

Tadeusz KACZOREK*

POSITIVITY OF DESCRIPTOR LINEAR SYSTEMS WITH REGULAR PENCILS

The positivity of descriptor continuous-time and discrete-time linear systems with regular pencils are addressed. Such systems can be reduced to standard linear systems and can be decomposed into dynamical and static parts. Two definitions of the positive systems are proposed. It is shown that the definitions are not equivalent. Conditions for the positivity of the systems and the relationship between two classes of positive systems are established. The considerations are illustrated by examples of electrical circuits and numerical examples.

1. INTRODUCTION

Descriptor linear systems with regular pencils have been considered in many papers and books [1-4, 7, 8, 17]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [1-3, 6, 17] and the realization problem for singular positive continuous-time systems with delays in [10]. The positive 1D and 2D linear systems have been considered in [17]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [21]. The fractional differential equations have been considered in monograph [20]. Fractional positive linear systems have been addressed in [5, 9] and in monograph [11]. Luenberger in [19] has proposed the shuffle algorithm to analysis of the singular linear systems. A method for the checking of positivity of descriptor linear systems by the use of the shuffle algorithm has been proposed in [12]. The positivity and reachability of fractional electrical circuits have been addressed in [13]. Modified version of the shuffle algorithm has been proposed for the reduction of the singular fractional system into dynamic and static parts in [14]. The descriptor fractional discrete-time and continuous-time linear systems have been investigated in [15].

In this paper two definitions of positive descriptor linear systems will be proposed and it will be shown that they are not equivalent. Conditions for the positivity of the descriptor systems and the relationships between two classes of positive systems will be established.

* Bialystok University of Technology.

The paper is organized as follows. In section 2 the preliminaries concerning the descriptor continuous-time and discrete-time linear systems and their reduction and decomposition are recalled. Conditions for the positivity of descriptor continuous-time linear systems and the relationship between two classes of positive systems are established in section 3. Similar problems for descriptor discrete-time linear systems are discussed in section 4. Concluding remarks are given in section 5.

The following notation will be used: \mathfrak{R} - the set of real numbers, $\mathfrak{R}^{n \times m}$ - the set of $n \times m$ real matrices, $\mathfrak{R}_+^{n \times m}$ - the set of $n \times m$ matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n \times 1}$, M_n - the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n \times n$ identity matrix. The addition to j -th row multiplied by scalar c to the i -th row will be denoted $L[i + j \times c]$ and interchanges of i -th and j -th rows by $L[i, j]$.

2. PRELIMINARIES AND THE PROBLEM FORMULATION

Consider the descriptor continuous-time linear system with regular pencil

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (2.1)$$

where $x(t) \in \mathfrak{R}^n$, $u(t) \in \mathfrak{R}^m$ are the state and input vectors and $E, A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$ and

$$\det[Es - A] \neq 0 \quad \text{for some } s \in C \quad (\text{the field of complex numbers}). \quad (2.2)$$

The descriptor system (2.1) satisfying (2.2) can be reduced by the use of the shuffle algorithm [14, 12] to the standard descriptor system

$$\dot{x}(t) = A_0x(t) + B_0u(t) + B_1u^{(1)}(t) + \dots + B_qu^{(q)}(t),$$

$$u^{(k)}(t) = \frac{d^k u(t)}{dt^k} \quad \text{for } k = 1, \dots, q \quad (2.3)$$

where $A_0 \in \mathfrak{R}^{n \times n}$, $B_k \in \mathfrak{R}^{n \times m}$, $k = 0, 1, \dots, q$.

Definition 2.1. The descriptor system (2.3) is called (internally) positive if $x(t) \in R_+^n$, $t \geq 0$ for every $x_0 \in R_+^n$ and all inputs $u^{(k)}(t) \in \mathfrak{R}_+^m$, $t \geq 0$, $k = 0, 1, \dots, q$.

