) for every $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that if $A \in \mathcal{B}$ and $A \cap I = \emptyset$, then $\mu(\{a + \tau \colon a \in A\}) \leq \beta \mu(A)$.

We have the following result due to [4].

Lemma 29. *([4]) ($\boldsymbol{H_4}$) implies ($\boldsymbol{H_3}$).*

Proposition 30. *([5]) Let $\mu, \nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_3}$) and let $f \in PAA(\mathbb{R}; X, \mu, \nu)$ be such that $f = g + h$, where $g \in AA(\mathbb{R}, X)$ and $h \in \mathcal{E}(\mathbb{R}, X, \mu, \nu)$. Then the set $\{g(t) \colon t \in \mathbb{R}\}$ is contained in the closure of the range of f, that is*

$$\{g(t) \colon t \in \mathbb{R}\} \subset \overline{\{f(t) \colon t \in \mathbb{R}\}}.$$

Corollary 31. *([5]) Assume that ($\boldsymbol{H_3}$) holds for some $\mu, \nu \in \mathcal{M}$. Then the decomposition of a (μ, ν)-pseudo almost automorphic function of the form $f = g + \phi$, where $g \in AA(\mathbb{R}; X)$ and $\phi \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique.*

The following corollary is a consequence of Corollary 31.

Proposition 32. *Assume that ($\boldsymbol{H_3}$) holds for some $\mu, \nu \in \mathcal{M}$. Then the decomposition of a (μ, ν)-pseudo-almost automorphic function $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$, is unique.*

Proof. Let $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA^{(n)}(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Then $\phi_1^{(i)} \in AA(\mathbb{R}; X)$ and $\phi_2^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$ for $i = 0, 1, \ldots, n$. As a consequence of Corollary 31, the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi^{(i)} = \phi_1^{(i)} + \phi_2^{(i)}$, where $\phi_1^{(i)} \in AA(\mathbb{R}; X)$ and $\phi_2^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu, \nu)$, is unique. We also have $PAA(\mathbb{R}; X, \mu, \nu, r) \subset PAA(\mathbb{R}; X, \mu, \nu)$. Thus, the decomposition of a (μ, ν)-pseudo-almost periodic function $\phi^{(i)} = \phi_1^{(i)} + \phi_2^{(i)}$ of class r, where $\phi_1^{(i)} \in AA(\mathbb{R}; X)$ and $\phi_2 \in \mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$, is also unique. Consequently, we get the desired result. $\square$

Definition 33. *Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2, and denote this as $\mu_1 \sim \mu_2$, if there exist constants $\alpha, \beta > 0$ and a bounded interval I (we do not exclude the case $I = \emptyset$) such that the following condition holds: if $A \in \mathcal{B}$ and $A \cap I = \emptyset$, then $\alpha\mu_1(A) \le \mu_2(A) \le \beta\mu_1(A)$.*

We know from [4] that the relation $\sim$ is a binary equivalence relation on $\mathcal{M}$. We denote by $cl(\mu)$ the equivalence class of a given measure $\mu \in \mathcal{M}$, that is

$$cl(\mu) = \{\varpi \in \mathcal{M} \colon \mu \sim \varpi\}.$$

Theorem 34. *Let $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then the spaces $PAA^{(n)}(\mathbb{R}; X, \mu_1, \nu_1, r)$ and $PAA^{(n)}(\mathbb{R}; X, \mu_2, \nu_2, r)$ coincide, that is*

$$PAA^{(n)}(\mathbb{R}; X, \mu_1, \nu_1, r) = PAA^{(n)}(\mathbb{R}; X, \mu_2, \nu_2, r).$$

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, there exist some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (we do not exclude the case $I = \emptyset$) such that $\alpha_1\mu_1(A) \le \mu_2(A) \le \beta_1\mu_1(A)$ and $\alpha_2\nu_1(A) \le \nu_2(A) \le \beta_2\nu_1(A)$ for each $A \in \mathcal{B}$ satisfying $A \cap I = \emptyset$. In particular

$$\frac{1}{\beta_2\nu_1(A)} \le \frac{1}{\nu_2(A)} \le \frac{1}{\alpha_2\nu_1(A)}.$$

