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Construction of Minimal Surfaces Using Flat Curves with Constant Complex Curvature

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Summary. Analytical description of isotropic lines and minimal surfaces by means of functions of complex variable is made. To find the isotropic lines analytical description parametric equations of a flat curve given by functions of natural parameter with constant complex curvature are used. Isotropic lines parametric equations are obtained from the condition of spatial curve differential arc equality to zero. Analytical description of minimal surfaces and associated minimal surfaces are made in complex space with isotropic lines of a transferring grid.

Demanding performing Cauchy-Riemann's conditions of differentiation for isotropic lines equations with constant complex curvature, analytical description of isotropic lines with real and imaginary parts of the complex variable was found. For stated isotropic lines analytical description of minimal surfaces and associated minimal surfaces was made. It was investigated that minimal surfaces and associated minimal surfaces formed from isotropic line with help of real part of complex variable is catenoid and right helicoid. Expressions of the first and second quadratic forms coefficients of generated minimal surfaces were found. It is shown that the mean of formed surfaces curvatures is zero at all points. Analytical description of one-parameter set of a associated minimal surfaces formed under their continuous bending, was made.

The proposed method of parametric equations of isotropic curves based on flat curves with constant complex curvature ($a \in R,\ b \in R,\ i -$ imaginary unit) allows to determine analytical description of flat lines defined by natural parameter functions of a in complex space.

Minimal surfaces parametric equations were found in the form of elementary functions, allowing to explore their geometric properties and differential characteristics to optimize the engineering methods of technical forms and architectural constructions design.

Key words: isotropic line, minimal surface, function of complex variable, constant complex curvature, Cauchy–Riemann equations, catenoid, right helicoid.

INTRODUCTION

Development of methods for geometric modeling is an important challenge to find the optimum solution to problems of transport logistics and design of surfaces of technical forms and architectural constructions according to postulated conditions. In particular, the graphs are used to study patterns of traffic [1] and for modeling parameters of technological solutions in construction [2]. Differential curves and surfaces characteristics were taken into account when designing technical surfaces forms in works [3-5]. Ability to find parameters of geometric models by means of computer-aided design is shown in the study [6].

Geometric models described by minimal surfaces can be used in CAD systems, while designing surfaces of technical forms and architectural constructions to solve the problems of finding the smallest surface area, which pass through a given flat or spatial curve.

Geometrical shape of a minimal surface, the mean curvature at all its points is equal to zero, ensures even distribution of efforts in the shell of surface and extra rigidity [7].

Setting minimum surface by the function $z = z(x;\ y)$, J. Lagrange was one of the first who concluded that the function $z = z(x;\ y)$ must satisfy the differential equation of Euler-Lagrange [8, p. 683] in partial derivatives, which generally are not integrated.

Therefore, one of the modern ways of minimal surfaces description is improvement of numerical methods for solving Euler-Lagrange differential equations [9, 10] and learning applications of designing of architectural structures surfaces [11, 12].

Analytical description of minimal surfaces can be obtained using complex variable in form of elementary functions, simplifying the research of geometrical and differential properties of formed minimal surfaces.

THE ANALYSIS OF RECENT RESEARCHES AND PUBLICATIONS

In works [13, 14] in some cases parametric equations for isotropic lines according to Schwartz and Weierstrass formulas were found and corresponding minimal surfaces using the properties of complex variable were built.

Modeling of minimal surfaces with the help of Bezier curves of the third order were reviewed in the work [15].

The method of analytical description of minimal surfaces using isotropic curves that lie on the surfaces of revolution assigned to isometric grid lines were represented in the works [16-18].

In the work [19] for analytical description of isotropic lines parametric equations of logarithmic spiral defined by natural parameter functions with real values of the curvature was used.

It is necessary to research an opportunity to study analytical description of isotropic lines and corresponding minimal surfaces using flat curves with constant complex curvature.

OBJECTIVE

Find analytical description of isotropic lines using parametric equations of flat curve given by functions of a natural parameter with constant complex curvature value $a+bi$ ($a\in R,\ b\in R,\ i-$ imaginary unit). With the help of found isotropic lines find analytical description of minimal surfaces.

THE MAIN RESULTS OF THE RESEARCH

Consider a plane curve, given by constant complex value of the curvature: $k(s)=a+bi,$ де $a\in R, b\in R,$ $i-$ imaginary unit, $s-$ length of the arc curve.

Then the curvature of the flat curve $k(s)$ is defined by the formula [20, p. 39]:

$$k(s)=\frac{d\varphi}{ds}, \qquad (1)$$

where: $\varphi-$ angle between the tangent to the curve and the abscissa.

Demanding meeting the condition $\varphi(0)=0,$ get parametric equations of a flat curve [20, p. 48] from the natural parameter s:

$$\begin{aligned} x(s)&=x(0)+\int_0^s\cos\left[\int_0^s k(s)ds\right]\cdot ds,\\ y(s)&=y(0)+\int_0^s\sin\left[\int_0^s k(s)ds\right]\cdot ds. \end{aligned} \qquad (2)$$

Substitute the value of complex curvature $k(s)=a+bi$ in (2), then under meeting the conditions $x(0)=0$ and $y(0)=0,$ obtain:

$$\begin{aligned} x(s)&=\frac{1}{a+bi}\cdot\sin[(a+bi)\cdot s],\\ y(s)&=-\frac{1}{a+bi}\cdot\cos[(a+bi)\cdot s]. \end{aligned} \qquad (3)$$

