OPTIMIZATION AND UNIFICATION OF TWO-STAGE HELICAL GEAR REDUCER

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Abstract

Gears were used in ancient times although in a primitive form. A turning point in the rapid development of gears was in the late nineteenth and early twentieth century. In this time the invention of high-performance tools, high-resistant materials and the creation of sufficient, accurate methods of heat and chemical treatment took place. Further improvement by optimizing the transmission structure, and the unification of its components is of great practical and economic importance. These problems are presented in this contribution. In particular, this paper deals with the problem of optimization and unification of two stage helical gear reducer with axial and opposite input and output shafts. The basic goal of this study is to unify the case because the case is one of the most expensive transmission elements. In connection with that, it is possible to put into the same case different sets of wheels to obtain predetermined different total gear ratios. An optimization procedure is very useful for a choice of the set of gear wheels. The total mass of gear wheels is assumed as an objective function. Equality and inequality restriction conditions are determined with respect to the standard recommendations and the literature. The center distance between the shafts is obtained through the optimization procedure imposed on the stage of the reducer which is subjected to the maximum load. The center distance after rounding is used next as a parameter in all variants of the reducer. The general formulation of the problem is presented for the considered engineering example. The numerical calculations are carried out for the specific data.

Keywords: *transmission, gear reducer, helical gear, optimization, unification of gears*

1. Introduction

A helical gear reducer belongs to a group of mechanical transmissions, whose main function is to transfer energy while changing its kinematic and dynamic parameters. The reducer is an ordered set of components and subsystems such as gears, shafts, bearings, seals, lubrication subsystems, housing, etc. Direct and indirect interrelations which occur in this complex system, are often difficult to present in a formal description. However, it is possible an objective description of the specific design features of the gear, which consists of geometric, material and dynamic features. The final design of the gear reducer thus determines the relationship between the all above mentioned features in such a way as to obtain the best structure under given conditions which is optimal in view of the criterion of optimization and restrictions imposed.

Currently, the general trend observed is to reduce the external dimensions of the gearbox, because it brings significant benefits. They include lower weight (reducing material consumption and the associated lower costs while maintaining similar materials and technology), lower inertia of rotating masses (reduction of dynamic loads), improving the operational indicators of machines (which include gears), easier installation, service and repair.

Gear design should also take into account the standard criteria, resulting mainly in reducing the number of expensive tools for cutting teeth (among other things, the standarization of the involute parameters and modules). Unification of rolling bearings, shafts and gears, using their interchangeability, as well as the unification of housings that allows to obtain in the same body different total gear ratio is now widely used. An example may be the solution of one of the leading manufacturers of gears, shown in Fig. 1.