Theorem 2.1. The descriptor continuous-time linear system (2.3) is positive if and only if

$$A_0 \in M_n, \quad B_k \in \mathfrak{R}_+^{n \times m}, \quad k = 0, 1, \dots, q. \quad (2.4)$$

It is also well-known [12, 14] that the descriptor system (2.1) satisfying (2.2) can be decomposed into the dynamical part

$$\dot{\bar{x}}_1(t) = A_1 \bar{x}_1(t) + B_{10}u(t) + B_{11}u^{(1)}(t) + \dots + B_{1,q_1}u^{(q_1)}(t), \quad \bar{x}_1(t) \in \mathfrak{R}^{n_1}, \quad u(t) \in \mathfrak{R}^m \quad (2.5a)$$

and the static part

$$\begin{aligned} \bar{x}_2(t) &= A_2 \bar{x}_1(t) + B_{20}u(t) + B_{21}u^{(1)}(t) + \dots + B_{2,q_2}u^{(q_2)}(t), \\ \bar{x}_2(t) &\in \mathfrak{R}^{n_2}, \quad u(t) \in \mathfrak{R}^m, \quad n = n_1 + n_2. \end{aligned} \quad (2.5b)$$

Definition 2.2. The descriptor system (2.5) is called positive if $\bar{x}_1(t) \in \mathfrak{R}_+^{n_1}$, $\bar{x}_2(t) \in \mathfrak{R}_+^{n_2}$, $t \geq 0$ for every $\bar{x}_{10} \in \mathfrak{R}_+^{n_1}$, $\bar{x}_{20} \in \mathfrak{R}_+^{n_2}$ and all inputs $u^{(k)}(t) \in \mathfrak{R}_+^m$, $t \geq 0$, $k = 0, 1, \dots, \max(q_1, q_2)$.

Theorem 2.2. The descriptor system (2.5) is positive if and only if

$$A_1 \in M_{n_1}, \quad A_2 \in \mathfrak{R}_+^{n_2 \times n_1}, \quad B_{i,j} \in \mathfrak{R}_+^{n_i \times m}, \quad i = 1, 2; \quad j = 0, 1, \dots, \max(q_1, q_2). \quad (2.6)$$

Now let us consider the descriptor discrete-time linear system with regular pencil

$$Ex_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0, 1, \dots\} \quad (2.7)$$

where $x_i \in \mathfrak{R}^n$, $u_i \in \mathfrak{R}^m$ are the state and input vectors and $E, A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$ and

$$\det[Ez - A] \neq 0 \quad \text{for some } z \in C. \quad (2.8)$$

The descriptor system (2.7) satisfying (2.8) can be reduced by the use of the shuffle algorithm [14, 12] to the standard descriptor system

$$x_{i+1} = A_0x_i + B_0u_i + B_1u_{i+1} + \dots + B_qu_{i+q} \quad (2.9)$$

where $A_0 \in \mathfrak{R}^{n \times n}$, $B_k \in \mathfrak{R}^{n \times m}$, $k = 0, 1, \dots, q$.

Definition 2.3. The descriptor systems (2.9) is called (internally) positive if $x_i \in R_+^n$, $i \in Z_+$ for every $x_0 \in R_+^n$ and all inputs $u_{i+k} \in \mathfrak{R}_+^m$, $i \in Z_+$, $k = 0, 1, \dots, q$.

Theorem 2.3. The descriptor discrete-time linear system (2.9) is positive if and only if

$$A_0 \in \mathfrak{R}_+^{n \times n}, \quad B_k \in \mathfrak{R}_+^{n \times m}, \quad k = 0, 1, \dots, q. \quad (2.10)$$

It is also well-known [12, 14] that the descriptor system (2.7) satisfying (2.8) can be decomposed into the dynamical part

$$\bar{x}_{1,i+1} = A_1 \bar{x}_{1,i} + B_{10}u_i + B_{11}u_{i+1} + \dots + B_{1,q_1}u_{i+q_1}, \quad \bar{x}_{1,i} \in \mathfrak{R}^{n_1}, \quad u_i \in \mathfrak{R}^m \quad (2.11a)$$

and the static part

$$\bar{x}_{2,i} = A_2 \bar{x}_{1,i} + B_{20} u_i + B_{21} u_{i+1} + \dots + B_{2,q_2} u_{i+q_2}, \quad \bar{x}_{2,i} \in \mathfrak{R}^{n_2}, \quad n = n_1 + n_2. \quad (2.11b)$$

Definition 2.4. The descriptor system (2.11) is called positive if $\bar{x}_{1,i}(t) \in \mathfrak{R}_+^{n_1}$, $\bar{x}_{2,i}(t) \in \mathfrak{R}_+^{n_2}$ for every $\bar{x}_{10} \in \mathfrak{R}_+^{n_1}$, $\bar{x}_{20} \in \mathfrak{R}_+^{n_2}$ and all inputs $u_{i+k} \in \mathfrak{R}_+^m$, $i \in Z_+$, $k = 0, 1, \dots, \max(q_1, q_2)$.