From condition [21, p. 14] $(x')^2+(y')^2+(z')^2=0$ define the expression $z(s)=i\cdot s$ and write the parametric equations of a spatial isotropic line:

$$\begin{aligned} x(s)&=\frac{1}{a+bi}\cdot\sin[(a+bi)\cdot s],\\ y(s)&=-\frac{1}{a+bi}\cdot\cos[(a+bi)\cdot s],\\ z(s)&=i\cdot s. \end{aligned} \qquad (4)$$

To find the equations of minimal and associated minimal surface it is necessary to change parametric equations of isotropic curve (4) [19]: $s=u+i\cdot v.$

Then, demanding meeting the conditions of Cauchy-Riemann equations [22, p. 22], we will obtain a parametric equations of minimal surfaces $X(u,v),\ Y(u,v), Z(u,v)$:

$$\begin{aligned} X(u,v)&=\mathrm{Re}\{x(u+i\cdot v)\},\\ Y(u,v)&=\mathrm{Re}\{y(u+i\cdot v)\},\\ Z(u,v)&=\mathrm{Re}\{i\cdot(u+i\cdot v)\}, \end{aligned} \qquad (5)$$

and associated minimal surface $X^*(u,v), Y^*(u,v),$ $Z^*(u,v)$:

$$\begin{aligned} X^*(u,v)&=\mathrm{Im}\{x(u+i\cdot v)\},\\ Y^*(u,v)&=\mathrm{Im}\{y(u+i\cdot v)\},\\ Z^*(u,v)&=\mathrm{Im}\{i\cdot(u+i\cdot v)\}. \end{aligned} \qquad (6)$$

Separating real and imaginary parts for each function (4), according to (5), (6) we obtain the minimal surface equations:

$$\begin{aligned} X(u,v)&=\frac{1}{a^2+b^2}\cdot(a\cdot\mathrm{ch}(bu+av)\cdot\sin(au-bv)+\\ &\quad +b\cdot\cos(au-bv)\cdot\mathrm{sh}(bu+av)),\\ Y(u,v)&=\frac{1}{a^2+b^2}\cdot(-a\cdot\cos(au-bv)\cdot\mathrm{ch}(bu+av)+\\ &\quad +b\cdot\sin(au-bv)\cdot\mathrm{sh}(bu+av)),\\ Z(u,v)&=-v, \end{aligned} \qquad (7)$$

and associated minimal surface equation:

$$\begin{aligned} X^*(u,v)&=\frac{1}{a^2+b^2}\cdot(-b\cdot\mathrm{ch}(bu+av)\cdot\sin(au-bv)+\\ &\quad +a\cdot\cos(au-bv)\cdot\mathrm{sh}(bu+av)),\\ Y^*(u,v)&=\frac{1}{a^2+b^2}\cdot(b\cdot\cos(au-bv)\cdot\mathrm{ch}(bu+av)+\\ &\quad +a\cdot\sin(au-bv)\cdot\mathrm{sh}(bu+av)),\\ Z^*(u,v)&=u. \end{aligned} \qquad (8)$$

In Fig. 1 a minimal surface is shown, Fig. 2 shows profile projection of this surface that is built on equations (7).

In Fig. 3 the associate minimal surface is shown, Fig. 2 shows the horizontal projection of this surface that is built on equations (8).

In Fig. 1, 2, 3, 4 images minimal surfaces and their projections built on equations (7), (8) in accordance with $a=0{,}8,\ b=0{,}1,\ u\in[-4;...4],\ v\in[-2;...2].$

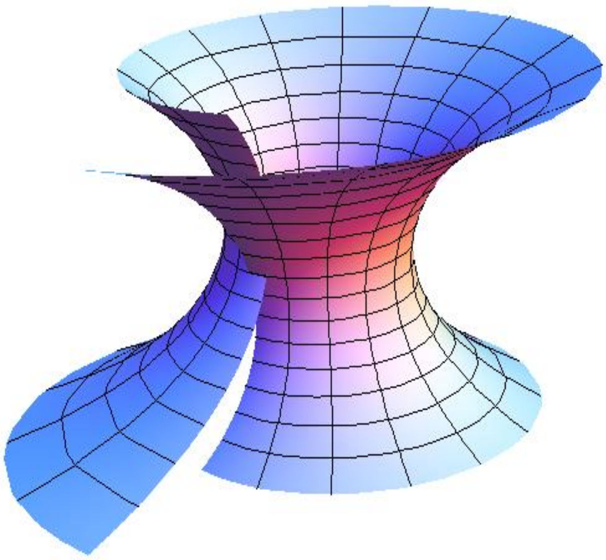

Fig. 1. Minimal surface built on equations (7)

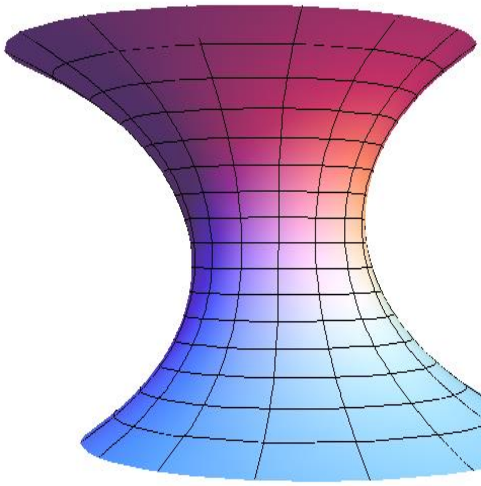

Fig. 2. Profile projection of the minimal surface built on equations (7)