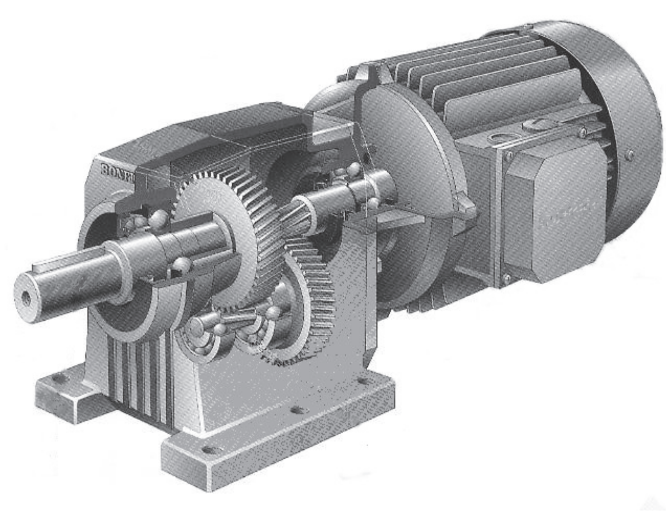

Fig. 1. Engineering example of the modern motoreducer

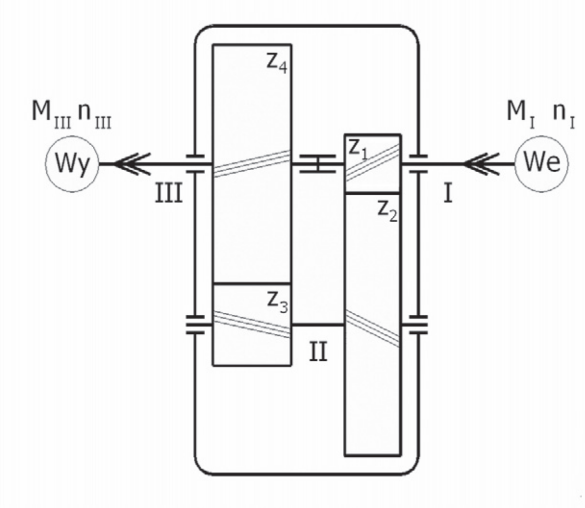

Fig. 2. Kinematical scheme of the considered gear reducer

This paper is dedicated to unification and optimization of two-stage helical gear reducer whose kinematic diagram is shown in Fig. 2. The main purpose of this work is the selection of certain design features of the gear reducer to meet the established criterion of optimization. The transmission should exhibit the highest degree of unification with respect to the number of gear wheels which must be used to obtain the assumed total gear ratios in the same housing.

2. Mathematical model of the gear reducer

The first step in the optimization process of each mechanism is to build a mathematical model of the structure [2, 4, 5]. Such a structure must be described using all design features. If a number (or a set of numbers) will be assigned to each of these features, the whole construction of the transmission may be presented clearly as a set of N numbers. As a result of such an approach the gear is reduced to a point $\boldsymbol{x}$ in N-dimensional Euclidean space

$$\boldsymbol{x} = \{x_1, \ldots, x_N\} \in \mathbf{R}^N . \tag{1}$$

The coordinates $x_1,\ldots,x_N$ (mathematical description of the gear construction) are usually divided into two groups. The first one is a set of design features specified in the design process. These features are called design variables. The second group is a set of design features imposed and determined, which can not be changed in the process of construction. These features are called parameters. Therefore, the considered structure can be represented by the design variables n and the parameters $P = N - n$.

In the case of multi-stage gears with helical cylindrical wheels sets, the design variables are typically the number of stages, gear ratios of the individual stage, modules, numbers of teeth, profile shift coefficients, helix angles, facewidths, center distance and sometimes properties of gear materials. The other quantities (describing gear reducers) are specified as parameters, which include the external dimensions, geometrical characteristics of the tooth profile, loads, ranges of durability and reliability, safety factors, etc. Some design features listed in the group of the design variables, are treated in some cases as parameters to be determined at an early stage of the optimization procedure. For example, the number of stages of the gear and the gear ratios of individual stages is determined mainly on the basis of recommendations from the literature [3], depending on the total gear ratios.

The engineering design variables cannot be unlimited. Structural conditions produce the limitations of their values. These restrictions are classified into two groups:

- Equality constraints called functional constraints resulting from physical or geometric relationships between design variables and parameters,
- Inequality constraints called side constraints which can be represented in the form $\varphi_i(x_1,...,x_n) \geq 0$, $i = 1,...,q$.

The equality constraints reduce the order of the problem by reducing the number of independent design variables. The side constraints may result, for example, from the smallest number of teeth due to tooth undercutting at the base or can be related to strength properties of the materials used to manufacture the gears. The design variable constraints are often referred to as side constraints and define what is commonly called feasible design space $\mathbf{\Phi} \in \mathbf{R}^n$, where $\mathbf{R}^n$ is n-dimensional Euclidean space.

The feasible design space $\mathbf{\Phi}$ includes a lot of technically reasonable structures, from which the best should be chosen, optimal with respect to the adopted criterion. To make this possible, it is necessary to establish an objective function $Q = Q(x_1,...,x_n)$ in the feasible design space. It is in general a function of the design variables usually defining the mass or the efficiency of the transmission.

3. Formulation of the gear optimization task

The task of gear design optimization is to find a system of design variables, i.e. the point $\boldsymbol{x} = \{x_1,...,x_n\}$ in the feasible design space $\mathbf{\Phi}$ for which the objective function Q reaches the extremum.

In order to simplify the above task, the gear is treated as a stationary object, i.e. the optimum state of it is assumed to be independent of time. It is understood that both the design variables and constraints in values are deterministic and the relationships between them are non-linear. In complex optimization problems, which include the optimization of the gear, the decomposition method (multi-level) is often used. It is based on the assumption that the original task may be replaced by another task consisting of a number of interrelated tasks of optimization of smaller dimension.

