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APPROXIMATION OF THE ELLIPSE OFFSET CURVES IN TURBO ROUNDABOUTS DESIGN

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Abstract. The elliptical turbo roundabouts are a safer and more effective alternative to the known multi-lane roundabouts. This paper contains the results of numerical analysis for the problem of approximation of the offset curves *off*(*el*(*a*, *b*); *s*) of an ellipse *el*(*a*, *b*) at distance *s* by ellipses $el_1(a+s, b+s)$ and $el_2(a-s, b-s)$. We considered the ellipses which can be used to shape the turbo roundabouts. It was tested for which parameter values *s* (the width of the lane) and *e* (eccentricity) the maximum deviation of the ellipse offset curve from the ellipse el_1 (el_2) does not exceed the accuracy of the delineating the curves in the terrain.

Keywords: ellipse offset curve, accuracy of delineating, turbo roundabout

1 Introduction

In this paper the studies from [1] and [2] are continued. The article contains mathematical facts helpful in designing elliptical turbo roundabouts. The possibility of approximating offset curves *off*(*el*(*a*, *b*); *s*) (of an ellipse *el*(*a*, *b*) at distance *s*) by ellipses $el_1(a+s, b+s)$ and $el_2(a\text{-}s, b\text{-}s)$ (for any values *a*, *b*, *s*) was numerically analyzed. The ellipses which can be used to shape the turbo roundabouts were considered in particular. It was tested for which parameter values *s*, *e* ($e=\sqrt{1-(b/a)^2}$) the maximum deviation of the ellipse offset curve from the ellipse el_1 (el_2) does not exceed the accuracy of the delineating the curves in the terrain.

Let us assume that *P* is any point on the ellipse *el* and *l* is the normal line to *el* at the point *P*. Points P_1 and P_2 lie on the normal *l* at distance *s* from *P*. Q_1, Q_2 are the intersection points of the normal line *l* with ellipses el_1 and el_2 respectively. In section 3 useful formulas for coordinates of points P_1, P_2, Q_1 and Q_2 (for any point *P*, i.e. for any angle *t*) were determined.

2 Elliptical turbo roundabouts

The ellipse offset curves are used, among other things, in the designing of two-lane turbo-

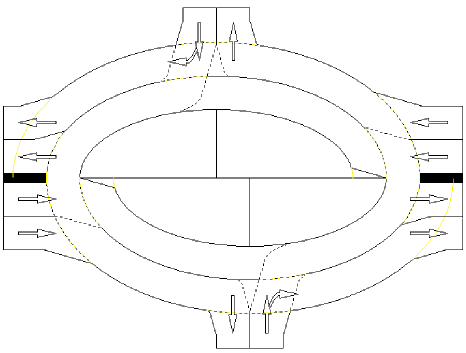

Figure 1: The typical turbo roundabout

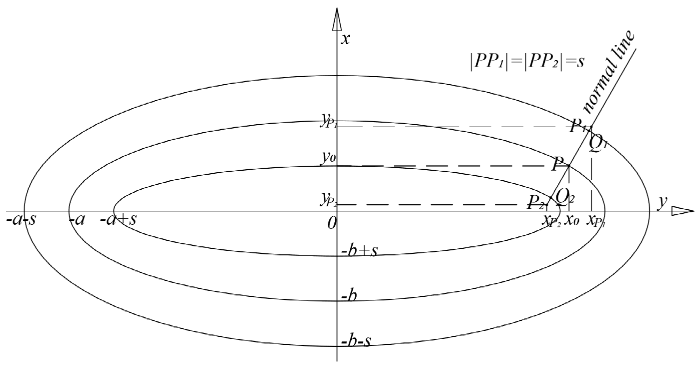

Figure 2: The ellipse offset curves

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roundabouts. The turbo roundabout was invented in the Netherlands in 1996 as a safer and more efficient alternative to the standard multi-lane roundabouts (cf. [7]). The vehicle entering the interior lane of such roundabout, after crossing the axis is automatically (without any collision) located in the outer lane (cf. [2]). The tracks of the vehicles do not intersect at the turbo roundabouts and giving right of way only occurs when the vehicle enters the roundabout. The shape of the central island (which we are analyzing) consists of two halves of an ellipse shifted by the width of the lane. The turbo roundabouts with the elliptical central island are especially recommended when one traffic direction is dominant in terms of the intensity (cf. [2]). The larger difference in the strength of the intensity of two traffic directions, the more flattened an ellipse should be used. The ellipse graph (shaping the central island) should be in the circle ring determined by the minimum and maximum radius of the circle allowed by the guidelines [9] for the central island of the given roundabout type (cf. [2]). The precision of the delineating of the curves in the terrain (by using modern electronic theodolites) is ±1cm.