Let $f \in C_b^n(\mathbb{R}, X)$. Since $\mu_1 \sim \mu_2$ and $\mathcal{B}$ is the Lebesgue σ-field, we obtain for τ sufficiently large that the quotient

$$\frac{\alpha_1\mu_1\Big(\Big\{t \in [-\tau, \tau] \setminus I \colon \sup\limits_{\theta\in[t-r,t]} |f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau, \tau] \setminus I)}$$

is not greater than

$$\frac{\mu_2\Big(\Big\{t \in [-\tau, \tau] \setminus I \colon \sup\limits_{\theta\in[t-r,t]} |f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\nu_2([-\tau, \tau] \setminus I)},$$

which in turn is not greater than

$$\frac{\beta_1\mu_1\Big(\Big\{t \in [-\tau, \tau] \setminus I \colon \sup\limits_{\theta\in[t-r,t]} |f^{(i)}(\theta)| > \varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau, \tau] \setminus I)}.$$

By using Theorem 27, we deduce that $\mathcal{E}^{(n)}(\mathbb{R}, X, \mu_1, \nu_1, r) = \mathcal{E}^{(n)}(\mathbb{R}, X, \mu_2, \nu_2, r)$. From the definition of a (μ, ν)-pseudo almost periodic function, we deduce that $PAA^{(n)}(\mathbb{R}; X, \mu_1, \nu_1, r) = PAA^{(n)}(\mathbb{R}; X, \mu_2, \nu_2, r)$. $\square$

Proposition 35. *([5]) Let $\mu, \nu \in \mathcal{M}$ satisfy $(\boldsymbol{H_4})$. Then $PAA(\mathbb{R}, X, \mu, \nu)$ is translation invariant, that is $f \in PAA(\mathbb{R}, X, \mu, \nu)$ implies $f_\alpha \in PAA(\mathbb{R}, X, \mu, \nu)$ for all $\alpha \in \mathbb{R}$.*

Corollary 36. *Let $\mu, \nu \in \mathcal{M}$ satisfy $(\boldsymbol{H_4})$. Then $PAA^{(n)}(\mathbb{R}, X, \mu, \nu)$ is translation invariant, that is $f \in PAA^{(n)}(\mathbb{R}, X, \mu, \nu)$ implies $f_\alpha \in PAA^{(n)}(\mathbb{R}, X, \mu, \nu)$ for all $\alpha \in \mathbb{R}$.*

For $\mu \in \mathcal{M}$ and $\alpha \in \mathbb{R}$, we denote by μ_α the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\alpha(A) = \mu(\{a + \alpha \colon a \in A\}). \tag{8}$$

Lemma 37. *([4]) Let $\mu \in \mathcal{M}$ satisfies $(\boldsymbol{H_3})$. Then the measures μ and μ_α are equivalent for all $\alpha \in \mathbb{R}$.*

Lemma 38. *([4]) The condition $(\boldsymbol{H_3})$ implies that*

$$\limsup_{\tau \to +\infty} \frac{\mu([-\tau - \sigma, \tau + \sigma])}{\mu([-\tau, \tau])} < +\infty$$

for every $\sigma > 0$.

We have the following result.

Theorem 39. *Assume that $(\boldsymbol{H_3})$ holds for some $\mu, \nu \in \mathcal{M}$ and*

$$\phi \in PAA_c^{(n)}(\mathbb{R}; X, \mu, \nu, r).$$

Then the function $t \mapsto \phi_t$ belongs to $PAA_c^{(n)}(C([-r, 0]; X), \mu, \nu, r)$.

Proof. Assume that $\phi = g + h$, where $g \in AA^{(n)}(\mathbb{R}; X)$ and $h \in \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Then we can see that $\phi_t^{(i)} = g_t^{(i)} + h_t^{(i)}$ for $i = 0, 1, \ldots, n$, and by [7], the function $t \mapsto g_t^{(i)}$ belongs to $AA_c(C([-r, 0]; X))$, which implies that

$$g_t \in AA_c^{(n)}(C([-r, 0]; X)).$$

Let $i \in \{0, 1, \ldots, n\}$, $\alpha \in \mathbb{R}$ and μ_α, ν_α be the positive measures defined by equation (8). Let us denote

$$M_\alpha(\tau) = \frac{1}{\nu_\alpha([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu_\alpha(t).$$