In the present paper, it is predetermined that the considered reducer is designed as shown in Fig. 2. Moreover, in the same gear housing the ratios of values: 7.96, 10.0, 12.6, 15.8, 19.9 and 25.1 must be obtained according to the Renard geometric series with quotient $\sqrt[10]{10} = 1.259$. Based on the graph shown in Fig. 3 the ratios were divided into individual stages, so that they were also the number of above series [3]. With respect to Fig. 3 a pair of wheels with ratios $u_1 = 2.52$, 3.98 and 6.31 is applied on the first stage and a pair of wheels with ratios $u_2 = 3.16$ and 3.98 is applied on the second stage.

The total mass of gear wheels is assumed as an objective function using a simplified formula:

$$Q = M = \frac{\pi\rho}{4}\left[\left(d_{w1}^2 + d_{w2}^2\right)b_{wI} + \left(d_{w3}^2 + d_{w4}^2\right)b_{wII}\right], \tag{2}$$

where:

d_{w1}-d_{w4} – pitch diameters of the wheels,
b_{wI}, b_{wII} – working facewidths of the first and the second stage,
ρ – density of wheel materials.

The optimization task is based on finding the smallest mass Q, that is, on minimizing the objective function (2). The design variables are selected to be: z_1, z_3 – numbers of teeth of pinions, m_{n1}, m_{n2} – normal modules of pair wheels, x_{n1}, x_{n3} – normal addendum modification coefficients of pinions and b_{wI}, b_{wII} – working facewidths of the first stage and the second stage. Other design features are treated as parameters.

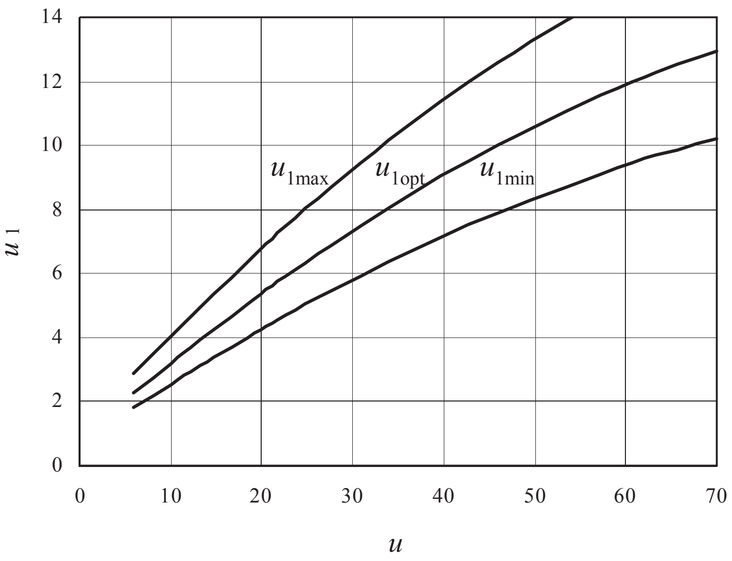

Fig. 3. Recommended by [3] separation of the two stage gear ratios: u – total ratio, u_I – ratio of the first stage

3.1. Equality constraints

The equality constraints generally reduce the order of the optimization task. One of them is the obvious relationship between the pitch diameters of the gears d_{w1}-d_{w4} and the center distance a_w:

$$d_{w1} + d_{w2} = d_{w3} + d_{w4} = 2a_w \,. \tag{3}$$

The centre distance a_w is determined by means of the optimization procedure applied to the maximum loaded stage of the gear and after rounding it is used as a parameter in all variants of the gear reducer.

To reduce the load on the second stage, the helix angle $\beta_I = 30^o$ is assumed for the wheels of the first stage. However, in order to eliminate the axial force on the shaft II (Fig. 2), the second pair of wheels should have the tooth helix angle β_{II} determined from the relation:

$$\beta_{II} = \mathrm{arctg}\left(\frac{1+u_1}{u_1+u}\,\mathrm{tg}\,\beta_I\right), \tag{4}$$

where:

u_1 – ratio of the first stage,
u – total ratio of the gear transmission.

The above angle is in the range of <7.6, 11.0> for the predetermined gear ratios. Due to the small difference in the extreme values of the angle β_{II}, the common value $\beta_{II} = 10°$ is adopted for all variants of the solution.