Theorem 2.4. The descriptor system (2.11) is positive if and only if

$$A_1 \in \mathfrak{R}_+^{n_1 \times n_1}, \quad A_2 \in \mathfrak{R}_+^{n_2 \times n_1}, \quad B_{i,j} \in \mathfrak{R}_+^{n_i \times m}, \quad i = 1, 2; \quad j = 0, 1, \dots, \max(q_1, q_2). \quad (2.12)$$

It will be shown that Definitions 2.1 and 2.2 (2.3 and 2.4) are not equivalent. The following four cases will be considered.

- Case 1. Both descriptor systems (2.3) and (2.5) ((2.9) and (2.11)) are positive.
- Case 2. The descriptor system (2.3) ((2.9)) is positive but the descriptor system (2.5) ((2.11)) is not positive.
- Case 3. The descriptor system (2.3) ((2.9)) is not positive but the descriptor system (2.5) ((2.11)) is positive.
- Case 4. Both descriptor systems (2.3) ((2.9)) and (2.5) ((2.11)) are not positive.

3. POSITIVE CONTINUOUS-TIME SYSTEMS

3.1. Case 1: Both descriptor systems are positive

By Theorem 2.1 the descriptor system (2.3) is positive if and only if the conditions (2.4) are met and by Theorem 2.2 the descriptor system (2.5) is positive if and only if the conditions (2.6) are satisfied. The details of the procedure will be shown on the following example.

Example 3.1. Consider the descriptor system (2.1) with matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (3.1)$$

To obtain the descriptor system (2.3) we perform on the array

$$[E \quad A \quad B] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \quad (3.2)$$

the elementary row operations $L[3+1 \times 1]$ and $L[2,3]$ and we obtain

$$\begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \quad (3.3)$$

After the shuffle of A_2 on (3.3) we get

$$\begin{bmatrix} E_1 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & -B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.4)$$

and the system (2.3) with the matrices

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (q = 0). \quad (3.5)$$

From (2.1) with (3.1) we obtain the descriptor system (2.5) of the form

$$\dot{\bar{x}}_1(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad (3.6a)$$

and

$$\bar{x}_2(t) = x_3(t) = 0. \quad (3.6b)$$

Both descriptor systems (2.3) with (3.5) and (3.6) are positive.

Theorem 3.1. Let the descriptor system (2.5) be positive, then the descriptor system (2.3) is positive if

$$A_2 A_1 \in \mathfrak{R}_+^{n_2 \times n_1}. \quad (3.7)$$

Proof. Differentiating with respect to time the equation (2.5b) and using (2.5a) we obtain

$$\begin{aligned} \dot{\bar{x}}_2(t) &= A_2 \dot{\bar{x}}_1(t) + B_{20} \dot{u}(t) + B_{21} \ddot{u}(t) + \dots + B_{2,q_2} u^{(q_2+1)}(t) \\ &= A_2 (A_1 \bar{x}_1(t) + B_{10} u(t) + B_{11} \dot{u}(t) + \dots + B_{1,q_1} u^{(q_1)}(t)) + B_{20} \dot{u}(t) + B_{21} \ddot{u}(t) + \dots + B_{2,q_2} u^{(q_2+1)}(t) \\ &= A_2 A_1 \bar{x}_1(t) + A_2 B_{10} u(t) + (B_{20} + A_2 B_{11}) \dot{u}(t) + \dots \\ &\quad + \begin{cases} (B_{2,q_1-1} + A_2 B_{1,q_1}) u^{(q_1)}(t) + \dots + B_{2,q_2} u^{(q_2+1)}(t) & \text{for } q_1 < q_2 + 1 \\ (B_{2,q_2} + A_2 B_{1,q_1}) u^{(q_1)}(t) & \text{for } q_1 = q_2 + 1 \\ A_2 B_{1,q_1} u^{(q_1)}(t) & \text{for } q_1 > q_2 + 1 \end{cases} \end{aligned} \quad (3.8)$$

From (3.8) it follows that all matrices are nonnegative if the condition (3.7) is met. The equations (2.5a) and (3.8) can be written in the form