3 Mathematical formulas

Let $c(t)=(x(t), y(t))$ ($t \in [\alpha, \beta]$) be a parametric representation of a planar curve (we write down functions $x(t)$, y(t) as x_t, y_t). The normal vector to the curve $c(t)$ at the point $P(x_t,y_t)$ is of the form $\mathrm{n} = [-y_t', x_t']$. The unit normal vector at the point $P(x_t,y_t)$ is as follows (cf. [8] p.335, [3])

$$\mathrm{n}_{\mathrm{ver}} = \frac{[-y_t', x_t']}{\sqrt{(x_t')^2 + (y_t')^2}} . \tag{1}$$

Lemma 1 (cf. [4]):

$$\arccos(x)+\arcsin(x) = \pi/2 \text{ for } x \in [-1, 1] \tag{2}$$

$$\text{If } x+y > 0 \text{ then } \arccos(x)+\arccos(y) = \arccos(xy - \sqrt{(1-x^2)(1-y^2)}\,). \tag{3}$$

Lemma 2: If $\arccos(x) = \arcsin(y)$ and $x+y > 0$ then $x^2 + y^2 = 1$.

Justification. Let $\arccos(x) = \arcsin(y)$. From (2) and (3) we obtain $\arccos(x)+\arccos(y) = \pi/2$ $\Rightarrow$ $\arccos(xy - \sqrt{(1-x^2)(1-y^2)}\,) = \arccos(0)$ $\Rightarrow$ $xy = \sqrt{(1-x^2)(1-y^2)}$ $\Rightarrow$ $x^2 + y^2 = 1$.

For a smooth planar curve c, we define an *offset curve* c_d *at distance* d in the following way. On each curve normal, we mark the two points that are at distance d from the curve c. The set of all of these points forms the offset $c_d = (c_d' \cup c_d'')$ (cf. [8] p. 335, [5]). The offset $c_d(t)$ at distance d to $c(t)$ is obtained as (cf. [8] p. 335) $c_d(t) = c(t) \pm d\mathrm{n}_{\mathrm{ver}}(t)$.

The curve c and its offset curves c_d' and c_d'' are not always of the same type. The offset c_d of an ellipse c is not a pair of ellipses. We can show this fact by drawing three concentric ellipses $el(a, b)$, $el_1(a+s, b+s)$, $el_2(a\text{-}s, b\text{-}s)$ and the normal l to the ellipse el at any point P (see Figure 2). The interesting offset surfaces (offset curves) are described in [5] and [6].

Let points P_1 and P_2 lie on the normal l at distance s from P. Q_1, Q_2 are the intersection points of the normal line l with ellipses el_1 and el_2 respectively. Non-zero distances $d_{P_1Q_1} = |\overline{P_1Q_1}|$ and $d_{P_2Q_2} = |\overline{P_2Q_2}|$ mean that ellipses el_1 and el_2 do not keep a constant distance s regarding the basic ellipse el at points not lying on the axes of the coordinate system.

3.1 The coordinates of points P_1 and P_2

The coordinates of points P_1 and P_2 lying on the normal l to the ellipse $e(t)$ (at the point $P(x_t, y_t)$) and distant from P by the length s were determined using the equation of offset curves. Let us take the parametric equations of the ellipse $e(t)$: $x = a\cos(t)$, $y = b\sin(t)$, $t \in [0,2\pi]$.
The unit normal vector to $e(t)$ at the point $P(x_t, y_t)$ is of the form

$$\mathrm{n}_{\mathrm{ver}} = \frac{[-b\cos(t), -a\sin(t)]}{\sqrt{b^2\cos^2(t) + a^2\sin^2(t)}} = \frac{[-b^2 x_t, -a^2 y_t]}{\sqrt{b^4 x_t^2 + a^4 y_t^2}} \quad \text{for} \quad \begin{cases} x_t = a\cos(t) \\ y_t = b\sin(t) \end{cases}. \tag{4}$$

