On Vladimir Markov type inequality in L^p norms on the interval [-1; 1]

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Abstract
We prove inequality $\|P^{(k)}\|_{L^p(-1,1)} \leq B_p \|T_n^{(k)}\|_{L^p(-1,1)} n^{\frac{2}{p}} \|P\|_{L^p(-1,1)}$; where B_p are constants independent of n = deg P with $1 \leq p \leq 2$, which is sharp in the case $k \geq 3$. A method presented in this note is based on a factorization of linear operator of k-th derivative throughout normed spaces of polynomial equipped with a Wiener type norm.

Key words: Vladimir Markov type inequality, Lp norms

1. Introduction.

Consider a normed space $(\mathcal{P}(\mathbb{C}),\ ||\cdot||)$ of polynomials of one variable equipped with a norm $||\cdot||$. The classical Vladiimir Markov inequality (cf. [8],[16]) is the following inequality for k-th derivative of a polynomial P of degree n

$$||P^{(k)}||_{[-1,1]} \leq T_n^{(k)}(1)||P||_{[-1,1]} = \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)}||P||_{[-1,1]}$$

$$\leq C^k\frac{n^{2k}}{k!}||P||_{[-1,1]} \tag{1.1}$$

with an absolute constant C. The meaning and its importance of the condition

$$||P^{(k)}|| \leq C^k\frac{(\deg P)^{km}}{(k!)^{m-1}}||P||$$

was discovered in [2]. Grzegorz Sroka in his paper [20], motivated by [1] has obtained the inequality

$$||P^{(k)}||_{L^p(-1,1)} \leq (C_k(p+1)k^2)^{1/p}||T_n^{(k)}||_{[-1,1]}||P||_{L^p(-1,1)},$$

where constants C_k are bounded and T_j are Chebyshev polynomials of the first kind (he showed that $C_k \leq \frac{12}{\sqrt[3]{2}}e^2$ for $k \geq 3$). As a corollary he derived the inequality of V. Markov's type

$$||P^{(k)}||_{L^p(-1,1)} \leq B_p^k\frac{1}{k!}n^{2k}||P||_{L^p(-1,1)}.$$

Let us note that looking for optimal bounds of a type

$$||P^{(k)}||_{p_1} \leq C_n(k,p_1,p_2)||P||_{p_2},\ n = \deg P$$

is a subject of many investigations (cf. [12], [19], [7]).

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2. Vladiimir Markov's type inequality

The main result of this note is the following improvement of [20] in the case $1 \leq p \leq 2$ (our arguments are quite different that used in [20]).

Theorem 2.1. *If* $1 \leq p \leq 2$*, then for any polynomial* P *of degree* $k \leq \deg P \leq n$ *we have inequalities*

$$||P^{(k)}||_p \leq B_p \max_{k \leq l \leq n} ||T_l^{(k)}||_p n^{\frac{2}{p}} ||P||_p = B_p ||T_n^{(k)}||_p n^{\frac{2}{p}} ||P||_p,$$

where

$$||P||_p = \left(\int_{-1}^{1} |P(x)|^p dx \right)^{1/p}, \quad B_p = (3e/\pi)^{1/p}(2p+2)^{1/p}.$$

Here T_n *are the classical Chebyshev's polynomials of the first kind.*

As a non-obvious corollary we obtain a version of V. Markov's property (it is a consequence of a fact that derivatives of T_n are related to other Jacobi polynomials). It was discussed in [4], mainly in the case $p = 2$.

Corollary 2.2. *For a fixed* $1 \leq p \leq 2$ *there exists a constant* C_p *such that for all* $k \geq 3$ *we have Vladimir Markov's type inequality*

$$||P^{(k)}||_p \leq C_p^k \frac{1}{k!} n^{2k} ||P||_p.$$

Remark 2.3. The corollary is also true in the case $k = 1, 2$ but can not be derived from Theorem 2.1 (cf. remarks in [1] related to Z. Ciesielski results from [10] who investigated the behavior of $||T_n'||_p$). In the case $k = 2$ and $1 < p \leq 2$ we can get a bound as in the corollary but with much worse constants.