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_2 A_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_{10} \\ A_2 B_{10} \end{bmatrix} u(t) + \begin{bmatrix} B_{11} \\ B_{20} + A_2 B_{11} \end{bmatrix} \dot{u}(t) + \dots \quad (3.9)$$

Note that

$$\begin{bmatrix} A_1 & 0 \\ A_2 A_1 & 0 \end{bmatrix} \in M_n, \quad \begin{bmatrix} B_{10} \\ A_2 B_{10} \end{bmatrix} \in \mathfrak{R}_+^{n \times m}, \quad \begin{bmatrix} B_{11} \\ B_{20} + A_2 B_{11} \end{bmatrix} \in \mathfrak{R}_+^{n \times m}, \quad \dots \quad (3.10)$$

if the conditions (3.7) are satisfied and by Theorem 2.1 the descriptor system is positive. \square

3.2. Case 2: The descriptor systems (2.5) is not positive, but the descriptor system (2.3) is positive

Theorem 3.2. Let the system (2.5) be not positive, then the system (2.3) is positive if

$$1) A_2 A_1 \in \mathfrak{R}_+^{n_2 \times n_1}, \quad A_2 B_{10} \in \mathfrak{R}_+^{n_2 \times m}, \quad A_2 B_{1,j} + B_{2,j-1} \in \mathfrak{R}_+^{n_2 \times m} \quad (3.11a)$$

for $j = 1, 2, \dots, \max(q_1, q_2)$

$$2) A_1 \in M_{n_1}, \quad B_{1,i} \in \mathfrak{R}_+^{n_1 \times m}, \quad i = 0, 1, \dots, q_1. \quad (3.11b)$$

Proof. In the same way as in proof of Theorem 3.1 we may obtain the equation (3.8), from which it follows that all its matrices are nonnegative if the conditions (3.11a) are met. If the conditions (3.11b) are also satisfied then (3.10) holds and the system (2.3) is positive. \square

Example 3.2. Consider the descriptor system (2.1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (3.12)$$

From (3.12) we have

$$\dot{\bar{x}}_1(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \quad (3.13a)$$

and

$$x_3(t) = x_1(t) - x_2(t) + u(t). \quad (3.13b)$$

By Theorem 2.2 the descriptor system (3.13) is not positive. Differentiating with respect to time (3.13b) and using (3.13a) we obtain

$$\dot{x}_3(t) = \dot{x}_1(t) - \dot{x}_2(t) + \dot{u}(t) = x_1(t) + 2x_2(t) + u(t) + x_2(t) + \dot{u}(t) = x_1(t) + 3x_2(t) + u(t) + \dot{u}(t) \quad (3.14a)$$

and

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{u}(t). \quad (3.14b)$$

By Theorem 2.1 the descriptor system (3.14) is positive since

$$A_0 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix} \in M_3, \quad B_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathfrak{R}_+^3, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathfrak{R}_+^3. \quad (3.15)$$

3.3. Case 3: The descriptor system (2.3) is not positive and the descriptor system (2.5) is positive

By Theorem 2.1 the descriptor system (2.3) is not positive if at least one of the conditions (2.4) is not satisfied. By inverting the idea used in the proof of Theorem 3.1 it is possible to obtain from nonpositive descriptor system (2.3) a positive descriptor system (2.5) if the matrix A_0 has at least one zero column.

Let the matrix A_0 have n_2 zero columns, i.e.

$$A_0 = \begin{bmatrix} A_{01} & 0 \\ A_{02} & 0 \end{bmatrix}, \quad \begin{bmatrix} A_{01} \\ A_{02} \end{bmatrix} \in \mathfrak{R}^{n \times (n-n_2)}. \quad (3.16)$$

Let by performing elementary row operations on the array

$$I_n \quad A_0 \quad B_0 \quad B_1 \quad \dots \quad B_q \quad (3.17)$$

be possible to obtain the array

$$\begin{array}{cccccccc} I_{n_1} & 0 & A_1 & 0 & B_{10} & B_{11} & \dots & B_{1,q} \\ -A_2 & I_{n_2} & 0 & 0 & 0 & B_{20} & \dots & B_{2,q} \end{array} \quad (3.18)$$

By back shuffle from (3.18) we obtain the equations (2.5a) and (2.5b) for $q_1 = q$ and $q_2 = q - 1$. Therefore, the following theorem has been proved.