The equation of the offset curves $off(e(t); s)$ is as follows

$$off(e(t); s) : [X, Y] = [x_t, y_t] \pm \frac{s[-b^2 x_t, -a^2 y_t]}{\sqrt{b^4 x_t^2 + a^4 y_t^2}} \quad \text{for} \quad \begin{cases} x_t = a\cos(t) \\ y_t = b\sin(t) \end{cases} \quad t \in [0,2\pi]. \tag{5}$$

Finally we obtain coordinates of points P_1 and P_2 lying on l and distant from P by the length s.

$$\begin{cases} x_{P_1} = x_t + \dfrac{sb^2 x_t}{\sqrt{b^4 x_t^2 + a^4 y_t^2}} = x_t + \dfrac{s}{\sqrt{1+v}} \\ y_{P_1} = y_t + \dfrac{sa^2 y_t}{\sqrt{b^4 x_t^2 + a^4 y_t^2}} = y_t + s\sqrt{\dfrac{v}{1+v}} \end{cases}, \quad \begin{cases} x_{P_2} = x_t - \dfrac{s}{\sqrt{1+v}} \\ y_{P_2} = y_t - s\sqrt{\dfrac{v}{1+v}} \end{cases}, \quad v = \left((a/b)\tan(t)\right)^2. \tag{6}$$

3.2 The coordinates of the point Q_1

The coordinates of the point Q_1 (the intersection of the normal line l to the ellipse $e(t)$ at the point $P(x_t, y_t)$ with the ellipse $el_1(a+s, b+s)$) were determined as follows. Let us write down the parametric equations of the normal line l to the ellipse $e(t)$ at the point P. Points P and P_1 lie on the normal line l. Hence

$$x = x_t + \frac{hs}{\sqrt{1+v}}, \quad y = y_t + \frac{hs\sqrt{v}}{\sqrt{1+v}}, \text{ where } x_t = a\cos(t), \ y_t = b\sin(t), \ t \in [0, \pi/2). \tag{7}$$

Let us set the parameter h giving the intersection points of the line l with ellipse $el_1(a+s, b+s)$.

$x = x_t + \dfrac{hs}{\sqrt{1+v}} = (a+s)\cos(\varphi)$, $y = y_t + \dfrac{hs\sqrt{v}}{\sqrt{1+v}} = (b+s)\sin(\varphi)$. From here

$$\varphi = \arccos\left(\left(x_t + \frac{hs}{\sqrt{1+v}}\right)\Big/(a+s)\right), \quad \varphi = \arcsin\left(\left(y_t + \frac{hs\sqrt{v}}{\sqrt{1+v}}\right)\Big/(b+s)\right). \tag{8}$$

Hence and from lemma 2: $\left(\left(x_t + \dfrac{hs}{\sqrt{1+v}}\right)\Big/(a+s)\right)^2 + \left(\left(y_t + \dfrac{hs\sqrt{v}}{\sqrt{1+v}}\right)\Big/(b+s)\right)^2 = 1$. Hence

$$Ah^2 + Bh + C = 0, \text{ where } A = \frac{s^2}{1+v}\left((b+s)^2 + (a+s)^2 v\right), \tag{9}$$

$$B = \frac{2s}{\sqrt{1+v}}\left((a+s)^2 y_t\sqrt{v} + (b+s)^2 x_t\right), C = ((a+s)y_t)^2 + ((b+s)x_t)^2 - ((a+s)(b+s))^2 \tag{10}$$

$$\varDelta = \frac{4s^2}{1+v}(a+s)^2(b+s)^2\left((a+s)^2 v + (b+s)^2 - \left(x_t\sqrt{v} - y_t\right)^2\right). \tag{11}$$

$\varDelta > 0$, $h_1 = (-B + \sqrt{\varDelta})/2A$ (the parameter for the point Q_1), $h_2 = (-B - \sqrt{\varDelta})/2A$. (12)

The coordinates of the point P_{k1} lying on the normal line l and distant from P by the length s-k

$$x_{P_{k1}} = x_t + \frac{s-k}{\sqrt{1+v}}, \; y_{P_{k1}} = y_t + \frac{(s-k)\sqrt{v}}{\sqrt{1+v}} \text{ (cf. (3)).} \quad (13)$$

Lemma 3: (a) The coordinates of the points P, P_{k1}, P_1 and Q_1 can be determined using the parametric equations (7) of the normal line l to the ellipse $el(a, b)$ at the point $P(x_t, y_t)$ for the parameter h equal respectively h=0, h=(s-k)/s, h=1 and h=h_1 (cf. (12)).
(b) $d_{P_1Q_1} \le k$ iff $h_1 \ge (s-k)/s$.