By using Lemma 37, it follows that μ_α and μ are equivalent, and ν_α and ν are also equivalent. Then, by using Theorem 34, we have $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu_\alpha, \nu_\alpha, r) = \mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Therefore $h^{(i)} \in \mathcal{E}(\mathbb{R}; X, \mu_\alpha, \nu_\alpha, r)$, that is

$$\lim_{\tau\to+\infty} M_\alpha(\tau) = 0.$$

On the other hand, for $r > 0$, we have

$$\begin{aligned}
& \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} \Big[\sup_{\xi\in[-r,0]} |h^{(i)}(\theta+\xi)| \Big] \Big) d\mu(t) \leq \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-2r,t]} |h^{(i)}(\theta)| \Big) d\mu(t) \leq \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-2r,t-r]} |h^{(i)}(\theta)| + \sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu(t) \leq \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau-r}^{+\tau-r} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu(t+r) + \\
+\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu(t) \leq \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau-r}^{+\tau+r} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu(t+r) + \\
+\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu(t).
\end{aligned}$$

The first component of the last sum can be written as

$$\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])} \cdot \frac{1}{\nu([-\tau-r,\tau+r])} \int_{-\tau-r}^{+\tau+r} \Big(\sup_{\theta\in[t-r,t]} |h^{(i)}(\theta)| \Big) d\mu(t+r),$$

and we see that it is not greater than

$$\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])} \cdot M_r(\tau+r).$$

Consequently, we obtain

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\Big[\sup_{\xi\in[-r,0]}|h^{(i)}(\theta+\xi)|\Big]\Big)d\mu(t)\leq\\
\leq\ &\frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\cdot M_r(\tau+r)+\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}|h^{(i)}(\theta)|\Big)d\mu(t),
\end{aligned}$$

which shows, by Lemmas 37–38, that $\phi_t^{(i)}\in PAA_c(C([-r,0];X),\mu,\nu,r)$. Thus, we obtain the desired result. $\square$

5. C^n-(μ,ν)-pseudo almost automorphic solutions of class r

In what follows, we will be looking at the existence of bounded integral solutions of class r of the equation (1).

Assume that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ hold and that the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. For $f\in BC(\mathbb{R};X)$ and $t\in\mathbb{R}$, we denote by $\Gamma f(t)$ the function from $C(\mathbb{R};X)$ defined as follows

$$\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds,$$

where Π^s and Π^u are the projections of C_0 onto the stable and unstable subspaces, respectively.

Proposition 40. *([9]) If $f\in BC(\mathbb{R};X)$, then there exists a unique bounded solution u of the equation* (1) *on $\mathbb{R}$, given by $u_t=\Gamma f(t)$.*

Proposition 41. *([7]) If $h\in AA_c(\mathbb{R},X)$, then the function $t\mapsto\Gamma h(t)(0)$ belongs to $AA_c(\mathbb{R},X)$.*

Corollary 42. *If $h\in AA_c^{(n)}(\mathbb{R};X)$, then the function $t\mapsto\Gamma h(t)(0)$ belongs to $AA_c^{(n)}(\mathbb{R},X)$.*

Proof. In fact, since $h\in AA_c^{(n)}(\mathbb{R};X)$, we have $h^{(i)}\in AA_c(\mathbb{R};X)$ for $i=0,1,\ldots,n$. Thus the function $t\mapsto\Gamma h^{(i)}(t)(0)$ belongs to $AA_c(\mathbb{R},X)$ for $i=0,1,\ldots,n$. $\square$

Theorem 43. *If $\mu,\nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_3}$) and $g \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$, then the function $t \mapsto \Gamma g(t)(0)$ belongs to $\mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$.*

Proof. In fact, since $g \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$, we have $g^{(i)} \in \mathcal{E}(\mathbb{R};X,\mu,\nu,r)$ for $i = 0,1,\ldots,n$. For $\tau > 0$, we get

$$\begin{aligned}
&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|\Gamma g^{(i)}(\theta)|ds\Big)d\mu(t) \leq \\
\leq\ & \overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}|\Pi^s|\ |g^{(i)}(s)|ds\Big)d\mu(t) + \\
+\ & \overline{M}\widetilde{M}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(\theta-s)}|\Pi^u|\ |g^{(i)}(s)|ds\Big)d\mu(t) \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) + \\
+\ & \overline{M}\widetilde{M}|\Pi^u|\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty}e^{\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t).
\end{aligned}$$

