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Transient behaviour of instantaneous and average availabilities in non-Markovian configuration

Keywords

instantaneous availability, average availability, exponential distribution, gamma distribution

Abstract

In this work, we calculate the exact instantaneous and average availabilities for a system in which the failure distribution is a gamma distribution with a rational shape parameter α, and the repair time distribution is exponential. Various regimes exist, for which the availabilities may or may not attain values below the asymptotic limit. This is an example of configurations where using the steady-state value may lead to an overoptimistic assessment of the availability of an equipment or system during its mission time.

1. Introduction

The availability of a repairable component is usually given by the ratio MTTF/(MTTF + MTTR), where MTTF is the Mean Time To Failure, and MTTR is the Mean Time To Repair (Henley & Kumamoto, 1991; Kuo & Zuo, 2003; Rausand & Høyland, 2004; Pham, 2006). The advantage of this expression lies in the ease of its computation for many lifetime and repair distributions. It is customary to consider the specific case of exponential failure and repair time distributions (we shall come back to this configuration in Section 2.2) for which calculations are easy, assuming as good as new repairs. In this case, the availability decreases monotonously from 1 to its steady-state value, which is a lower bound.

The assumption of exponential distributions for both failures and repairs has long been questioned. The availability for various pairs of distributions has been investigated, sometimes analytically, more frequently numerically. A large body of work has been devoted to confidence limits or bounds on the steady-state availability (Chandrasekhar & Natarajan, 1996, 1997; Butterworth & Nikolaisen, 1973; Zeiler et al., 2017) for a number of distributions (gamma, lognormal, Weibull, Inverse Gaussian, etc.). Few analytical results are available (Pham-Gia & Turkkan, 1999; Sarkar & Chaudhuri, 1999; Tillman et al., 1983; Tanguy et al., 2019), mostly in the case when one or both exponential distributions have been replaced by gamma distributions. The case of a constant repair time has also been considered, for instance by Butterworth and Nikolaisen (1973), and Rausand and Høyland (2004). Simulations are often the only way to obtain answers (Diallo & Aït-Kadi, 2010; Rao & Naikan, 2015; and references therein).

In a few instances (Butterworth & Nikolaisen, 1973; Pham-Gia & Turkkan, 1999; Zeiler et al., 2017), curves or tabulated values clearly show that the availability, starting from 1 at the time origin, reaches values below the asymptotic, steady-state limit. This feature is important, since it means that during its mission time, the device's availability may be smaller than the steady-state value, which is not a lower bound, and therefore cannot be considered as a conservative estimate of the availability anymore. Overestimating the availabilities of its elements could well lead to large errors in the assessment of a large system's

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availability.

Recent studies on transient availabilities in cluster configurations and communication channels (Distefano et al., 2010; Carnevali et al., 2015) have shown that their behaviours can be very different: (i) the limit may be attained after a few MTTF (ii) oscillations can be observed, and the asymptotic value is not a lower bound anymore. Similar strong oscillations of $A(t)$ have also appeared in various studies (Zeiler et al., 2017). In Figure (3c) of (Carnevali et al., 2015), a *conservative* approximation of the availability, computed using stochastic Time Petri Nets (sTPN), is defined by a 3-step function. The steady-state unavailability underestimates the transitory one by a factor larger than 3. Figure 4 of (Distefano et al., 2010) shows that the steady-state unavailability can be more than a factor two smaller than the transitory one.

Even though availability oscillations have been demonstrated in the above-mentioned studies, they do not seem to have attracted the attention they deserve. In a previous work, we proposed criteria allowing to determine whether the minimum, transitory availability is smaller than the steady-state value for general failure and repair distributions (Tanguy, 2020). In such cases, caution should be exercised when using the MTTF/(MTTF + MTTR) formula.

The aim of the present work is to generalize previous studies for a gamma failure time distribution and an exponential repair time distribution (Sarkar & Chaudhuri, 1999; Tillman et al., 1983; Tanguy et al., 2019), in which the shape parameter of the gamma distribution is not merely an integer, but can be a rational number. We also investigate the case of the average availability considered by Pham-Gia and Turkkan (1999), who computed it by numerically solving integral equations. The underlying reason of our study is to ascertain whether the time averaging of the availability between 0 and t can somehow smooth out the oscillations, so that the averaged availability remains *larger* than the steady-state value. This might have economic consequences in the design of Service Level Agreements (SLA), in which service warranties are sold at a premium. Since decreasing the unavailability of components of a system is often costly, considering an average availability instead of the instantaneous one may lead to important cost savings for manufacturers and service providers.

The chapter is organized as follows. Notations and definitions are recalled in Section 2. The exact expression of the instantaneous availability $A(t)$ is derived analytically in Section 3, when the shape parameter α of the failure time gamma distribution is a rational number. Section 4 is devoted to the determination of the average availability $\bar{A}(t)$ in the same conditions. We then survey in Section 5 the different regimes for the pair of availabilities as a function of α. In Section 6, we adapt our criteria defined in (Tanguy, 2020) to determine whether the average availability can temporarily be smaller than the steady-state value. We then conclude by a few recommendations.