Justification (b). The property results from the fact that points P_{k1}, P_1 and Q_1 lie on the normal l.

Example 1 a=25, b=17, s=3.5, k=0.01, t=35

h=0,	x_t=20.4788011072248	y_t=9.75079941796778
h=(s-k)/s=0.997142857142857,	xP_{k1}=22.9102122371578	yP_{k1}=12.2544647145356
h=1,	xP_1=22.9171790312264	yP_1=12.2616385406002
h=h_1=0.98465895377262,	xQ_1=22.8797717627745	yQ_1=12.2231196415504
dP_1Q_1=0.0536936617958302		

3.3 The coordinates of the point Q_2

The coordinates of the point Q_2 (the intersection of the normal line l to the ellipse $e(t)$ at the point $P(x_t, y_t)$ with the ellipse $el_2(a\text{-}s, b\text{-}s)$) were determined as follows. Let us write down the parametric equations of the normal line l to the ellipse $e(t)$ at the point P. Points P and P_2 lie on the normal line l. Hence

$$x = x_t - \frac{hs}{\sqrt{1+v}}, \; y = y_t - \frac{hs\sqrt{v}}{\sqrt{1+v}}, \text{ where } x_t = a\cos(t), \; y_t = b\sin(t), \; t \in [0, \pi/2) \quad (14)$$

Let us set the parameter h giving the intersection points of the line l with ellipse $el_2(a\text{-}s, b\text{-}s)$.

$$x = x_t - \frac{hs}{\sqrt{1+v}} = (a-s)\cos(\varphi), \; y = y_t - \frac{hs\sqrt{v}}{\sqrt{1+v}} = (b-s)\sin(\varphi). \quad (15)$$

By conducting analogous calculations as in 3.2 we get the equation

$$Ah^2 + Bh + C = 0, \text{ where } A = \frac{s^2}{1+v}\left((b-s)^2 + (a-s)^2 v\right), \quad (16)$$

$$B = \frac{-2s}{\sqrt{1+v}}\left((a-s)^2 y_t\sqrt{v} + (b-s)^2 x_t\right), \; C = ((a-s)y_t)^2 + ((b-s)x_t)^2 - ((a-s)(b-s))^2 \quad (17)$$

$$\Delta = \frac{4s^2}{1+v}(a-s)^2(b-s)^2\left((a-s)^2 v + (b-s)^2 - \left(x_t\sqrt{v} - y_t\right)^2\right). \quad (18)$$

For $\Delta > 0$, $h_1 = (-B + \sqrt{\Delta})/2A$, $h_2 = (-B - \sqrt{\Delta})/2A$ (the parameter for the point Q_2).(19)

Lemma 4: (a) The coordinates of the points P, P_{k2}, P_2 and Q_2 can be determined using the parametric equations (14) of the normal line l to the ellipse $el(a, b)$ at the point $P(x_t, y_t)$ for the parameter h equal respectively h=0, h=(s-k)/s, h=1 and h=h_2 (cf. (19)).
(b) $d_{P_2Q_2} \le k$ iff $h_2 \ge (s-k)/s$ (cf. (19)).

Justification (b). The property results from the fact that points P_2 and Q_2 lie on the normal l.

Example 2 a=25, b=17, s=3.5, k=0.01, t=65

h=0,	x_t=10.5654565435175	y_t=15.407232379623
h=(s-k)/s=0.997142857142857	xP_{k2}=9.51057699269641	yP_{k2}=12.0804728251527
h=1,	xP_2=9.50755441518689	yP_2=12.070940562819
h=h_2=0.989805472012358	xQ_2=9.51833925489295	yQ_2=12.1049525677987
dP_2Q_2=0.0356809367902093		

4 Parameter analysis for the ellipse offset curves

Fact 1. We shall say that the approximation of the ellipse offset curve *off*(*el*(a, b); s) by the ellipse el_1 (el_2) is satisfactory, if for any point P of the ellipse *el* there is $d_{P_1Q_1} \leq k$ ($d_{P_2Q_2} \leq k$) for k=0.01, i.e. when the deviation $d_{P_1Q_1}$ ($d_{P_2Q_2}$) does not exceed the accuracy of the delineating the curves in the terrain.