On the one hand, using Fubini's theorem, we have

$$\begin{aligned}
&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) \leq \\
\leq\ & \int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}e^{\omega r}\int_{-\infty}^{t}e^{-\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) \leq \\
\leq\ & e^{\omega r}\int_{-\tau}^{\tau}\int_{-\infty}^{t}e^{-\omega(t-s)}|g^{(i)}(s)|ds d\mu(t) \leq \\
\leq\ & e^{\omega r}\int_{-\tau}^{\tau}\int_{0}^{+\infty}e^{-\omega s}|g^{(i)}(t-s)|ds d\mu(t) \leq \\
\leq\ & e^{\omega r}\int_{0}^{+\infty}e^{-\omega s}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds.
\end{aligned}$$

For all $s \in \mathbb{R}^+$, we deduce by Corollary 36 that

$$\lim_{\tau\to+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t) = 0$$

and

$$\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t) \leq \frac{e^{-\omega s}\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\|g^{(i)}\|_{\infty},$$

where $\|g^{(i)}\|_\infty = \sup_{t\in\mathbb{R}} |g^{(i)}(t)|$. Since $g^{(i)}$ is a bounded function, the function $s \mapsto \dfrac{e^{-\omega s}\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\|g^{(i)}\|_\infty$ belongs to the space $L^1([0,+\infty))$. In view of the Lebesgue dominated convergence theorem, it follows that

$$e^{\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty} e^{-\omega s}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds = 0.$$

On the other hand, by Fubini's theorem, we also have

$$\begin{aligned}
&\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{\theta}^{+\infty} e^{\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) \leq \\
\leq\ &\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}\int_{t-r}^{+\infty} e^{\omega(t-s)}|g^{(i)}(s)|ds\Big)d\mu(t) \leq \\
\leq\ &\int_{-\tau}^{\tau}\int_{t-r}^{+\infty} e^{\omega(t-s)}|g^{(i)}(s)|dsd\mu(t) \leq \\
\leq\ &\int_{-\tau}^{\tau}\int_{-\infty}^{r} e^{\omega s}|g^{(i)}(t-s)|dsd\mu(t) \leq \\
\leq\ &\int_{-\infty}^{r} e^{\omega s}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds.
\end{aligned}$$

Since the function $s \mapsto \dfrac{e^{-\omega s}\mu([-\tau,\tau])}{\nu([-\tau,\tau])}\|g^{(i)}\|_\infty$ belongs to $L^1((-\infty,r])$, the same reasoning as above gives

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r} e^{\omega s}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}|g^{(i)}(t-s)|d\mu(t)ds = 0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in[t-r,t]}|(\Gamma g^{(i)})(\theta)|\Big)d\mu(t) = 0,$$

which implies that for $i = 0,1,\dots,n$ the function $t \mapsto \Gamma g^{(i)}(t)(0)$ belongs to $\mathcal{E}(\mathbb{R}; X, \mu, \nu, r)$. Thus, we obtain the desired result. $\square$

For the existence of C^n-(μ,ν)-pseudo almost periodic solution of class r, we make the following assumption:

(H$_5$) the function $f\colon \mathbb{R}\to X$ is C^n-(μ,ν)-compact pseudo almost automorphic of class r.

Proposition 44. *Assume* ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_3}$) *and* ($\boldsymbol{H_5}$) *hold. Then the equation* (1) *has a unique* C^n-(μ,ν)*-compact pseudo almost automorphic solution of class* r.