2. Notations and definitions

2.1. General definitions from renewal theory

We use the standard description used in renewal theory, in which the densities of probabilities of failure and repair are $f(t)$ and $g(t)$, respectively. The reliability $F(t)$ is given by

$$F(t) = 1 - \int_0^t f(\tau)\, d\tau \qquad (1)$$

with $F(0) = 1$. Another important quantity in the calculations is the Laplace transform of f (and g), defined by

$$\tilde{f}(s) = \int_0^\infty f(\tau)\ e^{-s\tau}\ d\tau. \qquad (2)$$

An integration by parts provides

$$\tilde{F}(s) = \frac{1-\tilde{f}(s)}{s}. \qquad (3)$$

An important quantity is the Mean Time To Failure (MTTF) that is found in many textbooks (Henley & Kumamoto, 1991; Kuo & Zuo, 2003; Rausand & Høyland, 2004; Pham, 2006), and defined by

$$\text{MTTF} = \int_0^\infty \tau\ f(\tau)\, d\tau = \int_0^\infty F(\tau)\ d\tau = \tilde{F}(0). \qquad (4)$$

Similarly, the Mean Time To Repair (MTTR) is defined by

$$\text{MTTR} = \int_0^\infty \tau\ g(\tau)\, d\tau = \int_0^\infty G(\tau)\, d\tau = \tilde{G}(0). \qquad (5)$$

The MTTF and MTTR are expectation values for lifetimes ranging from 0 to infinity.

Assuming that after each repair the system is as good as new, it is possible to compute the instantaneous availability $A(t)$ (also sometimes called point availability), or more exactly its Laplace transform $\tilde{A}(s)$ after the summation of all the possible histories of the system: working/failed/working/failed/working etc. Since the integral equations become algebraic in the Laplace transform domain,

$$\tilde{A}(s) = \frac{\tilde{F}(s)}{1 - \tilde{f}(s)\,\tilde{g}(s)} = \frac{\tilde{F}(s)}{s(\tilde{F}(s) + \tilde{G}(s) - s\,\tilde{F}(s)\,\tilde{G}(s))}. \quad (6)$$

The steady-state value of the availability, noted A_∞, can be computed easily because the Laplace transform of the constant function 1 is $1/s$. The well-known result is

$$A_\infty = \lim_{s \to 0} s\,\tilde{A}(s) = \frac{\tilde{F}(0)}{\tilde{F}(0) + \tilde{G}(0)} = \frac{\text{MTTF}}{\text{MTTF+MTTR}}. \quad (7)$$

Another definition of the availability mentioned in the Introduction is the so-called average availability $\bar{A}(t)$ – a particular case of the interval availability of Barlow and Proschan (1996) – defined by

$$\bar{A}(t) = \frac{1}{t}\int_0^t A(\tau)\,d\tau. \quad (8)$$

It is not difficult to show that

$$\frac{d}{ds}\tilde{\bar{A}}(s) = -\frac{1}{s}\,\tilde{A}(s). \quad (9)$$

Unfortunately, to get back to the temporal domain, inverse Laplace transforms are required, which are very tricky in the general case. This is possible in a few cases only, for instance when the failure and repair distributions are exponential, as will be shown in the next subsection.

2.2. Exponential distributions

In the case of exponential distributions with rates λ and μ, we have

$$F(t) = e^{-\lambda t} \qquad G(t) = e^{-\mu t}. \quad (10)$$

We deduce

$$\tilde{F}(s) = \frac{1}{s+\lambda} \qquad \tilde{G}(s) = \frac{1}{s+\mu} \quad (11)$$

leading to the familiar MTTF = $1/\lambda$ and MTTR = $1/\mu$. The instantaneous availability reads, after a simple inverse Laplace transform

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\,e^{-(\lambda+\mu)t}. \quad (12)$$

The availability $A(t)$ decreases monotonously to its steady-state or asymptotic limit ($t \to \infty$), so that

$$A_\infty = \frac{\mu}{\lambda+\mu}. \quad (13)$$

Obviously, $A(t) > A_\infty, \forall\, t$. In this particular case of exponential distributions, the average availability is

$$\bar{A}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{(\lambda+\mu)^2\,t}\left(1 - e^{-(\lambda+\mu)t}\right) \quad (14)$$

and

$$\bar{A}(t) > A(t) > A_\infty\,, \forall\, t. \quad (15)$$

We shall see in the following that such inequalities are not always satisfied for other pairs of lifetime and repair distributions. The approaches to the asymptotic limit A_∞ also differ: one is exponential, the other behaves as $\frac{1}{t}$.

2.3. Gamma distributions

Gamma distributions (Rausand & Høyland, 2004) are, after exponentials, among the most often used distributions in reliability theory. We consider in the following the gamma distribution defined by

$$f(t) = \frac{(\alpha\,\lambda)^\alpha\,t^{\alpha-1}}{\Gamma(\alpha)}\,e^{-\alpha\,\lambda\,t} \quad (16)$$

where α is the so-called shape parameter, and Γ is the Euler gamma function. The definition (16) ensures that the MTTF is still equal to $1/\lambda$. Its Laplace transform can be computed easily:

$$\tilde{f}(s) = \frac{(\alpha\,\lambda)^\alpha}{(s+\alpha\,\lambda)^\alpha}. \quad (17)$$

This expression can be inserted in the expression of $\tilde{A}(s)$, as will be done in the next subsection. The exponential case is recovered for $\alpha = 1$.