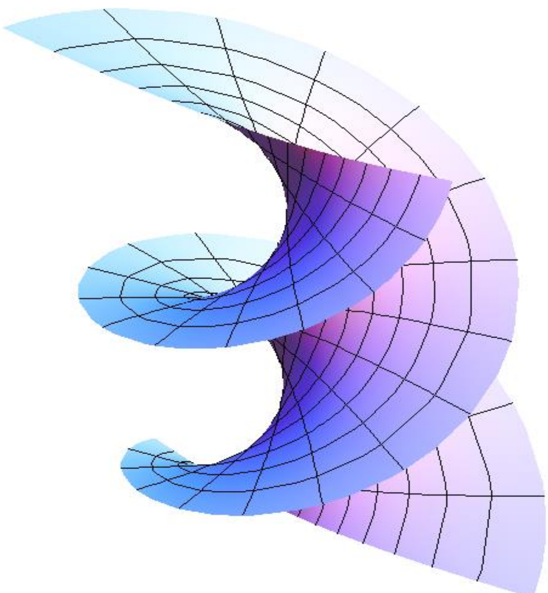

Fig. 3. Associated minimal surface built on equations (8)

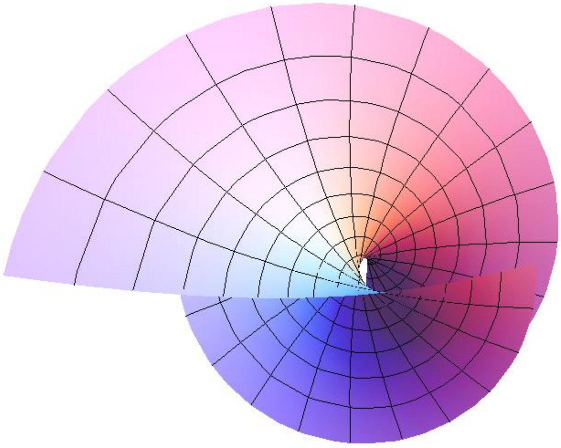

Fig. 4. Horizontal projection of the associated minimal surface built on equations (8)

We find coefficients expressions of the first quadratic surface form $X(u,v)$, $Y(u,v), Z(u,v)$ which define metric properties of the surface according to the formulas [20, p.183]:

$$\begin{aligned} E &= \left(X'_u\right)^2 + \left(Y'_u\right)^2 + \left(Z'_u\right)^2, \\ F &= X'_u \cdot X'_v + Y'_u \cdot Y'_v + Z'_u \cdot Z'_v, \\ G &= \left(X'_v\right)^2 + \left(Y'_v\right)^2 + \left(Z'_v\right)^2. \end{aligned} \qquad (9)$$

The coefficients of the first quadratic form of minimal surface (7) and the associated surface (8) equal to: $E = G = [\mathrm{ch}(bu + av)]^2$, $F = 0$.

Minimal surfaces built on equations (7) and (8) have the same expression of coefficients of the first quadratic form, that's why they allow continuous bending one above the other.

Equations of one-parameter set of associated minimal surfaces formed with continuous bending are of the form [19]:

$$\begin{aligned} X_\varphi(u,v) &= X(u,v)\cos\varphi + X^*(u,v)\sin\varphi, \\ Y_\varphi(u,v) &= Y(u,v)\cos\varphi + Y^*(u,v)\sin\varphi, \\ Z_\varphi(u,v) &= Z(u,v)\cos\varphi + Z^*(u,v)\sin\varphi, \end{aligned} \qquad (10)$$

where: $X(u,v); Y(u,v); Z(u,v)-$ parametric equations of minimal surface (7),

$X^*(u,v); Y^*(u,v); Z^*(u,v)-$ parametric equations of associated minimal surface (8),

$\varphi-$ bending parameter of surfaces, $\varphi \in \left[0; \dfrac{\pi}{2}\right]$.

It is obviously that $\varphi = 0$ equations (10) define the minimal surface (7), at $\varphi = \dfrac{\pi}{2}$ equations (10) define the associated minimal surface (8), for other values

$\varphi \in \left(0; \dfrac{\pi}{2}\right)$ equations (10) define associated minimal surfaces [19].

In Fig. 5, 6, 7, 8 images associated minimal surfaces built on equations (10) in accordance for $a = 0{,}8$; $b = 0{,}1$; $u \in [-4;\ldots 4]$; $v \in [-2;\ldots 2]$, at $\varphi = \dfrac{\pi}{8}$; $\varphi = \dfrac{\pi}{6}$; $\varphi = \dfrac{\pi}{4}$; $\varphi = \dfrac{3\pi}{8}$ respectively are built. These minimal surfaces are formed under continuous bending of a minimal surface (7) to associated minimal surface (8).