Section 4 contains the results of numerical analysis for the problem of approximation of the offset curves *off*(*el*(a, b); s) by ellipses el_1 and el_2. In order to check if the approximation of the given offset curve *off*(*el*(a, b); s) by the ellipse el_1 (el_2) is satisfactory, we have to solve one of the following problems.

Problem 1. We have the semi-minor axis b, the width of the lane s and k=0.01. For the ellipse offset curve *off*(*el*(a, b); s) we need to find

(A) such eccentricity e_{Max} that for $e \in [0, e_{Max}]$ and any point P of the ellipse *el* $d_{P_1Q_1}(e) \leq 0.01$.

(B) such eccentricity e_{Max} that for $e \in [0, e_{Max}]$ and any point P of the ellipse *el* $d_{P_2Q_2}(e) \leq 0.01$.

Problem 2. We have the semi-minor axis b, eccentricity e and k=0.01. For the ellipse offset curve *off*(*el*(a, b); s) we need to find

(C) such a distance s_{Max} that for $s \in [0.5, s_{Max}]$ and any point P of the ellipse *el* $d_{P_1Q_1}(s) \leq 0.01$.

(D) such a distance s_{Max} that for $s \in [0.5, s_{Max}]$ and any point P of the ellipse *el* $d_{P_2Q_2}(s) \leq 0.01$.

The problem 1 was resolved in the following way. For established values b, s and consecutive angles $t \in [0, \pi/2)$ we have determined the largest possible values of the eccentricity e_{max} such that the deviation $d_{P_1Q_1} \leq k$ ($d_{P_2Q_2} \leq k$). Next, from among the determined values e_{max} we selected the smallest one (the value e_{Max}). If (for given b and s) we choose $e \in [0, e_{Max}]$ ($a \in [b, a_{Max}]$) then the approximation of the offset curve *off*(*el*(a, b); s) by the ellipse el_1 (el_2) will be satisfactory. It turned out that for the ellipse offset curves (which can be used to shape the turbo roundabouts) the values e_{Max} occur for a specific angle t. The problem 2 was resolved similarly.

We introduce abbreviations: [RAR] – We start delineating the roundabout from the axis of the road. [RCI] – We start delineating the roundabout from the edge of the central island.

4.1 Parameter *t*

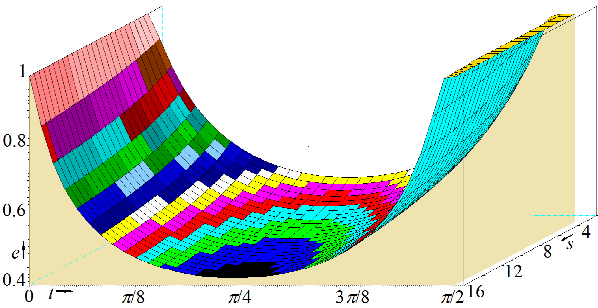
Test 1. The ellipse offset curves which can be used to shape the roundabouts (approved in [9]) were considered. In the case of a larger offset curve, the calculations were made for b=8.5, 9, …, 23.5m (i.e. each possible central island size (approved in [9]) was considered). In the case of a smaller offset curve, the calculations were made for b=16, 16.5, …, 32.5m (for each possible ellipse representing the road axis (or outer line) of the roundabout (approved in [9])). The width of the lane s was tested from 3.5m to 11m (every 0.5m). The eccentricity e was taken from 0 to 0.95 (every 0.05). The angle $t \in [0, \pi/2)$ was taken (every $\pi/36$). The following facts were checked. For a larger offset curve (for given b and s) the value e_{Max} occurs for the angle t=40° / 45° (cf. Table 1). For a smaller offset curve (for given b and s) the value e_{Max} occurs for the angle t=45°. For both offset curves (for given b and e) we can assume that the value s_{Max} (≥3.5) occurs for t=45°.

Table 1: For given b and s the value e_{Max} (such that for $e \in [0, e_{Max}]$ $d_{PQ_1}(e) \le 0.01$) was obtained for the angle t

	b=8.5	b=9	b=9.5	b=10	b=10.5-11.5	b=12-14	b=14.5-19.5	b=20-23.5
t=40°	$s \in [3.5, 6.5]$	$s \in [3.5, 6]$	$s \in [3.5, 5.5]$	$s \in [3.5, 5]$	$s \in [3.5, 4.5]$	$s \in [3.5, 4]$	s=3.5	-
t=45°	$s \in [7, 11]$	$s \in [6.5, 11]$	$s \in [6, 11]$	$s \in [5.5, 11]$	$s \in [5, 11]$	$s \in [4.5, 11]$	$s \in [4, 11]$	$s \in [3.5, 11]$

4.2 Parameter e_{Max}

Test 2. The ellipse offset curves $off(el(a, b); s)$ for parameter values s (s=3.5, 4, …, 16m), b=17m and $a = b/\sqrt{1-e^2}$ were analyzed. The eccentricity $e \in [0, 1)$ was taken (every 0.001).