In the proof of Theorem 2.1 we shall need the following important result. Let $x = \cos t = \frac{1}{2}(e^{it} + e^{-it})$ be element in the Wiener algebra of an absolute convergent trigonometric series $x = \sum_{n=-\infty}^{\infty} a_n e^{int}$ equipped with the l^1 Wiener norm $w_1(x) = \sum_{k=-\infty}^{\infty} |a_k|$.

Let $X_N = (\mathcal{P}_N, w_1(P(x)))$, $\mathcal{B}^N(x) = \{P \in \mathcal{P}_N : \ w_1(P(x)) \leq 1\}$, where $\mathcal{P}_N = \{P \in \mathcal{P}(\mathbb{C}) : \ \deg P \leq N\}$.

Proposition 2.4. *(Baran, Milówka, Ozorka* [5]*) For an arbitrary* $N \in \mathbb{N}$

$$\operatorname{extr}(\mathcal{B}^N(x)) = \{\eta_0 T_0, \ \ldots, \eta_N T_N : \ |\eta_j| = 1, \ j = 0, \ldots, N\}.$$

Here $\operatorname{extr}(\mathcal{B}^N(x))$ is the set of extreme points of the ball $\mathcal{B}^N(x)$ (cf. [17] for this very classical notion and its importance), T_j is j-th Chebyshev polynomial of the first kind.

Corollary 2.5. *If L is a linear operator on $\mathcal{P}_N$ then its norm between $(\mathcal{P}_N, w_1(P(x)))$ and $(\mathcal{P}_N, ||\cdot||_p)$ is equal to $\max_{0\le j\le N} ||LT_j||_p$ that means $||LP||_p \le \max_{0\le j\le N} ||LT_j||_p w_1(P(x))$ for $P \in \mathcal{P}_N$.*

Now we shall prove Theorem 2.1.

Proof. Let $P(\cos t) = \sum_{j=-n}^{n} c_j e^{ijt}$. We have, by the Hölder inequality,

$$\sum_{j=-n}^{n} |c_j| \le (2n+1)^{1/p} \left(\sum_{j=-n}^{n} |c_j|^q \right)^{1/q}$$

and applying the Hausdorff-Young inequality (c.f. [6],[22], which is a consequence of interpolation properties of spaces L^p), we shall get

$$\sum_{j=-n}^{n} |c_j| \le (2n+1)^{1/p} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p dt \right)^{1/p} .$$

Now we shall use the inequality like [13] (Lemma 3.1, p. 733)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p dt \le 2np(1+1/(np))^{np+1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p |\sin t| dt,$$

which gives

$$w_1(P(\cos t)) = \sum_{j=-n}^{n} |c_j|$$

$$\le (2n+1)^{1/p}(2p+1/n)^{1/p}(1+1/(np))^n n^{1/p} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(\cos t)|^p |\sin t| \right)^{1/p}$$

$$\le A_p n^{2/p} ||P||_p$$

with $A_p = (3e)^{1/p}(2p+1)^{1/p}$.

Now the crucial step is to apply Corollary 2.5, which gives the most important bound

$$||P^{(k)}||_p \le \max_{k\le l\le n} ||T_l^{(k)}||_p w_1(P(\cos t)).$$

Applying the bound for $w_1(P(\cos t))$ we finish the proof:

$$||P^{(k)}||_p \le \max_{k\le l\le n} ||T_l^{(k)}||_p B_p n^{2/p} ||P||_p$$

with $B_p = A_p/\pi^{1/p}$. □

Remark 2.6. Let us note the following surprising fact, which can be observed in the proof above: a bound of the form $w_1(P(\cos t)) \le B_p n^{2/p} ||P||_p$ is analogous to the bound (Nikolski type inequality) $||P||_{[-1,1]} \le C'_p n^{2/p} ||P||_p$, but $w_1(P(\cos t))$ can not be estimated by a product of a constant and $||P||_{[-1,1]}$.

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