2.4. Availability for gamma failure and exponential repair distributions for rational α

Using (6), (11), and (17) gives

$$\tilde{A}(s) = \frac{s+\mu}{s} \frac{(s+\alpha\,\lambda)^\alpha - (\alpha\,\lambda)^\alpha}{(s+\mu)(s+\alpha\,\lambda)^\alpha - \mu\,(\alpha\,\lambda)^\alpha}. \quad (18)$$

When α is an integer n, the denominator of $\tilde{A}(s)$ is a polynomial, the partial fraction decomposition of which leads to a result of the form (Sarkar & Chaudhuri,1999; Tanguy et al., 2019)

$$A(t) = \frac{\mu}{\lambda+\mu} + \sum_{s_i \neq 0} r_i\ e^{s_i\,t} \quad (19)$$

where s_i are the roots of

$$(s+\mu)(s+n\,\lambda)^n - \mu\,(n\,\lambda)^n = 0 \quad (20)$$

and the r_i the corresponding residues. Note that the first term of (19) corresponds to the contribution of the special root $s = 0$, and that the real parts of the non-zero s_i are negative. Sarkar and Chaudhuri (1999) limited their study to the exactly solvable cases $n = 2$ and $n = 3$, but did not mention that in the latter case the instantaneous availability can attain values smaller than the steady-state limit. The case $n = 1$ gives roots 0 and $-\mu - \lambda$, as expected (see (12)).

Such a simple inversion of the Laplace transform as given in (19) is not possible when α is not an integer, because the singularities of $\tilde{A}(s)$ are not isolated anymore. The proper derivation of $A(t)$ when α is rational, i.e., when $\alpha = \frac{p}{q}$ (with p and q integers), is given in the following Section.

3. Calculation of $A(t)$ for α rational ($\alpha = \frac{p}{q}$)

Equation (18) may be rewritten as

$$\tilde{A}(s) = \frac{1}{s} - \frac{1}{\mu}\ \frac{Y}{X-Y} \quad (21)$$

where

$$X = (s+\mu)(s+\alpha\,\lambda)^\alpha \quad (22)$$

$$Y = \mu\,(\alpha\,\lambda)^\alpha. \quad (23)$$

Since

$$\frac{X^q - Y^q}{X-Y} = \sum_{k=0}^{q-1} X^k\ Y^{q-1-k} \quad (24)$$

one can deduce

$$\tilde{A}(s) = \frac{1}{s} - \frac{1}{\mu}\ \frac{1}{X^q - Y^q} \sum_{k=0}^{q-1} X^k\ Y^{q-k}. \quad (25)$$

The first term on the right-hand side of (25) can be inverted easily: it corresponds to the constant 1 in the time domain. Furthermore,

$$X^q - Y^q = (s+\mu)^q \left(s + \frac{p}{q}\,\lambda\right)^p - \mu^q \left(\frac{p}{q}\,\lambda\right)^p \quad (26)$$

which is a polynomial of degree $p + q$ in s, the roots of which can be computed very simply. The only difficulty resides in the terms containing

$$X^k = (s+\mu)^k \left(s + \frac{p}{q}\,\lambda\right)^{\frac{p}{q}k}$$

because $\frac{p}{q}\,k$ is not an integer anymore.

At this point, we make a distinction between the cases $k = 0$ and $1 \leq k \leq q-1$.

3.1. Case $k = 0$

One must find the inverse Laplace transform of

$$-\frac{1}{\mu}\ \frac{\mu^q \left(\frac{p}{q}\lambda\right)^p}{(s+\mu)^q \left(s+\frac{p}{q}\lambda\right)^p - \mu^q \left(\frac{p}{q}\lambda\right)^p}. \quad (27)$$

Calling s_i the roots of the denominator, the partial fraction decomposition of (27) provides

$$-\frac{1}{\mu} \sum_{s_i} \frac{1}{\frac{q}{s_i+\mu} + \frac{p}{s_i+\frac{p}{q}\lambda}}\ \frac{1}{s-s_i}. \quad (28)$$

The inverse Laplace transform of (28) gives

$$-\frac{1}{\mu} \sum_{s_i} \frac{1}{\frac{q}{s_i+\mu} + \frac{p}{s_i+\frac{p}{q}\lambda}}\ e^{s_i\,t}\ . \quad (29)$$

3.2. Case $1 \leq k \leq q-1$

In the following, let us set $K = q - k$, so that $1 \leq K \leq q-1$. For each K, one must consider

$$-\frac{1}{\mu}\frac{(s+\mu)^q\left(s+\frac{p}{q}\lambda\right)^p}{(s+\mu)^q\left(s+\frac{p}{q}\lambda\right)^p-\mu^q\left(\frac{p}{q}\lambda\right)^p}\left(\frac{\mu}{s+\mu}\right)^K\left(\frac{\left(\frac{p}{q}\lambda\right)^{\frac{p}{q}}}{\left(s+\frac{p}{q}\lambda\right)^{\frac{p}{q}}}\right)^K. \quad (30)$$