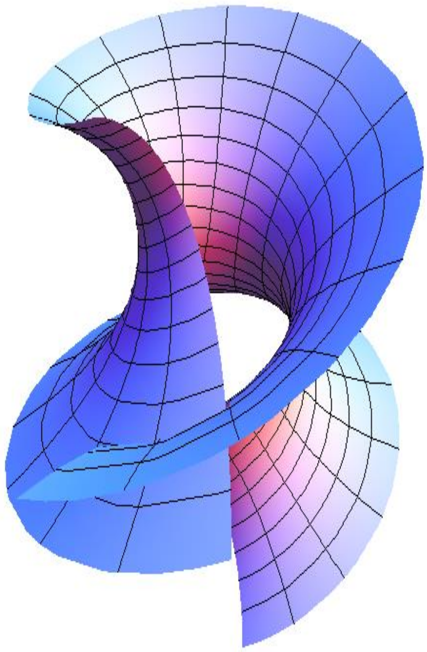

Fig. 5. Associated minimal surface built at $\varphi = \dfrac{\pi}{8}$

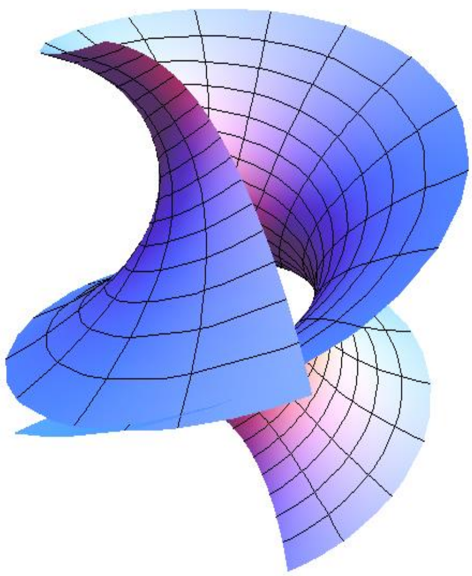

Fig. 6. Associated minimal surface built at $\varphi = \dfrac{\pi}{6}$

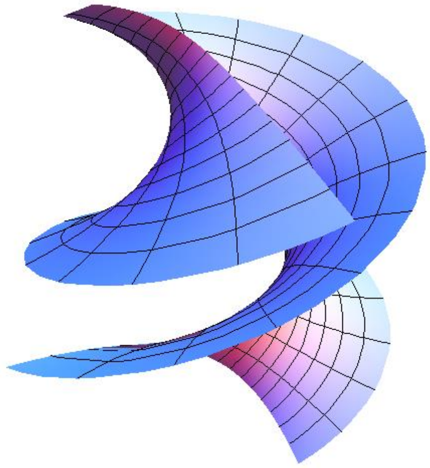

Fig. 7. Associated minimal surface built at $\varphi = \dfrac{\pi}{4}$

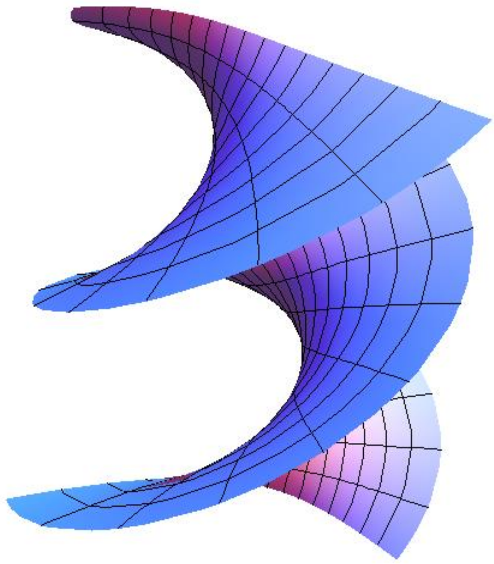

Fig. 8. Associated minimal surface built at $\varphi = \dfrac{3\pi}{8}$

We find expressions of a second surface quadratic form $X(u,v),\ Y(u,v), Z(u,v)$, that define the properties of surface curvature according to the formulas [20, p.192]:

$$L = \frac{1}{\sqrt{EG - F^2}} \cdot \begin{vmatrix} X''_{uu} & Y''_{uu} & Z''_{uu} \\ X'_u & Y'_u & Z'_u \\ X'_v & Y'_v & Z'_v \end{vmatrix},$$
$$M = \frac{1}{\sqrt{EG - F^2}} \cdot \begin{vmatrix} X''_{uv} & Y''_{uv} & Z''_{uv} \\ X'_u & Y'_u & Z'_u \\ X'_v & Y'_v & Z'_v \end{vmatrix}, \qquad (11)$$
$$N = \frac{1}{\sqrt{EG - F^2}} \cdot \begin{vmatrix} X''_{vv} & Y''_{vv} & Z''_{vv} \\ X'_u & Y'_u & Z'_u \\ X'_v & Y'_v & Z'_v \end{vmatrix},$$

The coefficients of the second quadratic form of minimal surface (7) equal to: $L=-N=a,\ \ M=-b.$

The coefficients of the second quadratic form of associated minimal surface (8) equal to: $L^*=-N^*=b,$ $M^*=a.$

The coefficients of the first and second quadratic forms of the constructed minimal surfaces (7) and (8), turn the expression of mean curvature $H=\dfrac{E\cdot N-2\cdot F\cdot M+G\cdot L}{2\left(E\cdot G-F^2\right)}$ for each of the specified surfaces to zero.

It should be noted that for isotropic line (4) a minimal surface was built (7) – "broken" catenoid, which in $a\neq 0, b=0$ is an ordinary catenoid. Associated minimal surface (8) is a right helicoid, an only minimal ruled surface.

Separate the real and imaginary parts of complex variable functions (3) performing Cauchy-Riemann conditions [22, p. 22]. For real functions (3) we get:

$$\begin{aligned} x_1(s)&=\mathrm{Re}\{x(s)\}=\frac{a\,\mathrm{ch}(bs)\sin(as)+b\cos(as)\mathrm{sh}(bs)}{a^2+b^2},\\ y_1(s)&=\mathrm{Re}\{y(s)\}=\frac{b\,\mathrm{sh}(bs)\sin(as)-a\cos(as)\mathrm{ch}(bs)}{a^2+b^2}.\end{aligned}\tag{12}$$