Theorem 3.3. Let the matrix A_0 have the form (3.16) and using elementary row operations it is possible to reduce the array (3.17) to the form (3.18). Then the descriptor system (2.5) is positive if the conditions (2.6) are satisfied.

Example 3.3. Consider the nonpositive descriptor system (2.3) with the matrices

$$A_0 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (q = 1). \quad (3.19)$$

The equations (3.22a) and (3.22b) can be written in the form

$$\begin{bmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 \\ 0 & -R_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e \quad (3.23)$$

Premultiplying the equation (3.23) by the inverse matrix

$$\begin{bmatrix} L_1 + L_3 & L_3 \\ L_3 & L_2 + L_3 \end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} L_2 + L_3 & -L_3 \\ -L_3 & L_1 + L_3 \end{bmatrix} \quad (3.24)$$

we obtain

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} -R_1(L_2 + L_3) & R_2L_3 \\ R_1L_3 & -R_2(L_1 + L_3) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} \frac{L_2}{L_1(L_2 + L_3) + L_2L_3} \\ \frac{L_1}{L_1(L_2 + L_3) + L_2L_3} \end{bmatrix} e \quad (3.25)$$

By Definition 2.2 the electrical circuit described by the equations (3.25) and (3.22c) is positive.

The equations (3.22) can be written in the form

$$\begin{bmatrix} L_1 + L_3 & L_3 & 0 \\ L_3 & L_2 + L_3 & 0 \\ 1 & 1 & -1 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 & 0 \\ 0 & -R_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e. \quad (3.26)$$

Premultiplying the equation (3.26) by the inverse matrix

$$\begin{bmatrix} L_1 + L_3 & L_3 & 0 \\ L_3 & L_2 + L_3 & 0 \\ 1 & 1 & -1 \end{bmatrix}^{-1} = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} L_2 + L_3 & -L_3 & 0 \\ -L_3 & L_1 + L_3 & 0 \\ L_2 & L_1 & -(L_1(L_2 + L_3) + L_2L_3) \end{bmatrix} \quad (3.27)$$

we obtain

$$\frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} -R_1(L_2 + L_3) & R_2L_3 & 0 \\ R_1L_3 & -R_2(L_1 + L_3) & 0 \\ -R_1L_2 & -R_2L_1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} \frac{L_2}{L_1(L_2 + L_3) + L_2L_3} \\ \frac{L_1}{L_1(L_2 + L_3) + L_2L_3} \\ \frac{L_1 + L_2}{L_1(L_2 + L_3) + L_2L_3} \end{bmatrix} e \quad (3.28)$$

By Definition 2.1 the electrical circuit described by the equation(3.28) is not positive. The array (3.17) for (3.28) has the form

$$\begin{array}{ccccccc}
I_3 & A_0 & B_0 = 0 & 1 & 0 & 0 & -\frac{R_1(L_2 + L_3)}{L_1(L_2 + L_3) + L_2L_3} & \frac{R_2L_3}{L_1(L_2 + L_3) + L_2L_3} & 0 & \frac{L_2}{L_1(L_2 + L_3) + L_2L_3} \\
& & & 1 & 0 & 0 & \frac{R_1L_3}{L_1(L_2 + L_3) + L_2L_3} & -\frac{R_2(L_1 + L_3)}{L_1(L_2 + L_3) + L_2L_3} & 0 & \frac{L_1}{L_1(L_2 + L_3) + L_2L_3} \\
& & & 0 & 0 & 1 & -\frac{R_1L_2}{L_1(L_2 + L_3) + L_2L_3} & -\frac{R_2L_1}{L_1(L_2 + L_3) + L_2L_3} & 0 & \frac{L_1 + L_2}{L_1(L_2 + L_3) + L_2L_3}
\end{array} \tag{3.29}$$

Performing on the array (3.29) the elementary row operation $L[3 + 1 \times (-1)]$, $L[3 + 2 \times (-1)]$ we obtain

$$\begin{array}{ccccccc}
1 & 0 & 0 & -\frac{R_1(L_2 + L_3)}{L_1(L_2 + L_3) + L_2L_3} & \frac{R_2L_3}{L_1(L_2 + L_3) + L_2L_3} & 0 & \frac{L_2}{L_1(L_2 + L_3) + L_2L_3} \\
0 & 1 & 0 & \frac{R_1L_3}{L_1(L_2 + L_3) + L_2L_3} & -\frac{R_2(L_1 + L_3)}{L_1(L_2 + L_3) + L_2L_3} & 0 & \frac{L_1}{L_1(L_2 + L_3) + L_2L_3} \\
-1 & -1 & 1 & 0 & 0 & 0 & 0
\end{array} \tag{3.30}$$

and the equations (3.25) and (3.22c) of the positive system.