The trick is now to perform a partial fraction decomposition of the terms of (30) bar the last one. After this decomposition, the following sum is obtained

$$-\frac{1}{\mu}\sum_{s_i}\frac{1}{\frac{q}{s_i+\mu}+\frac{p}{s_i+\frac{p}{q}\lambda}}\left(\frac{\mu}{s_i+\mu}\right)^K\frac{1}{s-s_i}\left(\frac{\left(\frac{p}{q}\lambda\right)^{\frac{p}{q}}}{\left(s+\frac{p}{q}\lambda\right)^{\frac{p}{q}}}\right)^K. \quad (31)$$

The inverse Laplace transform of $\frac{1}{s-s_i}\left(\frac{\alpha\lambda}{s+\alpha\lambda}\right)^{\alpha K}$ is simply (Erdélyi et al., 1954)

$$e^{s_i t}\left(\frac{\alpha\lambda}{s_i+\alpha\lambda}\right)^{\alpha K}\frac{\gamma(\alpha K,(s_i+\alpha\lambda)t)}{\Gamma(\alpha K)} \quad (32)$$

where $\gamma(\nu,z)$ is the lower incomplete gamma function defined by

$$\gamma(\nu,z) = \int_0^z e^{-\tau}\tau^{\nu-1}\,d\tau \quad (33)$$

3.3. Expression of $A(t)$

Using (29), (31) and (32), one obtains the final expression of $A(t)$

$$A(t) = 1 - \frac{1}{\mu}\sum_{s_i}\frac{e^{s_i t}}{\frac{q}{s_i+\mu}+\frac{p}{s_i+\frac{p}{q}\lambda}}\cdot$$

$$\cdot\left[1+\sum_{K=1}^{q-1}\left(\frac{\mu\left(\frac{p}{q}\lambda\right)^{\frac{p}{q}}}{(s_i+\mu)\left(s_i+\frac{p}{q}\lambda\right)^{\frac{p}{q}}}\right)^K\frac{\gamma\left(\frac{p}{q}K,\left(s_i+\frac{p}{q}\lambda\right)t\right)}{\Gamma\left(\frac{p}{q}K\right)}\right] \quad (34)$$

where the s_i's are all the roots (including $s_i = 0$) of

$$(s+\mu)^q\left(s+\frac{p}{q}\lambda\right)^p-\mu^q\left(\frac{p}{q}\lambda\right)^p = 0. \quad (35)$$

We recover the simple result when α is an integer n by setting $q = 1$ and $p = n$, so that the sum in (34) vanishes. In the following, we shall set

$$\zeta_i = \frac{\mu\left(\frac{p}{q}\lambda\right)^{\frac{p}{q}}}{(s_i+\mu)\left(s_i+\frac{p}{q}\lambda\right)^{\frac{p}{q}}} \quad (36)$$

which are qth roots of unity, i.e., ${\zeta_i}^q = 1$.

4. Calculation of $\bar{A}(t)$ for α rational

The knowledge of $A(t)$ enables the determination of $\bar{A}(t)$. For this, the two following integrals are needed

$$\frac{1}{t}\int_0^t\gamma(\nu,\alpha\tau)d\tau = \gamma(\nu,\alpha t)-\frac{1}{\alpha t}\gamma(\nu+1,\alpha t) \quad (37)$$

$$\frac{1}{t}\int_0^t e^{\beta\tau}\gamma(\nu,\alpha\tau)\,d\tau$$

$$= \frac{e^{\beta t}}{\beta t}\gamma(\nu,\alpha t)-\frac{1}{\beta t}\left(\frac{\alpha}{\alpha-\beta}\right)^\nu\gamma(\nu,(\alpha-\beta)t) \quad (38)$$

One may start with $A(t)$ rewritten as

$$1-\frac{1}{\mu}\sum_{s_i}r_i\,e^{s_i t}\left[1+\sum_{K=1}^{q-1}\frac{{\zeta_i}^K\gamma\left(\frac{pK}{q},\left(s_i+\frac{p}{q}\lambda\right)t\right)}{\Gamma\left(\frac{pK}{q}\right)}\right] \quad (39)$$

with

$$r_i = \frac{1}{\frac{q}{s_i+\mu}+\frac{p}{s_i+\frac{p}{q}\lambda}}. \quad (40)$$

The leading 1 will remain as such in the expression of $\bar{A}(t)$. For the remaining terms, one must distinguish between the root 0 and all the remaining roots $s_i \neq 0$ in order to take advantage of (37) and (38). The final result reads

$$\bar{A}(t)$$

$$= 1-\frac{\lambda}{q(\lambda+\mu)}\left(1+\sum_{K=1}^{q-1}\frac{\gamma\left(\frac{pK}{q},\frac{p}{q}\lambda t\right)-\frac{\gamma\left(\frac{pK}{q}+1,\frac{p}{q}\lambda t\right)}{\frac{p}{q}\lambda t}}{\Gamma\left(\frac{pK}{q}\right)}\right)+$$

$$-\frac{1}{\mu t}\sum_{s_i\neq 0}\frac{r_i\,e^{s_i t}}{s_i}\left(1+\sum_{K=1}^{q-1}\frac{{\zeta_i}^K\gamma\left(\frac{pK}{q},\left(s_i+\frac{p}{q}\lambda\right)t\right)}{\Gamma\left(\frac{pK}{q}\right)}\right)$$

$$+\frac{1}{\mu t}\sum_{s_i\neq 0}\frac{r_i}{s_i}\left(1+\sum_{K=1}^{q-1}\left(\frac{\mu}{s_i+\mu}\right)^K\frac{\gamma\left(\frac{pK}{q},\frac{p}{q}\lambda t\right)}{\Gamma\left(\frac{pK}{q}\right)}\right). \quad (41)$$