Proof. Since f is a C^n-(μ,ν)-compact pseudo almost automorphic function, f has a decomposition $f = f_1 + f_2$, where $f_1 \in AA_c^{(n)}(\mathbb{R};X)$ and $f_2 \in \mathcal{E}^{(n)}(\mathbb{R};X,\mu,\nu,r)$. By using Proposition 40, Corollary 42 and Theorem 43, we get the desired result. □

Our next objective is to show the existence of C^n-(μ,ν)-pseudo almost periodic solutions of class r for the following problem

$$u'(t) = Au(t) + L(u_t) + f(t,u_t), \quad t \in \mathbb{R}, \tag{9}$$

where the function $f\colon \mathbb{R}\times C \to X$ is continuous. With this aim, we make the following assumptions:

($\mathbf{H_6}$) the unstable space U satisfies: $U \equiv \{0\}$,

($\mathbf{H_7}$) the function $f\colon \mathbb{R}\times C \to X$ is uniformly C^n-(μ,ν)-pseudo compact almost automorphic, and for some $1 \le p < +\infty$, there exists a function $L_f \in L^p(\mathbb{R},\mathbb{R}^+)$ which satisfies Theorem 28 and the following condition:

$$|f(t,\varphi_1) - f(t,\varphi_2)| \le L_f(t)|\varphi_1 - \varphi_2|$$

for all $t \in \mathbb{R}$ and $\varphi_1,\varphi_2 \in C([-r,0];X_0)$.

Theorem 45. *Assume* ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$), ($\boldsymbol{H_4}$), ($\boldsymbol{H_6}$) *and* ($\boldsymbol{H_7}$) *hold. Then the equation* (9) *has a unique* C^n-(μ,ν)*-pseudo compact almost automorphic mild solution of class* r.

Proof. Let x be a function in $PAA_c^{(n)}(\mathbb{R};X,\mu,\nu,r)$. From Theorem 39, it follows that the function $t \mapsto x_t$ belongs to $PAA_c^{(n)}(C([-r,0];X),\mu,r)$. Hence, Theorem 28 implies that the function $g(.) := f(.,x_.)$ is in $PAA_c^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Since $U \equiv \{0\}$, we obtain $\Pi^u \equiv 0$. Consider now the mapping

$$\mathcal{H}\colon PAA_c(\mathbb{R};X,\mu,\nu,r) \to PAA_c(\mathbb{R};X,\mu,\nu,r)$$

defined as follows:

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds\Big](0), \quad t \in \mathbb{R}.$$

From Proposition 40, Corollary 42 and Theorem 43, we can infer that $\mathcal{H}$ maps $PAA_c^{(n)}(\mathbb{R};X,\mu,\nu,r)$ into $PAA_c^{(n)}(\mathbb{R};X,\mu,\nu,r)$. It suffices now to show that the operator $\mathcal{H}$ has a unique fixed point in $PAA_c^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Consider two cases.

Case 1: $L_f \in L^1(\mathbb{R})$ (that is $p=1$). Let $x_1, x_2 \in PAA_c^{(n)}(\mathbb{R};X,\mu,\nu,r)$. Then $x_1^{(i)}, x_2^{(i)} \in PAA_c(\mathbb{R};X,\mu,\nu,r)$ for $i=0,1,\dots,n$ and

$$\begin{aligned}
&|\mathcal{H}x_1^{(i)}(t)-\mathcal{H}x_2^{(i)}(t)| \leq \\
\leq\ & \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s}^{(i)})-f(s,x_{2s}^{(i)})))ds\Big| \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\ |x_1^{(i)}-x_2^{(i)}|\int_{-\infty}^{t}e^{-\omega(t-s)}L_f(s)ds \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\ |x_1^{(i)}-x_2^{(i)}|\int_{-\infty}^{t}L_f(s)ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&|\mathcal{H}^2x_1^{(i)}(t)-\mathcal{H}^2x_2^{(i)}(t)| \leq \\
\leq\ & \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,\mathcal{H}x_{1s}^{(i)})-f(s,\mathcal{H}x_{2s}^{(i)}))ds\Big| \leq \\
\leq\ & (\overline{M}\widetilde{M}|\Pi^s|)^2|x_1^{(i)}-x_2^{(i)}|\int_{-\infty}^{t}L_f(s)\int_{-\infty}^{s}L_f(\delta)d\delta ds \leq \\
\leq\ & \frac{(\overline{M}\widetilde{M}|\Pi^s|)^2}{2}\Big(\int_{-\infty}^{t}L_f(s)ds\Big)^2|x_1^{(i)}-x_2^{(i)}|.
\end{aligned}$$