From the conditions [21, p. 14] $(x_1')^2+(y_1')^2+$ $+(z_1')^2=0$ define the expression $z_1(s)=i\cdot\dfrac{\mathrm{sh}(bs)}{b}$ and write the spatial isotropic line parametric equations from arbitrary parameter t:

$$\begin{aligned} x_1(t)&=\frac{a\,\mathrm{ch}(bt)\sin(at)+b\cos(at)\mathrm{sh}(bt)}{a^2+b^2},\\ y_1(t)&=\frac{b\,\mathrm{sh}(bt)\sin(at)-a\cos(at)\mathrm{ch}(bt)}{a^2+b^2},\\ z_1(t)&=i\cdot\frac{\mathrm{sh}(bt)}{b}.\end{aligned}\tag{13}$$

Enter the replacement in isotropic curve parametric equations (13) [19]: $t=u+i\cdot v.$

Separating real and imaginary parts for each function (13), according to (5), (6) we obtain the minimal surface equations $X_1(u,v),\ Y_1(u,v), Z_1(u,v)$:

$$\begin{aligned} X_1&=\frac{\mathrm{ch}(bu)\sin(au)}{a^2+b^2}\left[a\cos(bv)\mathrm{ch}(av)+b\sin(bv)\mathrm{sh}(av)\right]+\\ &+\frac{\mathrm{sh}(bu)\cos(au)}{a^2+b^2}\left[b\cos(bv)\mathrm{ch}(av)-a\sin(bv)\mathrm{sh}(av)\right],\\ Y_1&=\frac{\mathrm{ch}(bu)\cos(au)}{a^2+b^2}\left[-a\cos(bv)\mathrm{ch}(av)-b\sin(bv)\mathrm{sh}(av)\right]+\\ &+\frac{\mathrm{sh}(bu)\sin(au)}{a^2+b^2}\left[b\cos(bv)\mathrm{ch}(av)-a\sin(bv)\mathrm{sh}(av)\right],\\ Z_1&=-\frac{\mathrm{ch}(bu)\cdot\sin(bv)}{b},\end{aligned}\tag{14}$$

and the associated minimal surface equations $X_1^*(u,v),$ $Y_1^*(u,v),\ Z_1^*(u,v)$:

$$\begin{aligned} X_1^*&=\frac{\mathrm{ch}(bu)\cos(au)}{a^2+b^2}\left[b\sin(bv)\mathrm{ch}(av)+a\cos(bv)\mathrm{sh}(av)\right]+\\ &+\frac{\mathrm{sh}(bu)\sin(au)}{a^2+b^2}\left[a\sin(bv)\mathrm{ch}(av)-b\cos(bv)\mathrm{sh}(av)\right],\\ Y_1^*&=\frac{\mathrm{ch}(bu)\sin(au)}{a^2+b^2}\left[b\sin(bv)\mathrm{ch}(av)+a\cos(bv)\mathrm{sh}(av)\right]-\\ &-\frac{\mathrm{sh}(bu)\cos(au)}{a^2+b^2}\left[a\sin(bv)\mathrm{ch}(av)-b\cos(bv)\mathrm{sh}(av)\right],\\ Z_1^*&=\frac{\cos(bv)\cdot\mathrm{sh}(bu)}{b}.\end{aligned}\tag{15}$$

The coefficients of the first quadratic form of the minimal surface (13) and the associated minimal surface (14), found according to formulas (9), equal to:

$$E=G=\frac{1}{2}\left(\cos(2bv)+\mathrm{ch}(2bu)\right)\cdot\left[\mathrm{ch}(bu+av)\right]^2,\ F=0.$$

The coefficients of the second quadratic form of the minimal surface (13), found according to formulas (11), equal to:

$$L=-N=a\cos(bv)\mathrm{ch}(bu),\ M=-a\sin(bv)\cdot\mathrm{sh}(bu).$$

The coefficients of the second quadratic form of associated minimal surface (14) equal to:

$$L^*=-N^*=-a\sin(bv)\cdot\mathrm{sh}(bu),\ M^*=a\cos(bv)\cdot\mathrm{ch}(bu).$$

The coefficients of the first and second quadratic forms of minimal surfaces (13) and (14), transform the mean curvature expression to zero.

Minimal surface built on equations (13), is Catenoid and associated minimal surface (14) is right helicoid which has common geometric properties with the surface shown in Fig. 3. Consider the imaginary part of the complex variable functions (3):

$$\begin{aligned} x_2(s)&=\mathrm{Im}\{x(s)\}=\frac{a\cos(as)\mathrm{sh}(bs)-b\,\mathrm{ch}(bs)\sin(as)}{a^2+b^2},\\ y_2(s)&=\mathrm{Im}\{y(s)\}=\frac{a\,\mathrm{sh}(bs)\sin(as)+b\cos(as)\mathrm{ch}(bs)}{a^2+b^2}.\end{aligned}\tag{16}$$

From condition [21, p. 14] $(x_1')^2+(y_1')^2+(z_1')^2=0$ define the expression $z_2(s)=i\cdot\dfrac{\mathrm{ch}(bs)}{b}$ and write the spatial isotropic line parametric equations from arbitrary parameter t:

$$\begin{aligned} x_2(t)&=\mathrm{Im}\{x(t)\}=\frac{a\cos(at)\mathrm{sh}(bt)-b\,\mathrm{ch}(bt)\sin(at)}{a^2+b^2},\\ y_2(t)&=\mathrm{Im}\{y(t)\}=\frac{a\,\mathrm{sh}(bt)\sin(at)+b\cos(at)\mathrm{ch}(bt)}{a^2+b^2},\\ z_2(t)&=i\cdot\frac{\mathrm{ch}(bt)}{b}.\end{aligned}\tag{17}$$