Furthermore, in the considered gear reducer normal addendum modification coefficients for both gear pairs must satisfy the formula:

$$S_{nI} = x_{n1} + x_{n2} = \frac{z_1+z_2}{2\mathrm{tg}\alpha_n}\left(\mathrm{inv}\,\alpha_{wtI} - \mathrm{inv}\,\alpha_{tI}\right), \quad S_{nII} = x_{n3} + x_{n4} = \frac{z_3+z_4}{2\mathrm{tg}\alpha_n}\left(\mathrm{inv}\,\alpha_{wtII} - \mathrm{inv}\,\alpha_{tII}\right), \tag{5}$$

where:

α_n – normal pressure angle (equal to 20°),
α_{tI}, α_{tII} – transverse pressure angles of the first and second gear stage,
α_{wtI}, α_{wtII} – pressure angles at the pitch cylinder of the first and second gear stage.

Moreover, it is assumed that the same determined material, namely surface-hardened steel C45, is adopted for the all gears.

3.2. Inequality constraints

Inequality constraints for the first pair of wheels z_1 and z_2 and the second pair of wheels z_3 and z_4 may be specified as technological, geometrical and strength constraints. They are presented in the paper only for the first pair of wheels, as for the other pair the inequality constraints are similar.

A solution is expected within the range of normal modules 1 – 10 mm, as for a typical general-purpose gears:

$$1 \le m_n \le 10\,, \tag{6}$$

and for the number of teeth on the first pinion:

$$10 \le z_1 \le 25\,. \tag{7}$$

Also, the working facewidth of wheels is reduced according to the following inequality:

$$b_{1\min} \le b_{wI} \le 1.0 \cdot d_1\,, \tag{8}$$

where:
$b_{1\min}$ – the minimum facewidth of the pinion, which allows for the measurement across n teeth.

The normal addendum modification coefficients x_{n1}, x_{n2} are calculated from equations (5_1) and (5_2), respectively. It should be noted that the two coefficients must be included in the ranges:

$$x_{\mathrm{nd1}} \le x_{n1} \le x_{\mathrm{ng1}}\,,\; x_{\mathrm{nd2}} \le x_{n2} \le x_{\mathrm{ng2}}\,, \tag{9}$$

where:

$$x_{\mathrm{nd}} = \max\left\{ \begin{array}{c} h^*_{\mathrm{an}} \dfrac{z_{\mathrm{v\,lim}} - z_v}{z_{\mathrm{v\,lim}}} \\ -h^*_{\mathrm{an}} \end{array} \right\},\; x_{\mathrm{ng}} = \min\left\{ \begin{array}{c} x_n^{\mathrm{sa}} \\ h^*_{\mathrm{an}} \end{array} \right\}, \tag{10}$$

$z_{\mathrm{v\,lim}}$ – minimum permissible virtual number of teeth without interference,
z_{v} – virtual number of teeth of a helical gear,
x_n^{sa} – normal addendum modification coefficient corresponding to the assumed minimum thickness at the top of the tooth.

Two strength constraints according to [1] can be reduced to a simplified condition on fatigue fracture of the pinion and gear tooth:

$$\sigma_{F1,2} = \frac{F_t K_F}{b_w m_n} Y_{\mathrm{FS1,2}} Y_\varepsilon Y_\beta \le \sigma_{\mathrm{FP}}\,, \tag{11}$$

where:
σ_F – tooth root stress,
σ_{FP} – permissible tooth root stress,
and a simplified condition on surface fatigue durability of the tooth (pitting) in the central point of meshing:

$$\sigma_H = Z_E Z_H Z_\varepsilon Z_\beta \sqrt{\frac{F_t K_H}{b_w d_1} \frac{u+1}{u}} \le \sigma_{\mathrm{HP}}\,, \tag{12}$$

where:
σ_H – calculated contact stress,
σ_{FP} – permissible contact stress,
F_t – nominal transverse tangential force at reference cylinder,
b_w – working facewidth.

The remaining coefficients in the expressions (11, 12) are in accordance with the standard [6] and are calculated according to this standard.

Similar restrictions apply to the second pair of gears (second stage).

4. Numerical example

The numerical calculations were carried out for a fixed gear parameters mentioned above (ratios, angles: α_n i β, etc.) supplemented by the following data:
$P = 15$ kW – input power of the gear reducer,
$n_1 = 1430$ rev/min – input rotational speed of the gear reducer,
$M_1 = 100.2$ Nm – input torque,
$a_w = 185$ mm – centre distance of the wheels,
$\sigma_{FP} = 324$ MPa – permissible tooth root stress,
$\sigma_{HP} = 990$ MPa – permissible contact stress,
basing on Excel spreadsheets using the standard optimization module called Solver.

A characteristic feature of the optimization is the appearance of a multitude of local minima that require constant monitoring and evaluation in order to determine which of them is global. An additional difficulty was the choice of the starting point due to the large number of constraints which were often violated.