Induction with respect to n, gives

$$|\ \mathcal{H}^nx_1^{(i)}(t)-\mathcal{H}^nx_2^{(i)}(t)| \leq \frac{(\overline{M}\widetilde{M}|\Pi^s|)^n}{n!}\Big(\int_{-\infty}^{t}L_f(s)ds\Big)^n|x_1^{(i)}-x_2^{(i)}|.$$

Therefore

$$|\ \mathcal{H}^nx_1^{(i)}-\mathcal{H}^nx_2^{(i)}| \leq \frac{(\overline{M}\widetilde{M}|\Pi^s|\ |L_f|_{L^1(\mathbb{R})})^n}{n!}|x_1^{(i)}-x_2^{(i)}|,$$

which implies that

$$\sum_{i=0}^{n}|\ \mathcal{H}^nx_1^{(i)}-\mathcal{H}^nx_2^{(i)}| \leq \frac{(\overline{M}\widetilde{M}|\Pi^s|\ |L_f|_{L^1(\mathbb{R})})^n}{n!}|x_1-x_2|_n.$$

Let n_0 be such that $(\overline{M}\widetilde{M}|\Pi^s|\ |L_f|_{L^1(\mathbb{R})})^{n_0}/n_0! < 1$. By the Banach fixed point theorem, $\mathcal{H}$ has a unique fixed point and this fixed point satisfies the integral equation

$$u_t = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,u_s))ds.$$

Case 2: $L_f \in L^p(\mathbb{R})$ for some $1 < p < \infty$. First, put

$$\lambda(t) = \int_{-\infty}^{t}(L_f(s))^p ds.$$

Then we define an equivalent norm over $PAA(\mathbb{R};X)$ by the formula

$$|f|_c = \sup_{t\in\mathbb{R}} e^{-c\lambda(t)}|f(t)|,$$

where c is a fixed positive number to be specified later. By using the Hölder inequality, we have

$$\begin{aligned}
& |\mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \leq \\
\leq\ & \Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s,x_{1s}^{(i)}) - f(s,x_{2s}^{(i)})))ds\Big| \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t} e^{-\omega(t-s)}L_f(s)|x_{1s}^{(i)} - x_{2s}^{(i)}|ds \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t} e^{-\omega(t-s)}e^{-c\mu(s)}e^{c\mu(s)}L_f(s)|x_{1s}^{(i)} - x_{2s}^{(i)}|ds \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\Big(e^{-\omega(t-s)}e^{c\mu(s)}L_f(s)\Big)\sup_{s\in\mathbb{R}}\Big(e^{-c\mu(s)}|x_1^{(i)}(s) - x_2^{(i)}(s)|\Big)ds \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t}\Big(e^{-\omega(t-s)}e^{c\mu(s)}L_f(s)ds\Big)|x_1^{(i)} - x_2^{(i)}|_c \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\Big(\int_{-\infty}^{t} e^{pc\mu(s)}(L_f(s))^p ds\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{t} e^{-\omega q(t-s)}ds\Big)^{\frac{1}{q}}|x_1^{(i)} - x_2^{(i)}|_c \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\Big(\int_{-\infty}^{t} e^{pc\mu(s)}\lambda'(s)ds\Big)^{\frac{1}{p}}\Big(\int_{-\infty}^{t} e^{-\omega q(t-s)}ds\Big)^{\frac{1}{q}}|x_1^{(i)} - x_2^{(i)}|_c \leq \\
\leq\ & \overline{M}\widetilde{M}|\Pi^s|\left(\frac{1}{(pc)^{\frac{1}{p}}}\cdot\frac{1}{(\omega q)^{\frac{1}{q}}}\right)e^{c\mu(t)}|x_1^{(i)} - x_2^{(i)}|_c.
\end{aligned}$$

Consequently

$$|\mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)|_c \quad \leq \quad \frac{\overline{M}\widetilde{M}|\Pi^s|}{(pc)^{\frac{1}{p}} \cdot (\omega q)^{\frac{1}{q}}}|x_1^{(i)} - x_2^{(i)}|_c,$$

which implies

$$\sum_{i=0}^{n} |\mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)|_c \quad \leq \quad \frac{\overline{M}\widetilde{M}|\Pi^s|}{(pc)^{\frac{1}{p}} \cdot (\omega q)^{\frac{1}{q}}}|x_1 - x_2|_{c,n},$$

where

$$|x|_{c,n} = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} |x^{(i)}(t)|_c \text{ for all } x \in C_b^n(\mathbb{R}, X).$$