5. Numerical comparison of $A(t)$ and $\bar{A}(t)$

It has been shown previously that the behaviour of $A(t)$ depends on α (Sarkar & Chaudhuri, 1999; Tanguy et al., 2019). Three regimes exist, and we shall review them in detail in this Section. We shall also add the variation with time of the average availability $\bar{A}(t)$.

5.1. Case $1 \leq \alpha \leq 2$

The first examples are displayed in Figures 1 to 3 for a system where $\lambda = 10^{-2}$ and $\mu = 1$.

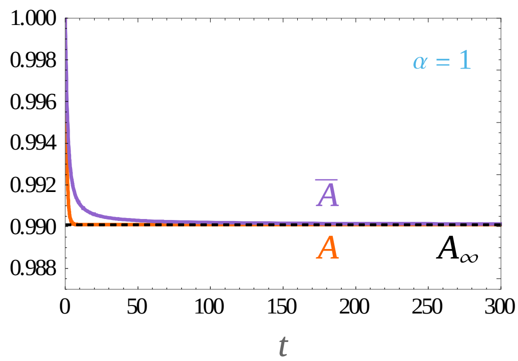

Figure 1. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = 1$, $\lambda = 10^{-2}$, and $\mu = 1$. This corresponds to a pair of exponential distributions.

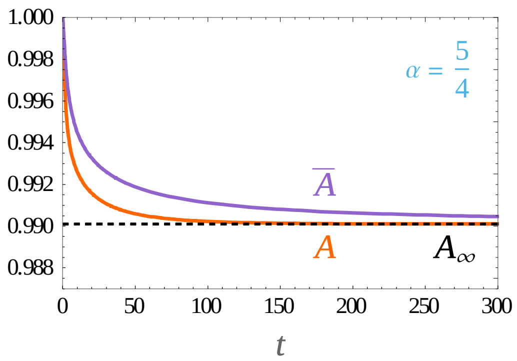

Figure 2. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = \frac{5}{4}$, $\lambda = 10^{-2}$, and $\mu = 1$.

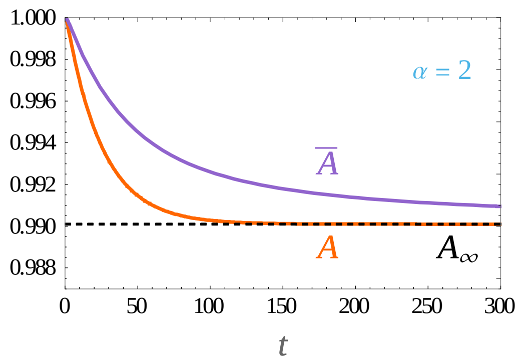

Figure 3. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = 2$, $\lambda = 10^{-2}$, and $\mu = 1$.

In each case, the instantaneous availability decreases monotonously with time towards its asymptotic value. It is worth noting that the characteristic timescale changes progressively from $1/\mu$ to $1/\lambda$. Furthermore, the approach to the steady-state value is not exponential for $\bar{A}(t)$, but follows a $1/t$ dependence.

5.2. Case $\alpha > 2$

When α is barely larger than 2, nothing seems to change (see Figure 4 for $\alpha = 9/4$), even though a minimum $A_{\min}$ is actually reached. For larger values of α, the instantaneous availability exhibits oscillations while the average availability decreases smoothly (see Figure 5 for $\alpha = 19/3$). However, as α increases, $\bar{A}(t)$ does not decrease monotonously anymore and displays oscillations as well, while remaining larger than A_∞ (see Figures 6 and 7 for $\alpha = 13$ and $\alpha = 25$, respectively). For a threshold α slightly larger than 15, it is also possible to observe periods of time during which $\bar{A}(t) < A(t)$.

It is also worth noticing that the minimum $A_{\min}$ of $A(t)$ is reached for $\lambda\, t_{\min} \approx 1 - \frac{1}{\alpha}$. A numerical study indicates that, for large α,

$$\lambda\, t_{\min} \approx 1 - \frac{1}{\alpha} + \frac{2\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)} \left(\frac{\alpha-1}{4}\right)^{\alpha} + \frac{\alpha^2-\alpha+1}{\alpha-1} \left(\frac{2\sqrt{\pi}}{\Gamma\left(\alpha+\frac{1}{2}\right)} \left(\frac{\alpha-1}{4}\right)^{\alpha}\right)^2 + \cdots \quad (42)$$

and that the occurrence of the mth secondary minima appears for

$$\lambda\, t_{\min}^{(m)} \approx m - \frac{1}{\alpha} + \frac{m+1}{m} \frac{\Gamma(1+m\,\alpha)}{\Gamma(1+(m+1)\alpha)} (m\,\alpha - 1)^{\alpha}. \quad (43)$$

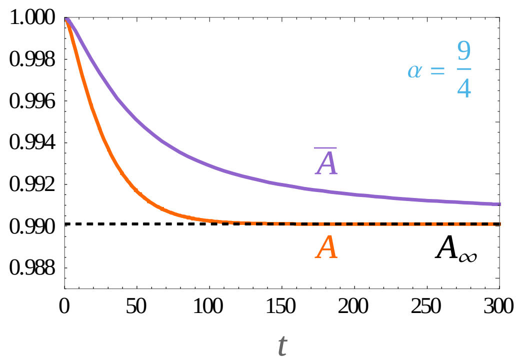

Figure 4. $A(t)$ and $\bar{A}(t)$ for $\alpha = \frac{9}{4}$, $\lambda = 10^{-2}$, and $\mu = 1$.