3.4. Case 4: Both descriptor systems are not positive

The descriptor system (2.3) is not positive if at least one of the conditions (2.4) is not met. Similarly, the descriptor system (2.5) is not positive if at least one of the conditions (2.6) is not satisfied.

Theorem 3.4. The descriptor system (2.3) is not positive if the matrix A_1 is not a Metzler matrix and at least one entry of the matrix $B_{i,j}$ for $i, j = 0, 1, \dots, q_1$ is negative.

Proof. In a similar way as in the proof of Theorem 3.1 we may obtain the equation (3.9). If the matrix A_1 is not a Metzler matrix and at least one entry of the matrices $B_{i,j}$ for $i, j = 0, 1, \dots, q_1$ is negative then the conditions (3.10) are not satisfied and by Theorem 2.1 the descriptor system (2.3) is not positive. \square

Example 3.5. Consider the electrical circuit shown on Fig. 2 with the given resistance R capacitances C_1, C_2, C_3 and voltage source e .

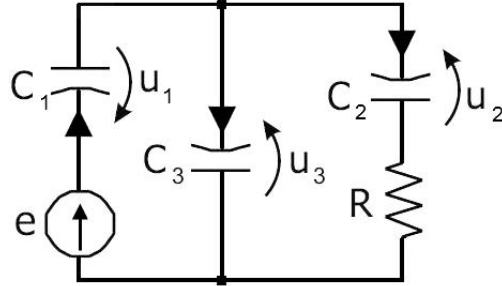


Fig. 2. Electrical circuit with resistance, capacitances and voltage source

Using the Kirchoff's laws we may write the equations

$$e = u_1 + u_3 \quad (3.31a)$$

$$e = u_1 + u_2 + RC_2 \frac{du_2}{dt} \quad (3.31b)$$

$$C_1 \frac{du_1}{dt} = C_2 \frac{du_2}{dt} + C_3 \frac{du_3}{dt} \quad (3.31c)$$

From (3.31) we have

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R(C_1 + C_3)} & -\frac{1}{R(C_1 + C_3)} \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R(C_1 + C_3)} \\ \frac{1}{RC_2} \end{bmatrix} e + \begin{bmatrix} \frac{C_3}{C_1 + C_3} \\ 0 \end{bmatrix} \dot{e} \quad (3.32a)$$

and

$$u_3 = e - u_1. \quad (3.32b)$$

By Definition 2.2 the electrical circuit described by the equations (3.32) is not positive.

Differentiating with respect to time the equation (3.31a) and using the equations (3.31b) and (3.31c) we obtain

$$\begin{bmatrix} -C_1 & C_2 & C_3 \\ 0 & RC_2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{e}. \quad (3.33)$$

Premultiplying the equation (3.33) by the inverse matrix

$$\begin{bmatrix} -C_1 & C_2 & C_3 \\ 0 & RC_2 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{RC_2(C_1 + C_3)} \begin{bmatrix} -RC_2 & C_2 & RC_2C_3 \\ 0 & C_1 + C_3 & 0 \\ RC_2 & -C_2 & RC_1C_2 \end{bmatrix} \quad (3.34)$$

we obtain

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R(C_1 + C_3)} & -\frac{1}{R(C_1 + C_3)} & 0 \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} & 0 \\ \frac{1}{R(C_1 + C_3)} & \frac{1}{R(C_1 + C_3)} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{R(C_1 + C_3)} \\ \frac{1}{RC_2} \\ -\frac{1}{R(C_1 + C_3)} \end{bmatrix} e + \begin{bmatrix} \frac{C_3}{C_1 + C_3} \\ 0 \\ \frac{C_1}{C_1 + C_3} \end{bmatrix} \dot{e} \quad (3.35)$$

By Definition 2.1 the electrical circuit described by the equation (3.35) is also not positive.