Since the function $c \mapsto \dfrac{1}{(pc)^{\frac{1}{p}}}$ converges to 0 when c tends to $+\infty$, it follows that for $c > 0$ large enough, we have $\dfrac{\overline{M}\widetilde{M}|\Pi^s|}{(pc)^{\frac{1}{p}} \cdot (\omega q)^{\frac{1}{q}}} < 1$. Thus $\mathcal{H}$ is a contractive mapping. By using the same argument as in Theorem 3.3 of [14], we conclude that there is a (μ, ν)-unique pseudo almost automorphic integral solution of the equation (9), which ends the proof. □

Proposition 46. *Assume ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_2}$), ($\boldsymbol{H_4}$) hold and f is a Lipschitz continuous function with respect to the second argument. If the Lipschitz constant $Lip(f)$ of f satisfies the inequality*

$$Lip(f) < \frac{\omega}{\overline{M}\widetilde{M}|\Pi^s|},$$

then the equation (9) *has a unique (μ, ν)-pseudo almost automorphic solution of class r.*

Proof. Let us denote $k = Lip(f)$. We have

$$\begin{aligned}
&|\mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \leq \\
\leq \quad & \Big| \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s, x_{1s}^{(i)}) - f(s, x_{2s}^{(i)})))ds \Big| \leq \\
\leq \quad & |\Pi^s|\overline{M}\widetilde{M}|x_1^{(i)} - x_2^{(i)}|k\Big(\int_{-\infty}^{t} e^{-\omega(t-s)}\Big) \leq \frac{|\Pi^s|\overline{M}\widetilde{M}k|x_1^{(i)} - x_2^{(i)}|}{\omega},
\end{aligned}$$

which implies

$$\sum_{i=0}^{n} |\ \mathcal{H}x_1^{(i)}(t) - \mathcal{H}x_2^{(i)}(t)| \quad \leq \quad \frac{|\Pi^s|\overline{M}\widetilde{M}k|x_1 - x_2|_n}{\omega}.$$

Consequently, if $k < \dfrac{\omega}{\overline{M}\widetilde{M}|\Pi^s|}$, then $\mathcal{H}$ is a strict contraction. $\square$

6. Application

For illustration, we will study the existence of solutions for the following model:

$$\begin{cases} \dfrac{\partial}{\partial t}z(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x) + \displaystyle\int_{-r}^{0} G(\theta)z(t+\theta,x)d\theta + \\ \qquad\qquad + \displaystyle\int_{-r}^{0} h(\theta, z(t+\theta,x))d\theta + \\ \qquad\qquad + \exp(\sin(\alpha t) + \sin(\beta t)) + \cos(t),\ t\in\mathbb{R},\ x\in[0,\pi], \\ z(t,0) = z(t,\pi) = 0,\ t\in\mathbb{R}, \end{cases} \tag{10}$$

where $\alpha, \beta \in \mathbb{R}$, the function $G\colon [-r,0] \to \mathbb{R}$ is continuous and the function $h\colon [-r,0]\times\mathbb{R} \to \mathbb{R}$ satisfies Lipschitz condition with respect to the second argument. To rewrite the equation (10) in the abstract form, we introduce the space $X = C_0([0,\pi];\mathbb{R})$ of continuous functions from $[0,\pi]$ to $\mathbb{R}^+$ equipped with the uniform norm topology.