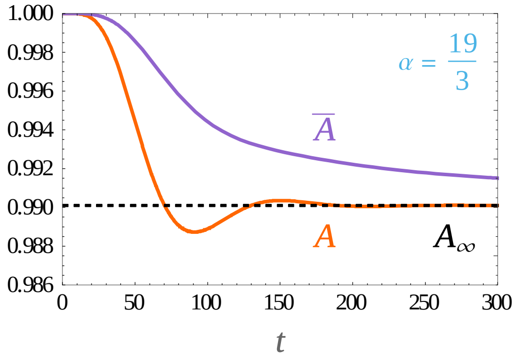

Figure 5. $A(t)$ and $\bar{A}(t)$ for $\alpha = \frac{19}{3}$, $\lambda = 10^{-2}$, and $\mu = 1$.

On the face of these observations, one might be tempted to suggest to equipment manufacturers and service providers to turn to the average availability instead of the instantaneous one when preparing Service Level Agreements (SLAs). We shall see that this is not always the case, however.

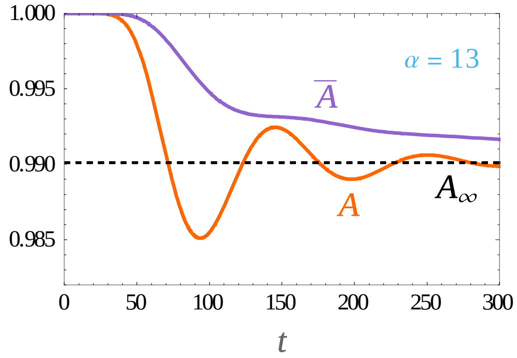

Figure 6. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = 13$, $\lambda = 10^{-2}$, and $\mu = 1$.

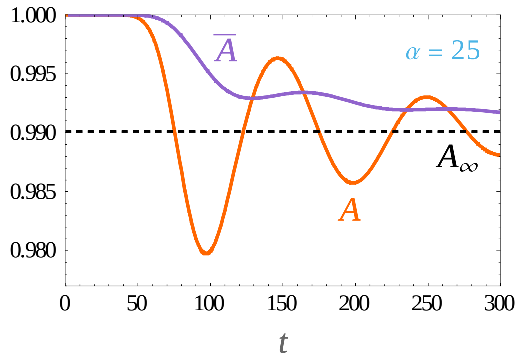

Figure 7. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = 25$, $\lambda = 10^{-2}$, and $\mu = 1$.

5.3. Case $0 < \alpha < 1$

5.3.1. General discussion

When α < 1, the behaviour of $A(t)$ is quite different: one observes a *single* minimum $A_{\min} < A_\infty$. The variations with time of $A(t)$ and $\bar{A}(t)$ are displayed in Figure 8 and Figure 9.

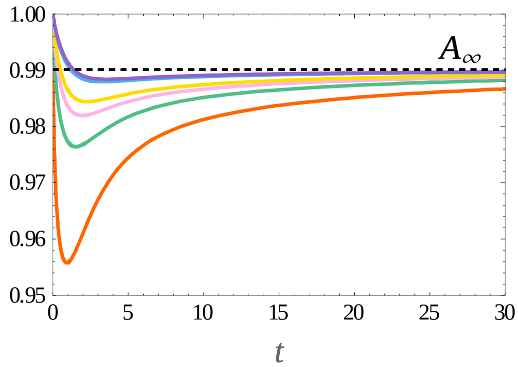

Figure 8. Variations of $A(t)$ for $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{9}{10}, \frac{11}{12}$ (the orange curve corresponds to $\alpha = \frac{1}{2}$), $\lambda = 10^{-2}$, and $\mu = 1$.

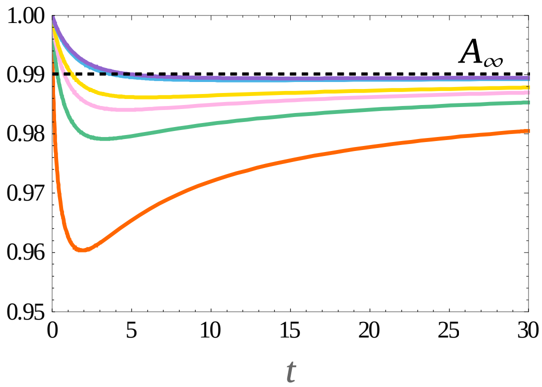

Figure 9. Variations of $\bar{A}(t)$ for $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{9}{10}, \frac{11}{12}$ (the orange curve corresponds to $\alpha = \frac{1}{2}$) , $\lambda = 10^{-2}$, and $\mu = 1$.