In isotropic curve parametric equations (17) enter the replacement [19]: $t = u + i \cdot v$. Separating real and imaginary parts for each function (17), according to (5), (6) we obtain the minimal surface equations $X_2(u,v),\ Y_2(u,v), Z_2(u,v)$:

$$\begin{aligned} X_2 &= \frac{\operatorname{ch}(bu)\sin au}{a^2+b^2}\left[-b\cos(bv)\operatorname{ch}(av)+a\sin(bv)\operatorname{sh}(av)\right]+ \\ &+\frac{\operatorname{sh}(bu)\cos au}{a^2+b^2}\left[a\cos(bv)\operatorname{ch}(av)+b\sin(bv)\operatorname{sh}(av)\right], \\ Y_2 &= \frac{\operatorname{ch}(bu)\cos au}{a^2+b^2}\left[b\cos(bv)\operatorname{ch}(av)-a\sin(bv)\operatorname{sh}(av)\right]+ \\ &+\frac{\operatorname{sh}(bu)\sin au}{a^2+b^2}\left[a\cos(bv)\operatorname{ch}(av)+b\sin(bv)\operatorname{sh}(av)\right], \\ Z_2 &= -\frac{\sin(bv)\cdot\operatorname{sh}(bu)}{b}. \end{aligned} \tag{18}$$

and the associated minimal surface equations $X_2^{\ *}(u,v)$, $Y_2^{*}(u,v),\ Z_2^{\ *}(u,v)$:

$$\begin{aligned} X_2^{*} &= \frac{\operatorname{ch}(bu)\cos au}{a^2+b^2}\left[a\sin(bv)\operatorname{ch}(av)-b\cos(bv)\operatorname{sh}(av)\right]+ \\ &-\frac{\operatorname{sh}(bu)\sin au}{a^2+b^2}\left[b\sin(bv)\operatorname{ch}(av)+a\cos(bv)\operatorname{sh}(av)\right], \\ Y_2^{*} &= \frac{\operatorname{ch}(bu)\sin au}{a^2+b^2}\left[a\sin(bv)\operatorname{ch}(av)-b\cos(bv)\operatorname{sh}(av)\right]+ \\ &+\frac{\operatorname{sh}(bu)\cos au}{a^2+b^2}\left[b\sin(bv)\operatorname{ch}(av)+a\cos(bv)\operatorname{sh}(av)\right], \\ Z_2^{*} &= \frac{\cos(bv)\cdot\operatorname{ch} au}{b}. \end{aligned} \tag{19}$$

In Fig. 9 a minimal surface is shown, in Fig. 10 horizontal projection of this surface is shown, they are built on equations (18), in accordance with $a = 0{,}8,\ b = 0{,}1,\ u \in [0;...\pi],\ v \in [-2;..2]$.

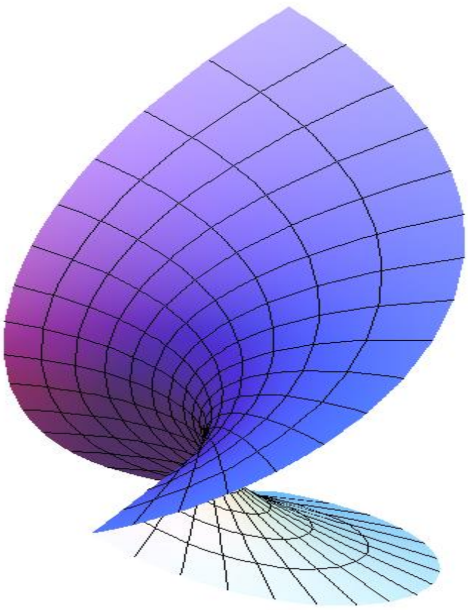

Fig. 9. Minimal surface built on equations (18)

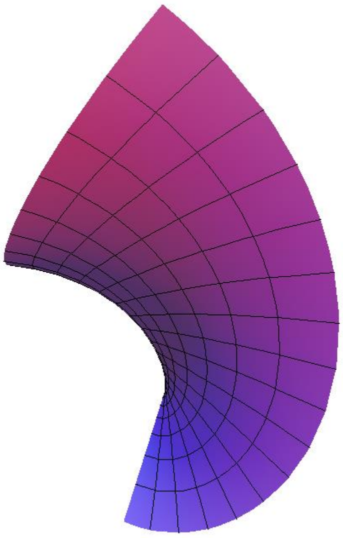

Fig. 10. Horizontal projection of the minimal surface built on equations (18)

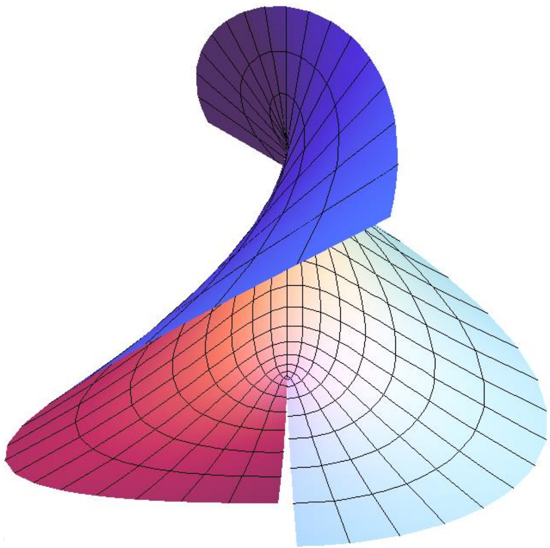

Fig. 11. Associated minimal surface built on equations (19)