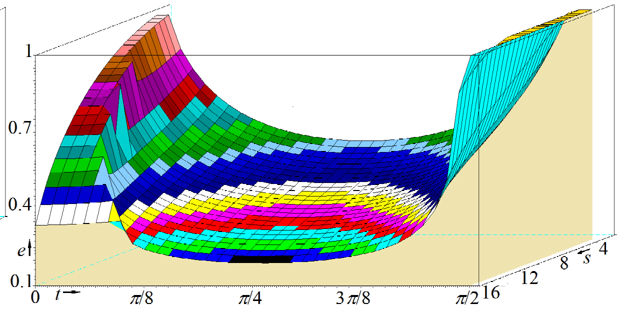


Figure 3: Such eccentricity e_{max} that for $e \in [0, e_{max}]$ $d_{PQ_1}(e) \le 0.01$ (the graph of the function $e_{max}=f_{(A)}(t, s)$)

Figure 4: Such eccentricity e_{max} that for $e \in [0, e_{max}]$ $d_{P_2Q_2}(e) \le 0.01$ (the graph of the function $e_{max}=f_{(B)}(t, s)$)

The angle $t \in [0, \pi/2)$ was taken (every $\pi/72$). For each angle t and for each distance s such value e_{max} was determined that for $e \in [0, e_{max}]$ $d_{PQ_1}(e) \le 0.01$ ($d_{P_2Q_2}(e) \le 0.01$).The results are shown in Figures 3 and 4. The selected calculations are given in Table 2 and Table 3.

Test 2.A. The values e_{Max} fulfilling the condition (A) were determined for the given values s.

Table 2: Such eccentricity e_{Max} that for $e \in [0, e_{Max}]$ $d_{PQ_1}(e) \le 0.01$ (determined for $t=\pi/4$)

The width of the lane s[m]													
s=3.5	s=4	s=5	s=6	s=7	s=8	s=9	s=10	s=11	s=12	s=13	s=14	s=15	s=16
e_{Max}=0.531	e_{Max}=0.519 a_{Max}=19.88	0.499	0.484	0.472	e_{Max}=0.462 a_{Max}=19.18	0.455	0.448	0.442	0.437	0.432	0.428	0.424	0.421

For the given parameter values b, k, s, t it was checked that the value e_{Max} (i.e. the smallest e_{max}) (the semi-axis length a_{Max}) occurs for the angle t=45° (only for s=3.5 e_{max}= 0,531124999999246 for t=40° and e_{max}=0,531189999999246 for t=45°) (Figure 3). Hence, the approximation of the offset curve $off(el(a, b); s)$ by the ellipse $el_1(a+s, b+s)$ will be satisfactory if we determine e_{Max} (a_{Max}) for t=45° and select $e \in [0, e_{Max}]$ ($a \in [b, a_{Max}]$).

Example 3. [RCI] We have b=17m, s=4m (2s=8m). We need to find a length a.

We choose $e \in [0, 0.462]$ ($a \in [b, 19.18]$) because 0.462=min{0.519, 0.462} (Table 2).

Test 2.B. The values e_{Max} fulfilling the condition (B) were determined for the given values s.

For the given parameter values b, k, s, t it was checked that the value e_{Max} (the semi-axis length a_{Max}) occurs for the angle $t=\pi/4$ (Figure 4). Hence, the approximation of the offset curve $off(el(a, b); s)$ by the ellipse $el_2(a\text{-}s, b\text{-}s)$ will be satisfactory if we determine e_{Max} (a_{Max}) for $t=\pi/4$ and select $e \in [0, e_{Max}]$ ($a \in [b, a_{Max}]$). Only offset curves (for which points P, P_2 and Q_2 belong to the same quadrant of the coordinate system) were included in the calculations.