The relevant time-scale is now of order $1/\mu$. As α gets closer to 1, $A_{\min}$ increases too, and its position is attained after a time that goes to infinity (the case of the exponential distributions).
The behaviour of $\bar{A}(t)$ is quite similar, even if slightly less pronounced. For this reason, the case $\alpha = \frac{1}{2}$ has been considered worth studying; the next subsection is devoted to this case.

5.3.2. Case $\alpha = \frac{1}{2}$

The special case $\alpha = \frac{1}{2}$ is worth studying because it can be solved analytically, which is a good opportunity to better understand the interplay between different parameters.
Writing the expression of $\tilde{A}(s)$ in (18) allows to

use the following Laplace transform, see for instance (Sneddon, 1972)

$$\frac{1}{s\,(s+b)}\,\sqrt{s+a} = \int_0^\infty \frac{1}{b}\left(\sqrt{a}\,\mathrm{erf}\left(\sqrt{a\tau}\right) + \right.$$

$$\left. - \, e^{-\,b\,\tau}\,\sqrt{a-b}\,\mathrm{erf}\left(\sqrt{(a-b)\tau}\right)\right) e^{-\,s\,\tau}\; d\tau \quad (44)$$

where erf is the error function defined by

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\; \int_0^z e^{-\tau^2}\, d\tau. \qquad (45)$$

After some manipulations, one can get an expression of $A(t)$ with several erf functions with various arguments. Alternatively, one can use

$$\gamma\left(\frac{1}{2}, z\right) \;=\; \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{z}\right) \qquad (46)$$

in (41).

The variations of $A(t)$ and $\bar{A}(t)$ are displayed in Figures 10 and 11 in order to show their different behaviours.

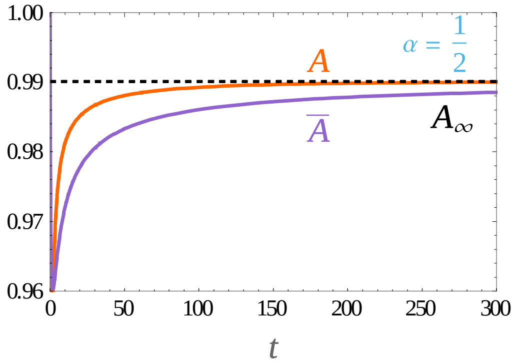

Figure 10. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = \frac{1}{2}$, $\lambda = 10^{-2}$, and $\mu = 1$.

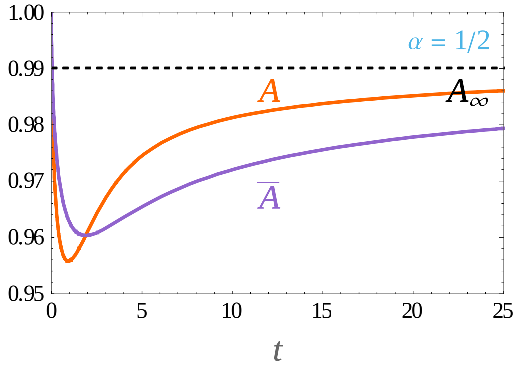

Figure 11. Variations of $A(t)$ and $\bar{A}(t)$ for $\alpha = \frac{1}{2}$, $\lambda = 10^{-2}$, and $\mu = 1$.

In Figure 10, the average availability is smaller than the instantaneous one, contrary to most situations. The situation is reversed when the timescale is of order $1/\mu$. The important fact remains that both are inferior to the steady-state value. At the minimum of $\bar{A}(t)$, the two availabilities are identical, namely, $A(t) = \bar{A}(t)$, as expected. Actually, it is possible to determine the position and value of the minima. The results for various values of λ are given in Table 1.

Table 1. Position of the minimum for $\mu = 1$ ($\alpha = \frac{1}{2}$)

λ	$\mu\, t_{\min}$	$A_{\min}$
10^{-2}	0.93145	0.955780
10^{-4}	0.86128	0.995673
10^{-6}	0.85475	0.999568
10^{-8}	0.85410	0.999957

In the limit $\lambda \to 0$, it is possible to show that $Z = \mu\, t_{\min} \approx 0.854032656598$. Likewise, one gets $\bar{Z} = \mu\, \bar{t}_{\min} \approx 1.6920310427$ for $\lambda \to 0$, where $\bar{t}_{\min}$ is the location of the minimum of $\bar{A}(t)$. Actually, Z is the solution of

$$\mathrm{erfi}\left(\sqrt{Z}\right) \;=\; \frac{e^{Z}}{\sqrt{\pi\, Z}} \qquad (47)$$

where

$$\mathrm{erfi}(z) \;=\; \frac{\mathrm{erf}(i\, z)}{i} \qquad (48)$$

while $\bar{Z}$ is the solution of

$$\mathrm{erfi}\,\left(\sqrt{\bar{Z}}\right) \;=\; \frac{e^{\bar{Z}}}{\sqrt{\pi\, \bar{Z}}}\,\frac{2\,\bar{Z}}{1+\bar{Z}}. \qquad (49)$$