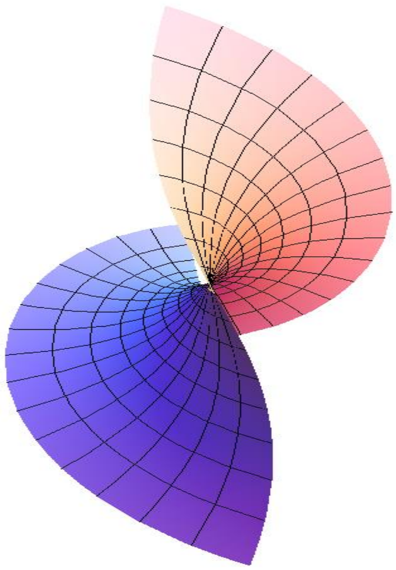

Fig. 12. Horizontal projection of the associated minimal surface built on equations (19)

In Fig. 11 the minimal surface is shown, in Fig. 12 horizontal projection of this surface is shown, they are built on equations (19), in accordance with $a = 0,8,\ b = 0,1,\ u \in [0;...\pi],\ v \in [-2;..2]$.

The coefficients of the first quadratic form of minimal surface (18) and the associated minimal surface (19), found according to formulas (9), equal to:

$$E = G = -\frac{1}{2}(\cos(2bv) - \mathrm{ch}(2bu)) \cdot [\mathrm{ch}(av)]^2,\ F = 0. \quad (19)$$

The coefficients of the second quadratic form of minimal surface (18), found according to formulas (11), equal: $L = -N = a\,\mathrm{sh}(bu)\cdot\cos(bv),\ M = -a\,\mathrm{ch}(bu)\cdot\sin(bv).$

The coefficients of the second quadratic form of associated minimal surface (19) equal to:

$$L^* = -N^* = a\,\mathrm{ch}(bu)\cdot\sin(bv),\ M^* = a\,\mathrm{sh}(bu)\cdot\cos(bv).$$

The coefficients of the first and second quadratic forms of minimal surfaces (18) and (19), transform the mean curvature expression to zero.

Minimal surfaces built on equations (18) and (19) have the same expression of coefficients of the first quadratic form, that's why they allow continuous bending one above the other.

Equations of one-parameter set of associated minimal surfaces formed with continuous bending are of the form [19]:

$$\begin{aligned} X_\varphi(u,v) &= X_2(u,v)\cdot\cos\varphi + X_2^*(u,v)\cdot\sin\varphi, \\ Y_\varphi(u,v) &= Y_2(u,v)\cdot\cos\varphi + Y_2^*(u,v)\cdot\sin\varphi, \\ Z_\varphi(u,v) &= Z_2(u,v)\cdot\cos\varphi + Z_2^*(u,v)\cdot\sin\varphi, \end{aligned} \quad (20)$$

where: $X_2(u,v), Y_2(u,v), Z_2(u,v)$ – minimal surface equation (18),

$X_2^*(u,v), Y_2^*(u,v), Z_2^*(u,v)$ – associated minimal surface equation (19),

$\varphi -$ bending parameter of surfaces, $\varphi \in \left[0;\ \frac{\pi}{2}\right].$

It is obviously that $\varphi = 0$ equations (20) define the minimal surface (18), at $\varphi = \frac{\pi}{2}$ equations (20) define the associated minimal surface (19), for other values $\varphi \in \left(0;\ \frac{\pi}{2}\right)$ equations (20) define associated minimal surfaces [19].

In Fig. 13, 14, 15, 16 images associated minimal surfaces built on equations (20) in accordance for $a = 0,8;$ $b = 0,1;$ $u \in [0;..\pi];$ $v \in [-2;..2],$ at $\varphi = \frac{\pi}{12};\ \varphi = \frac{\pi}{6};\ \varphi = \frac{\pi}{4};\ \varphi = \frac{3\pi}{8}$ respectively are built.

All associated minimal surfaces have the same expression (19) of coefficients of the first quadratic form.