Let $A\colon D(A) \to X$ be defined by

$$\begin{cases} D(A) = \{y \in X \cap C^2([0,\pi];\mathbb{R})\colon y'' \in X\}, \\ Ay = y''. \end{cases}$$

Then A satisfies the Hille-Yosida condition in X. Moreover, the part A_0 of A in $\overline{D(A)}$ is a generator of the strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied. If we define $f\colon \mathbb{R}\times C \to X$ and $L\colon C \to X$ as follows

$$\begin{aligned} f(t,\varphi)(x) &= \exp(\sin(\alpha t) + \sin(\beta t)) + \cos(t) + \int_{-r}^{0} h(\theta, \varphi(\theta)(x))d\theta, \\ L(\varphi)(x) &= \int_{-r}^{0} G(\theta)\varphi(\theta)(x)d\theta,\ \ x\in[0,\pi],\quad t\in\mathbb{R}, \end{aligned}$$

and we set $v(t) = z(t,x)$, then the equation (10) takes the following abstract form

$$v'(t) = Av(t) + L(v_t) + f(t, v_t), \quad t \in \mathbb{R}. \tag{11}$$

Consider the measures μ and ν whose Radon-Nikodym derivatives $\rho_1, \rho_2 \colon \mathbb{R} \to \mathbb{R}$ are defined as follows:

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0, \\ e^t & \text{for } t \leq 0, \end{cases} \qquad \rho_2(t) = |t|, \qquad t \in \mathbb{R}.$$

Then $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$, and

$$\mu(A) = \int_A \rho_1(t)dt, \quad \nu(A) = \int_A \rho_2(t)dt, \quad A \in \mathcal{B}.$$

From [4], we know that $\mu, \nu \in \mathcal{M}$ and that μ, ν satisfy $(\mathbf{H_4})$. Furthermore, by [12, Example 4.5], the function $t \mapsto \exp(\sin(\alpha t) + \sin(\beta t))$ is C^n-almost automorphic if α and β are incommensurate real numbers (i.e. α and β are relatively prime). We also have

$$\limsup_{\tau \to +\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau \to +\infty} \frac{\displaystyle\int_{-\tau}^{0} e^t dt + \int_0^{\tau} dt}{\displaystyle 2\int_0^{\tau} t dt} = \limsup_{\tau \to +\infty} \frac{1 - e^{-\tau} + \tau}{\tau^2} = 0 < \infty,$$

which implies $(\mathbf{H_2})$.

For all $t \in \mathbb{R}$ and $i = 0, 1, \ldots, n$, we have $|\cos^{(i)} t| \leq 1$, which implies

$$\begin{aligned}
& \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in [t-r,t]} |\cos^{(i)}(\theta)| d\mu(t) = \\
= \; & \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \Big(\int_{-\tau}^{0} \sup_{\theta \in [t-r,t]} |\cos^{(i)}(\theta)| e^t dt + \int_0^{\tau} \sup_{\theta \in [t-r,t]} |\cos^{(i)}(\theta)| dt \Big) \leq \\
\leq \; & \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \Big(\int_{-\tau}^{0} e^t dt + \int_0^{\tau} dt \Big) \leq \lim_{\tau \to +\infty} \frac{1 - e^{-\tau} + \tau}{\tau^2} = 0.
\end{aligned}$$

It follows that the function $t \mapsto \cos^{(i)} t$ belongs to $\mathcal{E}^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Consequently, the function f belongs to $PAA^{(n)}(\mathbb{R}; X, \mu, \nu, r)$. Moreover, L is a bounded linear operator from C to X. Let k be the Lipschitz constant of h. Then for every

$\varphi_1, \varphi_2 \in C$ and $t \geq 0$, we have

$$\begin{aligned} |f(t,\varphi_1) - f(t,\varphi_2)| &= r \sup_{0\leq x\leq \pi} |h(t,\varphi_1)(x) - h(t,\varphi_2)(x)| \leq \\ &\leq kr \sup_{\substack{-r<\theta\leq 0\\ 0\leq x\leq \pi}} |\varphi_1(\theta)(x) - \varphi_2(\theta)(x)|. \end{aligned}$$

Consequently, we conclude that f is Lipschitz continuous.

To show that $(\mathbf{H_6})$ holds, we need the following result established in [9].

Lemma 47. *([9]) If* $\displaystyle\int_{-r}^{0} |G(\theta)|d\theta < 1$, *then the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is hyperbolic and the unstable space* U *satisfies* $U \equiv \{0\}$.

Now, by Proposition 46, we deduce the following result.

Theorem 48. *Under the above assumptions, if Lip*(h) *is small enough, then the equation* (11) *has a unique* C^n*-*(μ,ν)*-pseudo almost automorphic solution* v *of class* r.

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