When λ is non-vanishingly small, the Taylor expansions are

$$A_{\min} = 1 \;-\frac{1}{\sqrt{2\,\pi\,Z}}\,\sqrt{\frac{\lambda}{\mu}}$$

$$-\;\frac{1}{2}\,\left(1-(1+Z)\,e^{-Z}\right)\,\frac{\lambda}{\mu} + \cdots \qquad (50)$$

$$\bar{A}_{\min} \;=\; 1 \;-\; \frac{1}{\sqrt{2\,\pi\,\bar{Z}}}\,\frac{2\,\bar{Z}}{1+\bar{Z}}\,\sqrt{\frac{\lambda}{\mu}}$$

$$-\;\frac{1}{2\,\bar{Z}}\,\left(\bar{Z}\,-2+(\bar{Z}\,+2)\,e^{-\bar{Z}}\right)\,\frac{\lambda}{\mu} + \cdots \; (51)$$

which translates into

$$A_{\min} \approx 1 - 0.4316908 \sqrt{\frac{\lambda}{\mu}} - 0.1053740 \frac{\lambda}{\mu} + \cdots \quad (52)$$

$$\bar{A}_{\min} \approx 1 - 0.3855353 \sqrt{\frac{\lambda}{\mu}} - 0.1098977 \frac{\lambda}{\mu} + \cdots. \quad (53)$$

Let us recall that $A_\infty = 1 - \frac{\lambda}{\lambda+\mu} \approx 1 - \frac{\lambda}{\mu}$, which shows that the minima may lie well below the steady-state value in the usual configuration $\lambda \ll \mu$.

6. A criterion for $\bar{A}_{\min} < A_\infty$

In a previous work (Tanguy, 2020), we established criteria determining the possible occurrence of a minimum availability satisfying $A_{\min} < A_\infty$. The gist of the method is to write

$$\tilde{A}(s) = \frac{A_\infty}{s} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Delta_n s^n \quad (54)$$

where

$$\Delta_n = \int_0^\infty (A(\tau) - A_\infty)\ \tau^n\ d\tau. \quad (55)$$

If at least one of the Δ_n's is negative, a minimum below the steady-state value necessarily occurs. From (9) and (54), one can deduce that, *formally*,

$$\tilde{\bar{A}}(s) = \frac{A_\infty}{s} - \Delta_0 \ln s + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \times n!} \Delta_n s^n + C \quad (56)$$

where C is a constant of integration.
One is led to set

$$\bar{\Delta}_n = \int_0^T (\bar{A}(\tau) - A_\infty)\ \tau^n\ d\tau \quad (57)$$

where T is a large value. The reason for this caveat is that the integral in (57) diverges when T goes to infinity. A simple confirmation can be obtained in the case of a pair of exponentials: (14) leads to a logarithmic divergence of $\bar{\Delta}_0$. If the integral converged for T going to infinity, we would have

$$\begin{aligned}\bar{\Delta}_0 &= \lim_{s \to 0^+} \int_0^\infty (\bar{A}(\tau) - A_\infty)\ e^{-s\tau}\ d\tau \\ &= \lim_{s \to 0^+} \left(\tilde{\bar{A}}(s) - \frac{A_\infty}{s}\right) \\ &= \lim_{s \to 0^+} (-\Delta_0 \ln s + C) \\ &= \Delta_0 \times \infty. \end{aligned} \quad (58)$$

This means that the sign of the divergence is that of Δ_0. The same calculation can be performed for $\bar{\Delta}_n, n \geq 1$:

$$\begin{aligned}\bar{\Delta}_n &= \lim_{s \to 0^+} (-1)^n \frac{d^n}{ds^n}\left(\tilde{\bar{A}}(s) - \frac{A_\infty}{s}\right) \\ &= \lim_{s \to 0^+} \left(\frac{\Delta_0}{s^n}(n-1)! - \frac{\Delta_n}{n}\right) \\ &= \Delta_0 \times \infty. \end{aligned} \quad (59)$$

The result is therefore independent of n. The sign of all the $\bar{\Delta}_n$'s is that of Δ_0. If the latter is negative, one should have $\bar{A}_{\min} < A_\infty$. In the case of the configuration gamma(α, λ)/exp(μ) studied in this chapter,

$$\Delta_0 = \frac{2\,\alpha\,\lambda + (\alpha - 1)\mu}{2\,\alpha\,(\lambda + \mu)^2}. \quad (60)$$

Δ_0 is therefore negative when $\alpha \leq \frac{\mu}{\mu + 2\,\lambda}$. This is coherent with the absence of a minimum smaller than A_∞ for $\alpha \geq 1$, as observed in Sections 5.1 and 5.2.

7. Conclusion

We have studied in detail the behaviours of the instantaneous and average availabilities for a gamma failure distribution and an exponential repair distribution, which can be ascertained analytically. We have shown that the actual transient variations greatly depend on the shape parameter of the gamma distribution. We have also proposed a general criterion for the possible apparition of a minimum of the average availability that would be smaller than the asymptotic limit.
An important lesson to learn is that in the transient regime, both definitions of the availability may attain values smaller than the steady-state one, A_∞. This may have important consequences for the design of systems and services, because their true

availability may be greatly overestimated during their mission time.

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