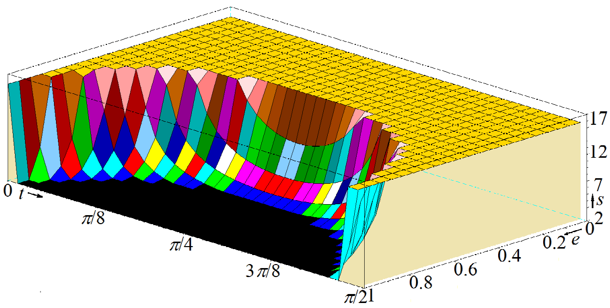
Table 3: Such eccentricity e_{Max} that for $e \in [0, e_{Max}]$ $d_{P_2Q_2}(e) \le 0.01$ (determined for $t=\pi/4$)

The width of the lane s[m]													
s=3.5	s=4	s=5	s=6	s=7	s=8	s=9	s=10	s=11	s=12	s=13	s=14	s=15	s=16
e_{Max}=0.488	e_{Max}=0.47 a_{Max}=19.26	0.44	0.414	0.391	0.371	0.351	0.332	0.314	0.295	0.275	0.252	0.226	0.19

Example 4. [RAR] We have b=17m, s=4m. We need to find a length a. $d_{P_1Q_1}(e) \le 0.01$ for $e \in [0,\ 0.519]$ ($a \in [b,\ 19.88]$) (cf. Table 2), $d_{P_2Q_2}(e) \le 0.01$ for $e \in [0,\ 0.47]$ ($a \in [b,\ 19.26]$) (cf. Table 3). Hence $e \in [0,\ 0.47]$ ($a \in [17,\ 19.26]$).

4.3 Parameter s_{Max}

Test 3. The ellipse offset curves $off(el(a,\ b);\ s)$ for parameter values s (s=3.5, 3.55, …, 16.5m), b=17m and $a = b/\sqrt{1-e^2}$ were analyzed. The eccentricity $e \in [0,\ 1)$ was taken (every 0.05). The angle $t \in [0,\ \pi/2)$ was taken (every $\pi/72$). For each angle t and for each value e such width of the lane s_{max} was determined that for $s \in [0.5,\ s_{max}]$ $d_{P_1Q_1}(s) \le 0.01$ ($d_{P_2Q_2}(s) \le 0.01$). The results are shown in Figures 5 and 6. The selected calculations are given in Tables 4 and 5.
Test 3.C. The values s_{Max} fulfilling the condition (C) were determined for the values $e \in [0,\ 1)$.



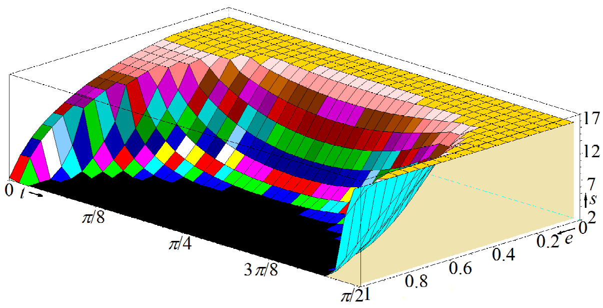


Figure 5: Such width s_{max} of the lane that for $s \in [0.5,\ s_{max}]$ $d_{P_1Q_1}(s) \le 0.01$ (the graph of the function $s_{max}=f_{(C)}(t,\ e)$)
Figure 6: Such width s_{max} of the lane that for $s \in [0.5,\ s_{max}]$ $d_{P_2Q_2}(s) \le 0.01$ (the graph of the function $s_{max}=f_{(D)}(t,\ e)$)

For the given parameter values b, k, e, t it was checked that for $e \in [0,\ 0.5]$ the values s_{max} occur for the angle $t=\pi/4$ (Figure 5). For $e \in [0.55,\ 1)$ and $t=\pi/4$ we have $s_{max}<3$ so the calculations were done for $t=\pi/4$. The approximation of the offset curve $off(el(a,\ b);\ s)$ by the ellipse $el_1(a+s, b+s)$ will be satisfactory if we determine s_{Max} for $t=\pi/4$ and select $s \in [0.5,\ s_{Max}]$ (Table 4).