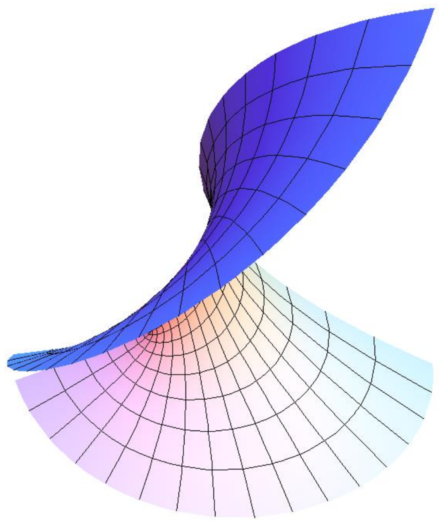

Fig. 13. Associated minimal surface built at $\varphi = \frac{\pi}{12}$

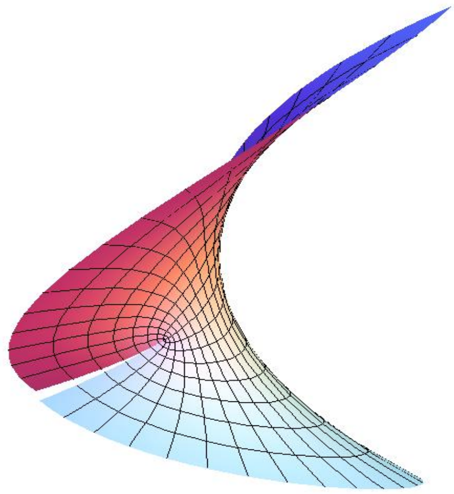

Fig. 14. Associated minimal surface built at $\varphi = \frac{\pi}{6}$

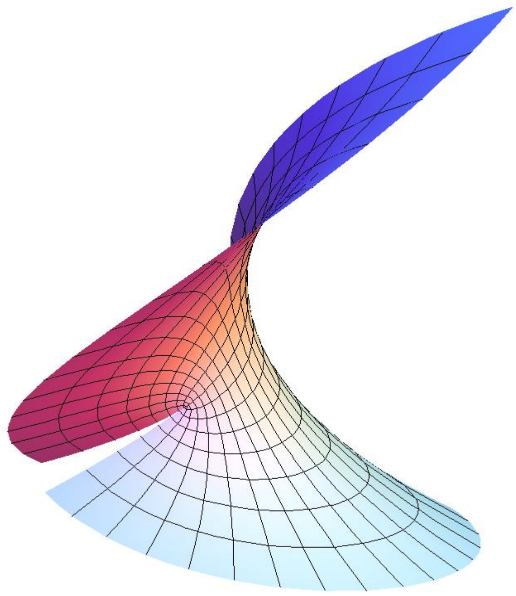

Fig. 15. Associated minimal surface built at $\varphi = \frac{\pi}{4}$

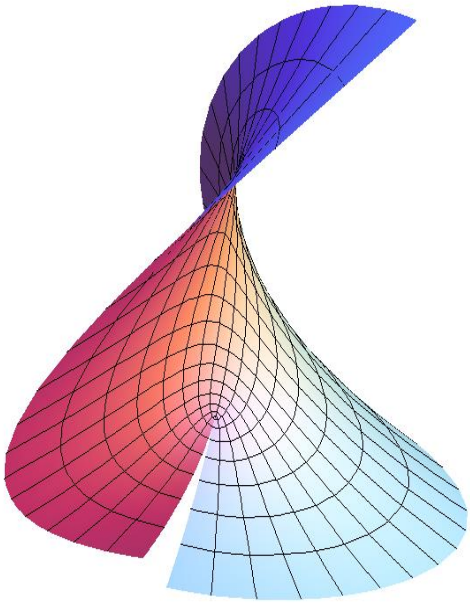

Fig. 16. Associated minimal surface built at $\varphi = \dfrac{3\pi}{8}$

These minimal surfaces are formed under continuous bending of a minimal surface (18) to associated minimal surface (19).

Use of flat curves given by parametric natural parameter equations with complex curvature, provides a relatively simple analytical description of minimal surfaces for further study of the geometrical properties.

CONCLUSIONS

1. The proposed method of finding isotropic curves parametric equations based on flat curves with constant complex curvature allows to determine analytical description of flat lines defined by natural parameter functions in complex space. The minimal surface (7), based on the specified isotropic line is "broken" Catenoid.

2. For isotropic line with constant complex curvature equations, it was found analytical description of isotropic lines with real and imaginary parts of the complex variable. For these isotropic lines analytical description of a minimal surfaces and associated minimal surfaces was made. It is investigated that minimal surface and associated minimal surface formed from isotropic line with the real part of complex variable is Catenoid and right helicoid. Coefficients expressions of the first and second generated minimal surfaces quadratic forms were found.

3. Minimal surface constructed using equations (18) can be used to design technical surfaces forms of soil loosening.

4. Minimal surfaces parametric equations were found in the form of elementary functions, which allows to explore their geometric properties and differential characteristics to optimize engineering methods to design technical forms and architectural constructions.

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КОНСТРУИРОВАНИЕ МИНИМАЛЬНЫХ ПОВЕРХНОСТЕЙ С ПОМОЩЬЮ ПЛОСКИХ КРИВЫХ С ПОСТОЯННОЙ КОМПЛЕКСНОЙ ВЕЛИЧИНОЙ КРИВИЗНЫ

Сергей Пилипака, Николай Муквич

Аннотация. Получено аналитическое описание изотропных линий и минимальных поверхностей с помощью функций комплексного переменного. Для нахождения уравнений изотропных линий использовано параметрические уравнения плоской кривой, заданной функциями натурального параметра с постоянной комплексной величиной кривизны. Параметрические уравнения изотропных линий получены из условия равенства нулю дифференциала дуги пространственной линии. Аналитическое описание минимальных поверхностей и присоединённых минимальных поверхностей осуществляется в комплексном пространстве с изотропными линиями сети переноса.

Используя условия дифференцируемости Коши-Римана для уравнений изотропной линии с комплексной величиной кривизны, получено аналитическое описание изотропных линий с помощью действительной и мнимой части функций комплексного переменного. Для указанных изотропных линий осуществлено аналитическое описание минимальных и присоединённых минимальных поверхностей. Показано, что минимальной поверхностью и присоединённой минимальной поверхностью, образованными с помощью действительной части функций комплексного переменного указанной изотропной линии, являются катеноид и прямой геликоид. Получено аналитическое описание одно-параметрического множества ассоциированных минимальных поверхностей, образованных с помощью их непрерывного изгибания.

Предложенный способ образования параметрических уравнений изотропных линий с помощью плоских кривых, кривизна которых есть комплексной величиной, позволяет определять аналитическое описание плоских линий, заданных функциями натурального параметра, в комплексном пространстве.

Параметрические уравнения минимальных поверхностей получены в виде элементарных функций, что позволяет исследовать их геометрические свойства и дифференциальные характеристики для оптимизации инженерных методов проектирования технических форм и архитектурных конструкций.

Ключевые слова: изотропная линия, минимальная поверхность, функция комплексной переменной, постоянная комплексная величина кривизны, условия Коши-Римана, катеноид, прямой геликоид.