4. POSITIVE DISCRETE-TIME SYSTEMS

Considerations for positive discrete-time linear systems are similar to the considerations in section 3 for positive continuous-time linear systems. In this section we shall concentrate our considerations only these which are different from those for continuous-time systems.

Theorem 4.1. The descriptor system (2.9) is positive if the descriptor system (2.11) is positive.

Proof. Using (2.11b) and (2.11a) we can write

$$\begin{aligned} \bar{x}_{2,i+1} &= A_2 \bar{x}_{1,i+1} + B_{20} u_{i+1} + B_{21} u_{i+2} + \dots + B_{2,q_2} u_{i+q_2+1} \\ &= A_2 (A_1 \bar{x}_{1,i} + B_{10} u_i + B_{11} u_{i+1} + \dots + B_{1,q_1} u_{i+q_1}) + B_{20} u_{i+1} + B_{21} u_{i+2} + \dots + B_{2,q_2} u_{i+q_2+1} \\ &= A_2 A_1 \bar{x}_{1,i} + A_2 B_{10} u_i + (B_{20} + A_2 B_{11}) u_{i+1} + \dots \\ &\quad + \begin{cases} (B_{2,q_2-1} + A_2 B_{1,q_1}) u_{i+q_1} + \dots + B_{2,q_2} u_{i+q_2+1} & \text{for } q_1 < q_2 + 1 \\ (B_{2,q_2-1} + A_2 B_{1,q_1}) u_{i,q_1} & \text{for } q_1 = q_2 + 1 \\ A_2 B_{1,q_1} u_{i+q_1} & \text{for } q_1 > q_2 + 1 \end{cases} \end{aligned} \quad (4.1)$$

If the conditions (2.12) are satisfied then all matrices in (4.1) are nonnegative. The equations (2.11a) and (4.1) can be written in the form

$$\begin{bmatrix} \bar{x}_{1,i+1} \\ \bar{x}_{2,i+1} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_2 A_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} + \begin{bmatrix} B_{10} \\ A_2 B_{10} \end{bmatrix} u_i + \begin{bmatrix} B_{11} \\ B_{20} + A_2 B_{11} \end{bmatrix} u_{i+1} + \dots \quad (4.2)$$

Note that

$$\begin{bmatrix} A_1 & 0 \\ A_2 A_1 & 0 \end{bmatrix} \in \mathfrak{R}_+^{n \times n}, \begin{bmatrix} B_{10} \\ A_2 B_{10} \end{bmatrix} \in \mathfrak{R}_+^{n \times m}, \begin{bmatrix} B_{11} \\ B_{20} + A_2 B_{11} \end{bmatrix} \in \mathfrak{R}_+^{n \times m}, \dots \quad (4.3)$$

This completes the proof. \square

Theorem 4.2. Let the descriptor system (2.11) be not positive, then the descriptor system (2.9) is positive if

$$\begin{aligned} A_1 \in \mathfrak{R}_+^{n_1 \times n_1}, A_2 A_1 \in \mathfrak{R}_+^{n_2 \times n_1}, A_2 B_{1,i} \in \mathfrak{R}_+^{n_2 \times m} \quad i = 0, 1, \dots, \max(q_1, q_2), \\ B_{j,k} \in \mathfrak{R}_+^{n_j \times m} \quad \text{for } j = 1, 2; i = 0, 1, \dots, \max(q_1, q_2). \end{aligned} \quad (4.4)$$

Proof is similar to the proof of Theorem 3.2.

Remark 4.1. If the descriptor system (2.11) is positive then the descriptor system (2.9) is also positive since the condition (3.7) is always satisfied for positive discrete-time systems.

5. CONCLUDING REMARKS

The positivity of descriptor continuous-time and discrete-time linear systems with regular pencils have been addressed. Starting from the well-known fact that such descriptor systems can be reduced to standard linear systems and can be decomposed into dynamical and static parts two definitions of the positive systems have been proposed. It has been shown that the definitions are not equivalent. Four cases of descriptor systems have been discussed. The conditions for the positivity of the descriptor continuous-time and discrete-time linear systems have been given. The relationship between positive systems have been also established. The considerations have been illustrated by examples of electrical circuits and numerical examples. The considerations can be extended to descriptor fractional continuous-time and discrete-time linear systems. An open problem is an extension of these considerations to descriptor linear systems with singular pencils.

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