Table 4: Such width s_{Max} of the lane that for $s \in [0.5,\ s_{Max}]$ $d_{P_1Q_1}(s) \le 0.01$ (determined for $t=\pi/4$) The [+] symbol means: for $s \in [3.5,\ 16.5]$ $d_{P_1Q_1}(s) \le 0.01$, the [-] symbol means: $d_{P_1Q_1}(s) > 0.01$ for $s \in [3.5,\ 16.5]$

The eccentricity e (the length of the semi-axis a[m])															
e=0 (a=17)	0.1 (17.08)	0.2 (17.35)	0.3 (17.82)	0.4 (18.55)	0.446 (19)	0.45 (19.04)	0.49 (19.5)	0.5 (19.63)	0.527 (20)	0.53 (20.5)	0.6 (21.25)	0.7 (23.80)	0.8 (28.33)	0.9 (39)	0.99 (120.5)
[+]	[+]	[+]	[+]	[+]	s_{Max}=10.18	9.66	5.59	4.95	3.67	3.54	[-]	[-]	[-]	[-]	[-]

Example 5. [RCI] We have b=17m, s=3.5m (2s=7m). We need to choose a length a from: a=19m, a=19,5m, a=20m.

For a=19m s_{Max}=10.18m, for a=19.5m s_{Max}=5.59m, for a=20m s_{Max}=3.67m (cf. Table 4). The approximation will be satisfactory if we choose a=19m because 2s=7m≤10.18m.

Test 3.D. The values s_{Max} fulfilling the condition (D) were determined for the values $e \in [0, 1)$.

For given parameter values b, k, e, t it was checked that the value s_{Max} (the smallest s_{max}) occurs for the angle $t=\pi/4$ (Figure 6). The approximation of the offset curve $off(el(a, b); s)$ by the ellipse $el_2(a\text{-}s, b\text{-}s)$ will be satisfactory if we determine s_{Max} for $t=\pi/4$ and select $s \in [0.5, s_{Max}]$.

Table 5: Such width of the lane s_{Max} that for $s \in [0.5, s_{Max}]$ $d_{P_2Q_2}(s) \le 0.01$ (determined for $t=\pi/4$)

The eccentricity e (the length of the semi-axis a[m])												
e=0 (a=17)	0.1 (17.08)	0.2 (17.35)	0.3 (17.82)	0.4 (18.55)	0.446 (19)	0.45 (19.04)	0.5 (19.63)	0.6 (21.25)	0.7 (23.80)	0.8 (28.33)	0.9 (39)	0.99 (120.51)
[+]	[+]	s_{Max}=15.77	11.73	6.62	4.77	4.65	3.21	[-]	[-]	[-]	[-]	[-]

5 Conclusions

The possibility of approximating the offset curves $off(el(a, b); s)$ by ellipses $el_1(a+s, b+s)$ and $el_2(a\text{-}s, b\text{-}s)$ for various values a, b, s was numerically analyzed. In section 4 we have checked for which parameter values b, s, e the approximation of the offset curve $off(el(a, b); s)$ by ellipse el_1 (el_2) is satisfactory. The ellipses which can be used to shape the turbo roundabouts (approved in [9]) were considered in particular. In section 3 the useful formulas for coordinates of points P_1, P_2, Q_1 and Q_2 (for any point P, i.e. for any angle t) were determined.

It was checked that we can get a greater flattening of a small elliptical two-lane turbo-roundabout (approved in [9]) when we begin to delineate it from the axis of the road.

The recommended (in [9]) width of the road on two-lane circular roundabout in the built-up area is 9.1-10m. The article contains a wider calculation for two reasons. The legal act (Journal of Laws No. 43) in force in Poland is not very precise in the field of the roundabouts designing and a designer can interpret it widely. “The guidelines” make it more precise, but they are not legally binding (cf. [10]). Secondly, the offset curves of the circle are circles and the ellipse offset curves are not ellipses.

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APROKSYMACJA KRZYWYCH OFFSETOWYCH ELIPSY W PROJEKTOWANIU ROND TURBINOWYCH

Eliptyczne ronda turbinowe są bardziej bezpieczną i efektywniejszą alternatywą dla standardowych rond wielopasmowych. W pracy przedstawiono wyniki analizy numerycznej dla problemu aproksymacji krzywych offsetowych $off(el(a, b); s)$ danej elipsy $el(a, b)$ o odległości s przez elipsę $el(a\text{-}s, b\text{-}s)$ oraz $el(a+s, b+s)$. Rozważamy elipsy, które mogą posłużyć do kształtowania rond turbinowych. Przetestowano, dla których wartości parametrów s (szerokość pasa ruchu) i e (mimośród) maksymalne odchylenie krzywej offsetowej elipsy od elipsy el_1 (el_2) nie przekracza dokładności tyczenia krzywych w terenie.