The examples of the calculation results for the ratios of the total u are summarized in Tab. 1-3.

Tab. 1. Optimum solutions for the real ratios of total u = 8.0 and 10.2

Quantity	$u = u_1u_2 = 2.55\times3.14 = 8.0$				$u = u_1u_2 = 2.55\times4.0 = 10.2$			
	First stage		Second stage		First stage		Second stage	
	Pinion(1)	Gear(2)	Pinion(3)	Gear(4)	Pinion(1)	Gear(2)	Pinion(3)	Gear(4)
z	22	56	14	44	22	56	18	72
d mm	101.614	258.653	85.296	268.073	101.614	258.653	73.111	292.443
x_n	0.40324	0.9	0.81848	0.77538	0.40324	0.9	-0.20572	0.78530
m_n mm	4		6		4		4	
b_w mm	40		50(35)		40		75(50)	
β °	30		10		30		10	
M kg	46.1				63.6			

Tab. 2. Optimum solutions for the real ratios of total u = 12.6 and 16

Quantity	$u = u_1u_2 = 4\times3.14 = 12.6$				$u = u_1u_2 = 4\times4.0 = 10.2$			
	First stage		Second stage		First stage		Second stage	
	Pinion(1)	Gear(2)	Pinion(3)	Gear(4)	Pinion(1)	Gear(2)	Pinion(3)	Gear(4)
z	13	52	14	44	13	52	18	72
d mm	75.056	300.222	85.296	268.073	75.056	300.222	73.111	292.443
x_n	-0.30962	-0.19622	0.81848	0.77538	-0.30962	-0.19622	-0.20572	0.78530
m_n mm	5		6		5		4	
b_w mm	24		50(35)		24		75(61)	
β °	30		10		30		10	
M kg	39.9				57.4			

Tab. 3. Optimum solutions for the real ratios of total u = 19.7 and 25.1

Quantity	$u = u_1u_2 = 6.27\times3.14 = 19.7$				$u = u_1u_2 = 6.27\times4.0 = 25.1$			
	First stage		Second stage		First stage		Second stage	
	Pinion(1)	Gear(2)	Pinion(3)	Gear(4)	Pinion(1)	Gear(2)	Pinion(3)	Gear(4)
z	11	69	14	44	11	69	18	72
d mm	50.807	318.697	85.296	268.073	50.807	318.697	73.111	292.443
x_n	0	0.06221	0.81848	0.77538	0	0.06221	-0.20572	0.78530
m_n mm	5		6		4		4	
b_w mm	25		50		25		75	
β °	30		10		30		10	
M kg	42.1				59.6			

The above tables contain basic data relating to the geometry of the gears for different gear ratios. One of them is the working facewidth of the wheel pairs. The values given in parentheses means the facewidths obtained from the optimization process, which had to be increased due to the unification of the pairs of wheels.

The presented calculations show that the total mass of the wheels (*M*) in the same case, is between 40-64 kg.

5. Final remarks

The results of the calculations lead to the conclusion that only three tools with modules $m_n = 4$, 5 and 6 mm are enough to make all the wheels. These wheels can be assembled into six pairs to create the six total gear ratios in a wide range of u = 7.96, 10.0, 7.8, 8.0, 8.5, and 25.1 in the same case. The calculations carried out for the assumed data also suggest to create only five pairs of wheels instead of twelve, which introduces a high degree of unification in the presented numerical example.

In the process of optimization, strength restrictions were active only in two cases, i.e. for the pair of wheels on the second stage with ratio $u_2 = 3.98$ and $u_2 = 3.16$, when the gear ratio of the first stage was $u_1 = 6.31$. Other optimal solutions are located far away from the strength constraints (for the assumed material of the gear wheels). Technological and geometric constraints had a significant impact on these solutions. The facewidth of the wheels have been adapted to the requirements of the unification, so that at least the measurement by n – teeth on pinions was possible. The total mass of the reducer wheels differs by less than 33%.

The pitch diameters of large wheels are in the range of 258.6-318.7 mm, wherein the extreme diameters are in sets (258.653 mm and 292.443 mm), and (318,697 mm and 268,073 mm). This allows for a simple oil bath lubrication in the gearbox.

In order to further unification of gear components for wheel sets presented in the paper, it is possible to use common supported shafts on ball bearings, which allows to avoid having to comply design of suitable systems of tension of angular contact ball bearings or tapered roller bearings.

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